Pro gradu -tutkielma
Teoreettisen fysiikan suuntautumisvaihtoehto

# Technicolor

Matti Antola

Ohjaaja: Professori Keijo Kajantie
Tarkastajat: Professori Keijo Kajantie
Kimmo Tuominen

Joulukuu 2007

HELSINGIN YLIOPISTO
FYSIKAALISTEN TIETEIDEN LAITOS
PL 64 (Gustaf Hällströmin katu 2)
00014 Helsingin yliopisto

# Acknowledgments

I would like to thank Ville Suur-Uski, Olli Taanila, and Samu Kurki for many physics-related discussions during my studies and related to this thesis. Also I would like to thank my friends and family for the support they have given me.

# Contents

# Introduction

The Standard Model of particle physics has been an exceptionally successful theory. Thirty-five years after its conception there has been no measurement to positively prove it wrong. Yet, pending experimental evidence, there is one dubiety: the Higgs sector. The theoretical problems associated with this sector together with the precedent of quantum chromodynamics naturally motivate the alternate concept of technicolor. In this thesis I introduce the concept of technicolor, in particular showing that walking technicolor models are not excluded if we assume that there is some separate dynamics to account for the masses of the top and bottom quarks.

Ever since Einstein introduced space-time symmetries, the idea of symmetries have been at the heart of theoretical physics. In quantum field theory, there is an important class of symmetries called internal symmetries. The first example of a symmetry of this type was the isospin symmetry of nucleons, which is based on a simple idea: what are the consequences of the fact that the strong nuclear force does not know if it is dealing with a proton or a neutron? Such questions can be answered in the framework of quantum field theory.

Ordinary matter is observed to be made of fermionic spin-1⁄2 particles. Forces are mediated between charged particles by spin-1 gauge bosons corresponding to the charge. Of the fundamental particles, leptons and quarks belong to a first category, and weak gauge bosons, gluons, and the photon belong to a second. In the Standard Model, the currently accepted quantum field theory describing nature, the strong and electroweak interactions are placed in the $\mathrm{SU}(3)_{\mathrm{strong}} \otimes \mathrm{SU}(2)_{\mathrm{L}} \otimes \mathrm{U}(1)_{\mathrm{Y}}$ local group. Locality means that, in our description of quantum field theory, we suppose that the group of transformations under which the Lagrangian is invariant need not be constant in space, i.e. the fields transform independently at each space-time point. This naturally leads to us to adopt the concept of a gauge field, the quanta of which are gauge bosons of the interaction. All known interactions are now described in terms of local symmetries. Thus it is also natural to classify theories according to their local symmetries.

However, because of the different $\mathrm{SU}(2)_{\mathrm{L}}$ charges of right- and left handed fields, gauge invariance precludes fermion masses. Additionally, gauge bosons are by construction massless, but we know that the weak interaction is weak, which implies mediation by massive gauge bosons. Thus the electroweak symmetry must be broken. Within the Standard Model, electroweak symmetry breaking was achieved by adding a scalar $\mathrm{SU}(2)_{\mathrm{L}}$ doublet with Yukawa couplings to fermions and vev-developing potential. This scalar is unlike any other fundamental particle; indeed, it has the very special role of generating all masses. After spontaneous symmetry breaking, all observed masses are accounted for in the Standard Model.

There are very few experimental facts that the Standard Model cannot explain. Ignoring gravity, the first observation of physics beyond the Standard Model is the need for neutrino masses. This is based on the observation of neutrino oscillations. Secondly, the obvious baryon-antibaryon asymmetry in the universe could be better explained by a new source of CP-violation. Also, the dark matter and cosmological constant energy components of the universe do not have a microphysical explanation. Lastly, the great range of masses of assumed elementary particles, from the top quark to the neutrino, remain unexplained by the Standard Model.

The most relevant of these observations is the last one, but frankly there is no experimental evidence to support the technicolor theory. The issues that have surfaced concern theoretical aspects of the Higgs mechanism. Although none of these problems are relevant in any practical calculations, many theorists find the following issues problematic in the Higgs sector:

1. The fine-tuning of parameters
2. Triviality
3. Flavor physics

The first two problems can be completely solved by introducing technicolor. It is a dynamical theory of electroweak symmetry breaking modeled on quantum chromodynamics. However, technicolor itself has no mechanism to generate ordinary fermion masses. For that, we have to introduce extended technicolor, a non-trivial theory of flavor.

These theories cause their own set of problems, which are not theoretical, but phenomenological. These are:

1. Discrepancies with precision electroweak measurements
2. Flavor-Changing Neutral Currents
3. A low top-quark mass

The primary solutions to these problems is to assume that technicolor is distinctly unlike quantum chromodynamics. This scenario, where the coupling constant evolves slowly across a large energy scale, is called walking technicolor.

In the first chapter I discuss quantum chromodynamics, and especially its low-energy dynamics. Then I shall review the problems in the Standard Model in relation to the scalar particle. In the third chapter, I will show that by replacing the scalar sector with a new theory that is like a scaled version of quantum chromodynamics, the most pressing problems can be solved. Finally, I will discuss the generation of fermion masses through extended technicolor interactions and the walking technicolor scenario.

The reader interested in an introduction to spontaneous symmetry breaking should read the appendix.

I use the following acronyms:
QCD: quantum chromodynamics
SSB: spontaneous symmetry sreaking
EWSB: electro-weak symmetry breaking
SM: standard model
TC: technicolor
ETC: extended technicolor

# 1
# QCD and the Chiral Symmetry

Because QCD exhibits confinement and chiral symmetry breaking, its low-energy effective theory is difficult to find analytically. However, this chapter should make it feasible that in the limit of vanishing quark masses, the low energy dynamics are described by the linear sigma-model. It is thus inferred that a QCD-like theory exhibits spontaneous chiral symmetry breaking. In the real world, quarks have non-vanishing masses. Then the Goldstone bosons of the spontaneously broken symmetry are light pseudo-Goldstone bosons, and are identified as the pions.

A perturbative analysis tells that the QCD coupling constant will grow infinite at small energies. This property is called asymptotic freedom, and it implies that there is a natural scale associated with QCD. The natural scale is defined as that at which the coupling grows to order one. For QCD this is $\Lambda_{QCD} \sim 200$ MeV.

The chiral symmetry of QCD is an example of an explicitly and dynamically broken symmetry. If a symmetry is explicitly broken, it means that the symmetry is not fully realized in the theory. In the case of QCD, the Lagrangian masses of quarks are this kind of small symmetry breaking term. However, the three lightest quarks have masses that are small compared to the natural scale of QCD, $\Lambda_{QCD}$, and as a result, QCD sees these quarks as approximately massless, and even after SSB there is an remaining approximate vector flavor symmetry.

The chiral condensate is a low energy phenomenon causing spontaneous chiral symmetry breaking. It is in analogy to the ordinary superconductor, where electrons form Cooper pairs through their interactions with lattice vibrations. The symmetry breaking is a fully dynamical effect in the sense that it is not visible in the Lagrangian, or in any order of perturbation theory. This is because at the limit of exact chiral symmetry, the Lagrangian is decomposed into left and right-handed parts, that cannot couple perturbatively.

First I introduce QCD and its flavor symmetries. Then I show that at low energies, there is an approximate vector flavor symmetry, by showing that the particle content of the theory can be placed in representations of the symmetry. Then I introduce the sigma-models and make it plausible that the dynamics are similar to that of the low-energy regime of QCD. Finally I consider the possibility that the cause of spontaneous symmetry breakdown is a dynamical chiral condensate. I derive the Goldberger-Treiman relation and first order corrections to vanishing pion masses, and consider chiral perturbation theory.

## 1.1 QCD Introduced

The chiral symmetry of QCD is a global flavor symmetry. Flavor means that the symmetry acts in the space of quark flavors - a flavor is up, down, charm etc. In a different parametrization, the left and right symmetries are called the vector and axial symmetries. It can be said that the left and right symmetries are broken to their vectorial subgroup at low energies - actually the axial group is broken, but the left and right symmetries are more useful concepts in other contexts. Because of this unbroken vectorial subgroup, we can apply symmetry analysis to classify hadrons.

As a side note, leptons can also be assigned a flavor. However, this is much less useful than with quarks, because the lepton flavor symmetry is gauged: it corresponds to the $\mathrm{SU}(2)_{\mathrm{L}}$ group. Thus gauge bosons change lepton flavor, i.e. $e \to \nu_e + W$. In the last chapter of this thesis, while studying extended technicolor, we will also gauge the quark flavor symmetry. This will result in exactly similar flavor-changing currents, which are not observed for quarks.

### QCD Lagrangian

Consider QCD with two massless flavors of quarks. The Lagrangian for these massless quarks is very simply

$$\mathcal{L} = \bar{Q} i \gamma_\mu D^\mu Q \ , \tag{1.1}$$

where $Q = (u, d)$. The covariant derivative acts on the color degrees of freedom, and the local $\mathrm{SU(3)_{color}}$ symmetry does not matter for this discussion. We immediately note a global U(2) symmetry, because $\not{D} = \gamma_\mu D^\mu$ does not see the flavor space. However, there is more to it. Under $Q \to U(\gamma_5)Q$, we see

$$\bar{Q} i \gamma_\mu D^\mu Q \quad \to \quad \bar{Q} i U^\dagger(-\gamma_5) U(-\gamma_5) D^\mu \gamma_\mu Q = \bar{Q} i \gamma_\mu D^\mu Q$$

i.e. the term is invariant. Thus there is also another global U(2) symmetry, which is the axial symmetry.

### Vector and Axial Symmetries

So, because U(2) = SU(2)$\otimes$U(1), we can actually find four flavor symmetries, two SU(2) and two U(1):

(i) $Q \to e^{-i\alpha} Q$
(ii) $Q \to e^{-i\beta\gamma_5} Q$
(iii) $Q \to e^{-i\alpha\cdot\tau/2} Q$
(iv) $Q \to e^{-i\beta\cdot(\tau/2)\gamma_5} Q$

(i) is related to quark number conservation. (ii) is the axial quark number transformation. (iii) is called the flavor isospin transformation, and (iv) is the axial flavor isospin transformation. The first symmetry is unbroken and commutes with the others, so we may dismiss it. The conserved quantity is actually quark number. The second symmetry does not correspond to any known symmetry. We will find that the quark masses do not break this symmetry much, so there should be a corresponding pseudo-Goldstone, but there isn't. This is named the U(1) problem, and is an example of quantum corrections anomalously breaking a symmetry. The third and fourth symmetries are the ones we are currently interested in.

### Notation

Here and later, $\tau$ should be understood as a three-vector, i.e. $\tau = (\tau_1, \tau_2, \tau_3)$. I will not always distinguish vectors from scalars, but summation over suppressed indices is marked by a dot: $v_i u^i = v \cdot u$.

### Left and Right Symmetries

We may rewrite the Lagrangian (1.1) with the projection operators

$$P_{R,L} \quad = \quad \frac{1}{2}(1 \pm \gamma_5) \ .$$

First note that, since $\gamma_5$ is hermitian and anticommutes with $\gamma_\mu$,

$$\bar{q} P_L = q^\dagger \gamma_0 P_L = q^\dagger P_R \gamma_0 = (P_R q)^\dagger \gamma_0 = \bar{q}_R$$

then

$$\begin{aligned}
\mathcal{L} &= \bar{Q} i \not{D} (P_L + P_R)^2 Q = \bar{Q} i \not{D} (P_L P_L + P_R P_R + P_L P_R + P_R P_L) Q \\
&= Q P_R i \not{D} P_L Q + \bar{Q} P_L i \not{D} P_R Q \\
&= \bar{Q}_L i \not{D} Q_L + \bar{Q}_R i \not{D} Q_R \ ,
\end{aligned}$$

where we also used $P_L P_R = 0 = P_R P_L$.

We can now identify an $\mathrm{SU}(2)_\mathrm{L}\otimes\mathrm{SU}(2)_\mathrm{R}$ symmetry where the left and right fields respectively transform with $Q_{L,R} \to e^{-i\alpha\cdot\tau/2}Q_{L,R}$. Then it is clear that the mass term

$$\bar{Q}mQ = \bar{Q}_L m Q_R + \bar{Q}_R m Q_L \ ,$$

breaks this symmetry. Here $m$ is a diagonal matrix. From the first form we can see it specifically breaks the axial symmetry.

We can write $\mathrm{SU}(2)_\mathrm{L}\otimes\mathrm{SU}(2)_\mathrm{R}$=$\mathrm{SU}(2)_\mathrm{V}\otimes\mathrm{SU}(2)_\mathrm{A}$, where $\mathrm{V} = \mathrm{R} + \mathrm{L}$ and $\mathrm{A} = \mathrm{R} - \mathrm{L}$. We can easily calculate, from (A.3), the conserved vector and axial currents:

$$\begin{aligned} v^\mu &= i\bar{Q}\gamma^\mu\tau Q \\ a^\mu &= i\bar{Q}\gamma^\mu\gamma_5\tau Q \end{aligned}$$

and defining the corresponding charge operators $V = \int \mathrm{d}^3x v^0(x)$, $A = \int \mathrm{d}^3x a^0$, we may find (using $\left\{Q^\dagger(x), Q(x')\right\} = i\delta(x - x')$) that

$$\begin{aligned} [V, Q] &= i\int \mathrm{d}^3x \left[Q_i^\dagger(x)\tau_{ij}Q_j(x), Q_k(x')\right] \\ &= \tau Q \\ [A, Q] &= \tau Q \ . \end{aligned}$$

These commutators would be diagonal in the $\mathrm{L} - \mathrm{R}$ basis.

It is believed that the axial symmetry is broken by a chiral condensate, while the vector symmetry remains unbroken at low energies. Thus particles can be classified according to how they transform under the vector group.

### Low Energy Particle Content

QCD is believed to be a confining theory, which means that we can only observe color singlets. A physically intuitive explanation is that the color force does not decrease in strength with distance, so that if we try to pull colored particles apart, it becomes energetically cheaper to produce a new colorless particle.

These particles binded by the strong force are now called hadrons. In early times, when the energy reach of particle reactors grew, particle physicists detected an increasing number of seemingly fundamental particles interacting through this strong force. However, a pattern was found in these particles, and it led to the realization that hadrons consist of a number of quarks, in such a way that the hadron is colorless. Now we understand that three quarks, the up, down, and charmed, are relatively light, and thus the QCD Lagrangian has an *approximate* $\mathrm{SU}(3)_\mathrm{V}$ symmetry. The corresponding $\mathrm{SU}(2)_\mathrm{V}$ symmetry is even better realized.

So there are two symmetries in play. First, any physical particle must be a color singlet. This means that we cannot see the objects transforming in the fundamental representation of flavor $\mathrm{SU}(2)_\mathrm{V}$ or $\mathrm{SU}(3)_\mathrm{V}$. However, any higher number of quarks can essentially combine in a colorless way; for example, mesons consist of two quarks in color + anticolor combinations.

These particles are grouped by how they transform under the flavor group. It is useful to compare to addition of spin. Two electrons can either align or anti-align their spin. If they are aligned, a rotation will certainly alter the wave function differently than if they were anti-aligned. Just as particles are labeled by their spin, it is useful to label hadrons by how they transform under the global flavor group - or equivalently, by what their quark content is.

### SU(2)

There are two quarks transforming in the fundamental representation **2** of $\mathrm{SU}(2)_\mathrm{V}$,

$$Q = \begin{pmatrix} u \\ d \end{pmatrix} \ .$$

To see what combinations of these fields we can produce, it is useful to use Young's tableaux [4].

Mesons are quark-antiquark bound states. Antiquarks transform under the conjugate representation $\bar{\mathbf{2}}$. Thus under $\mathrm{SU}(2)_\mathrm{V}$, mesons can be found in the following representations:

$$\mathbf{2} \times \bar{\mathbf{2}} = \mathbf{2} \times \mathbf{2} = \mathbf{1} + \mathbf{3}$$

Note that spin has not been considered here. This means that there is a spin 1 and a spin 0 triplet with the same quark combinations but the quarks having different spin alignment. Thus there is one triplet which are the $\mathbf{0}^-$ pseudoscalar pions, and another triplet which are the $\mathbf{1}^-$ pseudovector $\rho$-mesons. Technicolor will produce similar particles. The quark contents are

$$\pi^+, \rho^+ \sim \bar{d}u \quad \pi^0, \rho^0 \sim (\bar{u}u - \bar{d}d)/\sqrt{2} \quad \pi^-, \rho^- \sim \bar{u}d \quad .$$

There are also $\mathrm{SU}(2)_\mathrm{V}$ singlets. These include the pseudoscalar $\eta \sim \bar{u}u + \bar{d}d$.

The lightest baryons, which contain three quarks, are grouped as

$$\mathbf{2} \times \mathbf{2} \times \mathbf{2} = \mathbf{2} \times (\mathbf{1} + \mathbf{3}) = \mathbf{2} + \mathbf{2} + \mathbf{4} \; .$$

One of the **2**'s contains the proton and the neutron.

### SU(3): The Eightfold-Way

Adding the third quark, strange, which is also quite light, we have $\mathrm{SU}(3)_\mathrm{V}$ flavor symmetry. Now quarks and antiquarks transform under **3** or $\bar{\mathbf{3}}$ respectively. Thus from

$$\mathbf{3} \times \bar{\mathbf{3}} = \mathbf{1} + \mathbf{8}$$

we find that there are mesons transforming in $\mathrm{SU}(3)_\mathrm{V}$ singlets and octets. The singlet can be either the $\mathbf{0}^-$ or $\mathbf{1}^-$ and is $\bar{u}u + \bar{d}d + \bar{s}s$. The $\mathbf{0}^-$ octets are

$$\pi^+ \sim \bar{d}u \quad \pi^0 \sim (\bar{u}u - \bar{d}d)/\sqrt{2} \quad \pi^- \sim \bar{u}d$$

$$K^+ \sim \bar{s}u \quad K^0 \sim \bar{s}d \quad \bar{K^0} \sim \bar{d}s \quad K^- \sim \bar{u}s$$

$$\eta^0 \sim (\bar{u}u + \bar{d}d - 2\bar{s}s)/\sqrt{6}$$

For the baryons, which are quark-quark-quark states, we need the additional multiplications

$$\begin{aligned} \mathbf{3} \times \mathbf{3} &= \bar{\mathbf{3}} + \mathbf{6} \\ \mathbf{3} \times \mathbf{6} &= \mathbf{8} + \mathbf{10} \end{aligned}$$

which together imply

$$\mathbf{3} \times \mathbf{3} \times \mathbf{3} = \mathbf{3} \times (\bar{\mathbf{3}} + \mathbf{6}) = \mathbf{1} + \mathbf{8} + \mathbf{8} + \mathbf{10} \; .$$

Baryons are not as important for technicolor, as the number of technicolors is usually at least 4, which means that technibaryons have 4 or more techniquarks and are thus much heavier. They will probably decay quickly to heavy fermions, if it is allowed as in the ETC scenario.

The most important low-energy particles for QCD are, of course, the pions and the nucleons. These particles transform under the $\mathrm{SU}(2)_\mathrm{V}$ because the strange quark is already much heavier than either up or down. Approximate quark hard Lagrangian masses are given below, in MeV:

$$\begin{array}{lll} u \sim 3 & c \sim 1 \times 10^3 & t \sim 2 \times 10^6 \\ d \sim 7 & s \sim 1 \times 10^2 & b \sim 4 \times 10^3 \end{array} \tag{1.2}$$

## 1.2 The Sigma-models

Now examine another model that contains a $\mathrm{SU}(2)_L \otimes \mathrm{SU}(2)_R$ symmetry [4]. It is this model that we hope encompasses the features of a low-energy effective Lagrangian for QCD.

### Linear Sigma Model

Consider the following Lagrangian:

$$\mathcal{L}_\sigma = \frac{1}{2}\left[(\partial_\mu \sigma)^2 + (\partial_\mu \pi)^2\right] + \bar{N} i \gamma^\mu \partial_\mu N + g\bar{N}(\sigma + i\tau \cdot \pi\gamma_5)N - V(\sigma^2 + \pi^2) ,$$

where $\pi = (\pi_1, \pi_2, \pi_3)$ is a pion pseudoscalar isotriplet, $N = (p, n)$ an isodoublet of fermion nucleons, $\sigma$ is an isoscalar, and the potential is

$$V(\phi^2) = -\frac{\mu^2}{2}\phi^2 + \frac{\lambda}{4}\phi^4 .$$

### Conserved Currents

The Lagrangian is invariant under the transformations

$$\begin{aligned} \sigma &\rightarrow \sigma' = \sigma \\ \pi &\rightarrow \pi' = \pi + \alpha \times \pi \\ N &\rightarrow N' = N + i\alpha \cdot \frac{\tau}{2} N . \end{aligned}$$

In the current case (A.1) generalizes to $\delta\psi_r = \alpha_i T^{(i)}_{rs}\psi_s$ for $i = (1, 2, 3)$, because $\alpha$ is a 3-vector and $r = (\sigma, \pi_i, N_i)$. We have

$$\begin{aligned} T^{(i)}_{\sigma\sigma} &= 0 \\ T^{(i)}_{\pi_r\pi_s} &= \epsilon_{irs} \\ T^{(i)}_{NN} &= i\frac{\tau_i}{2} , \end{aligned}$$

and from these, remembering that $N$ is a fermion field, the conserved currents (up to a multiplicative constant)

$$v^{(i)}_\mu = \epsilon_{irs}\pi_r\partial_\mu\pi_s + \bar{N}\gamma_\mu\frac{\tau_i}{2}N$$

and the corresponding generators $V_a = \int \mathrm{d}^3x v^{(a)}_0(x)$. Similarly the Lagrangian is invariant under another set of transformations,

$$\begin{aligned} \sigma &\rightarrow \sigma' = \sigma + \beta \cdot \pi \\ \pi &\rightarrow \pi' = \pi - \beta\sigma \\ N &\rightarrow N' = N + i\beta \cdot \frac{\tau}{2}\gamma_5 N \end{aligned}$$

which gives the conserved currents

$$a^{(i)}_\mu = \pi_i\partial_\mu\sigma - \sigma\partial_\mu\pi_i + \bar{N}\gamma_\mu\frac{\tau_i}{2}\gamma_5 N$$

and generators $A_a$. The second set of currents are called axial currents. Using the commutation relations

$$\begin{aligned} [\phi(x,t), \partial_0\phi(y,t)] &= i\delta^3(x-y) \\ \left\{\psi(x,t), \psi^\dagger(y,t)\right\} &= i\delta^3(x-y) \end{aligned}$$

for $\psi = \{n, p\}$ and $\phi = \{\pi, \sigma\}$, we get, after careful calculation:

$$\begin{aligned} [V_a, V_b] &= i\epsilon_{abc}V_c \\ [V_a, A_b] &= i\epsilon_{abc}A_c \\ [A_a, A_b] &= i\epsilon_{abc}V_c \end{aligned}$$

If we name $Q_L = \frac{1}{2}(V - A)$ and $Q_R = \frac{1}{2}(V + A)$ we see that

$$\begin{aligned}
\left[Q_L^a, Q_L^b\right] &= i\epsilon^{abc}Q_L^c \\
\left[Q_L, Q_R\right] &= 0 \\
\left[Q_R^a, Q_R^b\right] &= i\epsilon^{abc}Q_R^c
\end{aligned}$$

We have thus recovered two independent transformation groups, with generators obeying the SU(2) Lie algebra: name them SU$(2)_L$ and SU$(2)_R$. How can we understand what the transformations are? Counting the number of generators helps. As we have seen, both SU(2)'s have 3 independent generators. SO(4)would have 6 in analogue to the Lorentz group SO$(1,3)$, where we have 3 corresponding to space-space rotations and 3 to space-time rotations. Thus it might seem plausible that SU(2)$\otimes$SU(2)$\simeq$SO(4). Mathematically this means that there would be a one-to-one correspondence between elements of both groups. Physically, we should have a rotation of 4 combinations of fields that is a symmetry of the Lagrangian. In the mesonic sector we easily see that $\sigma^2 + \pi^2$ is conserved if we simply rotate the fields among themselves. It is trivial to check that this happens under both the vector and axial groups, at least to the first order.

This symmetry is spontaneously broken if $\mu^2 > 0$. The potential has SO(4) symmetry and thus vacuum configurations correspond to points on a sphere. By convention we choose that $\langle 0|\pi|0\rangle = 0$ and $\langle 0|\sigma|0\rangle = v$.

### Goldstone Theorem

At this point, it is useful to contemplate on the Goldstone theorem, proved in the appendix for the general case. From our current definitions, it is easy to find $[A^a, \pi^b(0)] = i\delta^{ab}\sigma$ and $[A^a, \sigma(0)] = -i\pi^a$. Thus when we assume that $\sigma$ develops a vev, we can write $v = \langle 0|\sigma|0\rangle = -i\frac{1}{3}\langle 0|[A^a, \pi^a(0)]|0\rangle$, which we can insert to (A.5). After inserting a full set of states we have

$$3iv = \sum_n (2\pi)^3\delta^3(p_n)\left\{\left\langle 0|a_a^0(0)|n\right\rangle\langle n|\pi_a(0)|0\rangle e^{-iE_nt} - \langle 0|\pi_a(0)|n\rangle\left\langle n|a_a^0(0)|0\right\rangle e^{iE_nt}\right\} .$$

It is shown that $v$ must be constant in time; thus $|n\rangle$ is massless. Now we have to conclude that $|n\rangle = |\pi_a(p_n)\rangle$. With the usual 1-particle normalization

$$\sum_n |n\rangle\langle n| = \int\frac{\mathrm{d}^3p}{(2\pi)^3 2p_0}|p\rangle\langle p|$$

we have

$$3iv = \sum_a\int\frac{d^3p}{2p_0}\delta^3(p)\left\{\left\langle 0|a_a^0(0)|\pi_a(p)\right\rangle\langle\pi_a(p)|\pi_a(0)|0\rangle - \langle 0|\pi_a(0)|\pi_a(p)\rangle\left\langle\pi_a(p)|a_a^0(0)|0\right\rangle\right\} .$$

The equation is satisfied for $\left\langle 0|a_a^0(0)|\pi_b(p)\right\rangle = ivp_0\delta_{ab}$ if we normalize $\langle\pi_a(p)|\pi_a(0)|0\rangle = 1$. Then, by relativistic covariance, we must have

$$\langle 0|a_a^\mu(0)|\pi_b(p)\rangle = ivp^\mu\delta_{ab} \tag{1.3}$$

and the matrix element of the current divergence is

$$\langle 0|\partial_\mu a_a^\mu(0)|\pi_b(p)\rangle = vm_\pi^2\delta_{ab} .$$

Current conservation implies either $v = 0$ or $m_\pi = 0$; this is the Goldstone theorem in a concrete form.

**Broken Phase**

Now proceed by defining $\sigma = \varsigma + v$ to find the Lagrangian

$$\begin{aligned}\mathcal{L}_{\sigma,\text{broken}} &= \frac{1}{2}\left[(\partial_\mu\varsigma)^2 + (\partial_\mu\pi)^2\right] + \bar{N}i\gamma^\mu\partial_\mu N - gv\bar{N}N + g\bar{N}(\varsigma + i\tau\cdot\pi\gamma_5)N \\ &\quad +\mu^2\varsigma^2 - \lambda v\varsigma(\varsigma^2+\pi^2) - \frac{1}{4}\lambda(\varsigma^2+\pi^2)^2 + \text{const},\end{aligned}$$

where $v^2 = \mu^2/\lambda$. Notice how the masses of the particle spectrum change radically: the fermion field becomes massive, as does the isoscalar, but the three pions remain massless. Since the $\mathrm{SU}(2)_\mathrm{V}$ symmetry did not affect $\sigma$, we can deduce the Lagrangian is still invariant under the symmetry. The 'vacuum choice' respected the $\mathrm{SU}(2)_\mathrm{V}$ symmetry, i.e. the vacuum is invariant under the corresponding transformations. However, the vacuum is not invariant under $\mathrm{SU}(2)_\mathrm{A}$, and the symmetry is sponteneously broken. But although the symmetry is not apparent in the Lagrangian, the axial current is conserved: $\partial_\mu a_a^\mu = 0$. This is why a spontaneously broken symmetry is sometimes called a hidden symmetry. For an explicitly broken symmetry, $\partial_\mu J_i^\mu \neq 0$.

Thus we may postulate that this Lagrangian is some kind of hadronic version of the QCD Lagrangian with two massless quarks. However, the current model does not include the effect of explicit small quark masses. The effect will be incorporated in the nonlinear sigma model studied next.

**Nonlinear Sigma Model**

Consider adding of a term $c\sigma$ to $\mathcal{L}_\sigma$. The vectorial current will be conserved, because it did not affect $\sigma$, but what happens to the axial current? Denoting the new current by $J$, we see from (A.2) that

$$\begin{aligned}\delta(\mathcal{L}_\sigma + c\sigma) &= c\delta\sigma + 0 = \beta_i\partial_\mu J_i^\mu = \beta_i c\pi_i \\ \partial_\mu J_i^\mu &= c\pi_i\end{aligned}$$

The additional term will affect the minimum of the potential as well. With the same convention, $\langle 0|\sigma|0\rangle = v$, we find the minimum is at $v(\mu^2 - \lambda v^2) = c$. We can see that if c vanishes, we get the preceding $v$. Now if we define $\sigma = \varsigma + v$, we find a term $\pi^2(\mu^2 - \lambda v^2)/2$, which is a mass term for the pion:

$$(\mu^2 - \lambda v^2) = \frac{c}{v} \equiv m_\pi^2 \neq 0$$

Looking at the particle content and interactions, this might be a realistic model of low-energy hadron dynamics. The $\pi N$ interaction is the 'one pion exchange' potential postulated by Yukawa, with incorporation of isospin. The pions and nucleons can have realistic masses. The only immediate problem is that $\sigma$ is a $0^+$ meson, but no such particle is found in nature.

**Chiral Circle**

It turns out we can eliminate the particle if we assume its mass is very large. This means assuming $\sqrt{2}\mu$ is large while $v$ is finite. The parameter $c$ is by definition small, so from $\mu^2 = m_\pi^2 + \lambda v^2$ we conclude that in this case, $\lambda$ must be very large. The potential has the form

$$V(\phi^2) = -\frac{\lambda v^2}{2}\phi^2 + \frac{\lambda}{4}\phi^4 - m_\pi^2\phi^2\ .$$

As $\lambda$ goes to infinity, near the value $\phi^2 = v^2$, $-\lambda v^4/2$ becomes a very negative constant. This encourages us to use a nonperturbative semiclassical limit of enforcing

$$\sigma^2 + \pi^2 = v^2$$

because other values will be greatly suppressed. We are still assuming $\sigma$ to develop the vev; but think of $\sigma = \sqrt{v^2 - \pi^2}$ as the dynamical variable, thus effectively removing one degree of freedom. Since $m_\pi$ is not altered, the pions remain massive. Now the derivative term for $\sigma$ becomes

$$\frac{1}{2}[\partial_\mu \sqrt{v^2 - \pi^2}]^2 = \frac{1}{2}(-\pi \cdot \partial_\mu \pi)^2 (v^2 - \pi^2)^{-1} = \frac{1}{2v^2}(\pi \cdot \partial_\mu \pi)^2 + \mathcal{O}(\pi^6) \ .$$

The first term makes a simple prediction on pion-pion scattering. The final full Lagrangian is

$$\mathcal{L}_{\text{final}} = \frac{1}{2}(\partial_\mu \pi)^2 - \frac{1}{2}m_\pi^2 \pi^2 + \bar{N}\left(i\partial\!\!\!/ + ig\tau \cdot \pi\gamma_5 - gv\right) N + \frac{1}{2v^2}(\pi \cdot \partial_\mu \pi)^2 + \ldots$$

which is a somewhat complete model of the low-energy dynamics of QCD. Here we have introduced the Feynman slash notation, $a\!\!\!/ = a_\mu \gamma^\mu$.

We note that all terms are no longer renormalizable. This should not bother us as long as the involved momentums are below the scale $v$. To see this, consider for example the pion-pion scattering term. It is essentially the $\phi^4$ interaction term with $\lambda/4! \sim k^2/2v^2$, which is small if $k^2 \ll v^2$.

Of course, in the current model we have no explanation for the cause of SSB. It is surprising that given the complexity of the problem, we do not even need to know: the low-energy behavior can be discerned by simply assuming the symmetry is broken. For example, had we simply written a derivative term of dimension six we would have found the pion-pion scattering term, since $v^2$ is the relevant dimensionful parameter. This is the key to a general modern approach on low-energy dynamics, called effective field theory.

## 1.3 Dynamically Broken Chiral Symmetry

Since the Sigma model seems to apply to the low-energy regime of QCD, we may ask why is the chiral symmetry of QCD broken. Because there are no scalar operators, the symmetry must be broken by a composite operator. It is believed that the reason is a quark condensate in the vacuum state.

Thus, let us continue on the case of QCD with massless up and down quarks [1]. Superconductivity occurs because a small electron-electron attraction leads to a condensate of electron pairs in the ground state of the system. In QCD, there are very strong attractive interactions, and now if we assume the masses are zero, it is very economical to create bound quark-antiquark pairs. Thus the QCD vacuum could well include a quark condensate. But such a condensate, parametrized by the expectation value of a scalar operator

$$\left\langle \bar{Q}Q \right\rangle = \left\langle \bar{Q}_L Q_R + \bar{Q}_R Q_L \right\rangle \neq 0$$

is *not* invariant under the axial SU(2). Thus the $\mathrm{SU}(2)_\mathrm{A}$ is spontaneously broken and we have three related Goldstone bosons, that can be created by the axial current. Inspired by (1.3), write

$$\langle 0 \,| a_i^\mu(x) |\, \pi_j(p) \rangle = -ip^\mu f_\pi \delta_{ij} e^{-ipx} \ . \tag{1.4}$$

The particle $\pi$ has the quantum numbers of the axial current $a$ and should be massless at the limit of vanishing quark masses. We are factually provided with three such pseudoscalar particles, the pions. Then the constant appearing here, $f_\pi$, can be measured from the rate of $\pi^+$ decay through the weak channel, and is found $f_\pi = 93$ MeV. It is thus called the pion decay constant. Contracting (1.4) with $p_\mu$ we find the pion is massless, as is expected in this approximation.

Note that this condensate mixes quarks of different helicities. This means that the quarks are allowed an effective mass! However, the situation is complicated by confinement. As we cannot see quarks directly, what this really means is that quarks look heavier than their Lagrangian mass as they move inside a bound state. We will get an estimate of this difference from considering the pion mass.

One might consider more elaborate condensates, such as $\left\langle \bar{Q}_R Q_R + \bar{Q}_L Q_L \right\rangle$, which would break the ordinary vectorial isospin. However, the Vafa-Witten theorem states that the flavor $\mathrm{SU}(N)_\mathrm{V}$

cannot be broken spontaneously. Additionally, the small Lagrangian masses of quarks force the usual condensate, exactly like a small external magnetic field forces the direction of magnetization in ferromagnets.

### Goldberger-Treiman Relation

Let us now derive the Goldberger-Treiman relation. The result will relate the coupling constant of a $\pi N$ term with the mass of the nucleon and $f_\pi$. Begin by considering the matrix element

$$\langle N(p')\,|a^\mu(q)|\,N(p)\rangle = \bar{u}(p')\left[\gamma^\mu F_1(q^2) + \frac{i\sigma^{\mu\nu}q_\nu}{2m}F_2(q^2) + q^\mu F_3(q^2)\right]\gamma_5\sigma u(p)\ ,$$

where we have written the most general possible axial structure. Remember, the $a$ and $\sigma$ have one hidden index. At $q = 0$ the two latter terms in the brackets disappear, but $F_1(0)$ need not be zero. Conventionally we define $F_1(0) = g_A$. Now we use the conservation of the axial current when quark masses vanish:

$$\begin{aligned} 0 &= \bar{u}(p')\left[\not{q}F_1(q^2) + q^2F_3(q^2)\right]\gamma_5\sigma u(p) \\ &= \bar{u}(p')\left[(\not{p}' - \not{p})F_1(q^2) + q^2F_3(q^2)\right]\gamma_5\sigma u(p) \\ &= \bar{u}(p')\left[2m_N F_1(q^2) + q^2F_3(q^2)\right]\gamma_5\sigma u(p)\ , \end{aligned}$$

where on the last line we have to use the equations of motion, $\bar{u}(p')(\not{p}' - m_N) = 0$, and $(\not{p} - m_N)u(p) = 0$, commuting $\not{p}$ over the $\gamma_5$. Thus we must have

$$g_A = -\lim_{q\to 0}\frac{q^2F_3(q^2)}{2m_N}\ ,$$

which would vanish unless $F_3$ contains a pole at $q = 0$. Such a pole would correspond to a massless particle, one just like the pion!

For calculating $F_3$, we must assume some form for the pion-nucleon term. We could take the interaction term from our Sigma-model, but conventionally there is an additional factor of 2:

$$\mathcal{L}_{\pi NN} = 2ig_{\pi NN}\bar{N}\gamma_5(\pi\cdot\tau)N\ .$$

From (1.4) we see that we must have a term of the form $f_\pi\partial_\mu\pi \in a_\mu$. The contribution of this term is of the form $q_\mu f_\pi\,\langle N\,|\pi(q)|\,N\rangle$, which gives

$$(-iq^\mu f_\pi)(\frac{i}{q^2})(-2g_{\pi NN}\bar{u}\tau\gamma_5 u)\ .$$

The first part is just $\langle 0\,|a_i^\mu(x)|\,\pi_j(p)\rangle$, then we have the pion propagator, and the last part is the pion-NN vertex. Thus this contribution to $F_3$ dominates at small q:

$$F_3 = -\frac{2f_\pi g_{\pi NN}}{q^2}\ ,$$

and so we have

$$g_A = \frac{f_\pi g_{\pi NN}}{m_N}$$

which is the famous Goldberger-Treiman relation. It is satisfied experimentally to a 5% accuracy. This greatly increases confidense that our interpretation of the SSB pattern is correct.

### Pion Masses

We might still be interested in finding the pion masses when the quark masses are not exactly zero. We can find how they are related quite simply. With nonzero quark masses, we have an additional term $\mathcal{L}_m = \bar{Q}mQ$, where $m = \mathrm{diag}(m_u, m_d)$. Then we can calculate

$$\begin{aligned} \partial_\mu a^\mu &= \delta\mathcal{L} + \delta(\bar{Q}mQ) \\ &\propto \bar{Q}\gamma_5\{m,\sigma\}Q\ . \end{aligned}$$

Thus from (1.4) we find

$$\begin{aligned}\langle 0\left|\partial_\mu a_i^\mu(x)\right|\pi_j(p)\rangle &\propto \delta_{ij}p^2 f_\pi \propto \langle 0\left|\bar{Q}\left\{m,\sigma_i\right\}\gamma_5 Q\right|\pi_j(p)\rangle \\ &\propto tr\left[\left\{m,\sigma_i\right\}\sigma_j\right] \\ &\propto \delta_{ij}(m_u+m_d)\ ,\end{aligned}$$

where the second line follows from index structure and the last line from the fact that the part of $m$ proportional to the identity is $m_u + m_d$. Thus we may write a proportionality constant and get

$$m_\pi^2 = (m_u + m_d)\frac{M^2}{f_\pi}\ .$$

Note that $m_\pi^2$ is proportional to the explicit symmetry breaking term ($m_u$). Compare this to the form of the nonlinear sigma-model, where we had $m_\pi^2 = \frac{c}{v}$.

The parameter $M$ can be estimated to be around 400 MeV. Thus for $m_\pi = 140$ MeV one needs $m_u + m_d \sim 10$ MeV. This would indeed imply a small perturbation compared to the strong interaction scale $\Lambda_{QCD} \sim 200$ MeV. Thus if the quarks acquire a dynamical mass from strong interactions, the dynamical mass will be much larger than their explicit symmetry-breaking Lagrangian mass. All in all, it would seem plausible that the QCD chiral symmetry is an approximate symmetry which is dynamically broken by a quark condensate.

As long as the effective mass is much larger than the explicit mass, the quarks inside hadrons will behave in the first approximation as they were massless, and thus degenerate in mass. Thus the isospin symmetry does not constitute anything fundamental about the relations of the quark masses; it is simply the statement that for those quarks with masses much less than 200 MeV, QCD does not see the masses. In other words, that the chiral flavor symmetry has consequences is not related to the fact that the lightest quarks have small mass differences, but to the fact that they are light compared to the natural scale of QCD.

### Chiral Perturbation Theory

We now look at the low-energy problem from another angle, namely that of effective field theory. We will need the result when discussing the technicolor scale.

At low energies, only the lightest particles matter. Thus we can exclude all quarks except for the up and down quarks. We know, since there are no parity doublets, that the flavor SU(2) is somehow broken. We want to find, from a top-down perspective, what the low-energy degrees of freedom of QCD are. Of course these are the pions, which are interpreted Goldstone bosons of the broken SU(2) chiral flavor symmetry, and nucleons. We want to find the Lagrangian of the pions.

Under SSB, the formula is to replace the transformations that a field is invariant under with a dynamic field representing that transformation, and then calculate the Lagrangian around the vev. For example, with the complex scalar field considered in the appendix, we have invariance under $\varphi \to e^{i\alpha}\varphi$. Thus we write in the Lagrangian

$$\varphi \to (\langle\varphi\rangle + \phi)\mathrm{exp}i\frac{\theta(x)}{v}\ ,$$

and take the few first terms of the expansion of the exponent.

In QCD, the SU(2) symmetry is $Q \to e^{-i\alpha\cdot\tau/2 - i\gamma_5\beta\cdot\tau/2}Q$. The situation is different in the respect that we cannot simply insert something of this form to the Lagrangian, because the quarks are confined. The implicit solution has been effective field theory. Related to QCD, the method is often called chiral perturbation theory. The general scheme is to write the most general Lagrangian with the symmetries of the parent field theory.

Now consider the condensate. Assign a definite value, say

$$\langle\bar{Q}Q\rangle = -v^3\ .$$

Note that this assignment is a diagonal matrix equation in flavor space. Now the basic idea is the same as with the scalar field. There, the Goldstone bosons replaced transformations around the specific chosen vacuum in the direction of invariance. Here, Goldstone bosons should represent fluctuations around this choice:

$$\langle \bar{Q}(x)Q(x) \rangle = -v^3 U(x) \ .$$

We require $\det U = 1$ because the $\mathrm{U}(1)_{\mathrm{A}}$ symmetry is badly broken by the anomaly. Otherwise we would only require $U$ to be unitary. In that case, we would have an additional particle appearing in the low-energy particle spectrum. Thus, write

$$U = \exp\left(i\pi \cdot \tau / f_\pi\right) \ . \tag{1.5}$$

The fields $\pi$ are, of course, identified with the pions. Now the low-energy Lagrangian should be the most general possible built from this field. However, the lowest-dimensional nontrivial term built out of these objects is

$$\mathcal{L} = -\frac{1}{4} f_\pi^2 \mathrm{tr}\left[ \left(\partial^\mu U\right)^\dagger \left(\partial_\mu U\right) \right] \ .$$

This will give the same result for the pion-pion scattering as the previous section. Continuing on these lines, and taking all the terms that are allowed by symmetries, we can arrive at the low-energy effective Lagrangian. This would, for example, include fourth derivatives of $U$, and also other particles, e.g. the proton and neutron. This is indeed a lengthy task, and so we have simply assumed the Sigma-model Lagrangian and tried to justify it corresponds with the physical setting.

# 2

# Probing Higgs' Problems

In the SM the Higgs field causes EWSB. It is a vev-developing scalar field with Yukawa couplings to the fermions. The electroweak gauge part SU(2)⊗U(1) is then broken to the U(1) of electromagnetism, and the $W$ and $Z$ bosons acquire masses along with the the fermions.

Unnaturalness is the most pressing problem in the Higgs solution. Based on renormalization groups, it is possible to argument that every theory has a physical cutoff. Taking this viewpoint can actually explain why we consider only renormalizable field theories. Then the fine-tuning problem is, that in this scheme of renormalization, we need unnatural fine-tuning of parameters.

This problem also manifests in another way. We can find that the scalar particle is related to the partition function of statistical systems. In this picture we see that requiring $m \sim 0$ is the same as requiring $T \sim T_c$ in a statistical system to a great accuracy. In a lab, an experimentalist can carefully tune the temperature. But in the case of the universe, one would have to resort to the anthropic principle, which should always be avoided since there is no counterargument!

Trivialness is the exact statement that a weakly interacting scalar particle is likely always related to an effective theory, because there is a definite maximum range of validity of the theory. If we either push the cutoff scale high or examine the coupling at low energies, we find that the coupling constant vanishes, i.e. the theory becomes the trivial free field theory. Since the coupling is related to the physical Higgs particle mass, we can get a mass limit for the Higgs particle from this analysis.

The last problem is just the statement that we would like a dynamic explanation for every energy scale. The energy scale of QCD comes beautifully and naturally without great sensitivity to the cutoff scale, fully by dynamical effects. Compared to this, the simple Higgs picture seems lacking. This is called the hierarchy problem. Also, although the Yukawa couplings do allow for the observed masses, there is no explanation for the various masses. This is called the flavor problem.

There is not much more to the hierarchy and flavor problems, and they seem to be overlooked by many theoretical physicists. Although it is obviously true that we would enjoy a better theory, since there is no concrete problem, especially the flavor problem gets a philosophical connotation. In addition its difficulty makes it unpopular to study. Extended technicolor attacks the problem directly.

I begin this chapter by deriving tree-level relations for masses in the SM. Then I will show why the cutoff scheme of renormalization is always relevant, and derive the relation between the bare and dressed mass of the scalar particle. This relation will explicate what is meant by unnaturalness: that the dressed mass is not proportional to the bare mass, but has an additive contribution proportional to the cutoff scale. I will also derive the trivialness property by considering the scalar contribution to an effective potential. Then I find mass bounds on the Higgs particle.

## 2.1 The Standard Model Masses

In technicolor, our ultimate wish is to explain mass by dynamical means. Within the SM, the Goldstone bosons from the SSB of the electroweak symmetry give a mass to the gauge bosons proportional to the vev. Similarly the Yukawa terms $g\phi\bar{\psi}\psi$ give mass terms proportional to the vev. Our disposition is to write the most general possible couplings, and then use the assumed symmetries to simplify the Lagrangian.

**Gauge Boson Masses**

The scalar part of the SM Lagrangian reads

$$\mathcal{L}_{\text{scalar}} = |D_\mu \phi|^2 - V(\phi) \ .$$

where

$$D_\mu = \partial_\mu + ig\left(\frac{\sigma}{2} \cdot W_\mu\right) + ig'\frac{Y}{2}B_\mu \ .$$

The couplings are $g$ for $\mathrm{SU}(2)_\mathrm{L}$ and $g'$ for $\mathrm{U}(1)$, and $Y$ is the hypercharge operator. The generators are fully determined by the group algebra in the correct representation. We get the gauge boson masses from evaluating this at the scalar field vev, which can be arbitrarily chosen as

$$\phi = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v \end{pmatrix} \ .$$

Then, expanding the covariant derivative with the $\mathrm{SU}(2)$ and $\mathrm{U}(1)$ gauge boson fields, we get

$$\begin{aligned} \Delta\mathcal{L} &= \frac{1}{2}\begin{pmatrix} 0 & v \end{pmatrix}\left(\frac{1}{2}g\sigma \cdot W_\mu + \frac{1}{2}g'B_\mu\right)^2 \begin{pmatrix} 0 \\ v \end{pmatrix} \\ &= \frac{1}{2}v^2\left[\frac{1}{2}g^2 W_+ W_- + \frac{1}{4}\left(-gW_3 + g'B\right)^2\right] \\ &= \left(\frac{gv}{2}\right)^2 W_+ W_- + \frac{1}{2}\left(\frac{\sqrt{g^2+g'^2}v}{2}\right)^2 Z_\mu^2 + 0 \cdot A^2 \end{aligned}$$

where

$$W_\pm = \tfrac{1}{\sqrt{2}}\left(W_1 \mp iW_2\right) \quad Z = \tfrac{gW_3 - g'B}{\sqrt{g^2+g'^2}} \quad A = \tfrac{g'W_3 + gB}{\sqrt{g^2+g'^2}}$$

The field $A$ is chosen to be orthogonal to $Z$ in the sense that

$$\begin{pmatrix} A \\ Z \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} B \\ W_3 \end{pmatrix}$$

where $\tan\theta = g'/g$. We thus have the standard mass term of a charged particle, $m_\mathrm{W} = gv/2$ and $m_\mathrm{Z} = \sqrt{g^2+g'^2}v/2$. The photon remains massless. We also have a nontrivial relation,

$$\frac{m_W}{m_Z} = \frac{g}{(g^2+g'^2)^{1/2}} = \cos\theta_W \ .$$

**Lepton Masses**

Since there are no right handed neutrinos, the only possible Yukawa couplings of the Higgs doublet to leptons is [1]

$$\mathcal{L}_{hll} = -\lambda^{ij}\bar{E}_L^i \cdot \phi e_R^j + \text{h.c.} \ ,$$

where $E$ is the lepton doublet, and $\phi$ the Higgs doublet. $i, j$ are generation indices. This term is of the form

$$-\begin{pmatrix} \bar{\nu}_L \lambda e_R \\ \bar{e}_R \lambda e_r \end{pmatrix} \cdot \phi$$

This coupling can be diagonalized as follows. $\lambda\lambda^\dagger$ is unitary. Thus it is a unitary rotation of its eigenvalues, write $\lambda\lambda^\dagger = UD^2U^\dagger = UDW^\dagger WD^\dagger U^\dagger$. Thus $\lambda = UDW^\dagger$ where $W$ and $U$ are unitary and $D$ diagonal.

Subsequently do the change of variables $E^i_L \to U^{ij} E^j_L$, $e^i_R \to W^{ij} e^j_R$. Since we are transforming both elements of the doublet similarly, the covariant derivative terms with the SU(2) matrices are unaltered. Thus we get terms of the form

$$\mathcal{L}_{hll} = -\lambda^i \bar{E}^i_L \cdot \phi e^i_R + \text{h.c.} \; .$$

which are exactly electron, muon and tau mass terms of the form $v\lambda_i$.

### Quark Masses

The most general SU(2) singlet gauge-invariant renormalizable coupling with zero net hypercharge is

$$\mathcal{L}_{hqq} = -\lambda^{ij}_d \bar{Q}^i_L \cdot \phi d^j_R - \lambda^{ij}_u \epsilon^{ab} \bar{Q}^i_{La} \phi^\dagger_b u^j_R + h.c. \; .$$

We want to diagonalize these to-be mass terms like we did with the lepton masses. Do exactly the same trick, i.e. write $\lambda_d = U_d D_d W^\dagger_d$ and $\lambda_u = U_u D_u W^\dagger_u$. Then the change of variables $d_R \to W_d \cdot d_R$ and $u_R \to W_u \cdot u_R$ will eliminate the $W$'s from the theory, as the right handed fields are singlets under SU(2) and thus the interaction terms are of the form $\bar{q}_R i \not{D} q_R$. Now we have terms of the form ($\epsilon \equiv [\epsilon_{ij}]$)

$$-\begin{pmatrix} \bar{u}_L U_d D_d d_R \\ \bar{d}_L U_d D_d d_R \end{pmatrix} \cdot \phi - \begin{pmatrix} \bar{u}_L U_d D_d u_R \\ \bar{d}_L U_d D_d u_R \end{pmatrix} \cdot \epsilon \cdot \phi^\dagger$$

and we see that we can rotate $U$ away from only one component. In the unitary gauge, we can choose the lower component of $\phi$ to develop the vev. In this case, clearly we can diagonalize the mass-matrix by choosing $d_L \to U_d \cdot d_L$ and $u_L \to U_u \cdot u_L$. Each mass-parameter thus has a corresponding coupling.

Since QCD is chiral-blind i.e. the $L$ and $R$ couple equivalently, this transformation will not affect the QCD Lagrangian. But the weak interaction terms of SU(2) are not chiral-blind. The current that couples quarks to the $W_-$ boson transforms as follows:

$$J^+_\mu = \frac{1}{\sqrt{2}} \bar{u}_L \cdot \gamma_\mu d_L \to \frac{1}{\sqrt{2}} \bar{u}_L \cdot (U^\dagger_u U_d) \cdot \gamma_\mu d_L \; .$$

This shows that there is mixing among the triplet $u^i_L$ and or the $d^i_L$, and this mixing is given by the unitary matrix $V = U^\dagger_u U_d$. A $3 \times 3$ unitary matrix has one overall complex phase. Since this phase will appear only when coupling to all three generations, it is a possible explanation for the weak CP-violation observed in nature. This could explain e.g. the lifetime difference of the $K_L$ and $K_S$.

One more observation deserves attention. When making the chiral transformations on the quarks, we assumed implicitly that the path integral measure would stay invariant. This is not strictly true. In fact, we induce a total derivative term that cannot be removed! This is because it induces a nontrivial topology to the vacuum. But such a term is also shown to come from instanton effects. This is known as the strong CP problem, and its usual solution involves a dynamical scalar degree of freedom called the axion; however, this particle has not been observed.

### Higgs' Mass

The Higgs sector's potential is

$$V_{Higgs} = -\frac{1}{2}\mu^2 \phi^* \phi + \frac{\lambda}{4!} (\phi^* \phi)^2 \; .$$

In the unitary gauge we can choose,

$$\left\langle \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} \right\rangle = \begin{pmatrix} 0 \\ v \end{pmatrix}$$

and writing $\phi^0 = v + h$ we get $m^2_h = \lambda v^2/2$. Because $\lambda$ is a free parameter, the SM does not predict the Higgs mass.

## 2.2 The Unnatural Higgs

Unnaturalness is by far the strongest argument against fundamental scalar field theories. We will go through the argument from two different perspectives.

### Analogy with Statistical Physics

For a quantum field theory with Lagrangian $\mathcal{L}$, the generating functional of correlation functions is

$$\mathcal{Z}[J] = \int \mathcal{D}\phi e^{iS[J]} ,$$

where $S[J] = \int \mathrm{d}^4x \, (\mathcal{L} + J\phi)$. The time variable integration in the exponent goes from $-T$ to $T$, with $T \to \infty(1 - i\epsilon)$. The infinitesimal $\epsilon$ gives the correct Feynman propagator. The Wick trick is to rotate the contour integral so that $T \to -i\infty$, i.e. $x_0 \to -ix_4$. Since the integration path is deformed through an area with no poles, it should not alter the integral. Then we get

$$\mathcal{Z}[J] = \int \mathcal{D}\phi e^{-S_E[J]} ,$$

where $S_E[J] = \int \mathrm{d}^4x_E \, (\mathcal{L}_E - J\phi)$. Take $\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4$. Then

$$\mathcal{L}_E = \frac{1}{2}(\partial_E^\mu\phi)^2 + \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4 \tag{2.1}$$

because there is an $-i$ from the measure of integration. But $\mathcal{L}_E$ looks very much like an energy; it has a large value for large amplitudes or gradients. Actually $\mathcal{Z}[J]$ is precisely the statistical partition function of a 4-dimensional macroscopic system, described approximately by treating fluctuations as a continuum field.

In Landau theory of phase transitions, the parameter $m^2 \sim T - T_c$. Thus our SM is in analogue to a 4-d statistical system that is just slightly below the critical temperature. But just like the Landau theory is a phenomenological theory, the Higgs picture provides no understanding of why $m^2$ would be so very small and negative. In fact, this argument requires that $m^2$ is fine-tuned to be very small, in analogue to fine-tuning $T \sim T_c$. The relevant question is, who is doing the fine-tuning?

A related fact is that in quantum field theory, the effective degrees of freedom are those particles with a mass much smaller than the energy scale in question. Thus we always want theories with massless or almost massless particles. Then, according to the previous argument, a scalar field theory can never be fundamental! Some particles, of course, are naturally massless, such as the chiral fermion, the gauge boson, and the Goldstone boson. These are thus the natural building blocks of quantum field theory. Of course, this paragraph begs to mention supersymmetry, in which the chiral symmetry of a superpartner fermion protects the small mass of the scalar, which then becomes natural.

### A Physical Cut-off

Renormalization is a procedure in which we write all physical quantities as a function of unphysical divergent quantities. These divergent quantities appear when we consider the perturbative interactions (loops). If the theory is renormalizable, physical quantities will stay finite in each order of perturbation theory, with only a finite number of different types of divergences. The divergent quantities can be calculated in many different ways, e.g. with a cutoff or by dimensional regularization.

The SM is a renormalizable theory. In fact, this requirement greatly reduces the amount of possible theories, and can thus be viewed as a simplifying principle. All this points to us that perhaps, in correct quantum fiel theories, there is no physical cutoff. In condensed matter physics, cutoffs appear at the scale at which the continuous description of space breaks down, so maybe this means our space truly is continuous. Here one can retreat to saying that since the interpretation does not affect physics, the question should not be asked.

However, Wilson showed that there is a way to explain why we require renormalizable theories. The bottom line is that a field theory at a scale $E$ compared to the same theory at the scale $\Lambda$ will have non-renormalizable dimension D operators modified by a factor of $(E/\Lambda)^{D-d}$, where $d$ is the dimension of space-time. Thus non-renormalizable operators have vanishing effects at lower energy-scales, and requiring renormalizable theories is equivalent to requiring theories that are valid to much higher energy scales. The relevant concept is that of the renormalization group.

Wilson's treatment is intriguing. We will take a scalar field theory with a cutoff, integrate away the high-momenta parts, and then transform the path integral back into the original form, except with the replacement $\mathcal{L} \to \mathcal{L}_{\text{eff}}$. Consider a real scalar field [1]. Define the path integral measure $\mathcal{D}\phi = \Pi_k \phi(k)$. Then, in Euclidean space, and setting $J = 0$

$$\mathcal{Z} = \int \mathcal{D}\phi e^{-S_E} \ .$$

Apply a cutoff $\Lambda$. This means $\phi(k) = 0$ for $k > \Lambda$. For concreteness, assume the previous $\mathcal{L}_E$ (2.1) and omit the subindex $E$.

Now we want to integrate over high-energy modes. Let $\phi \to \bar{\phi} + \hat{\phi}$, where $\bar{\phi}(k)$ vanishes for $k \geq b\Lambda$, and $\hat{\phi}(k)$ vanishes unless $k \in ]b\Lambda, \Lambda[$, $b < 1$. Then clearly $\mathcal{D}\phi \to \mathcal{D}\bar{\phi}\mathcal{D}\hat{\phi}$, and

$$\mathcal{L}[\phi] \to \mathcal{L}[\bar{\phi}] + \mathcal{L}[\hat{\phi}] + \lambda \left( \frac{1}{6}\bar{\phi}^3\hat{\phi} + \frac{1}{4}\bar{\phi}^2\hat{\phi}^2 + \frac{1}{6}\bar{\phi}\hat{\phi}^3 \right) \ .$$

The $\bar{\phi}\hat{\phi}$-term disappeared because different Fourier modes are orthogonal (remember the integration limits).

Next, perform the integral over $\hat{\phi}$. To do this, we first note that we are interested in the limit $m \ll \Lambda$ so that we can treat all other terms as perturbations except $\frac{1}{2}(\partial_\mu \hat{\phi})^2$. This gives $\hat{\phi}$ the trivial scalar propagator $\sim 1/k^2$. The other terms are regarded as perturbations, and we may use diagrammatic techniques. The calculation is not very relevant to the rest of this argument, so we just write the result:

$$\mathcal{L}[\phi] = \mathcal{L}[\bar{\phi}] + (\text{sum of connected diagrams}) \equiv \mathcal{L}_{\text{eff}}[\bar{\phi}] \ .$$

Of course, the RHS has diagrams with $\bar{\phi}$. Now comes the trick. Write

$$\begin{aligned} \mathcal{L}_{\text{eff}}[\bar{\phi}] &= \frac{1}{2}(1+\Delta Z)(\partial_\mu \bar{\phi})^2 + \frac{1}{2}(m^2 + \Delta m^2)\bar{\phi}^2 + \frac{1}{4!}(\lambda + \Delta\lambda)\bar{\phi}^4 \\ &\quad + \Delta C (\partial_\mu \bar{\phi})^2 + \Delta D \bar{\phi}^6 + \ldots \end{aligned}$$

If we rescale distances and momenta with

$$k' = \frac{k}{b} \quad , \quad x' = xb$$

The first term will be $\frac{1}{2}b^{2-d}(1+\Delta Z)(\partial_\mu \bar{\phi})^2$ because of the $\mathrm{d}^4 x$ and partial derivatives. This suggests we also scale

$$\phi' = \sqrt{b^{2-d}(1+\Delta Z)}\bar{\phi} \ .$$

Because the primed quantities are integrated over, we may rename them without the primes. Now we have written

$$\mathcal{Z} \approx \int \mathcal{D}\phi e^{-S_{\text{eff}}[\phi]}$$

where

$$\begin{aligned} S_{\text{eff}}[\phi] &= \int \mathrm{d}^4 x \left[ \frac{1}{2}(\partial_\mu \phi)^2 + \frac{1}{2}m'^2\phi^2 + \frac{\lambda'}{4!}\phi^4 \right. \\ &\quad \left. + C'(\partial_\mu \phi)^4 + D'\phi^6 \right] \end{aligned}$$

and

$$\begin{array}{rcl} m'^2 &=& (1+\Delta Z)^{-1}(m^2+\Delta m^2)b^{-2} \ , \\ \lambda' &=& (1+\Delta Z)^{-2}(\lambda+\Delta\lambda)b^{d-4} \ , \\ C' &=& (1+\Delta Z)^{-2}(C+\Delta C)b^{d} \ , \\ D' &=& (1+\Delta Z)^{-3}(D+\Delta D)b^{2d-6} \ , \end{array} \tag{2.2}$$

i.e. we have written a transformation of the Lagrangian. In our case $C$ and $D$ were zero, but they are now included for generality.

Now compare calculating a correlation function at low external momenta. We can use either formulation. With the original formulation, high-momentum fluctuations of the field are suddenly turned on as we compute loop diagrams. With our current $\mathcal{L}_{\text{eff}}$, however, these effects have been absorbed into constants of the Lagrangian, and no infinities appear.

If we keep $b \approx 1$, our perturbative treatment of the $\Delta$ quantities is valid as long as the coupling constants stay small. However, we will have to iterate the transformation many times to get to a lower energy scale. This limit, in which the transformation is continuous, is the basic idea of the renormalization group.

Look at (2.2) more closely. If $m^2 = \lambda = C = D = 0$, to zeroth order all primed quantities vanish as well. Thus $\mathcal{L}_0 = 1/2(\partial_\mu\phi)^2$ is a fixed point of the renormalization group transformation. If the parameters are very close to zero, we have the simple transformation laws

$$m'^2 = b^{-2}m^2 \ , \quad \lambda' = b^{d-4}\lambda \ , \quad C' = b^d C \ , \quad D' = b^{2d-6}D \quad .$$

Taking $d = 4$, this suggests that the coefficient of an operator with $N$ powers of $\phi$ and $M$ derivatives transforms as

$$C'_{N,M} = b^{N+M-4}C_{N,M} \ .$$

Actually, the coefficient is exactly what we would expect from naive dimensional analysis! The preceding operator at scale $\Lambda$ should have a coefficient $g\Lambda^{4-N-M}$. At scale $E$, $gE^{4-N-M}$. Thus the dimensionless ratio multiplying the low-scale operator should be $(E/\Lambda)^{N+M-4} \equiv b^{N+M-4}$.

Thus the result is that, at least in the viscinity of the free Lagrangian (and at least for the $\phi^4$-theory), exactly the renormalizable terms remain of any theory defined at a much higher scale. If this is a fact, then it seems plausible to assume that every theory has a physical cutoff, and that we require renormalizability is just the statement that we require the theory to be valid to a high scale, but not necessarily to an infinitely high scale.

### Unnaturalness as Fine-tuning

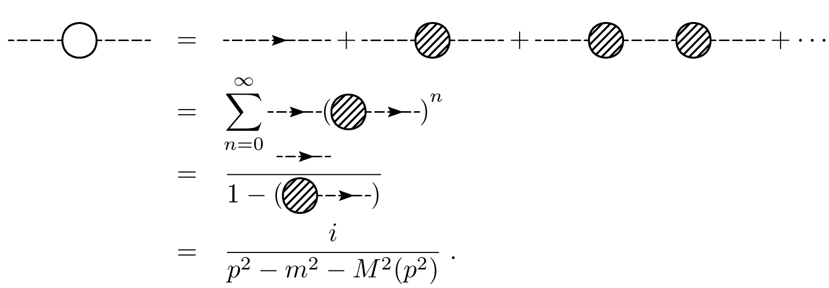
For the *grande finale* of the unnaturalness argument, consider a generic scalar field theory with a fermion:

$$\mathcal{L} = i\bar{\psi}\partial\!\!\!/\psi + |\partial_\mu\phi|^2 - m_s^2|\phi|^2 - (\lambda\phi\bar{\psi}\psi + \text{h.c.}) \ . \tag{2.3}$$

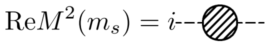
If we assume, for discussion's sake, that $\phi$ develops a vacuum expectation value:

$$\phi = \frac{1}{\sqrt{2}}(\varphi + v) \ ,$$

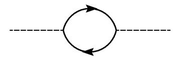
then the particle spectrum becomes massive: $m_f = \lambda v/\sqrt{2}$ and $m_h = m_s$. Now we want to find the one-loop correction to the masses. As usual, define $M^2(p^2)$ from the geometric series of the two-point function:



It is clear from here that

$$\delta m^2 = \mathrm{Re}M^2(m_s) = i$$  .

Relevant for this discussion is the fermion loop

 .

We want to find the leading contribution to the mass in terms of a cutoff $\Lambda$. We can write down this integral with the usual Feynman rules. To calculate it, we need Feynman parametrization, a simple change of variables, the Wick rotation, and the formula $\int \mathrm{d}^4 k_E f(k_E^2) = \pi^2 \int_0^{\Lambda^2} y \mathrm{d}y f(y)$:

$$\begin{aligned}
M^2(p^2 = m_h) &= i\left(\frac{-i\lambda}{\sqrt{2}}\right)^2 (-1)\int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{i^2 \mathrm{tr}\left[(\not{k} + m_f)(\not{k} - \not{p} + m_f)\right]}{\left[k^2 - m_f^2\right]\left[(k-p)^2 - m_f^2\right]} \\
&= i\left(\frac{-i\lambda}{\sqrt{2}}\right)^2 \int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{\mathrm{tr}\left[(\not{k} + m_f)(\not{k} - \not{p} + m_f)\right]}{\left[(1-x)k^2 + x(k-p)^2\right]^2} \\
&= i\left(\frac{-i\lambda}{\sqrt{2}}\right)^2 \int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^4 k'}{(2\pi)^4} \frac{4(k'^2 - x(2-x)m_h^2 + m_f^2)}{\left[k'^2 + x(1+x)m_h^2 - m_f^2\right]^2} \\
&= i\left(\frac{-i\lambda}{\sqrt{2}}\right)^2 \int_0^1 \mathrm{d}x \int \frac{i\mathrm{d}^4 k_E}{(2\pi)^4} \frac{4(-k_E^2 - x(2-x)m_h^2 + m_f^2)}{\left[-k_E^2 + x(1+x)m_h^2 - m_f^2\right]^2} \\
&= m_s^2 \left(\frac{-i\lambda}{\sqrt{2}}\right)^2 \int_0^1 \mathrm{d}x \pi^2 \int_0^{\frac{\Lambda^2}{m_h^2}} \frac{y\mathrm{d}y}{(2\pi)^4} \frac{4(y + x(2-x) + \alpha)}{\left[y - x(1+x) + \alpha\right]^2} \\
&= -m_s^2 \lambda^2 \frac{1}{8\pi^2} \int_0^1 \mathrm{d}x \left\{\frac{\Lambda^2}{m_h^2} + \ldots\right\}
\end{aligned}$$

where the $\ldots$ denotes terms with $\Lambda$ inside a logarithm, or in the denominator. Thus we have discovered a leading contribution to the mass renormalization:

$$\delta m_{h,leading}^2 = -\frac{\lambda^2}{8\pi^2}\Lambda^2$$

For the fermion mass, the relevant graph is the one with a h-loop. A perfectly similar calculation gives the corresponding result:

$$\delta m_{f,leading} = -\frac{3\lambda^2 m_f}{64\pi^2} \ln\left(\frac{\Lambda^2}{m_F^2}\right)$$

Also, the same calculation for the SM gives similar results.

We earlier found that a fermion mass term broke the chiral symmetry. The Yukawa term is of the same form. If $\lambda$ is small, we may consider the Lagrangian to be approximately chirally invariant. In case of the exact symmetry, we have decoupled left- and right handed degrees of freedom that perturbative corrections cannot recouple. Thus in either limit $m_f \to 0$ or $\lambda \to 0$ we must have $\delta m_f \to 0$. Then it seems plausible, by dimensional arguments, that the leading correction is proportional to $m_f$ and the dependence on $\Lambda$ is logarithmic. This keeps the correction small. However, there is no symmetry that would in this way protect the small mass of the scalar.

Now, in accordance to the previous section, interpret $\Lambda$ as a physical cutoff. This could be as large as $\Lambda \sim 10^{16}$GeV. In the SM, in order to keep the WW-scattering from violating unitarity, we must have the physical Higgs mass around 1 TeV. Thus

$$m_h^2 = m_{h0}^2 + \delta m_{h,1}^2 \ ,$$

where the unphysical bare mass must be adjusted to a precision of roughly $1$ in $10^{16}$ in order to cancel the quadratically divergent correction term. This adjustment must be made in each order of perturbation theory, and is quite unnatural indeed.

Here we finally get a formal definition of an unnatural mass. We say the scalar mass with quadratically divergent mass renormalization is unnatural, while the fermion mass with logarithmic divergence is natural.

## 2.3 Trivialness

Pure scalar field theories are trivial, and it is likely that all other models with fundamental scalar fields are be trivial also. Trivialness means that the scalar degree of freedom decouples from the theory at low energy-scales. To see this behavior, we need to use the renormalization group equation. The renormalization group improvement of the coupling constant $\lambda$ comes from the vertex counterterm. However, we will take a another path and use the effective action formalism.

**Effective Action and Running Coupling Constant**

Take massless scalar field theory, with the simplest Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{\lambda}{4!}\phi^4 + \text{counterterms.}$$

The effective potential is a quantum corrected classical potential. Calculating the effective potential is an arduous task, but luckily the calculation is generalizable [14]. The one-loop contribution of scalar loops to the effective potential is given by

$$V_{\text{scalar}} = \frac{1}{64\pi^2} tr \left\{ \bar{M}^4(\phi_c) \ln \frac{\bar{M}^2(\phi_c)}{M^2} \right\} \ ,$$

where

$$\bar{M}^2_{ab} = \frac{\partial^2 V_0}{\partial \phi_a \partial \phi_b}$$

and the trace is over indices $a$, $b$. The subindex $c$ denotes the classical field, and $M$ is an arbitrary energy-scale (not necessarily the cutoff). Now, with just one scalar, $\bar{M}^2 = \lambda\phi^2/2$ and we have the one-loop corrected potential:

$$V_{0+1} = \frac{\lambda}{4!}\phi_c^4 + \frac{\lambda^2}{256\pi^2}\phi_c^4 (\ln \frac{\phi_c^2}{M^2} + \ln \frac{\lambda}{2}) + \text{counterterms} \ .$$

The counterterms take care that the $\ln\frac{\lambda}{2}$ term disappears (require $\partial^4 V|_M = \lambda$). The effective action can be expanded in powers of momentum, about the point where external momenta vanish:

$$\Gamma = \int \mathrm{d}x \left\{ -V_c + \frac{1}{2}(\partial_\mu \phi_c)^2 Z_c + \dots \right\}$$

The parameter M was arbitrary, so the effective action should not depend on it:

$$M\frac{\mathrm{d}\Gamma}{\mathrm{d}M} = \left\{ M\frac{\partial}{\partial M} + \beta\frac{\partial}{\partial \lambda} + \gamma \int \mathrm{d}x \phi_c \frac{\delta}{\delta \phi_c} \right\} \Gamma = 0$$

This is the Callaman-Sumanzik renormalization group equation. It defines $\beta$ and $\gamma$.

We need the one-loop corrected wave-function renormalization $Z$; in our simple case it turns out to be $Z = 1$. It is useful to work with the dimensionless function,

$$V^4(t, \lambda) = \frac{\partial^4 V}{\partial \phi_c^4} = \lambda + \frac{3\lambda^2 t}{16\pi^2}$$

where $t = \ln(\phi_c/M)$, because the dimensionless $V^4$ and $Z$ can only depend on dimensionless variables. Then writing the part proportional to $Z_c$, which must vanish independently:

$$\left\{ -(1-\gamma)\frac{\partial}{\partial t} + \beta\frac{\partial}{\partial \lambda} + 2\gamma \right\} Z = 0$$

we find $\gamma = 0$. The other independent equation is

$$\left\{-\frac{\partial}{\partial t} + \beta\frac{\partial}{\partial\lambda}\right\} V^4 = 0 \ ,$$

from which we get, in the lowest order in $\lambda$,

$$\beta = \frac{3\lambda^2}{16\pi^2} \ .$$

This has the solution

$$\lambda(t) = \frac{\lambda_0}{1 - \frac{3\lambda_0 t}{16\pi^2}}$$

where $\lambda_0 = \lambda(t = 0)$.

**Trivialness and Higgs mass bounds**

Now evaluate this at $\phi_c = E$ and $M = \Lambda$, and interpret these so that $E$ is an arbitrary energy scale and $\Lambda$ is the physical cutoff. Then $t = \ln(E/\Lambda) \leqslant 0$. As $t \to -\infty$, $\lambda(t) \to 0$ independent of $\lambda_0$, the self coupling at a high energy. Thus at low $E/\Lambda$, the scalar field theory becomes a trivial free field theory. Other interactions modify this relation, but at the limit of small other couplings, the property is probably true for all Higgs models [12].

The equation has another interesting feature as well, basically reverting the previous statement. It is clear that $\lambda(t) \leq \lambda_0$. But what if $\lambda_0$ is very large? In this limit, we see

$$\lambda(t) = -\frac{16\pi^2}{3t} \ .$$

We can solve for $\Lambda$. Then the scale at which the theory becomes strongly coupled, $\Lambda_s$, is

$$\Lambda_s < \Lambda = E\mathrm{exp}\frac{16\pi^2}{3\lambda} \ .$$

So if the theory becomes strongly coupled before the cutoff of new physics, this relation approximately applies. The larger the self coupling, the lower this scale is forced to be. Our treatment of the SM implicitly assumes that the theory is not strongly interacting, and thus this relation says that the SM must be an effective theory.

Requiring the theory to stay weakly interacting to some scale $\Lambda_w$, this relation provides an upper bound for the Higgs mass. Assume $\lambda = (m_{Higgs}/v)^2$ and set $E = m_{Higgs}$. Then

$$m_{Higgs} \lesssim \frac{2\pi v}{\sqrt{3\ln(\frac{\Lambda_w}{m_{Higgs}})}} \ .$$

Clearly our cutoff $\Lambda_w$ must be greater than the Higgs mass for some range of validity of the effective theory[24]. Lattice-based arguments give a tighter limit, $\Lambda_w \gtrsim 2\pi m_{Higgs}$. Then we get the triviality bound $m_{Higgs} \lesssim 500$ GeV. If the Higgs were heavier than this, we could surely say that there was new physics at a few TeV. If the Higgs is much lighter, the scale of new physics is pushed higher quite easily.

While this basic argument is correct, in the SM also interactions with other particles could have a significant contribution. This raises the triviality bound.

# 3 Technicolor

To recapitulate what we know about strongly interacting theories, consider QCD. Since there is no sign of parity doubling of the particle spectrum, we know that the chiral symmetry must be broken. Since the Goldberger-Treiman relation holds so accurately, we can deduce that it is a spontaneously broken symmetry, and the since there is no fundamental scalar operator to break the symmetry, it must be a composite operator. Thus we have the picture that an asymptotically free theory will produce a fermion condensate at low energies. Note that this is truly dynamical symmetry breaking, because the symmetry breaking is not visible in the QCD Lagrangian, or in any perturbative expansion.

If the Higgs sector is removed from the SM, all particles are apparently massless. However, SSB of the quark chiral flavor symmetry propelled by a quark condensate still produces the would-be pions. These Goldstone bosons give gauge bosons masses! However, the masses will turn out to be much too small, and also there are no physical pions in the theory.

The naturalness, trivialness and hierarchy problems are solved in one stroke by assuming a TC type of theory. TC is essentially a copy of QCD with a much bigger characteristic energy scale $\Lambda_{TC}$. We remove the Higgs sector from the SM, and replace it with a TC-sector which has an exact chiral symmetry, broken by the chiral condensate in the low-energy regime. The related Goldstone bosons disappear and the weak gauge bosons become massive. The only immediate problem is that fermions remain massless.

I begin this chapter by considering the SM without the Higgs sector. I show that EWSB would still occur because of the chiral condensate of QCD, but at a much lower scale. Using this idea, I introduce the minimal technicolor model by replacing the Higgs sector with a strongly interacting technicolor sector. I show that the minimal technicolor model is anomaly free, and that it resolves the hierarchy problem, and examine the low energy particle content. I will find that the particle spectrum, while sensitive to the specific model, has some predictability from comparison to QCD. The lightest particles are the Goldstone bosons not consumed by the gauge bosons. Some of these particles gain a small mass from SM interactions.

Finally, just to give a general idea, the scales relevant to this thesis are approximately:

$$\begin{array}{ccc} \Lambda_{QCD} & \sim & 100\text{MeV} \\ v_{weak} & \sim & 10^2\text{GeV} \\ \Lambda_{TC} & \sim & 10^2\text{GeV} \\ \Lambda_{ETC} & \sim & 10^4\text{GeV} \\ \Lambda_{GUT} & \sim & 10^{16}\text{GeV} \\ M_p & \sim & 10^{18}\text{GeV} \end{array}$$

## 3.1 The Standard Model Without Higgs

Consider the SM without the Higgs sector. In this case all particles look massless at tree level. However, the gauge bosons will not remain so. For simplicity, consider only the two lightest quarks.

It is a general result that any gauge boson coupled to the current of a spontaneously broken symmetry will acquire a mass. Here the reason is in the hadronic contributions to gauge boson propagators. Such graphs give contributions that modify the propagator [7]:

$$\frac{g^{\mu\nu} - q^\mu q^\nu / q^2}{q^2} \longrightarrow \frac{g^{\mu\nu} - q^\mu q^\nu / q^2}{q^2(1 + \Pi(q^2))} .$$

If the functional form of $\Pi(q^2)$ is smooth near the origin, the particles will remain massless. If however, there is a pole, then the propagator will be modified to that of a massive particle. Such is the case currently as we have the massless pions! We will consider the condensate, then find how the chiral symmetry current couples to the gauge bosons, and finally calculate the masses of these gauge bosons.

### Dynamical Symmetry Breaking

SSB is signaled by a non-vanishing value of a scalar operator:

$$\langle \bar{Q}Q \rangle \neq 0 .$$

This says that there is a quark condensate, i.e. the vacuum contains quark-antiquark pairs. The operator $\bar{Q}Q$ does not transform correctly under SU(2)$_{\mathrm{A}}$. Thus we have 3 massless Goldstone bosons. The conserved currents corresponding to the broken symmetries will couple the vacuum to these pions. Thus we may write

$$\langle 0 \,| a_i^\mu |\, \pi_j \rangle = i f_\pi q^\mu \delta_{ij}$$

where $a_i^\mu$ are the three currents. Note that $f_\pi$ defined in this way is a purely strong interaction quantity, and cannot depend on the parameters of the weak interactions.

### Global Symmetry Currents and Gauge Bosons

We must somehow see how to connect the local symmetry (gauge bosons) with this global symmetry current. The coupling will provide us with exactly the vertex needed to get massive gauge bosons.

Take an arbitrary Lagrangian $\mathcal{L}_0$ with a global symmetry $G$. Then we have the corresponding Noether current $J$ (A.3), that satisfies $\delta\mathcal{L}_0 = \alpha_a \delta J^a = 0$. If we now let $\alpha$ be $x$-dependent, we will have an additional term $\delta\mathcal{L}_0 = 0 + (\partial_\mu \alpha_a) J^{a\mu}$.

If we require $\mathcal{L}$ to be the Lagrangian invariant under such a local transformation, it is trivial to see that

$$\mathcal{L} = \mathcal{L}_0 - g A_\mu^a J^{a\mu}$$

satisfies this requirement to the first order in $g$. In general, terms of higher order in $A$ can be arranged to cancel higher-order terms in the gauge-transformation.

In effect, adding this term promotes the global symmetry $G$ to its local counterpart. In our current case, we can think that since the axial current contains at least the pion operator, it in effect contains quark operators, and this is exactly the way it must thus couple to the weak gauge bosons to be gauge-invariant.

### Electroweak Boson Masses

According to the previous the argument, we write an interaction term $-\frac{1}{2} g W_\mu^\pm a_\mp^\mu$, where $g$ is the SU(2)$_L$ gauge coupling constant. The currents $a_\pm^\mu$ couple to the $\pi^\pm$ as before. Thus the vertex is

$$-i\frac{1}{2} g f_\pi k_\mu \tag{3.1}$$

where $k_\mu$comes from the derivative.

Ward's identity tells us that the vacuum polarization amplitude is necessarily transverse; i.e.

$$= i\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)\Pi_{ab}(k^2)\ ,$$

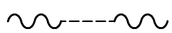
where $a$, $b$ are SU(2) or U(1) indices. Assuming a term

$$\Delta\mathcal{L} = \frac{1}{2}m_{ab}^2 A_\mu^a A_\mu^b$$

we get a vertex proportional to $g_{\mu\nu}m_{ab}^2$, and a contribution

$$\Pi_{ab} \in m_{ab}^2\ .$$

Thus if there is some contribution surviving in the $k \to 0$ limit of the vacuum polarization amplitude, there must be a corresponding mass term to preserve the Ward identity. Generally such contributions do not exist, but now we have the massless pion:



$$\begin{aligned} &= \left(-i\frac{1}{2}gf_\pi k_\mu\right)\left(\frac{i}{k^2}\right)\left(-i\frac{1}{2}gf_\pi k_\nu\right) \\ &= -i\left(\frac{k_\mu k_\nu}{k^2}\right)\left(\frac{1}{2}gf_\pi\right)^2 \end{aligned}$$

Thus we have $m_W = \frac{1}{2}gf_\pi$. Since the coupling of $B$ is similar but $g \to g'$, we can easily generalize this to a mass matrix in the case of the $B$ and $W_3$ fields:

$$M^2 = \begin{pmatrix} g^2 & gg' \\ gg' & g'^2 \end{pmatrix}\frac{f_\pi^2}{4}$$

The eigenvalues are

$$\begin{aligned} m_\gamma^2 &= 0 \\ m_Z^2 &= \frac{1}{4}(g^2 + g'^2)f_\pi^2 \end{aligned}$$

and the states that diagonalize the matrix are identified as the usual photon and weak neutral boson. Importantly, we retain the phenomenologically extremely successful relation

$$\frac{m_W}{m_Z} = \frac{g}{(g^2 + g'^2)^{1/2}} = \cos\theta_W \tag{3.2}$$

where $\theta_W$ is the usual weak mixing angle defined in terms of coupling constants. This result is the same as the tree level result in the normal Higgs scenario.

Of course, there is a reason for this coincidence. In either model, taking the limit $g' \to 0$ there is a custodial global SU(2) symmetry in the symmetry breaking sector. Under this custodial symmetry, the Goldstone bosons and the $\mathrm{SU(2)_L}$ gauge currents transform in the same **3** representation. This means that in this limit, the SU(2) bosons must be degenerate in mass [17]. As there is only one chargeless $\mathrm{SU(2)_L}$ boson, if we require $m_\gamma = 0$, we have (3.2).

In the Higgs case, the custodial symmetry arises if we write the two complex fields as four real fields. Then the vev breaks O(4) →O(3) ≃SU(2). In our current case we have the quarks in an isospin $\mathrm{SU(2)_L}$ doublet. The gauged $\mathrm{SU(2)_L}$ is broken, but a global SU(2) remains unbroken!

In the foregoing discussion we only considered the potential or elsewhise symmetry breaking sector. Generally there are interactions. In the SM they bring corrections to (3.2) of order 1%. In our current case, it is natural to question if the interactions produce larger corrections. However, since the relation follows from a symmetry of the strong interactions, it must be valid up to electroweak radiative corrections. It turns out there are much stronger phenomenological implications of TC than these modifications.

Thus, even simply removing the Higgs sector produces promising results. However, there are also crucial shortcomings. With this model the masses of the gauge bosons would be of the order of 10 MeV, while they should be about 100 GeV. Also, we do not have any physical pions in the theory, as the pions went to the longitudinal degrees of freedom of the gauge bosons. However, it is clear that these problems can be avoided if we assume a similar theory to QCD at a much larger scale. Although fermions will remain massless, it is the basic idea of technicolor.

## 3.2 The Minimal Model

Suppose we have the SM but without the Higgs sector. This also entails that particles are massless at tree level. To break the electroweak symmetry, add an electroweak doublet $T_L$ of fermions that interacts through a new strong interaction, technicolor. The TC gauge group $TC$ is assumed to be SU($N_T$).

$$T = \begin{pmatrix} A \\ B \end{pmatrix}$$

$T_L = P_L T$ is a doublet while $A_R$ and $B_R$ are singlets under $\mathrm{SU(2)_L}$, i.e. $I_3(A_L, B_L, T_R) = (\frac{1}{2}, -\frac{1}{2}, 0)$. $T$ is a color-singlet. In the next-to-minimal model, we add $N_D$ of these doublets. We will assume $N_D = 1$ unless otherwise stated.

**Gauge Anomalies**

In some cases quantum corrections can violate a symmetry. These are found as a nonzero divergence of the corresponding Noether current. The current is then called anomalous.

Anomalies arise when both axial and vector symmetry currents are present, since it is not possible to find a regularization scheme which preserves both currents. They are visibile in triangle graphs with three external gauge boson legs, or essentially one-loop corrections to vertices. The $\eta$-meson is very heavy exactly because of an anomaly.

Anomalies coupled to gauge symmetries cannot be tolerated, since current conservation is a necessity in the construction of gauge field theory. Gauge anomalies are proportional to [3]

$$D_{\alpha\beta\gamma} = \frac{1}{2}\mathrm{tr}\left[\{T_\alpha, T_\beta\}\, T_\gamma\right] \ ,$$

where $T_\alpha$ is the representation of the gauge algebra acting on a left handed fermion or antifermion field, and the trace is over the fields. The indices label different gauge groups.

We require that gauge anomalies vanish. The full gauge group is $\mathrm{SU}(N_{TC})\otimes\mathrm{SU}(3)\otimes\mathrm{SU}(2)\otimes\mathrm{U}(1)$. Let $Y(T_L, A_R, B_R) = (a, b, c)$, and see if these constants are determined from requiring the gauge anomalies to cancel.

While evaluating the triangle graphs, we only need to check if they vanish. There are two simplifications. First, consider drawing an electron that emits a Z-boson and changes to a muon. Then let this muon emit a photon. Now at the third vertex, the muon must change back into an electron, and it cannot do this by emitting a photon. This happens with any non-U$(1)$ gauge group, and thus we only need to consider combinations in which the product $T_\alpha T_\beta T_\gamma$ is a singlet under the SU$(n)$ groups. The other simplification is that anomalies vanish under some set of groups if under those groups the fermions furnish a real representation. Thus

- SU($N_T$)-SU($N_T$)-SU($N_T$) vanishes because $N_T + N_T + \bar{N}_T + \bar{N}_T$ is a real representation of the technifermions.
- SU($N_T$)-SU($N_T$)-SU$(2)$ vanishes because we cannot draw a diagram with only one SU$(2)$ leg.
- SU($N_T$)-SU($N_T$)-U$(1)$ vanishes if, when we sum over all left-handed fields which transform under TC, the net hypercharge vanishes:

$$N_D\left[Y(A_L) + Y(B_L) + Y(A_R^*) + Y(B_R^*)\right] = N_D\left[2a + b + c\right] = 0$$

- SU($N_T$)-X-X where X is not SU($N_T$) vanishes because we cannot draw a diagram with only one SU($N_T$) leg.
- Any anomaly with SU(3) vanishes because $T$ is color-singlet and the SM anomalies are assumed to cancel.
- SU(2)-SU(2)-SU(2) vanishes in general because SU(2) has only real or pseudoreal representations.
- SU(2)-SU(2)-U(1) vanishes if $N_D a = 0$ because the SM anomalies are assumed to cancel.
- U(1)-U(1)-U(1) vanishes if $N_D \left[2a^3 + b^3 + c^3\right] = 0$.

Then we see we must choose $Y(T_L, A_R, B_R) = (0, b, -b)$, and for definiteness let $b = 1$. Thus electric charges defined by $Q = I_3 + \frac{Y}{2}$ give

$$Q(A, B) = (\frac{1}{2}, -\frac{1}{2}) \ .$$

There is one more type of anomaly, namely the Witten global SU(2) anomaly. It can be qualitatively described as follows. Gauge transformations that vanish at spacial infinity correspond to all gauge transformations on a 4-sphere. But SU(2) is isomorphic to the 3-sphere, and maps from the 4-sphere to the 3-sphere form two disconnected subsets: $\pi_4(S^3) = Z_2$. For an odd number of SU(2) doublets, the contributions from these two domains exactly cancel in the path integral defining the theory. Thus the the Witten SU(2) anomaly vanishes if the number of $\mathrm{SU(2)_L}$ doublets is even. As a corollary, we require $N_D N_T$ to be even.

### Technicolor Scale

Technicolor, like QCD, is a strongly interacting theory. The related phenomenom of the non-perturbative condensate is difficult to study analytically. However, approximating QCD by the Nambu-Jona-Lasino model, we can get results that become exact in the limit of a large number of colors. The model connects the high-energy asymptotically free regime with the low-energy confined regime, and produces rules that characterize QCD and are also used extensively in TC. They can be written [8]

$$f_\pi \sim \sqrt{N_c}\Lambda_{QCD} \quad \langle \bar{q}_i q_j \rangle \sim \delta_{ij} N_c \Lambda^3_{QCD} \quad m_0 \sim \Lambda_{QCD}$$

where $m_0$ is a typical dynamical mass. These rules apply at the renormalization scale of $\Lambda_{QCD}$. From the first rule, we have

$$F_\pi \sim f_\pi \sqrt{\frac{N_T}{3}} \left(\frac{\Lambda_{TC}}{\Lambda_{QCD}}\right) \ .$$

However, the technipion decay constant $F_\pi$ is essentially controlled by the masses of the gauge bosons. Thus we should be able to solve $\Lambda_{TC}$ from this equation in terms of the true variables $N_T$, $N_D$, and known parameters including

$$v \equiv v_{weak} = v_0/\sqrt{2} = 175\mathrm{GeV}$$

which is the EWSB scale.

We start by considering the coupling of the pions and weak gauge bosons. If we are in the broken phase, the SM Higgs field can be written

$$H = \frac{1}{\sqrt{2}} \exp\left(\frac{i\pi\cdot\tau}{v_0}\right) \begin{pmatrix} v_0 + h \\ 0 \end{pmatrix} \ .$$

This is exactly the analogue to $\varphi = (v + \phi(x))\exp(i\theta(x)/v)$ in the case of the complex scalar field, which is discussed in the appendix. In the case of TC, we had (3.1) the coupling of the global symmetry current and the local gauge bosons: $\frac{1}{2}gF_\pi W^i_\mu k_\mu \pi_i$. Calculating $(D_\mu H)^\dagger(D_\mu H)$ and comparing the relevant terms we easily find $v_0 = F_\pi$.

However, we want to include the effect of $N_D > 1$. For this, we use our result (1.5):

$$U = \exp\left(i\pi\cdot\tau/f_\pi\right) \ .$$

Then the previous $W-\pi$ term is recovered from $\frac{F_\pi^2}{4}\mathrm{tr}\left(|D_\mu U|^2\right)$. Since each doublet separately should produce this kind of term, for many doublets we write

$$\mathcal{L} = \frac{N_D F_\pi^2}{4}\mathrm{tr}\left(|D_\mu U|^2\right) \ .$$

Then, expanding to find the boson mass term, we find $v_0 = \sqrt{N_D}F_\pi$.

Hence we can write

$$v_0 \sim f_\pi\sqrt{\frac{N_D N_T}{3}}\left(\frac{\Lambda_{TC}}{\Lambda_{QCD}}\right) \ .$$

Invert this relation to find the result:

$$\Lambda_{TC} \sim v\frac{\Lambda_{QCD}}{f_\pi}\sqrt{\frac{3}{2N_D N_T}} \sim \frac{460}{\sqrt{N_D N_T}}\mathrm{GeV} \ .$$

Here and later we will use the values $v = 175$ GeV, $f_\pi = 93$ MeV, $\Lambda_{QCD} = 200$ MeV. If we also set $N_D = 2$ and $N_T = 4$, we find $\Lambda_{TC} \approx 160$ GeV.

### Hierarchy Problem Solved

Let us now see how the hierarchy problem is solved in this type of scenario naturally. Since the TC scale is larger than QCD scale, assuming unification at some high energy tells us that the $\beta$-function of TC must be more negative. Using the previous estimate, we want $\Lambda_{TC}/\Lambda_{QCD} \sim 10^3$. The $\beta$-function of an asymptotically free theory is given by:

$$\beta \equiv \mu\frac{\mathrm{d}\alpha}{\mathrm{d}\mu} = -\frac{\alpha^2}{\pi}(\frac{11N}{6} - \frac{n_f}{3}) \equiv -\frac{\alpha^2}{\pi}b_1 \ .$$

Integrating and assuming grand unification, which means that the coupling constants are the same at GUT-scale, we get

$$\frac{\Lambda_{TC}}{\Lambda_{QCD}} = \exp\left[\frac{\pi}{\alpha_{GUT}}\left(-\frac{1}{b_1^{TC}} + \frac{1}{b_1^{QCD}}\right)\right] \ .$$

With 3 colors and 6 flavors, $b_1^{QCD} = 7/2$. Assume $\alpha_{GUT} \sim 1/30$. Then for $N_D = 1$,

| $N_T$ | 2 | 3 | 4 |
|---|---|---|---|
| $\Lambda_{TC}/\Lambda_{QCD}$ | $10^{-2}$ | $10^3$ | $10^5$ |

and for $N_D = 2$,

| $N_T$ | 3 | 4 | 5 |
|---|---|---|---|
| $\Lambda_{TC}/\Lambda_{QCD}$ | $10^1$ | $10^5$ | $10^6$ |

Thus, although the estimate is very sensitive to the number of techniflavors, we see that assuming strong dynamics, the large separation of scales can exist completely naturally. These results seem to imply that $N_D = 1$, $N_T = 3$ would be the natural choice, but that theory has the Witten global anomaly.

## 3.3 Particle Content

**Pions**

At low energies, we will have both the techniquark and the quark condensate breaking the axial vector current. If the full axial current is defined $J^5$, then

$$\begin{aligned} \langle 0 \left| J_\mu^5 \right| \pi_{QCD} \rangle &= i f_\pi q^\mu \\ \langle 0 \left| J_\mu^5 \right| \pi_{TC} \rangle &= i F_\pi q^\mu \end{aligned}$$

The parameter $F_\pi$ is defined so that weak gauge bosons have right mass.

The weak bosons couple to the pions through the axial current. Thus the relevant eigenstates are the physical pion which has no coupling and its orthogonal component, which will be absorbed by the gauge bosons. We have

$$\begin{aligned} \langle 0 \left| J_\mu^5 \right| \pi_{phys} \rangle &= 0 \\ \langle 0 \left| J_\mu^5 \right| \pi_W \rangle &= \sqrt{F_\pi^2 + f_\pi^2} q^\mu \end{aligned}$$

and the correct eigenstates are:

$$\begin{aligned} |\pi_{phys}\rangle &= \frac{F_\pi |\pi_{QCD}\rangle - f_\pi |\pi_{TC}\rangle}{\sqrt{F_\pi^2 + f_\pi^2}} \\ &= |\pi_{QCD}\rangle + \mathcal{O}\left(\frac{f_\pi}{F_\pi}\right) |\pi_{TC}\rangle \\ |\pi_W\rangle &= \frac{F_\pi |\pi_{TC}\rangle + f_\pi |\pi_{QCD}\rangle}{\sqrt{F_\pi^2 + f_\pi^2}} \\ &= |\pi_{TC}\rangle + \mathcal{O}\left(\frac{f_\pi}{F_\pi}\right) |\pi_{QCD}\rangle \end{aligned}$$

Since $F_\pi \gg f_\pi$, the physical pion is mostly the QCD-pion and the absorbed pion mostly the TC-pion. Since the longitudinal component of the bosons is strongly interacting, we expect this would show in $e^+e^- \to W^+W^-$ cross-sections. Longitudinal production would show a complicated form factor with resonances, while transverse $W$ would be similar to the SM $W$.

**Mass Gap and Technibaryons**

In QCD, the quark condensate induces a dynamical mass for the quarks. We earlier found that for the lightest quarks, this dynamical mass was much larger than the hard Lagrangian mass. We now want to estimate the corresponding techniquark constituent mass. Using the scaling rules,

$$m_T \sim \Lambda_{TC} \sim v \frac{\Lambda_{QCD}}{f_\pi} \sqrt{\frac{3}{2N_D N_T}} \sim v \frac{m_N}{f_\pi} \sqrt{\frac{1}{2N_D N_T}}$$

where $m_N$ is the nucleon mass. Thus there will be technibaryons with masses of order

$$m_{technibaryon} \sim N_T m_T \sim v \frac{m_N}{f_\pi} \sqrt{\frac{N_T}{2N_D}} \sim 2 \text{ TeV}.$$

These technibaryons are fermions or bosons depending on the number of techniquark flavors. In the current model, these technibaryons would be stable. However, when we introduce ETC, techniquarks will be able to decay to quarks or leptons. The heavy technibaryons will then decay into complex final states with many quarks and leptons. At very high energies, $E \gg 10\text{TeV}$ we would expect 'techni-jets' similarly as in QCD.

### Goldstone Bosons and Techniaxions

The spectrum of the lightest mesons is predictable by analogy to QCD. With $N_D = 1$, the quark condensate produces exactly the minimum of pions to make the gauge bosons massive. However, we also broke the $\mathrm{U}(1)_\mathrm{A}$. In QCD there is an anomaly breaking this symmetry explicitly, making the corresponding particle very heavy. An estimate of the $\eta_{TC}$ mass is [8] (using $N_D = 1$)

$$m_{\eta TC} \sim \frac{3}{N_T}\sqrt{\frac{8}{N_T N_D}}\frac{v}{f_\pi} m_{\eta QCD} \sim 2 \text{ TeV}$$

which likely makes it hidden from direct searches in the near future. There are also $1^-$ vector-mesons, discussed in the next subsection.

Now consider increasing the number of techniquark doublets. First ignore $\mathrm{U}(1)$. The total flavor symmetry is $\mathrm{SU}(2N_D)_\mathrm{V}\otimes\mathrm{SU}(2N_D)_\mathrm{A}$. The $\mathrm{SU}(2N_D)_\mathrm{A}$ group has $4N_D^2 - 1$ generators, or $4N_D^2$ if we include the identity which corresponds to the $\mathrm{U}(1)_A$ symmetry, or the $\eta$-particle. When the quark condensate breaks this symmetry, there will be $4N_D^2$ Goldstone bosons, some of which may gain masses through SM interactions. This corresponds to the situation in QCD where we assumed two massless quarks, and thus had three almost massless pions. Since the fundamental representation is not allowed, the particles will be in the adjoint representation, also corresponding to the Goldstone bosons. In QCD, three quarks could be assumed approximately massless; currently all techniquarks are massless.

The global version of $\mathrm{SU}(2)_\mathrm{L}$ mixes the electroweak doublet elements. The flavor subgroup that acts on weak doublets horizontally is $\mathrm{SU}(N_D)$. If we include identity operators, their direct product has $(3 + 1)(N_D^2 - 1 + 1) = 4N_D^2$ generators. Thus it is palpable we can classify the generators associated with the Goldstone bosons according to their transformation properties under $\mathrm{SU}(2)_\mathrm{L}\otimes\mathrm{SU}(N_D)$.

Let $I_2$ be the $\mathrm{SU}(2)$ identity and $\tau^a$ the generators. Let $I_N$ be the $\mathrm{SU}(N_D)$ identity and $\lambda^A$ the generators. Then, the grouping according to how the particles transform under the corresponding groups becomes:

- $I_2 \otimes I_N$: this is the $\eta$ that was already discussed. It becomes heavy because of the axial anomaly.

- $I_2 \otimes \lambda^A$: these are dubbed 'techni-axions'. The corresponding generators commute with $\mathrm{SU}(2)$. The electrically neutral ones remain massless apart from electroweak instanton effects. Since they correspond to a residual symmetry, they are sterile, or undetectable save for their gravitational interactions. Their axial currents have electroweak anomalies, like the $\pi^0$, but no QCD anomaly. These axions should decay with strength $F_\pi$. However, astrophysical arguments constrain $F_{axion} \geqslant 10^8$GeV. This problem is cured in ETC.

- $\tau^a \otimes I_N$: these are the three Goldstone bosons that become the longitudinal component of the weak gauge bosons.

- $\tau^a \otimes \lambda^A$: these are pseudo-Goldstone bosons. They carry $\mathrm{SU}(2)$ charge and thus acquire a small mass when we turn on the $\mathrm{SU}(2)$ gauge interactions, similarly to how the $\pi^\pm/\pi^0$ mass difference is caused by electromagnetic effects in QCD. To estimate the mass, rescale the $\pi^\pm - \pi^0$ mass difference. This estimate gives $m \sim 6$ GeV. Such light charged scalars are obviously experimentally ruled out. There might be further corrections when we put $\mathrm{U}(1)$ on, but they should be small.

Now include $\mathrm{U}(1)_\mathrm{Y}$. The particles whose generators do not commute with $Y \otimes I_N$ have electric charge, i.e. they are electrically charged technipions. They belong to each group: some are eaten by the gauge bosons, some have $\mathrm{SU}(2)$ charge, and some do not. All will gain small masses from the interactions, of order 5 GeV according to the previous estimate. These masses are unacceptably small to have avoided detection.

### Techni-vector Mesons

Assume $N_D = 1$. The $0^-$ triplet mesons (pions) are eaten by the gauge bosons. The physical spectrum will still contain the $\rho$-triplet and the $\omega$, which are the $1^-$ mesons. In QCD we have the process

$$\rho \to \pi\pi \ .$$

Thus in technicolor, we have

$$\rho_T \to WW \ .$$

These vector mesons could provide resonance structures in processes like

$$\left\{ \begin{array}{c} pp \\ e^+e^- \end{array} \right\} \to W^+W^- \ .$$

Their masses can be estimated by scaling,

$$m_{\rho TC} \sim m_{\omega TC} \sim \frac{F_T}{f_\pi} \sim m_\rho \frac{v}{f_\pi}\sqrt{\frac{6}{N_T N_D}} \sim 1.3 \text{ TeV}.$$

# 4

# More Technicolor

As we have now seen, technicolor is the natural model to solve the unnaturalness and trivialness problems. Unfortunately, the minimal model we considered predicts light technihadronic particles that have not been seen. Also, the SM fermions remained massless. Answering these challenges is the direction of further model-building. We cannot revert to scalar fields, except possibly under the context of supersymmetry.

Luckily, there is a natural way to provide the masses. Assume a high scale $\Lambda_{ETC}$ associated with the spontaneous symmetry breaking of a gauge group $ETC$. The gauge bosons that correspond to the broken generators become massive. At low energies, their propagators may be replaced by $1/\Lambda_{ETC}^2$ - effectively creating a 4-point vertex. Now if these gauge bosons connect ordinary fermions to technifermions that have a dynamical mass, usual fermions will become massive as well. A corresponding interaction will also raise the masses of unwanted techniparticles.

This simple model has one major drawback. The same gauge group must also connect ordinary fermions to each other. This produces flavor-changing neutral currents that are phenomenologically nonexistant. To solve this issue, we introduce the concept of walking technicolor, which means that the gauge coupling evolves slowly over a large hierarchy. This enhances the condensate but does not affect $F_\pi$, i.e. we get a large enhancement of ordinary fermion masses.

I first consider extended technicolor, and how this new interaction, defined at high energies, gives masses to ordinary fermions. I show that flavor-changing neutral currents give an upper limit on the possible masses that these interactions can generate, and thus that it is not possible to produce large enough masses to have even the second generation quarks. To generate larger masses, I introduce walking technicolor. In walking technicolor, the theory is assumed to be near a conformal point at $\Lambda_{ETC}$. It is then shown that in a certain nonperturbative approximation the anomalous dimension is a constant, $\gamma_m \sim 1$, which enhances the condensate value greatly, effectively enhancing masses.

## 4.1 Extended Technicolor

It is easy to write down mass terms for the fermions in the current model. A term of the form

$$m\bar{f}f = \langle \bar{F}F \rangle \bar{f}f$$

corresponding to a four-point interaction. Thus we want to have such interactions between technifermions and ordinary leptons and quarks. We know that the weak interactions lead to such a vertex at low energies; thus in analogy to the weak currents we need currents of the form

$$\bar{F}\gamma_\mu f \ ,$$

coupled to the new gauge bosons. Here and later $F$ denotes a technifermion and $f$ an ordinary fermion.

In a full theory we must assume a large gauge group $ETC$ that has all the desired currents. Clearly $ETC$ must be large enough to furnish representations with both $F$'s and $f$'s. For example, we might simply blockwise embed SU($N_{TC}$) into SU($N_{ETC}$) with $N_{ETC} > N_{TC}$.

### Minimal Model

Consider the SM. We have 3×3=9 electroweak quark-doublets, and 3 electroweak lepton-doublets. In addition, their 12 corresponding righthanded fields are electroweak singlets. $ETC$ should couple $F$'s to all of these. Then we should have each generation falling into 3 doublets of quarks and one of leptons under the gauged $\mathrm{SU}(2)_{\mathrm{L}}$, and 3 doublets of quarks and one of leptons under the global $\mathrm{SU}(2)_{\mathrm{R}}$. Set $N_D = 1$ and $N_T = 4$. Then we still need 4 lefthanded techniquark $\mathrm{SU}(2)_{\mathrm{L}}$ doublets and 4 righthanded $\mathrm{SU}(2)_{\mathrm{R}}$ doublets. Counting e.g. the lefthanded fields, we have (4 doublets) × (3 generations) from the SM and 4 techniquark doublets: $4 \times 3 + 4 = 16$. Thus the natural gauge group is

$$ETC = \mathrm{SU}(16) \otimes \mathrm{SU}(2)_{\mathrm{L}} \otimes \mathrm{SU}(2)_{\mathrm{R}} \otimes \mathrm{U}(1) \ .$$

Note that now both the color and technicolor symmetries are embedded into $\mathrm{SU}(16)$. Under this gauge group, the fermions form two **16** multiplets:

$$\left(Q_1, Q_2, Q_3, Q_4, q_r^1, q_g^1, q_b^1, q_r^2, q_g^2, q_b^2, q_r^3, q_g^3, q_b^3, l^1, l^2, l^3\right)_{L,R}$$

where the superscript denotes generation, $Q$'s are techniquarks, $q$'s normal quarks and $l$'s leptons. All the multiplet elements are $\mathrm{SU}(2)$ doublets. This means that the $L$ multiplet is a singlet under $\mathrm{SU}(2)_{\mathrm{R}}$, and doublet under $\mathrm{SU}(2)_{\mathrm{L}}$, and vice versa for the $R$ multiplet.

Now, picture $\mathrm{SU}(16)$ as a horizontal symmetry and $\mathrm{SU}(2)$ as a vertical symmetry. Compare this to QCD, where we have the color symmetry in place of $\mathrm{SU}(16)$. Then we had physically motivated reason to believe the quark condensate broke the axial weak isospin. Now it suffices to say the techniquark condensate breaks both $\mathrm{SU}(2)$ symmetries.

### Low-energy Theory

Starting from the theory with gauge group $ETC$ defined at some high energy, we must arrive at a low-energy theory in which the only active symmetry gauge groups are $TC \otimes SM$. We thus require the parent theory $ETC$ to undergo symmetry breaking at a scale $\Lambda_{ETC}$, for example:

$$ETC \overset{E\sim\Lambda_{ETC}}{\longrightarrow} TC \otimes SM \ .$$

This elevates the coset gauge bosons of $ETC/TC \otimes SM$ to masses of order $g\Lambda_{ETC}$, in accordance to the Higgs mechanism, while keeping the technicolor and standard model gauge bosons intact. This reason for this symmetry breaking is not explained in the current context; in this sense, technicolor is a bottom-up approach to model building.

The symmetry breaking pattern can, in general, be much more complicated. Some interesting ideas involve these considerations. One is that of 'Tumbling Gauge Theories', in which the symmetry breaking is sequential in the sense that

$$ETC \to G_1 \to G_2 \to \ldots \to G_{n-1} \to TC \ .$$

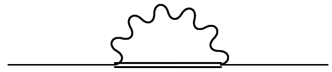
At each stage, there will be coset gauge bosons gaining masses of order $\Lambda_n$. This symmetry breaking pattern can be made dynamic, and in principle, may generate the mass hierarchy for quarks and leptons of different flavors, in accordance to what will be discussed next.

### Low Energy Relic Interactions

Although we have required the correct low energy residual symmetries, the low energy phenomenology still includes effects caused by the broken $ETC$ generators, or essentially the massive gauge bosons. In the high energy theory we, in general, have three kinds of currents coupled to the ETC gauge bosons:

$$\bar{F}\gamma_\mu F \quad \bar{F}\gamma_\mu f \quad \bar{f}\gamma_\mu f$$

These are connected to each other by gauge boson exchange. Taking the middle term, we have terms in the Lagrangian of the form $gA_\mu^a \bar{F}\gamma_\mu T_a f$, where $T_a$ is an $ETC$ generator. For example, we could have the following graph, where an ordinary fermion changes into a technifermion and back:



Because the gauge bosons corresponding to these generators gain a large mass of order $\Lambda_{ETC}$, their propagator is simplified at low energies:

$$\frac{1}{q^2-\Lambda_{ETC}^2} \sim -\frac{1}{\Lambda_{ETC}} \ .$$

In the above diagram, the gauge boson line will be replaced by an effective 4-point vertex. As will be explicitly discussed later, the technifermion will receive a dynamical mass, like quarks in QCD. Thus this is the diagram that will generate fermion masses.

In general, when the broken generators become massive, we get effective terms of the form

$$\bar{\alpha}_{ab}\frac{\bar{F}\gamma_\mu\bar{T}_a F\bar{F}\gamma_\mu\bar{T}_b F}{\Lambda_{ETC}^2} + \bar{\beta}_{ab}\frac{\bar{F}\gamma_\mu\bar{T}_a f\bar{f}\gamma_\mu\bar{T}_b F}{\Lambda_{ETC}^2} + \bar{\gamma}_{ab}\frac{\bar{f}\gamma_\mu\bar{T}_a f\bar{f}\gamma_\mu\bar{T}_b f}{\Lambda_{ETC}^2} \ ,$$

where now $\bar{T}_a$ correspond to the broken generators, but also include any chiral $P_L$, $P_R$ factors. We want to use the Fierz-transformations to rearrange technifermions on one side. In general, a Fierz arrangement is of the form

$$\bar{u}_1 G_a u_2 \bar{u}_3 G_b u_4 = \Sigma_{c,d}\alpha_{abcd}\bar{u}_1 G_c u_4 \bar{u}_3 G_d u_2 \ .$$

Thus, taking chiral factors separately, we have sums over all the original, non-chiral $ETC$ generators. We must also include the identity in the generators. Mass terms mix the $L$ and $R$ components so we can pick up the most interesting terms:

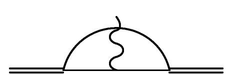
$$\alpha_{ab}\frac{\bar{F}T^a F\bar{F}T^b F}{\Lambda_{ETC}^2} + \beta_{ab}\frac{\bar{F}_L T^a F_R \bar{f}_R T^b f_L}{\Lambda_{ETC}^2} + \gamma_{ab}\frac{\bar{f}_L T^a f_R \bar{f}_R T^b f_L}{\Lambda_{ETC}^2} \ .$$

**The $\alpha$-terms: Techniaxion Masses**

Here we again need results that come from the Nambu-Jona-Lasinio model. Earlier we wrote $\langle\bar{q}_i q_j\rangle \sim \delta_{ij}N_c\Lambda_{QCD}^3$. Thus similarly $\langle\bar{F}_i F_j\rangle \sim \delta_{ij}N_{TC}\Lambda_{TC}^3$. Now we need an expression for the operator $\bar{F}_i F_j$ at lower energies. A beliavable explanation parallels the one that led to (1.5). At low energy scales, when the techniquarks have condensed, the relevant degrees of freedom are the pions. Thus, one is motivated to make the replacements

$$\bar{F}_{aR}F_L^b \to cN_{TC}\Lambda_{TC}^3\Sigma_a^b \quad \equiv cN_{TC}\Lambda_{TC}^3 \exp(i\pi^c\tilde{T}^c/F_\pi)_a^b$$

where $b$ is an $\mathrm{SU(2)_L}$ index, $a$ is an $\mathrm{SU(2)_R}$ index, and $\tilde{T}$'s are the $TC$ generators. Now because of the $\alpha$ terms, we have a 4-technifermion vertex. We can draw such a vertex between two loops, starting from a technipion. Thus the following diagram



where an incoming technipion splits into two techniquarks which interact through a heavy virtual gauge boson, contributes

$$\alpha_{ab}\frac{c^2 N_{TC}^2\Lambda_{TC}^6}{\Lambda_{ETC}^2}\mathrm{tr}\left(\Sigma T^a\Sigma^\dagger T^b\right) \ .$$

Now we can expand the $\Sigma$'s to find the pion mass terms:

$$\alpha_{ab}\frac{c^2 N_{TC}^2\Lambda_{TC}^6}{\Lambda_{ETC}^2}\mathrm{tr}\left(\left[\pi^c\tilde{T}^c, T^a\right]\left[\pi^d\tilde{T}^d, T^b\right]\right)$$

Now clearly if

$$\left[\tilde{T}^a, T^b\right] = 0 \ ,$$

then the corresponding Goldstone $\pi^a$ will have vanishing mass contributions from ETC. If this is not the case, then the corresponding Goldstone will, in general, receive a mass contribution of the order $N_{TC}\Lambda_{TC}^3/\Lambda_{ETC}^2$.

In QCD we have a similar situation. There the charged pion $\pi^\pm$ is heavier than the neutral $\pi^0$. This is precisely because the neutral pion is associated with a generator that commutes with electric charge, while the charged pions' generators do not commute with electric charge, thereby receiving electromagnetic contributions to their mass.

As stated before, the techniaxions were sterile under other than TC interactions. Thus this interaction is the only way for them to receive masses. A typical result is

$$m^2_{\text{axion}} \sim \frac{1}{N_{TC}} (\text{GeV})^2 \ ,$$

which is still very small. Our previous estimate was that the charged pseudo-Goldstone bosons received a mass of order 5 GeV from SM interactions, which is then still the leading contribution. Also any colored states will have larger masses from the strong interaction.

This problem is, in principle, solved in the walking technicolor scenario. The condensate will receive a significant boost, of order $\Lambda_{ETC}/\Lambda_{TC} \sim 10^3$. Thus the techniaxions masses, for example, are in the TeV scale, out of reach of today's experiments.

#### The $\beta$-terms: Quark and Lepton Masses

The terms with $\beta$-coefficients will give masses and mixing angles to the ordinary quarks and leptons. At the TC scale, by our scaling rules we have the condensate $\langle \bar{Q}Q \rangle \sim N_{TC}\Lambda_{TC}^3$. Thus the natural scale for ETC-induced quark and lepton masses is

$$m_{q,l} \sim \beta \frac{N_{TC}\Lambda_{TC}^3}{\Lambda_{ETC}^2} \ .$$

Note that this relation is reciprocal in $\Lambda_{ETC}$: higher masses require a smaller $ETC$-breaking scale.

For $\Lambda_{ETC} \gtrsim \Lambda_{TC}$, this relation would seem to allow $m \sim \Lambda_{TC}$. Assume $\beta N_{TC} \lesssim 1$ and $\Lambda_{TC} \sim 100$ GeV. Then

| | | | |
|---|---|---|---|
| $m$(MeV) | $10$ | $10^3$ | $10^5$ |
| $\Lambda_{ETC}$(GeV) | $10^4$ | $10^3$ | $10^2$ |

The first mass is approximately that of the up-quark, the second that of the charm-quark, and the third that of the top-quark. Since $\Lambda_{TC} \sim 10^2$ GeV, the heavier particles place an upper bound on the value of $\Lambda_{ETC}$.

The pattern and scale of masses and mixing angles is naturally sensitive to the symmetry breaking pattern of $ETC$ through the different components of $\beta$. It is thus a formidable task to find a realistic model, but in principle it is possible. Thus we see how ETC models can potentially solve the flavor problem.

## 4.2 Problems

#### The $\gamma$-terms: Flavor-changing Neutral Currents

Since the observation is that particles of different generations have different mass but otherwise same charges under the SM, and ETC is now proposed to generate these masses, then ETC must couple differently to fermions of different generations. The mass eigenstates are not, in general, the same as interaction eigenstates, so that we have flavor changing neutral currents. The most severe restriction comes from the strangeness-changing $\Delta S = 2$ graphs.

The corresponding term is

$$\frac{(\bar{s}\gamma_5 d)(\bar{s}\gamma_5 d)}{\Lambda_{ETC}^2} \ .$$

This gives a contribution to the well-measured mass difference of the kaons $K_L$ and $K_S$. These particles are weak-interaction eigenstates consisting of $K^0$ and $\bar{K}^0$. Because kaons decay through the weak interactions, these eigenstates have definite lifetimes, with the $K_L \sim d\bar{s} + \bar{d}s$ being long lived.

The above term induces a mass difference [8]

$$\frac{\delta m^2}{m_K^2} \sim \gamma \frac{f_K^2}{\Lambda_{ETC}^2} \lesssim 10^{-14} \ ,$$

where $f_K \sim f_\pi \sim 100$ MeV is the decay constant of the kaon. From here,

$$\Lambda_{ETC} \gtrsim 10^3 \mathrm{TeV}.$$

Now apply this to the previous mass-estimate:

$$m_{q,l} \lesssim 10^{-14} \frac{\beta}{\gamma} N_{TC} \frac{\Lambda_{TC}^3}{f_K^2} \ .$$

Using $\Lambda_{TC} \lesssim 10^3$ GeV,

$$m_{q,l} \lesssim 10^{-14} \frac{\beta}{\gamma} N_{TC} \frac{\Lambda_{TC}^3}{f_K^2} \sim \frac{\beta}{\gamma} N_{TC} \mathrm{MeV}, \qquad (4.1)$$

which means that even if $\frac{\beta}{\gamma} \sim 10$ and $N_{TC} \sim 10$ we have $m_{q,l} \lesssim 100$ MeV, which allows the first generation of SM particles but already the second-generation up type quark $m_{charm} \sim 1$ GeV is too heavy. Moreover, we are missing the top quark $m_{top} \sim 100$ GeV by 3 orders of magnitude.

**Precision Electroweak Measurements**

Precision electroweak measurements actually limit the technicolor theories, not extended technicolor [12]. The basic parameters of the SM, $\alpha(M_Z)$, $M_Z$, $\sin^2\theta_W$ are known so precisely that they can limit physics at much higher scales, up to 100 GeV. The quantities most sensitive to such new physics are defined as correlators of electroweak currents:

$$\int \mathrm{d}^4 x e^{-iq\cdot x} \left\langle 0 \left| T \left( j_i^\mu(x) j_j^\nu(0) \right) \right| 0 \right\rangle = i g^{\mu\nu} \Pi_{ij}(q^2) + a q^\mu q^\nu$$

Once one has accounted for all the SM physics, including the single physical Higgs boson with an assumed mass $m_H$, new high energy physics brings additional contributions. The S-parameter, which measures splitting between $m_W$ and $m_Z$ induced by weak-isospin conserving effects, is defined

$$S = 16\pi \left[ \Pi'_{ZZ}(0) - \Pi'_{Z\gamma}(0) \right] \ ,$$

where the $\Pi$'s are self-energy corrections between the particles labeled by the subindices, and the prime means derivative. The experimental limit is [25]

$$S = -0.13 \pm 0.10(-0.08) \ .$$

The central value corresponds to $m_H = 117$ GeV, and the value in parentheses gives the change for $m_H = 300$ GeV. For QCD-like technicolor, Peskin and Takeuchi found

$$S = 4\pi \left( 1 + \frac{m_{\rho T}^2}{m_{\alpha_1 T}^2} \right) \frac{F_\pi^2}{M_{\rho T}^2} \approx 0.25 N_D \frac{N_{TC}}{3} \ ,$$

where $m_{\rho T}^2$ and $m_{\alpha_1 T}^2$ are the masses of the corresponding particles. This is surely of order 1, and shows that if a viable model of technicolor exists, it can not be based on simple rescaling of QCD. In the walking technicolor scenario discussed next, there is reason to believe the $S$-parameter is small, but no reliable way to estimate it.

## 4.3 Walking Technicolor

Until now, we have assumed that technicolor is like QCD. This means that at high energies, the theory is asymptotically free. As we come to lower energies, the renormalization group equation tells that the coupling constant grows. The scale $\Lambda_{QCD}$ is defined as the scale where the coupling becomes large. However, the perturbative expansions which indicate that $\alpha$ grows infinite cannot be trusted.

It is theorized that before this scale, the coupling becomes equal to a critical value $\alpha_c$, at which the vacuum state will include quark-antiquark pairs. In QCD the coupling constant 'runs' in the sense that it declines very quickly with increasing energy. However, there is a priori no reason to assume the same behavior for TC. A slowly evolving, or walking, asymptotically free theory can greatly enhance the condensate at the ETC scale. In the graphs that generate masses, the dominant loop contributions come from the ETC-scale momenta, and thus this enhancement directly enhances our previous mass estimate.

The condensate value can depend on the scale and the coupling $\alpha$. For notational comfort, define $\langle\bar{Q}Q\rangle_{ETC} \equiv N(\mu')$, $\langle\bar{Q}Q\rangle_{TC} \equiv N(\mu)$. Then their dimensionless ratio must, by dimensional analysis, depend on other dimensionless ratios:

$$\frac{N(\mu')}{N(\mu)} \equiv G(\alpha, \frac{\mu'}{\mu})$$

Apply the operator $\mu'\mathrm{d}/\mathrm{d}\mu'$ and evaluate the equation. at $\mu = \mu'$:

$$\frac{1}{N(\mu)}\mu\frac{\mathrm{d}N(\mu)}{\mathrm{d}\mu} = \frac{\mathrm{d}}{\mathrm{d}z}G(\alpha, z \to 1)$$

Define the right hand side as $\gamma_m(\alpha)$. Then the solution is

$$N(\mu) = \exp\left(\int_{\Lambda}^{\mu}\frac{\mathrm{d}\mu'}{\mu'}\gamma_m(\alpha(\mu'))\right)N(\Lambda)$$

or

$$\langle\bar{Q}Q\rangle_{ETC} = \exp\left(\int_{\Lambda_{TC}}^{\Lambda_{ETC}}\frac{\mathrm{d}\mu}{\mu}\gamma_m(\alpha(\mu))\right)\langle\bar{Q}Q\rangle_{TC} \ .$$

In QCD, asymptotic freedom sets in quickly above $\Lambda_{QCD}$. Applying this to technicolor, the technicolor coupling constant $\alpha \propto 1/\ln\mu$ above the TC scale. Then with the anomalous dimension $\gamma_m \propto \alpha$, we have the factor $(\ln(\Lambda_{ETC}/\Lambda_{TC}))^{\gamma_m}$.

Now assume that $\alpha = \alpha^*$ is approximately constant between $\Lambda_{TC}$ and $\Lambda_{ETC}$. Then the corresponding ratio is

$$(\Lambda_{ETC}/\Lambda_{TC})^{\gamma_m(\alpha^*)}$$

This can be a much larger renormalization effect.

#### Anomalous Dimension

To find the anomalous dimension of the techniquark operator $\bar{Q}Q(\mu)$ we must relate it to the fermion self-energy $\Sigma(\mu)$. Assume we have a scale-invariant form for the operator $\Sigma(\mu, k)$:

$$\Sigma(\mu, k) = a\mu\left(\frac{\mu^2}{k^2}\right)^b \ .$$

Here $\mu$ is the renormalization scale and $k$ the momenta. Normally $m = \lim_{\to 0}\Sigma(k)$, but now we have a physical infrared cutoff corresponding to the TC chiral condensate. Thus assume that at the scale $\Lambda_{TC}$ we have the scale-invariant operator

$$\Sigma(\Lambda_{TC}, \Lambda_{TC})\bar{Q}Q(\Lambda_{TC}) \ .$$

Then at the scale $\Lambda_{ETC}$ we must have

$$\Sigma(\Lambda_{ETC}, \Lambda_{TC})\bar{Q}Q(\Lambda_{ETC}) \ .$$

The operator $\bar{Q}Q$ has mass dimension 3, thus the natural scaling is $\bar{Q}Q \propto \mu^3$. The deviation from this gives the anomalous dimension:

$$\frac{\langle\bar{Q}Q\rangle_{ETC}}{\langle\bar{Q}Q\rangle_{TC}} = \left(\frac{\Lambda_{ETC}}{\Lambda_{TC}}\right)^3 \frac{\Sigma(\Lambda_{TC}, \Lambda_{TC})}{\Sigma(\Lambda_{ETC}, \Lambda_{TC})} = \left(\frac{\Lambda_{ETC}}{\Lambda_{TC}}\right)^{3-(1+2b)} = \left(\frac{\Lambda_{ETC}}{\Lambda_{TC}}\right)^{2-2b} ,$$

i.e.

$$\gamma_m = 2 - 2b \ .$$

**Schwinger-Dyson Analysis**

The point of this analysis is to find the value of the anomalous scaling $\gamma_m$. Since the techniquarks condense near the $\Lambda_{TC}$ scale, we must have $\alpha^* \sim \alpha_c$ constant.

The Schwinger equations connect Green's functions to each other. The beginning point for these equations is the assumption, that the generating functional

$$\int \mathcal{D}\phi \exp(iS(\phi) + iA\phi)$$

is invariant if we change the integrand by $\phi \to \phi + \epsilon$, where $\epsilon$ disappears quickly enough. This has an obvious analogy to a usual integral over all space, which is invariant if the variable of the integrand is translated. Now by a Taylor expansion, the integral must satisfy

$$\int \mathcal{D}\phi \frac{\delta}{\delta\phi_x} \exp(iS(\phi) + iA\phi) = 0 \ .$$

By careful calculation we may arrive at the Dyson equation for any Green's function. From the two-point Green's function i.e. the full propagator, we can find an expression for the self energy $\Sigma$ in any specific model. In general, the defining equation is of the form

$$\Sigma = \Delta^{-1} - G^{-1} ,$$

where $G$ is the full propagator, $\Delta$ the zeroth order propagator, and this equation defines $\Sigma$. The exact calculation is impossible, so we have to implement some approximation. In general, the ladder approximation consists of approximating full vertices (third or higher-order Green's functions) in this equation by their tree-level counterparts. Our current case is that of a zero bare mass fermion. While the defining equation is nonlinear, we write the linearized equation in the traditional rainbow approximation in the Landau gauge:

$$\Sigma(p) = \frac{3C_2(R)}{4\pi} \int_0^\infty \frac{\mathrm{d}k^2}{M^2} \alpha(k)\Sigma(k) \ .$$

where $C_2(R)$ is the quadratic Casimir of the complex technifermion representation $R$, $M^2 = \max(k^2, p^2)$, and all operators are renormalized at a scale $\mu$. Deriving this equation we have assumed that the largest contribution comes from $k \sim p$. Thus approximate $\alpha(k) \sim \alpha(p)$ constant, and take it out of the integral. Assume the scale-invariant form

$$\Sigma(k) = a\mu \left(\frac{\mu^2}{k^2}\right)^b \ .$$

Then

$$\begin{aligned} 1 &= \frac{3C_2(R)}{4\pi}\alpha_T \left[p^{2(b-1)} \int_0^{p^2} \mathrm{d}x x^{-b} + p^{2b} \int_{p^2}^\infty \mathrm{d}x x^{-(b+1)}\right] \\ &= \frac{3C_2(R)}{4\pi}\alpha_T \frac{1}{b(1-b)} \ . \end{aligned}$$

Thus we have two solutions,

$$b^{\pm} \equiv \frac{1}{2}\left(1 \pm \sqrt{1-\alpha_T/\alpha_c}\right) \ ,$$

where $\alpha_c = \pi/3C_2$. Thus, since $\alpha_T = \alpha_c$, we have $b^{\pm} = 1/2$, and

$$\Sigma(k,\mu) = C\mu\frac{\mu}{k} \ .$$

where $C$ is some constant. Now, in conjunction with our previous result we have $\gamma_m \sim 1$. Note that this treatment has been nonperturbative. A perturbative analysis would give

$$\gamma_m \sim 1 - \sqrt{1-\alpha_T/\alpha_c} \ ,$$

which corresponds to the other solution $b^{+}$. However, in the perturbative analysis we could not assume $\alpha_T$ to be large.

Theories with this type of scale-invariant behavior occur naturally if the theory is close to a conformal fixed point. Then all these theories, according to this analysis, will produce a significant enchancement of the condensate. Thus it may not be unnatural to assume such behavior for technicolor.

### The Top-quark Mass

Thus the techniquark condensate at the ETC scale has been enhanced by a factor of $\Lambda_{ETC}/\Lambda_{TC}$. This does not affect the term that caused the flavor changing neutral currents, but affects the fermion mass terms. The kaon decay gave $\Lambda_{ETC} \gtrsim 10^3\text{TeV}$. Using $\Lambda_{TC} \sim 1$ TeV, our previous estimate is modified to

$$m_{q,l} \lesssim \frac{\beta}{\gamma} N_{TC}\,\text{GeV},$$

which means that if $\frac{\beta}{\gamma} \sim 1$ and $N_{TC} \sim 10$ we have $m_{q,l} \lesssim 10$ GeV. Comparing to (1.2), we see that all quarks except for the third generation are included in these limits. Also all leptons, with $m_\tau \sim 2$ GeV being the heaviest, are included. This means that in principle, if we assume some other mechanism to generate the third generation quark masses, walking technicolor models are not currently outruled.

# Conclusions

I began this thesis by considering QCD and especially the low-energy dynamics resulting from two phenomena: confinement and chiral symmetry breaking. I found that the chiral symmetry is spontaneously broken, and that the resulting low-energy dynamics can be modeled by the sigma models.

I then examined and explicated the problems of the Higgs sector in the Standard Model. By showing that we can always assume a physical cutoff, I showed that the Higgs scalar particle is unnatural, i.e. its bare mass must be tuned to an absurd precision of $1 : 10^{16}$ to get the correct physical mass. I also showed that the scalar theory is trivial, or that at large energies, the scalar self-coupling vanishes. From this I deduced the Higgs triviality bound $m_{Higgs} \lesssim 500$ GeV, which can still be modified by other interactions.

To solve these problems I removed the Higgs sector and introduced the minimal technicolor model. The model is essentially a replica of QCD at a higher scale. I showed that the model is anomaly free and that it also solves the hierarchy problem. I found that the low energy particle spectrum contains massless or very light technipions as well as technibaryons and techni-vector mesons with a high mass of over 1 TeV.

Finally, I introduced the technicolor theory of flavor: extended technicolor. I showed that it gives Standard Model fermions small masses. I also discussed flavor-changing neutral currents and precision electroweak measurements. To reconcile these phenomenological aspects, and to provide larger masses to Standard Model fermions and technipions, I introduced walking technicolor. I showed that all standard model fermion masses except for the top and bottom quarks can be explained by this type of model, and that technipions are so heavy that they cannot be detected at current particle accelerators. Thus in this thesis it has been shown that, if we do not take the high masses of the third generation quarks into account, walking technicolor models should not be ruled out as extensions of the standard model.

There are many interesting topics that I did not consider. In QCD, the strong CP problem occurs because of a nonperturbative term in the Lagrangian that we cannot overlook. Assuming a natural coefficient would produce a very large outruled dipole moment for the neutron. The weak CP problem is related to the Cabibbo-Kobayashi-Maskawa matrix and its overall complex phase.

With the top quark mass and the CP problem in mind, it would be natural to investigate topcolor assisted technicolor theories. In these theories, one would basically assume the top quark to be the first techniquark. ETC interactions give the top quark a mass of the order of bottom. Most of the top mass is then dynamically generated by the top quark condensate. Topcolor assisted technicolor models could also naturally provide a dynamic picture of CP non-conservation arising from the vacuum alignment in ETC theories. In these models, the strong CP problem can be evaded without the need for any additional particles like the axion. Thus the topcolor assisted technicolor scenario might seem attractive.

Assuming supersymmetry also repairs the unnaturalness of the Higgs particle. Compared to supersymmetry, technicolor theories provide a much more specific low-energy phenomenology. However, they also have less in common with string theory. Neither model can currently be ruled out, but as the Large Hadron Collider at CERN becomes operational in 2008, nature may soon reveal the truth.

# A
# Spontaneous Symmetry Breaking

Symmetries are important not only because of the existence of visible symmetries but also because sometimes they $don't$ exist although we have reason to suspect so. Such a symmetry is called spontaneously broken. In this appendix we will introduce the topic of spontaneous symmetry breaking and give an example of the Higgs mechanism.

The essence of SSB is that the vacuum state, i.e. ground state of the system, does not respect a symmetry of the Lagrangian. Such a situation can present itself in various ways. We can have a Lagrangian with a potential that respects the symmetry originally, but not after it is developed into a power series in the fields around the true minimum. This is concretely choosing the ground state. Physically, such situations can arise if certain parameters of the Lagrangian depend on e.g. temperature.

SSB can also happen more concealedly. This is because the ground state can be modified by the dynamics of the full Lagrangian. This means that the SSB would not be apparent at tree level, i.e. in the Lagrangian, contrary to the previous case. The term dynamical symmetry breaking can be used to refer to SSB that does not show at tree level. The quark condensate and the related symmetry breaking is an example of dynamical symmetry breaking.

If a local symmetry is spontaneously broken, the particle spectrum behaves differently. What happens is that the gauge bosons corresponding to the generators of the broken symmetry become massive. This is the famous Higgs mechanism, and is exactly how we understand the electroweak group breaking into electromagnetism.

## A.1 Prelude

The laws of physics concerning an ordinary chair are rotationally symmetric, but a solution, the chair, is obviously not. By the same reasoning, we expect it is possible that the lowest energy solution of the quantum theory, the vacuum, can be asymmetrical under a transformation that keeps the Lagrangian invariant. [2, 3]

### Multiplets

Consider two states $|A\rangle$ and $|B\rangle$, where $\phi_i^\dagger |0\rangle = |i\rangle$. Furthermore let $[Q, H] = 0$, i.e. $Q$ is a generator of a symmetry group. Assume $[Q, \phi_A^\dagger] = \phi_B^\dagger$. Then defining $U \equiv \exp(i\epsilon Q) \approx 1 + i\epsilon Q$,

$$U^\dagger \phi_A^\dagger U = (1 - i\epsilon Q)\phi_A^\dagger(1 + i\epsilon Q) \approx \phi_A^\dagger - i\epsilon \left[Q, \phi_A^\dagger\right] = \phi_A^\dagger - i\epsilon\phi_B^\dagger \ ,$$

so under the transformation $U$, $|A\rangle$ is rotated towards $|B\rangle$.

To see a multiplet structure, the particles must have the same energy. Thus assume $E_A = E_B$. Then

$$E_A \phi_B^\dagger |0\rangle = E_B |B\rangle = H\phi_B^\dagger |0\rangle = H(Q\phi_A^\dagger - \phi_A^\dagger Q) |0\rangle \ .$$

Rearranging, we get

$$0 = \left[E_A(Q\phi_A^\dagger - \phi_B^\dagger) - H\phi_A^\dagger Q\right] |0\rangle = (E_A - H)\phi_A^\dagger Q |0\rangle \ .$$

We may now conclude either $Q|0\rangle \propto |0\rangle$ or $Q|0\rangle = 0$. We will later see, with the Fabri-Picasso theorem, that the correct conclusion is the latter one. Thus, under the transformation $U$, the vacuum is rotated into itself: $U|0\rangle = |0\rangle$.

This means that a multiplet structure will arise when the vacuum is invariant under the symmetry transformation between the states. Then, when $Q|0\rangle \neq 0$, we see no multiplet structure, and we have spontaneous symmetry breaking.

**Spontaneous Symmetry Breaking and the Infinite Volume Limit**

Suppose we have a Hamiltonian $\mathcal{H}(\phi)$ of a real scalar field satisfying the symmetry $\mathcal{H}(\phi) = \mathcal{H}(-\phi)$. Then the potential $V$ has the property $V(\phi) = V(-\phi)$. The ground state is the state with lowest energy. If $\mathcal{L} = T - V$, then $\mathcal{H} = T + V$, and the ground state corresponds to a constant field minimizing $V$. If the minimum of $V(\phi)$ happens to be at a nonzero value, and we assume so currently, then because of symmetry there are two unequivalent ground states, and we shall call them $|+\rangle$ and $|-\rangle$. Of course,

$$|+\rangle \xrightarrow{\phi\to-\phi} |-\rangle \xrightarrow{\phi\to-\phi} |+\rangle$$

Both states thus correspond to a state of broken symmetry.

However, it is not yet correct to conclude that we have a case of spontaneously broken symmetry. The counterargument is that we might have a linear combination of these states as the true ground state.

Using the aforementioned symmetries, we find

$$\langle +|\mathcal{H}(\phi)|+\rangle = \langle -|\mathcal{H}(\phi)|-\rangle \equiv a$$

$$\langle +|\mathcal{H}(\phi)|-\rangle = \langle -|\mathcal{H}(\phi)|+\rangle \equiv b \ ,$$

where $a$ and $b$ must be real. Assuming usual normalization, we find

$$\mathcal{H}(\phi)|+\rangle = a|+\rangle + b|-\rangle \quad \mathcal{H}(\phi)|-\rangle = a|-\rangle + b|+\rangle \quad ,$$

so the eigenstates of $\mathcal{H}$ are $|+\rangle \pm |-\rangle$, with eigenvalues $a \pm b$. Note that the eigenstates are invariant up to a sign under $\phi \to -\phi$.

But finding the value of $b$ involves an integration of field configurations that tunnel from one minimum to another. Of course calculating this value is a complicated task. But according to my favorite analogue, field theory is just an infinite number of harmonic oscillators connected to each other. In this metaphore, each of these harmonic oscillators must tunnel from one ground state to the other. This must be small probability, and decreasing with volume; in fact, $b$ is smaller than $a$ by a factor proportional to $e^{-volume/L^3}$. Thus for any macroscopic size, the eigenstates are essentially degenerate.

We might also have a perturbation satisfying $\mathcal{H}_p(-\phi) = -\mathcal{H}_p(\phi)$. In this case

$$\langle +|\mathcal{H}_p(\phi)|+\rangle = -\langle -|\mathcal{H}_p(\phi)|-\rangle \equiv c$$

and the off-diagonal elements are very small by the same token as $b$ is. Thus

$$\mathcal{H}_p(|+\rangle \pm |-\rangle) = c|+\rangle \pm (-c)|-\rangle = c(|+\rangle \mp |-\rangle) \ .$$

This means that the eigenstates of $\mathcal{H}$ are not eigenstates of $\mathcal{H}_p$; in fact, they are maximally mixed. Since the ground states of $\mathcal{H}$were degenerate, the true ground state will be very close to one of the eigenstates that diagonalize the perturbed Hamiltonian, and not the eigenstates invariant under $\phi \to -\phi$! Note that this effect becomes exact as the volume goes to infinity, as the non-diagonal contributions tend to zero. In any case, while the perturbation is small, it is impossible to tell which minimum we are located at, because we have $\frac{c}{a} \ll 1$.

## A.2 Theorems

First we must investigate general properties of symmetries. In quantum physics, the Hamiltonian guides the time-evolution of a system. Thus if an operator commutes with the Hamiltonian, its eigenstates will stay eigenstates over time, i.e. we have conservation of something. The situation is similar in quantum field theory, and there is a very important general theorem, the Noether theorem. This theorem states that there is a conserved current. The zeroth component of this current is the corresponding charge, and it is constant in time.

In the case of spontaneous symmetry breaking, this charge becomes badly defined. The Fabri-Picasso theorem makes this notion exact. In that case, there are Goldstone bosons - a massless particle corresponding to each broken generator.

**Noether's theorem**

Suppose our Lagrangian $\mathcal{L}$ is invariant under an infinitesimal transformation [2]

$$\delta\psi_r = -i\epsilon T_{rs}\psi_s \ . \tag{A.1}$$

Then we have

$$0 = \delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\psi_r}\delta\psi_r + \frac{\partial\mathcal{L}}{\partial(\partial^\mu\psi_r)}\partial^\mu(\delta\psi_r) = \partial^\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial^\mu\psi_r)}\delta\psi_r\right) \tag{A.2}$$

where we have used the (classical) Euler-Lagrange equations of motion for the first term. So we have found a conserved current,

$$j_\mu = \frac{\partial\mathcal{L}}{\partial(\partial^\mu\psi_r)}T_{rs}\psi_s \ . \tag{A.3}$$

The corresponding total charge is $Q = \int \mathrm{d}^3x \left\langle 0|j^0(x)|0\right\rangle$. This is constant in time. To see why, note that

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = \int \mathrm{d}^3x \left\langle 0|\partial_0 j^0(x)|0\right\rangle = -\int \mathrm{d}^3x \left\langle 0|\partial_i j^i(x)|0\right\rangle$$

and $j^i$ is proportional to the field which is assumed to vanish smoothly at infinity, so the whole expression vanishes.

If the symmetry is explicitly broken, there is no conserved current. But sometimes the symmetry is broken 'a little'. Let the symmetry be broken by a small additional term in the Lagrangian, $\Delta$. Let $\mathcal{L}$ be the part with a vanishing variation. Then

$$\delta(\mathcal{L}+\Delta) = -i\epsilon\partial_\mu J^\mu = \delta\Delta \ ,$$

where $J$ is the corresponding current. From this we can find an expression for the divergence of the current.

If the symmetry is spontaneously broken, the corresponding current is still conserved. This is the crucial difference between a spontaneously and explicitly broken symmetry.

**The Fabri-Picasso Theorem**

Suppose $\mathcal{L}$ is invariant under a one-parameter continuous global internal symmetry with a conserved Noether current $j^\mu$ and charge $Q$ [2]. We may then prove that either $Q|0\rangle = 0$ or $Q|0\rangle$ has no norm.

Because $Q$ is an internal symmetry, $[P^\mu, Q] = 0$, and $P\,|0\rangle = 0\,|0\rangle$ for the lowest energy eigenstate. Then

$$\left\langle 0|j^0(x)Q|0\right\rangle = \left\langle 0|e^{iP\cdot x}j^0(0)e^{-iP\cdot x}Q|0\right\rangle = \left\langle 0|e^{iP\cdot x}j^0(0)Qe^{-iP\cdot x}|0\right\rangle = \left\langle 0|j^0(0)Q|0\right\rangle$$

and we have the norm of$Q|0\rangle$:

$$\langle 0|QQ|0\rangle = \int \mathrm{d}^3x \left\langle 0|j^0(x)Q|0\right\rangle = \int \mathrm{d}^3x \left\langle 0|j^0(0)Q|0\right\rangle \ .$$

This diverges in the infinite volume limit unless $Q|0\rangle = 0$. If $Q|0\rangle = 0$, we say the vacuum respects the invariance of the Lagrangian. If this is not the case, we say the symmetry is spontaneously broken.

**The Goldstone Theorem**

The Goldstone theorem tells us that in the case of SSB, to every broken generator there is a corresponding massless particle with the same quantum numbers. Consider the transformation of a generic field $\phi$ [4].

$$\phi' = e^{i\epsilon Q}\phi e^{-i\epsilon Q} \approx \phi + i\epsilon[Q, \phi] \ .$$

Although $Q$ may have an infinite norm, the commutator exists always. Current conservation $\partial_\mu j^\mu = 0$ implies

$$\begin{aligned} 0 &= \int \mathrm{d}^3x \, [\partial_\mu j^\mu(x,t), \phi(0)] \\ &= \partial^0 \int \mathrm{d}^3x \left[j^0(x,t), \phi(0)\right] + \int \mathrm{d}^3 S_i \left[j^i(x,t), \phi(0)\right] \ . \end{aligned}$$

The second term vanishes as the integration is taken to infinity because of the inevitable large space-like separation. Hence we can recover

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[Q(t), \phi(0)\right] = 0 \ . \tag{A.4}$$

The commutator is some combination of fields, so we may ask what is its vev,

$$\langle 0| \left[Q(t), \phi(0)\right] |0\rangle \quad \equiv \quad v \ .$$

Inserting a complete set of states after opening the commutator and trading the p-part of the exponent for a delta function we have

$$\begin{aligned} v &= \int \mathrm{d}^3x \left\langle 0\right| \left[j^0(x,t), \phi(0)\right] |0\rangle \\ &= \sum_n (2\pi)^3 \delta^3(p_n) \\ &\quad \left\{\left\langle 0|j^0(0)|n\right\rangle \langle n|\phi(0)|0\rangle \, e^{-iE_n t} - \langle 0|\phi(0)|n\rangle \left\langle n|j^0(0)|0\right\rangle e^{iE_n t}\right\} \end{aligned} \tag{A.5}$$

If we now ask that $v \neq 0$, it should nonetheless be constant in time . The positive and negative frequency parts can not cancel at all times, so we are forced to conclude that the state $|n\rangle$ defined by $p_n = 0$, $E_n = 0$ must have non-vanishing matrix elements $\left\langle 0|j^0(0)|n\right\rangle$ and $\langle n|\phi(0)|0\rangle$. This state is called the Goldstone boson, and because $E_n = 0$, it is massless.

## A.3 The Complex Scalar Field

The scalar field facilitates the simplest example of spontaneous symmetry breaking. We can also demonestrate the Higgs mechanism with it. The Standard Model scalar is a doublet in SU$(2)$, but the basic mechanism remains unchanged.

### Global Spontaneous Symmetry Breaking

Consider the most general renormalizable Lagrangian of a complex scalar field

$$\mathcal{L}_{\text{scalar}} = \partial^\mu \varphi^* \partial_\mu \varphi + m^2 \varphi^* \varphi - \lambda(\varphi^* \varphi)^2 \ . \tag{A.6}$$

The $\phi$ field must occur in combinations of $\phi^* \phi$ because $\mathcal{L}$ must be real. No higher powers can be allowed by renormalizability, since in natural units [action]=[dimensionless], and [distance]=[mass]$^{-1}$, so [$\phi$]=[mass], and any higher power would produce a parameter with negative powers of mass. We must have $\lambda > 0$ or the potential would be unbound from below.

If $m^2 > 0$, the potential has the form of a Mexican hat with a negative true minimum. We can expand around any point on this circle we like, but in doing so we are choosing that value to be the ground state of our system. The original (global) symmetry of $\varphi \to e^{i\beta}\varphi$ will be hidden, as the new fields are small oscillations in the viscinity of the minimum, i.e.

$$\frac{\partial V}{\partial \phi} \mid_{\phi=0} = 0 \ .$$

Parametrize

$$\varphi = (v + \phi(x)) \exp i(\frac{\theta(x)}{v}) \ ,$$

where we assume $\phi$, $\theta$ to be real. The parameter $v$ can be found from requiring the term proportional to $\phi$ to vanish, giving $v^2 = m^2/2\lambda$. Then

$$\mathcal{L}_{\text{broken}} = (\partial^\mu \phi)^2 + (\partial^\mu \theta)^2 + 2m^2\phi^2 + \ldots$$

where the $\ldots$ means interaction terms, i.e. products of at least three fields. Note that we have a massless Goldstone boson $\theta$ and a massive scalar $\phi$. Here we can get intuition about the Goldstone bosons. The Goldstone theorem states that there must be one massless particle for every broken generator. This can be understood because those generators lead us from one minimum to the other, and since the value of the potential does not change for a symmetry transformation, the second derivative of the Lagrangian in that direction must be zero. This means that the field with fluctuations in that direction must be massless.

$\mathcal{L}_{\text{broken}}$ is a result of redefinition of fields and should be perfectly equivalent to (A.6). However, although it does describe the same dynamics, there is a semantic difference; now we have expanded in the fields assuming we are near the minimum and thus $\phi$ and $\theta$ must be considered small. Thus it does not make sense to speak of a symmetry $\theta \to \theta$+const, because this would imply changing the ground state.

In this simple model SSB happened because we had $m^2$ is a varying function of some, e.g. thermodynamic, variables. Then when $m^2 < 0$ we have the symmetric phase. But as $m^2 > 0$, we have spontaneous symmetry breaking and the broken phase. There is always an order parameter connected to such a phase transition; in this case it is precisely $m^2$.

### Higgs Mechanism

Now add a gauge field to the scalar field Lagrangian:

$$\mathcal{L}_{\text{gauge}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

where $F_{\mu\nu}$=$\partial_\mu A_\nu - \partial_\nu A_\mu$. This term is invariant under a gauge transformation $A_\mu \to A_\mu - \partial_\mu \eta(x)$. Adding a mass term to the photon would ruin this symmetry. However, if we replace the derivative of the scalar field with the covariant derivative, $\partial_\mu \to D_\mu = \partial_\mu - ieA_\mu$, our full Lagrangian is invariant under a local group $\text{U}(1)_{\text{local}}$:

$$\begin{aligned} A_\mu &\to A_\mu - \partial_\mu \eta(x) \\ \varphi &\to e^{-ie\eta(x)}\varphi \end{aligned}$$

Yet the same discussion applies for the potential, i.e. it is invariant under the *global* U(1). It does not matter, however, for the following discussion: if we parametrize the $\varphi$ field as before, and choose the ground state, both the global and local symmetries are broken, and the transformations make no sense thereafter. So we might as well speak only of the local symmetry. After calculation, we have

$$\begin{aligned}\mathcal{L}_{\text{scalar+gauge,broken}} &= \mathcal{L}_{\text{gauge}} + (\partial^\mu \phi)^2 + 2m^2\phi^2 + (\partial^\mu \theta)^2 \\ &\quad -2evA^\mu \partial_\mu \theta + e^2 v^2 A_\mu A^\mu + \dots\end{aligned}$$

Now our $\mathrm{U}(1)_{\text{local}}$ is broken by the previous reasoning. Note that the mass of the vector field is proportional to $v$. This is usually true; after SSB, the massive gauge bosons will have masses of the order of the symmetry breaking scale.

To remove the mixing term, remember that $A_\mu$ has unphysical degrees of freedom manifesting in the gauge freedom. Thus fix the gauge by choosing $A_\mu \to A_\mu + \beta\partial_\mu\theta$. Note that this is not a gauge transformation of the $\mathrm{U}(1)_{\text{local}}$. It does not leave the Lagrangian formally invariant. The resulting Lagrangian is

$$\mathcal{L}_{\text{final}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (\partial^\mu \phi)^2 + 2m^2\phi^2 + e^2v^2A_\mu A^\mu + \dots$$

The unphysical $\theta$-field has disappeared completely. We say the Goldstone boson has been 'eaten', and it has become the extra degree of freedom of the now massive gauge field. Also we have a remaining massive scalar field $\phi$.

This is an example of the Higgs mechanism, and $\phi$ is the physical Higgs particle. As we saw, a vev of the Higgs broke a local symmetry, and the corresponding Goldstone boson disappeared while the corresponding gauge boson became massive. This is how the Higgs mechanism works in general. In the SM, the scalar field's vacuum expectation value is also used to give fermion fields masses through Yukawa couplings $g\phi\bar{\psi}\psi$.

# Bibliography


[1] M. Peskin and D. Schroeder, "An Introduction to Quantum Field Theory" (Westview Press, 1995), Ch. 11, 12, 19, 20.

[2] I. Aitchison and A. Hey, "Gauge Theories in Particle Physics " (Institute of Physics Publishing, London, 2004), Vol. 2, Part 7.

[3] S. Weinberg, "The Quantum Theory of Fields" (Cambridge University Press, 1996), Vol. 2, Ch. 18, 19, 22.

[4] Ta-Pei Cheng and Ling-Fong Li, "Gauge Theory of Elementary Particle Physics" (Oxford University Press, 1988), Ch. 4.

[5] M. Srednicki, "Quantum Field Theory" (Cambridge University Press, 2007), Ch. 83.

[6] V. Miransky, "Dynamical Symmetry Breaking in Quantum Field Theories" (World Scientific Publishing, Singapore, 1993), Ch. 12, 14.

[7] E. Farhi and L. Susskind, "Technicolor," Phys. Rept. **74**, 277 (1981).

[8] C. T. Hill and E. H. Simmons, "Strong dynamics and electroweak symmetry breaking," Phys. Rept. **381**, 235 (2003) [Erratum-ibid. **390**, 553 (2004)] [arXiv:hep-ph/0203079].

[9] K. D. Lane, "An Introduction To Technicolor," arXiv:hep-ph/9401324.

[10] K. D. Lane, "Technicolor," arXiv:hep-ph/9501249.

[11] K. D. Lane, "Technicolor 2000," arXiv:hep-ph/0007304.

[12] K. Lane, "Two lectures on Technicolor," arXiv:hep-ph/0202255.

[13] J. D. Wells, "TASI lecture notes: Introduction to precision electroweak analysis," arXiv:hep-ph/0512342.

[14] S. R. Coleman and E. Weinberg, "Radiative Corrections As The Origin Of Spontaneous Symmetry Breaking," Phys. Rev. D **7** (1973) 1888.

[15] L. Susskind, "Dynamics Of Spontaneous Symmetry Breaking In The Weinberg-Salam Theory," Phys. Rev. D **20**, 2619 (1979).

[16] S. Dawson, "The MSSM and why it works," arXiv:hep-ph/9712464.

[17] R. S. Chivukula, "Models of electroweak symmetry breaking," arXiv:hep-ph/9803219.

[18] S. Eidelman *et al.* [Particle Data Group], "Review of particle physics," Phys. Lett. B **592** (2004) 1.

[19] E. Eichten and K. D. Lane, "Dynamical Breaking Of Weak Interaction Symmetries," Phys. Lett. B **90**, 125 (1980).

[20] T. W. Appelquist, D. Karabali and L. C. R. Wijewardhana, "Chiral Hierarchies and the Flavor Changing Neutral Current Problem in Phys. Rev. Lett. **57**, 957 (1986).

[21] T. Appelquist and L. C. R. Wijewardhana, "Chiral Hierarchies And Chiral Perturbations In Technicolor," Phys. Rev. D **35**, 774 (1987).

[22] T. Appelquist, K. D. Lane and U. Mahanta, "ON THE LADDER APPROXIMATION FOR SPONTANEOUS CHIRAL SYMMETRY BREAKING," Phys. Rev. Lett. **61**, 1553 (1988).

[23] U. Mahanta, "WALKING TECHNICOLOR BEYOND THE LADDER APPROXIMATION," Phys. Rev. Lett. **62**, 2349 (1989).

[24] D. Kominis and R. S. Chivukula, "Triviality bounds in two doublet models," Phys. Lett. B **304** (1993) 152 [arXiv:hep-ph/9301222].

[25] J. Erler and P. Langacker, "Electroweak model and constraints on new physics," arXiv:hep-ph/0407097.