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Noncommutative Gravitation as a Gauge Theory of Twisted Poincaré Symmetry

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Chapter 1

Introduction

1.1 Spacetime noncommutativity

1.1.1 The idea of spacetime noncommutativity

The traditional concept of spacetime describes space and time as a continuum of points, or more precisely as a differentiable manifold which is locally isomorphic to $\mathbb{R}^4$. On a spacetime manifold $\mathcal{M}$ every point can be locally identified with a finite number of real coordinates $x^\mu \in \mathbb{R}^4$. The differentiability of the spacetime manifold insures that the local coordinates are continuous and smooth — two points with infitesimally differing coordinates will be infinitesimally close to each other on the manifold.[1]

Although this description of spacetime has been very successful, it is widely believed in the physics community that the manifold structure of spacetime should break down at very short distances of the order of the *Planck length*

$$l_{\mathrm{P}} = \sqrt{\frac{\hbar G}{c^3}} \approx 1.6 \cdot 10^{-35}\,\mathrm{m} \tag{1.1}$$

which corresponds to the *Planck energy* [2]

$$E_{\mathrm{P}} = \frac{\hbar c}{l_P} \approx 1.2 \cdot 10^{19}\,\mathrm{GeV}\ , \tag{1.2}$$

where $\hbar$ is the Planck constant, c is the speed of light and G is the Newtonian constant of gravitation. At these super-short distances physical phenomena are believed to be *nonlocal* — opposed to the locality of traditional geometrical theories of gravitation and quantum and gauge field theories of particle physics. In order to capture this nonlocality, the mathematical concepts used to describe spacetime in high energy physics should be revised.

The idea of spacetime noncommutativity has been adopted to meet this demand. Spacetime noncommutativity is a way to deform the classical spacetime, so that nonlocality becomes its characteristic feature. Noncommutativity of spacetime

[1] Of course the concept of distance requires that a metric is defined on the manifold.

[2] The energy equivalent of the *Planck mass* m_{P}: $E_{\mathrm{P}} = m_{\mathrm{P}} c^2$.

means that the notion of a point is no longer well-defined and therefore noncommutative spacetime is literally pointless. On such a noncommutative spacetime physical phenomena are naturally nonlocal. Formally this can be achieved by defining the coordinate operators $\hat{x}^\mu$ of noncommutative spacetime to satisfy the commutation relations

$$\left[\hat{x}^\mu, \hat{x}^\nu\right] = i\theta^{\mu\nu} , \tag{1.3}$$

where in the simplest case $\theta^{\mu\nu}$ is an antisymmetric constant matrix of dimension length squared. This kind of spacetime cannot be described with traditional differential geometry. Thus new mathematics is needed. The branch of mathematics which is used to describe noncommutative spaces is called *noncommutative geometry* and it is based on the algebraic approach to the geometry of spaces.

In the context of quantum mechanics the noncommutativity of coordinates (1.3) implies that the spacetime has to be replaced by a Hilbert space of states, i.e. the spacetime itself becomes a quantum object. The noncommutativity of coordinate operators induces the uncertainty relations for coordinates

$$\Delta\hat{x}^\mu \, \Delta\hat{x}^\nu \geq \frac{1}{2} \left|\theta^{\mu\nu}\right| , \tag{1.4}$$

so that a spacetime point is replaced by a Planck cell of dimension given by the Planck area. One may think of ordinary spacetime coordinates x^μ as macroscopic order parameters obtained by coarse-graining over scales smaller than the fundamental scale of order $\sqrt{|\theta|}$.

In addition to the inherent nonlocality, noncommutativity of spacetime coordinates has several important and interesting implications in theoretical high energy physics. These include especially violation of the Lorentz invariance and formulation of gauge symmetries in noncommutative spacetime, both of which we will discuss in this work.

Because the idea of spacetime noncommutativity is far from trivial and because its implications are so profound, we will motivate the idea by briefly reviewing the main arguments for spacetime noncommutativity.

1.1.2 Quantum mechanics

The idea behind spacetime noncommutativity is very much inspired by quantum mechanics. Noncommutativity is the central mathematical concept expressing uncertainty in quantum mechanics, where it applies to any pair of conjugate variables, such as position and momentum.

The classical phase space of canonical position and momentum variables x^i, p_j is quantized by replacing the variables with Hermitean operators $\hat{x}^i, \hat{p}_j$ which obey the Heisenberg commutation relations

$$[\hat{x}^i, \hat{p}_j] = i\hbar\delta^i_j . \tag{1.5}$$

As a result the classical phase space dissapears and it is replaced by a Hilbert space of states. The noncommutativity of canonically conjugated operators implies the

Heiseberg uncertainty principle of quantum mechanics, which states that noncommuting observables cannot be exactly measured simultaneously. Generally for a pair of noncommuting observables $\hat{A}$ and $\hat{B}$, the commutation relation

$$[\hat{A}, \hat{B}] = i\hat{C}$$

implies the general Heisenberg uncertainty principle

$$\Delta\hat{A}\,\Delta\hat{B} \geq \frac{1}{2}\left|\langle\hat{C}\rangle\right| , \tag{1.6}$$

where $\Delta\hat{A}$ is the standard deviation for the observable $\hat{A}$

$$\Delta\hat{A} = \sqrt{\langle(\hat{A} - \langle\hat{A}\rangle)^2\rangle} .$$

For the canonical position and momentum operators the uncertainty relation (1.6) is written as

$$\Delta\hat{x}^i\,\Delta\hat{p}_j \geq \frac{\hbar}{2}\delta^i_j . \tag{1.7}$$

So, in the scale defined by the Planck constant $\hbar$ the quantum phase space becomes smeared out and the notion of point is replaced with that of a Planck cell. In the (classical) limit $\hbar \to 0$ one recovers the classical phase space.

It was von Neumann who first attempted to rigorously describe such a "quantum space" and he dubbed this study "pointless geometry", referring to the fact that the notion of a point in a quantum phase space is meaningless. This led to the theory of von Neumann algebras and it was essentially the birth of noncommutative geometry, referring to the study of topological spaces whose commutative C^*-algebras of functions are replaced by noncommutative algebras [1, 2, 3, 4].

In this setting the idea to define the noncommutativity relation among coordinate operators (1.3) arises quite naturally. Already in the pioneering days of Quantum Field Theory (QFT) it was suggested by Heisenberg that one could use a noncommutative structure for spacetime coordinates at very small length scales to introduce an effective ultraviolet cutoff. This suggestion was motivated by the need to regularize divergences which had troubled QFT from the very beginning. It was H. S. Snyder who wrote the first paper on the subject [5], where he introduced a Lorentz invariant "quantized spacetime" whose coordinates x_μ are operators obeying the commutation relations

$$[x_\mu, x_\nu] = i\frac{a^2}{\hbar}L_{\mu\nu} ,$$

where a is a fundamental length unit and $L_{\mu\nu}$ are the generators of the Lorentz group. Few months later C. N. Yang tried to restore the translational invariance broken by Snyder's model [6]. Back then as well as today, noncommuting spacetime coordinates were used in the hope of improving the renormalizability of QFT and of understanding the nonlocality of physics at the Planck scale.

However, the research of noncommutative spacetime did not take off well and it was soon forgotten for several decades. The difficult theoretical problems caused by the *a priori* nonlocality and by the violation of Lorentz invarince in noncommutative field theories were too undesirable and unfruitful at that time. Success of the renormalization programme in controlling divergences of QFT also furthered the forgetting of spacetime noncommutativity.

1.1.3 Noncommutative geometry

The correspondence between geometric spaces and commutative algebras is a well known and basic idea of algebraic geometry. Noncommutative geometry generalizes this correspondence to noncommutative algebras. In the physical applications of noncommutative geometry discussed in this work, we are interested in the correspondence between noncommutative algebras of functions on a space and the geometry of the underlying noncommutative space.

The ideas of noncommutative geometry were revived in the 1980's by the mathematicians Connes, Drinfel'd and Woronowicz. They generalized the notion of a differential structure to the noncommutative setting [7, 8, 9, 10], i.e. to arbitrary C^*-algebras, and also to quantum groups and matrix pseudo-groups. Along with the definition of a generalized integration [11], this led to an operator algebraic description of noncommutative spacetimes — based entirely on algebras of functions — and it enabled one to define Yang-Mills gauge theories on a large class of noncommutative spaces. For quite some time, the physical applications were based on geometric interpretations of the standard model and its various fields and coupling constants (the so-called Connes–Lott model) [12, 13, 14]. Gravity was also eventually introduced in a unifying way [15, 16, 17, 18, 19]. Unfortunately this approach suffered from many weaknesses — most glaring was the problem that quantum radiative corrections could not be incorporated in order to give satisfactory predictions — and eventually it died out. Nevertheless, thanks to these mathematicians, the idea of spacetime noncommutativity became again very much alive.

1.1.4 General relativity and quantum mechanical measurements

More evidence for the spacetime noncommutativity came from the works of Doplicher, Fredenhagen and Roberts [20, 21]. They showed that combining quantum mechanical measurements obeying Heisenberg's uncertainty principle (1.6) with Einstein's theory of classical gravitation, leads to the conclusion that ordinary spacetime loses all operational meaning at short distances. Their argument was: Measuring a spacetime coordinate x with high accuracy Δx causes an uncertainty in a conjugated momentum of the order $\hbar/\Delta x$. Neglecting rest masses, an energy of the order $\hbar c/\Delta x$ is transmitted to the system and concentrated at some time in the localization region of the measurement. The energy–momentum tensor $T_{\mu\nu}$ associated to the energy concentration generates a gravitational field which, in principle, should be determined by solving Einstein equations for the metric tensor $g_{\mu\nu}$

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu} \ .$$

The smaller the uncertainties Δx^μ in the measurement of coordinates, the stronger will be the gravitational field generated by the measurement. When the gravitational field becomes so strong that it prevents light or any other signals from leaving the localization region, an operational meaning can no longer be attached to the localization.

Analysing the limitations of localization measurements — for the sake of possible gravitational collapse of the localization region —, using a semiclassical approximation for an unknown theory of quantum gravitation, leads to uncertainty relations among spacetime coordinates

$$\Delta x^0 \sum_{j=1}^{3} \Delta x^j \gtrsim l_{\mathrm{P}}^2 \ , \qquad \sum_{1\geq j\geq k\geq 3} \Delta x^j \Delta x^k \gtrsim l_{\mathrm{P}}^2 \ .$$

These relations can be traced back to the commutation relations among coordinates (though not uniquely)

$$[x^\mu, x^\nu] = iQ^{\mu\nu} \ ,$$

where Q is a tensor whose components $Q^{\mu\nu}$ commute with all coordinates. Thus, the presence of classical gravitation makes the spacetime effectively noncommutative and this feature should be present in any quantum theory of gravitation.

The paper [21] also introduced the fundamentals of QFT on noncommutative spacetimes.

1.1.5 String theory with constant background field

String theory is the best candidate for a quantum theory of gravitation. Therefore it has an important role in the study of the structure of spacetime.

String theory has the built-in characteristics of nonlocality and uncertainty of coordinate measurements at short distances. It is the finite mean length of strings l_{s} that necessarily makes physics nonlocal and forces the shortest length that can be observed by using the strings as probes. Hence, it was not a big suprise when noncommutative spacetime coordinates began to repeatedly emerge in the research of string theory.

String theory is one of the strongest reasons why spacetime noncommutativity and noncommutative gravitation has been studied so much during the last decade. In the end of the 1990s it was discovered that certain limits of string theory and M-theory will directly lead to noncommutative gauge theories [22, 23], which are simpler than the original theories but still preserve some of their stringy features like nonlocality. N. Seiberg and E. Witten developed the idea by elegantly proving that when the end points of strings in a theory of open strings are constrained to move on D-branes in a constant (supergravity) B-field background and the theory is taken in a certain low-energy limit, then the full dynamics of the theory is described by a (supersymmetric) gauge theory on a noncommutative spacetime [24]. In this Seiberg-Witten (low-energy) limit the open string modes completely decouple from the closed string modes and only the end point degrees of freedom for the open strings are left to live on a noncommutative spacetime defined by the coordinate commutation relations (1.3). Thus noncommutative gauge theory emerges as a low-energy limit of open string theory with constant antisymmetric background field.

Because the closed string modes decouple in the Seiberg-Witten limit, the resulting gauge theories do not have graviton, the quantum of gravitation. Nevertheless, noncommutative gravitation can be studied in the Seiberg-Witten limit by considering first order corrections for the closed string modes. This approach has already provided us important information about noncommutative gravitation and twisted symmetries [25, 26]. In string theory gravitational interactions have much richer dynamics than in some other noncommutative deformations of GR [26] — especially than the ones based on the invariace under the naive twisted diffeomorhisms [27, 28].

1.2 Gravitation as a gauge theory

1.2.1 The big picture of interactions

Our best experimentally verified knowledge about the fundamental interactions of nature can be summarized in the Standard Model of elementary particle physics and in the theory of General Relativity.

The Standard Model of elementary particle physics (SM) is a gauge field theory based on the gauge symmetry group

$$SU(3)_{\mathrm{C}} \times SU(2)_{\mathrm{L}} \times U(1)_{\mathrm{Y}} \ ,$$

where "C" refers to the color symmetry of quarks in quantum chromodynamics, "L" refers to the doublets of left-handed fermions in the electroweak theory and "Y" refers to the (weak) hypercharge. SM is defined on the Minkowski spacetime, where it can be consistently quantized. The spacetime in SM is an invariable background where all events take place.

The General Relativity (GR) on the other hand is a classical geometrical theory of gravitation. It describes gravitation geometrically by coupling the curvature of a spacetime manifold with the energy-momentum tensor of matter and radiation fields. Hence, in GR the density and movement of matter determine how spacetime curves and the curvature of spacetime determines how matter moves in space and time. Due to the zero torsion condition of spacetime in GR the curvature is uniquely defined by the metric tensor. Thus the fundamental dynamical variable of the theory is the metric of spacetime, i.e. the spacetime itself. Unfortunately, GR cannot be consistently quantized because of its pathological renormalization characteristics.

GR describes gravitation at the macroscopic and cosmic levels and SM describes the world of particles in the sub-atomic level. Hence the areas of applicability for these theories are far apart. Unfortunately, the weakeness of gravitation compared to the other three interactions has prevented us from observing gravitational interactions between elementary particles. Thus, we do not know well how gravitation behaves at the sub-atomic level, which makes the construction of quantum theories of gravitation harder.

Both GR and SM describe and predict physical phenomena in their separate areas of applicability with unparalleled accuracy and success. The problem is that the theoretical frameworks of SM and GR are so contradictory that they cannot be unified. This theoretical conflict between SM and GR is severe and well known. It has been the most fundamental problem in theoretical physics for decades.

1.2.2 Gauge symmetry approach

In the gauge theories of particle physics both the existence of the gauge fields and the ways they couple to matter fields are necessary consequences of the local gauge symmetry of the theory. Also the structure of the free Lagrangean for the gauge fields is defined by the gauge symmetry, so that the key characteristics of the gauge fields are also implied by the gauge invariance. Since the role of gauge fields is to mediate interactions between matter fields, a local gauge symmetry de facto defines characteristics of all interactions of a gauge theory. This is the reason why the concept of a gauge symmetry is so powerful and without dispute one of the most important concepts in modern theoretical physics.

In order to understand the contradiction between GR and SM, it is important to understand that gravitation — like all other fundamental interactions — can be formulated as a gauge theory. In the pioneering work by R. Utiyama [29], gauge theories were elegantly generalized and finally gravitation was considered as a gauge theory of the Lorentz symmetry. Few years later T. W. B. Kibble developed the idea by constructing a gauge theory of the Poincaré symmetry [30] and by rediscovering the Einstein-Cartan theory of gravitation in a form more familiar to most physicists. Since then the idea of gravitation as a gauge theory has been elaborated by many people. An incomplete list of references on the subject is [31] [32] [33] [34, 35, 36] [37] [38] [39] [40] [41]. The paper by F. W. Hehl et al. [37] contains an excellent historical guide to the literature on the subject, including a complete list of references. This study adopts the more recent point of view by first considering gauge theory of gravitation on Minkowski spacetime [39, 42] and by later intepreting the theory geometrically.

By understanding gravitation as a gauge theory we achieve several advantages compared to GR. A gauge theory of gravitation explains both the existence of gravitation and its properties as necessary consequences of a single symmetry principle. The second major advantage is the unification of the conceptual basis of theories of fundamental interactions. This enables an elegant interpretation where all interactions are results of two gauge symmetries — an external one for gravitation and an internal one for SM. Further advantage is the weakening of the theoretical connection between gravitation and the geometry of spacetime, especially when an internal-like gauge symmetry is used. All this enables us to better study the similarities and differences between SM and the theory of gravitation. The unification of a gauge theory of gravitation and SM has only succeeded in the (trivial) case when the gauge symmetry generators for gravitation and SM commute. Nevertheless, the conceptual unification enabled by the gauge symmetry approach to gravitation is a promising implication on the underlying unity of fundamental interactions.

1.3 Noncommutative gravitation

Noncommutative theories of gravitation have been under intense research over two decades. Research on noncommutative gravitation can be roughly classified to three categories based on the focus and the research strategy. The first category contains studies which are mainly based on noncommutative geometry and are directed to the search of consistent noncommutative deformations of Einstein's GR or more generally noncommutative deformations of Riemannian geometries. The second line of study is concentrating to seek noncommutative gravitation by studying string theories in certain low-energy limits — namely the Seiberg-Witten limit. The third approach is seeking for a grand symmetry principle in the framework of canonical noncommutative field theory that could lead us to a consistent theory of noncommutative gravitation. There are also some studies that utilize more than one of these methods. This work fits best to the last category.

The critical challenges in the construction of any noncommutative theory of gravitation are to find some dynamical principle to follow while deforming GR and to consistently implement the concept of a general coordinate transformation in the framework of noncommutative field theory. The history of field theories has taught us that the best guiding dynamical principles are symmetries, for example spacetime isometries and gauge symmetries. We believe that the right guiding symmetry principle can also provide us the information of how to implement the general coordinate covariance in noncommutative gravitation.

The main purpose of this work is to bring the idea of spacetime noncommutativity and the idea of gravitation as a gauge theory together in order to study the possibility to construct a theory of gravitation in noncommutative spacetime as a gauge theory of twisted Poincaré symmetry. We will use the twisted Poincaré symmetry [43, 44] as the candidate gauge symmetry for noncommutative gravitation because it is the closest analogy for the Poincaré symmetry in noncommutative spacetime and because we prefer the Poincaré group as a gauge symmetry group for the classical gauge theory of gravitation.

It is not yet known whether the twisted Poincaré symmetry can or cannot be consistently generalized to a local gauge symmetry in noncommutative spacetime. If this is possible we want to know if the resulting gauge theory is a viable theory of gravitation in noncommutative spacetime. If noncommutative gravitation cannot be formulated as a gauge theory of twisted Poincaré symmetry, we will hopefully learn something on how it should be formulated instead.

1.4 Structure of this study

We will begin our journey by reviewing the essential concept of invariance under Poincaré transformations in traditional commutative field theories. Poincaré symmetry will be presented as an internal-like symmetry for maximal similarity with non-Abelian gauge theories like SM. We will generalize the global Poincaré symmetry to a local gauge symmetry and thereby construct a classical gauge theory of the Poincaré symmetry, which we will show to be a viable theory of gravitation. This construction and the resulting theory will serve us as a classical reference when we later move to the framework of noncommutative spacetime.

Next we will discuss noncommutative spacetime and introduce the concepts that are needed in order to define and study noncommutative quantum and gauge field theories. Special emphasis is given to the concept of twisted Poincaré symmetry, which provides a new concept of relativistic invarince for noncommutative field theories [43, 44]. We will, for instance, discuss how imposing the twisted Poincaré symmetry defines the same noncommutative algebra of functions — i.e. geometry of spacetime — which is generated by the noncommutativity of coordinates (1.3).

We will continue by discussing the formulation of gauge theories in noncommutative spacetime, because they are essentially important in building realistic models. Since gauge symmetries are local and noncommutative spacetime is inherently nonlocal, care has to be taken when formulating noncommutative gauge theories. We will consider different ways to implement gauge symmetries in noncommutative spacetime.

Finally we will try to put it all together in order to generalize the gauge theory approach to gravitation in the noncommutative setting. We will also discuss how this approach is related to other proposed noncommutative theories of gravitation.

Chapter 2

Gravitation as a classical gauge theory of the Poincaré symmetry

2.1 Importance of the Poincaré symmetry

One cornerstone of all local relativistic field theories is the concept of relativistic invariance, which is traditionally implemented by demanding invariance of the action under the group of Poincaré transformations.[1] Therefore, every field in theories like SM has to be a representation of the Poincaré group. Moreover, the fields in the action of a relativistic field theory have to appear in specific combinations that are invariant under Poincaré transformations. This has a major effect on the characteristics of the fields and on the ways the fields interact with each other.

Because the Poincaré symmetry has such an important role in relativistic physics and especially in this work, we will first introduce the Poincaré group in some detail and then discuss the global Poincaré symmetry. Next we will develop a gauge theory of the Poincaré symmetry. Finally we will give a geometrical interpretation of the theory and compare it with GR.

In this chapter we use the natural high energy units with the Planck constant and the speed of light set to 1, $\hbar = c = 1$, in order to emphasize the essential content of the formulae.

2.2 Poincaré group

2.2.1 Definition and structure

The Poincaré group is the maximal symmetry group of the Minkowski spacetime. In other words, the Poincaré group is the complete group of isometries of the Minkowski spacetime.

The Poincaré group is a 10-dimensional noncompact *Lie group*. The group of spacetime translations is a *normal subgroup* of the Poincaré group. The group of Lorentz transformations is a subgroup of the Poincaré group and it is a 6-dimensional

[1] In the axiomatic (algebraic) approach to QFT the Poincaré invariance is one of the axioms of the theory.

non-Abelian Lie group. The Poincaré group can be constructed as a semidirect product of the translation group $\mathcal{T}_4$ and of the homogeneous Lorentz group $SO(1,3)$

$$\text{Poincaré group} = SO(1,3) \ltimes \mathcal{T}_4 \ .$$

Finite Poincaré transformations of spacetime coordinates x^μ are defined by [2]

$$x^\mu \longrightarrow x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu \quad ; \ \Lambda^\rho{}_\mu \Lambda^\sigma{}_\nu \eta_{\rho\sigma} = \eta_{\mu\nu} \ , \tag{2.1}$$

where the matrix $\Lambda^\mu{}_\nu$ provides the Lorentz transformation and a^μ are the translation parameters. If $\det \Lambda = 1$, we speak about *proper* Lorentz transformations.[3]

The ten generators of the Poincaré group constitute a *Lie algebra* named the Poincaré algebra $\mathcal{P}$ [4]

$$[P_\mu, P_\nu] = 0 \tag{2.2a}$$

$$[M_{\mu\nu}, P_\rho] = -i(\eta_{\mu\rho} P_\nu - \eta_{\nu\rho} P_\mu) \tag{2.2b}$$

$$[M_{\mu\nu}, M_{\rho\sigma}] = -i(\eta_{\mu\rho} M_{\nu\sigma} - \eta_{\mu\sigma} M_{\nu\rho} - \eta_{\nu\rho} M_{\mu\sigma} + \eta_{\nu\sigma} M_{\mu\rho}) \ , \tag{2.2c}$$

where the M-matrix is antisymmetric $M_{\mu\nu} = -M_{\nu\mu}$. The generators $M_{\mu\nu}$ form a closed subalgebra, which is the Lie algebra of the Lorentz group.[5] The generators of the Lorentz group can be divided into the familiar generators for boosts $K_i = M_{0i}$ and for spatial rotations $J_i = \frac{1}{2}\epsilon_{ijk} M_{jk}$, where the Latin indices take the three "spatial" values $\{1,2,3\}$ and ϵ is the fully antisymmetric permutation symbol. The generators of translations P_μ form a commutative subalgebra of the Poincaré algebra, reflecting the fact that the translation subgroup is Abelian. We have introduced the imaginary unit i in the Poincaré algebra (2.2), which is a common practice in the gauge theories of particle physics and in quantum mechanics in general, so that the generators of the Poincaré group are Hermitian.

The relation between the elements of the Poincaré group and the Lie algebra (2.2) is provided by the exponential map

$$U(\varepsilon) = \exp(i\varepsilon^\mu P_\mu) \tag{2.3a}$$

$$U(\omega) = \exp\left(\frac{i}{2}\omega^{\mu\nu} M_{\mu\nu}\right) \ , \tag{2.3b}$$

where ε^μ and $\omega^{\mu\nu} = -\omega^{\nu\mu}$ are the ten parameters of the Poincaré group manifold. On unitary reprsentations of the Poincaré group, the group elements are represented by unitary operators (2.3).

2.2.2 Representation theory

Basics

In defining representation of the Poincaré group — the algebra of smooth (C^∞) functions on the Minkowski spacetime — the generators of infinitesimal Poincaré

[2] The Greek indices take the four values $\{0,1,2,3\}$. A sum over repeated indices is always implied unless otherwise stated.

[3] $\det \eta = -1$ and (2.1) imply $(\det \Lambda)^2 = 1$.

[4] Also named $\mathfrak{iso}(1,3)$ after the inhomogeneous proper Lorentz group which is just another name for the Poincaré group.

[5] Named $\mathfrak{so}(1,3)$ after the proper Lorentz group.

transformations are the momentum operators and the generalized orbital angular momentum operators

$$P_\mu = i\partial_\mu \ , \tag{2.4a}$$
$$M_{\mu\nu} = i(x_\mu\partial_\nu - x_\nu\partial_\mu) \ , \tag{2.4b}$$

respectively.

The theory of *unitary representations* for the Poincaré group is divided to two parts. For physical reasons these are called massive and massless representations. Additional representations of interest in relativistic physics are finite dimesional matrix and tensor representations of the Lorentz group. A complete classification of irreducible representations of the Poincaré group was first introduced by E. P. Wigner in [45].

Quadratic *Casimir operators* of a Lie algebra are distinguished elements of the centre of the *universal enveloping algebra* of the Lie algebra.[6]. The quadratic Casimir operators commute with all generators of the Lie algebra and therefore they can be used to classify representations of the Lie algebra.

First we discuss the representation theory of the Lorentz group — emphasizing the importance of the rotation Lie group $SU(2) \cong SO(3)/\mathbb{Z}_2$ generated by the angular momentum operators [7] — and then we present a physically motivated treatment of the representation theory of the Poincaré group.

For a more accessible and comprehensive introduction to the group theory, to the representation theory of groups and to their physical applications, particularly in the quantum theory, see e.g. [46, 47].

Finite dimensional representations of the Lorentz group

The Lorentz algebra (2.2c) of the generators of rotations $J_i = \frac{1}{2}\epsilon_{ijk}M_{jk}$ and boosts $K_i = M_{0i}$

$$\begin{aligned} [J_i, J_j] &= i\epsilon_{ijk}J_k \ , \\ [K_i, J_j] &= i\epsilon_{ijk}K_k \ , \\ [K_i, K_j] &= -i\epsilon_{ijk}J_k \ , \end{aligned} \tag{2.5}$$

can be rewritten as two independent rotation $SU(2)$ algebras by introducing the combinations

$$X_i^\pm = \frac{1}{2}(J_i \pm iK_i) \tag{2.6}$$

as the generators of the two $SU(2)$ Lie algebras

$$\begin{aligned} [X_i^\pm, X_j^\pm] &= i\epsilon_{ijk}X_k^\pm \ , \\ [X_i^+, X_j^-] &= 0 \ . \end{aligned}$$

The rotation algebra (2.5) has a single quadratic Casimir operator $\boldsymbol{J}^2 = J_iJ_i$ which has the eigenvalues $j(j+1)$, $j = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots$ and therefore the irreducible representations of $SU(2)$ are labelled with the integer and half-integer valued quantum

[6] We will discuss universal enveloping algebras later in the section 3.3.

[7] The group: $\mathbb{Z}_2 = (\cdot, \{1, -1\}) \cong$ integers modulo 2.

number j. The fundamental $j = \frac{1}{2}$ representation of $SU(2)$ is given by the Pauli σ-matrices. Thus the irreducible representations of the Lorentz group are labelled by the pair (j_+, j_-) of these quantum numbers $j_\pm = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots$. Hence, the Lorentz group inherits the representation content of $SU(2)$.

Because of the imaginary unit in the definition (2.6) of $X_i^\pm$, it is a certain complexification of the Lorentz group that is locally isomorphic to $SU(2) \times SU(2)$, not the Lorentz group itself (in a strict sense). Indeed, the proper Lorentz group is isomorphic to $SL(2, \mathbb{C})/\mathbb{Z}_2$, i.e. to the group of complex-valued 2×2 matrices with unit determinant/$\mathbb{Z}_2$.

Massive representations

In quantum mechanics we are primarily interested in unitary representations of the Poincaré group. Unitary representations are linear representations on a complex Hilbert space whose elements are unitary operators acting on the Hilbert space of states.

The quadratic Casimir operators for the Poincaré group are the quadratic momentum operator

$$P^2 = P_\mu P^\mu$$

and the quadratic spin operator

$$W^2 = W_\mu W^\mu \ ,$$

where W_μ is the covariant Pauli-Lubanski vector defined by

$$W_\mu = -\frac{1}{2} \epsilon_{\mu\nu\rho\sigma} M^{\nu\rho} P^\sigma \ . \tag{2.7}$$

Acting on a rest-frame eigenstate of a massive particle the momentum operator effectively reduces to the rest-frame momentum eigenvalue

$$P_\mu \rightarrow \bar{p}_\mu = (m, \mathbf{0})$$

and the Pauli-Lubanski vector (2.7) reduces to

$$W_\mu \rightarrow -\frac{1}{2} \epsilon_{\mu\nu\rho\sigma} M^{\nu\rho} \bar{p}^\sigma = \frac{m}{2} \epsilon_{ijk} M^{jk} = m J_i \ .$$

Thus the quadratic Casimir operators for the Poincaré group on a rest-frame eigenstate are the mass squared

$$P^2 = m^2$$

and the mass squared times the quadratic angular momentum operator

$$W^2 = -m^2 \boldsymbol{J}^2 \ .$$

A rest-frame eigenstate does not have orbital angular momentum, so the quadratic angular momentum operator is indeed the quadratic spin operator. This means that we can classify the massive irreducible representations of the Ponicaré group by using the mass m and the spin s. The Hilbert space of states spanning a representation has the mass and the spin as quantum numbers

$$P^2 |m, s; \ldots\rangle = m^2 |m, s; \ldots\rangle \quad ; \; m \neq 0$$
$$W^2 |m, s; \ldots\rangle = -m^2 s(s+1) |m, s; \ldots\rangle \quad ; \; s = \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots \; .$$

The additional quantum numbers needed to label states in a massive representation of the Poincaré group are eigenvalues of operators which have to commute with each other, so that they can be simultaneously diagonalized. Since the momentum operators commute (2.2a) we can use their eigenvalues to label the states. The mass is already fixed by the representation, so it is sufficient to specify the spatial momentum $\boldsymbol{p}$. Any component J_i of the angular momentum operator $\boldsymbol{J}$ alone cannot be used because they do not commute with the momentum operators

$$[P_i, J_j] = i\varepsilon_{ijk} P_k \; .$$

Instead we can use the projection of the angular momentum on the momentum

$$\frac{\boldsymbol{J} \cdot \boldsymbol{P}}{|\boldsymbol{P}|}$$

which is called the *helicity operator*. As any component of the angular momentum the helicity has the eigenvalues $\lambda \in \{-s, -s+1, \ldots, s-1, s\}$. The helicity is by definition invariant under spatial rotations, but it does change under boosts.

Thus, the infinite-dimensional unitary vector space representations of the Poincaré group are labelled by the mass m and the spin s and spanned by the momentum eigenstates $|m, s; \boldsymbol{p}, \lambda\rangle$ which have the following eigenvalues

$$P_i |m, s; \boldsymbol{p}, \lambda\rangle = p_i |m, s; \boldsymbol{p}, \lambda\rangle \; ,$$
$$\frac{\boldsymbol{J} \cdot \boldsymbol{P}}{|\boldsymbol{P}|} |m, s; \boldsymbol{p}, \lambda\rangle = \lambda |m, s; \boldsymbol{p}, \lambda\rangle \; .$$

Because the rest-frame states have the rotation group $SO(3)$ (or $SU(2)$) as their *stability group* — the subgroup that leaves the states invariant —, the Lorentz transformation $\Lambda_{p' \leftarrow p}$ that transforms the momentum eigenstate $|m, s; \boldsymbol{p}, \lambda\rangle$ to another momentum eigenstate $|m, s; \boldsymbol{p'}, \lambda\rangle$ is induced by an irreducible matrix representation $D_{\lambda'\lambda}(R_{\mathrm{W}})$ of the rotation group,[8]

$$\Lambda_{p' \leftarrow p} |m, s; \boldsymbol{p}, \lambda\rangle = \sum_{\lambda'} D_{\lambda'\lambda}(R_{\mathrm{W}}) |m, s; \boldsymbol{p'}, \lambda'\rangle \; .$$

The corresponding Wigner rotation R_{W} depends on Lorentz transformations $\Lambda_{p \leftarrow 0}$, $\Lambda_{p' \leftarrow p}$ and $\Lambda_{p' \leftarrow 0}$,

$$R_{\mathrm{W}} = \Lambda^{-1}_{p' \leftarrow 0} \Lambda_{p' \leftarrow p} \Lambda_{p \leftarrow 0} \; .$$

[8] The general idea is that instead of first boosting a rest-frame state to $\boldsymbol{p}$ and then Lorentz transforming it by $\Lambda_{p' \leftarrow p}$ we can directly rotate the rest-frame state to align with $\boldsymbol{p'}$ and then boost the rest-frame state to $\boldsymbol{p'}$. Since a rest-frame state transforms under a rotation as a pure angular momentum eigenstate, the aligning rotation of the rest-frame state and the boost to $\boldsymbol{p'}$ commute and hence the full Lorentz transformation is equivalent to a rotation of the $\boldsymbol{p'}$ momentum state.

Massless representations

Massless eigenstates with $p^2 = 0$ do not have a rest-frame — since it would have to move with the speed of light —, so the stability group for a canonical momentum eigenstate is not the same $SU(2)$ as for the massive eigenstates.

Let us consider a momentum $\bar{p}_\mu = \varepsilon(1, \boldsymbol{e}_3) = \varepsilon(1, 0, 0, 1)$ eigenstate of a massless particle. It is fairly easy to see that the stability group for this eigenstate is isomorphic to the group of isometries of the two-dimensional Euclidean space i.e. the two-dimensional Euclidean group $E(2)$. Acting on the massless eigenstate the Pauli-Lubanski vector reduces to

$$W_\mu \to -\frac{1}{2}\varepsilon_{\mu\nu\rho\sigma} M^{\nu\rho} p^\sigma = \varepsilon(J_3, J_1 + K_2, J_2 - K_1, J_3) = \varepsilon(J_3, L_1, L_2, J_3) \ ,$$

So the quadratic Casimir operator of massless representations reduces to

$$W^2 = -\varepsilon^2 \left(L_1^2 + L_2^2\right) \ .$$

The chosen angular momentum component J_3 and the new generator combinations L_1 and L_2 form the algebra [9]

$$[J_3, L_1] = iL_2 \ , \tag{2.8a}$$
$$[J_3, L_2] = -iL_1 \ , \tag{2.8b}$$
$$[L_1, L_2] = 0 \ . \tag{2.8c}$$

The first two relations tell us that the combinations

$$L_\pm = L_1 \pm iL_2$$

are the raising and lowering operators for the angular momentum J_3

$$[J_3, L_\pm] = \pm J_\pm \ .$$

Thus J_3 has a spectrum with constant spacing between eigenvalues. The all-important difference between the familiar angular momentum algebra and the algebra (2.8) is the commutativity of L_1 and L_2, which implies that there is no relation between J_3 and the quadratic Casimir operator W^2. As a result the eigenvalues of J_3 do not have to be integers or half-integers, unless we force the states of a representation to be null-states of the generators L_1 and L_2,

$$L_1 \left|\ldots\right\rangle = L_2 \left|\ldots\right\rangle = 0 \ .$$

This is the case for all physically interesting massless representations. The quantum numbers needed to label the eigenstates of a massless representation are the momentum p_μ and the eigenvalue λ of J_3.

$$P_\mu \left|p, \lambda\right\rangle = p_\mu \left|p, \lambda\right\rangle \ ,$$
$$J_3 \left|p, \lambda\right\rangle = \lambda \left|p, \lambda\right\rangle \ .$$

[9] In $E(2)$: L_1 and L_2 are the translation generators for the Euclidean plane and J_3 generates rotations around a point.

The spin quantum number λ is again called helicity, though from the mathematical point of view it is a different entity compared to the helicity of a massive representation. The helicity of a massless eigenstate is invariant under Poincaré transformations.

In the massless case the Wigner rotation is a simple phase change

$$\Lambda_{p' \leftarrow p} |p, \lambda\rangle = e^{-i\Phi_{\mathrm{W}}\lambda} |p', \lambda\rangle \ ,$$

where Φ_{W} is a real-valued Wigner angle.

Elementary particles as representations of the Poincaré group

In relativistic field theories, the elementary particles and matter they comprise are described by field representations of the Poincaré group. In order to obtain the generators of a multi-dimensional field representation of the Poincaré group, one has to combine the standard scalar field representation (2.4) of the Poincaré group and a finite-dimensional representation of the Lorentz group, whose generators $\Sigma_{\mu\nu}$ commute with the scalar field generators (2.4).

In addition to the properties discussed above many elementary particles need additional parity (P), charge and charge conjugation (C) quantum numbers to label their representations. The charge quantum numbers are related to the gauge symmetries. Eigenstates of more complex particles with inner structure (e.g. quark compounds, hadrons) can have even more quantum numbers. These additional quantum numbers are the reason why we have to use *reducible representations* for some elementary particles. For example, the helicity is a pseudoscalar that changes sign under parity transformations. Thus, if a massless particle participates in an interaction that conserves parity it has to possess two helicity states $\pm s$, so that it can compensate the change of sign in helicity by reversing the helicity state. This implies that we have to use reducible representations for massless particles that participate in parity conserving interactions. An example of such a particle is the photon which has ± 1 helicity states.

We present a short inventory of important field representations of the Poincaré group in high-energy physics. The only spin 0 particle in SM is the yet-to-be-found Higgs boson required by the mechanism that gives masses to elementary particles through spontaneous symmetry breaking in the electroweak theory. The spin $\frac{1}{2}$ particles, the matter fields of SM, come in two flavors. The massless neutrinos are described by the two-dimensional $(j_+, j_-) = (\frac{1}{2}, 0), (0, \frac{1}{2})$ irreducible representations. The Dirac spinors describe massive spin $\frac{1}{2}$ particles. They live in the four-dimensional reducible representation $(\frac{1}{2}, 0) \oplus (0, \frac{1}{2})$ and their Lorentz transformations are generated by the matrices

$$\Sigma_{\mu\nu} = \frac{i}{4}\left[\gamma_\mu, \gamma_\nu\right] \ ,$$

where γ_μ are the γ-matrices. The γ-matrices are a four-dimensional realisation of the Clifford algebra

$$\{\gamma_\mu, \gamma_\nu\} \equiv \gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2\eta_{\mu\nu} \ .$$

The vector bosons are spin 1 particles which mediate the interactions in SM. The photon and the gluons are massless gauge fields and the weak Z and W vector bosons are massive. If the graviton — the hypothetical mediator of gravitation — exists, it must have spin 2, because gravitation is described by a second-rank tensor field. In supersymmetric models the particle content is doubled by the superpartners, whose spin differs by $\pm\frac{1}{2}$ from that of their partners.

Lorentz tensors are multidimensional objects that are covariant under Lorentz transformations. The four-dimensional representation $(\frac{1}{2},\frac{1}{2})$ of the Lorentz group corresponds to its defining representation, the four-vectors. These spacetime vectors, i.e. $(1,0)$-tensors, V^σ live in the representation

$$(\Sigma_{\mu\nu})^\rho{}_\sigma = i(\delta^\rho_\mu \eta_{\nu\sigma} - \delta^\rho_\nu \eta_{\mu\sigma}) \tag{2.9}$$

and the dual vectors, i.e. $(0,1)$-tensors, V_σ live in the representation

$$(\Sigma_{\mu\nu})_\rho{}^\sigma = i(\eta_{\mu\rho}\delta^\sigma_\nu - \eta_{\nu\rho}\delta^\sigma_\mu) \ . \tag{2.10}$$

The generalization to arbitary tensor representations is straightforward. For (n,m)-tensors $V^{\mu_1\cdots\mu_n}{}_{\nu_1\cdots\nu_m}$ we have

$$\begin{aligned} -\frac{i}{2}\omega^{\rho\sigma}\Sigma_{\rho\sigma}V^{\mu_1\cdots\mu_n}{}_{\nu_1\cdots\nu_m} = \omega^{\mu_1}{}_\rho V^{\rho\cdots\mu_n}{}_{\nu_1\cdots\nu_m} + \cdots + \omega^{\mu_n}{}_\rho V^{\mu_1\cdots\rho}{}_{\nu_1\cdots\nu_m} \\ + \omega_{\nu_1}{}^\rho V^{\mu_1\cdots\mu_n}{}_{\rho\cdots\nu_m} + \cdots + \omega_{\nu_m}{}^\rho V^{\mu_1\cdots\mu_n}{}_{\nu_1\cdots\rho} \ . \end{aligned} \tag{2.11}$$

2.3 Global Poincaré symmetry as an internal-like symmetry

2.3.1 Global Poincaré invariance

Let us consider a generic relativistic field theory defined on Minkowski spacetime. The action functional of the theory is constructed from a local Lagrangian, $L_\mathrm{M}\big(u_i(x),\partial_\mu u_i(x)\big)$, for a set of *matter fields* $u_i(x)$, $i=1,2,\ldots,n$,

$$S_\mathrm{M}[u_i] = \int_\Omega \mathrm{d}^4x\, L_\mathrm{M}\big(u_i(x),\partial_\mu u_i(x)\big) \ . \tag{2.12}$$

Relativistic invariance is implemented by requiring that the action (2.12) is invariant under global Poincaré transformations. For this to be possible the set of fields $u(x)$ has to be a representation of the Poincaré group. We have excluded the explicit coordinate dependence from the Lagrangian L_M, because it is clearly forbidden by the invariance under coordinate translations.[10]

The traditional way to introduce Poincaré transformations is to understand them as simultaneous transformations in spacetime coordinates and in the fields

[10]Variation under any infinitesimal translation, $x^\mu \to x^\mu + a^\mu$, has to vanish: $\delta S_\mathrm{M} = \int_\Omega \mathrm{d}x^4\,\delta x^\mu \frac{\partial L_\mathrm{M}}{\partial x^\mu} = a^\mu \int_\Omega \mathrm{d}x^4 \frac{\partial L_\mathrm{M}}{\partial x^\mu} = 0 \Rightarrow \frac{\partial L_\mathrm{M}}{\partial x^\mu} = 0$, i.e. there is no explicit coordinate dependence in L_M.

of the theory. Since the Poincaré transformations are continuous it is sufficient to consider infinitesimal transformations

$$x^\mu \longrightarrow x'^\mu = x^\mu + \delta x^\mu = x^\mu + \varepsilon^\mu + \omega^\mu{}_\nu x^\nu \tag{2.13a}$$

$$u(x) \longrightarrow u'(x') = u(x) + \delta u(x) = u(x) - \frac{i}{2}\omega^{\mu\nu}\Sigma_{\mu\nu}u(x) \ , \tag{2.13b}$$

where ε^μ and $\omega^{\mu\nu} = -\omega^{\nu\mu}$ are the ten infinitesimal parameters of Poincaré transformations and $\Sigma_{\mu\nu}$ are the generators of Lorentz transformations for the finite dimensional representation where the components of the $u(x)$ fields live. The fields $u(x)$ can be thought as a column vector with the components $u_i(x)$ and the generators $\Sigma_{\mu\nu}$ are matrices which act on the fields by matrix multiplication.

We take an alternative approach by writing the Poincaré transformations as internal-like transformations, that only affect the fields $u(x)$ but not the coordinates. In other words, we consider the global Poincaré symmetry as an internal-like symmetry. The external Poincaré transformations (2.13) can be revised to internal-like transformations by writing the transformed fields in the same spacetime point as the original fields

$$u'(x) = u'(x' - \delta x) = u'(x') - \delta x^\mu \partial_\mu \big[u(x) + \delta u(x)\big] = u'(x') - \delta x^\mu \partial_\mu u(x) \ .$$

After evaluating this transformation, we leave the coordinates unchanged

$$x^\mu \longrightarrow x'^\mu = x^\mu \ .$$

In the case of the Poincaré transformations (2.13) the complementary internal-like transformations are

$$u(x) \longrightarrow u'(x) = u(x) + \delta u(x) = u(x) - (\varepsilon^\mu + \omega^{\mu\nu}x_\nu)\partial_\mu u(x) - \frac{i}{2}\omega^{\mu\nu}\Sigma_{\mu\nu}u(x) \ . \tag{2.14}$$

We can further rewrite these internal-like Poincaré transformations in a more familar form

$$u(x) \longrightarrow u'(x) = Uu(x) \approx (1 + \Theta)u(x) \ , \tag{2.15}$$

where Θ is an anti-Hermitian operator representing the infinitesimal global Poincaré transformations on the algebra of fields in Minkowski spacetime,

$$\begin{aligned} \Theta &= -(\varepsilon^\mu + \omega^{\mu\nu}x_\nu)\partial_\mu - \frac{i}{2}\omega^{\mu\nu}\Sigma_{\mu\nu} \\ &= i\varepsilon^\mu P_\mu - \frac{i}{2}\omega^{\mu\nu}M_{\mu\nu} \ . \end{aligned} \tag{2.16}$$

These are the global Poincaré transformations and we require that the action (2.12) is invariant under them. From the last form of the transformation operator (2.16), we can see that the operator can be decomposed in terms of the generators of the group of Poincaré transformations in the $u(x)$ field representation

$$P_\mu = i\partial_\mu \ , \tag{2.17a}$$

$$M_{\mu\nu} = i(x_\mu\partial_\nu - x_\nu\partial_\mu) + \Sigma_{\mu\nu} \ . \tag{2.17b}$$

Thus the unitary gauge transformation operator U introduced in (2.15) is indeed the element of the Poincaré group (2.3) acting on the fields $u(x)$. The finite-dimensional part of the generators of Lorentz transformations $\Sigma_{\mu\nu}$ is always chosen to match the representation spanned by the fields the generators act on.

An important difference between the global Poincaré transformations (2.15) and the truly internal gauge transformations used in the theories of particle physics is that the generators of Poincaré transformations contain partial derivative operators — in addition to matrix-valued generators. This has a major implication for the gauge theory of Poincaré symmetry: The infinitesimal global Poincaré transformations (2.15) are not linear with respect to the fields $u(x)$.

For partial derivatives of the fields $u(x)$, the infinitesimal Poincaré transformations can be obtained by utilizing the fact that the transformations (2.15) do not affect the coordinates,

$$\delta\partial_\mu u(x) = \partial_\mu \delta u(x) = \omega_\mu{}^\nu \partial_\nu u(x) - (\varepsilon^\nu + \omega^{\nu\rho} x_\rho)\partial_\nu \partial_\mu u(x) - \frac{i}{2}\omega^{\nu\rho}\Sigma_{\nu\rho}\partial_\mu u(x) \ . \tag{2.18}$$

Though we were able to write the global Poincaré transformations in a form similar with internal gauge transformations, we cannot completely escape the fact that the Poincaré symmetry is by its origin an external symmetry. As a result the invariance of the Lagrangian under global Poincaré transformations is not equivalent with the invarince of the corresponding action functional — As is the case in the familiar internal gauge symmetries, like the one in SM. The reason behind this is expressly the nonlinearity of the transformations (2.15).

Let us consider the transformation of the Lagrangian density L_{M} under the global Poincaré transformation (2.15) [11]

$$\delta L_{\mathrm{M}} \equiv \frac{\partial L_{\mathrm{M}}}{\partial u(x)}\delta u(x) + \frac{\partial L_{\mathrm{M}}}{\partial(\partial_\mu u(x))}\delta\partial_\mu u(x) \ . \tag{2.19}$$

Due to the Stokes theorem the change of the action

$$\delta S_{\mathrm{M}} \equiv \int_\Omega \mathrm{d}^4 x \, \delta L_{\mathrm{M}} \tag{2.20}$$

vanishes if the transformation of the Lagrangian either vanishes or if it is a pure divergence

$$\delta L_{\mathrm{M}} = \partial_\mu f^\mu$$

and the fields f^μ vanish on the boundary of spacetime $x \in \partial\Omega$. Since the transformations of the fields (2.15) are nonlinear, (2.19) does not vanish for any nontrivial Lagrangian. Thus the action is invariant if and only if (2.19) is a pure divergence and the fields $u(x)$ have appropriate boundary conditions. By inspection of the transformation rules (2.15) and (2.18) it can be seen that the transformation of the Lagrangian has to be

$$\delta L_{\mathrm{M}} = -(\varepsilon^\mu + \omega^{\mu\nu}x_\nu)\partial_\mu L_{\mathrm{M}} = -\partial_\mu\big((\varepsilon^\mu + \omega^{\mu\nu}x_\nu)L_{\mathrm{M}}\big) \tag{2.21}$$

for the action (2.12) to be invariant under the global Poincaré transformations (2.15). Thus (2.21) and the vanishing of the Lagrangian on the boundary of spacetime are the correct Poincaré invariance conditions for the Lagrangian.

[11] $\frac{\partial}{\partial u(x)}$ is a row vector with components $\frac{\partial}{\partial u_i(x)}$.

2.3.2 Conservation laws

According to the Noether theorem we can rewrite the invariance condition (2.21) as conservation equations of currents by utilizing the equations of motion for the fields

$$\frac{\partial L_{\mathrm{M}}}{\partial u(x)} - \partial_\mu \frac{\partial L_{\mathrm{M}}}{\partial(\partial_\mu u(x))} = 0 \tag{2.22}$$

and by using the infinitesimal Poincaré transformations (2.15) and (2.18). In this way we obtain the conservation equations

$$\partial_\nu T^\nu{}_\mu = 0 \; , \tag{2.23a}$$

$$\partial_\rho S^\rho{}_{\mu\nu} = 0 \; , \tag{2.23b}$$

where we have defined the canonical energy-momentum and angular momentum tensors

$$T^\nu{}_\mu = \frac{\partial L_{\mathrm{M}}}{\partial(\partial_\nu u(x))}\partial_\mu u(x) - \delta^\nu_\mu L_{\mathrm{M}} \tag{2.24a}$$

$$S^\rho{}_{\mu\nu} = T^\rho{}_\mu x_\nu - T^\rho{}_\nu x_\mu + \frac{\partial L_{\mathrm{M}}}{\partial(\partial_\rho u(x))} i\Sigma_{\mu\nu} u(x) \; , \tag{2.24b}$$

respectively, as the conserved currents of the theory. Thus the full conserved current for the global Poincaré symmetry is

$$J^\rho = \varepsilon^\mu T^\rho{}_\mu + \frac{1}{2}\omega^{\mu\nu} S^\rho{}_{\mu\nu} \; . \tag{2.25}$$

The local conservation equations (2.23), together with the appropriate boundary conditions, imply that the corresponding globally conserving charges are the total energy-momentum

$$p_\mu = \int \mathrm{d}^3x \, T^0{}_\mu$$

and the total angular momentum

$$m_{\mu\nu} = \int \mathrm{d}^3x \, S^0{}_{\mu\nu} \; .$$

The fundamental conservation laws (2.23) of the global Poincaré internal-like symmetry are identical with the conservation laws of the traditionally formed Poincaré symmetry. This proves that these two complementary definitions of the global Poincaré symmetry — based on the two concepts of Poincaré transformations (2.13) and (2.15) — are indeed physically equivalent.

2.4 Local Poincaré gauge symmetry

2.4.1 Local Poincaré transformations and gauge invariance

In order to construct a full Poincaré gauge symmetry, we must first generalize the Poincaré transformations (2.15) to local gauge transformations. We accomplish this by letting the parameters of the transformation group to take different values in each point of spacetime or more precisely by letting the parameters to be arbitary real-valued functions of spacetime points x,

$$\varepsilon^{\mu} \equiv \varepsilon^{\mu}(x) \ , \quad \omega^{\mu\nu} \equiv \omega^{\mu\nu}(x) \ . \tag{2.26}$$

This is the only change we have to make in the elements of the group of global Poincaré transformations (2.3). No modification in the form of the global Poincaré transformations (2.15) is needed in order to obtain the full Poincaré gauge transformations. Only the transformation operator (2.16) has to be redefined as a local operator

$$\begin{aligned}\Theta(x) &= -\big(\varepsilon^{\mu}(x) + \omega^{\mu\nu}(x)x_{\nu}\big)\partial_{\mu} - \frac{i}{2}\omega^{\mu\nu}(x)\Sigma_{\mu\nu} \\ &= i\varepsilon^{\mu}(x)P_{\mu} - \frac{i}{2}\omega^{\mu\nu}(x)M_{\mu\nu} \ .\end{aligned} \tag{2.27}$$

Since the generators (2.17) and the parameters (2.26) of the Poincaré gauge transformations do not commute, the gauge transformations of partial derivatives are no longer of the form (2.18). Instead, the partial derivatives of the fields have the gauge transformations [12]

$$\begin{aligned}\delta\partial_{\mu}u = -(\partial_{\mu}\varepsilon^{\nu} + \partial_{\mu}\omega^{\nu\rho}x_{\rho})\partial_{\nu}u + \omega_{\mu}{}^{\nu}\partial_{\nu}u - (\varepsilon^{\nu} + \omega^{\nu\rho}x_{\rho})\partial_{\nu}\partial_{\mu}u \\ - \frac{i}{2}\partial_{\mu}\omega^{\nu\rho}\Sigma_{\nu\rho}u - \frac{i}{2}\omega^{\nu\rho}\Sigma_{\nu\rho}\partial_{\mu}u \ .\end{aligned} \tag{2.28}$$

Because these transformations contain partial derivatives of the gauge group parameters, a Lagrangian which satisfies the invariance condition (2.21) under global transformations cannot do so under local gauge transformations. Moreover, any Lagrangian constructed from the matter fields u alone cannot satisfy the invariance condition under the gauge transformations. This is an essential observation in all gauge theories.

Nevertheless, we require that the action of the theory has to be invariant under the local Poincaré gauge transformations provided by the operator (2.27). The local gauge invariance is achieved by introducing a set of gauge fields and by using them together with the matter fields to construct a Lagrangian which satisfies the invariance condition (2.21) under the gauge transformations. This construction is in many ways similar to the construction of gauge theories for semisimple non-Abelian gauge groups. For an introduction to such gauge theories see [48, 49] and for the original idea of a non-Abelian gauge symmetry and some of its early developments see the classics [50] and [29].

2.4.2 Covariant derivative and gauge fields

In order to enable the construction of a gauge invariant action, we need to define a covariant derivative which preserves its form under local Poincaré gauge transformations. Hence, we require that the covariant derivative ∇_{μ} has to satisfy the

[12] From now on we do not write the x-dependence of fields $u(x)$ explicitly.

covariance condition

$$\nabla'_\mu U(x) = U(x)\nabla_\mu \ , \tag{2.29}$$

where ∇'_μ is a gauge-transformed covariant derivative and $U(x)$ is the local unitary operator providing the Poincaré gauge transformations, or equivalently for infinitesimal gauge transformations

$$\nabla'_\mu\big(1+\Theta(x)\big) = \big(1+\Theta(x)\big)\nabla_\mu \ . \tag{2.30}$$

By replacing the partial derivatives of the matter fields u into the Lagrangian with such covariant derivatives we can make the Lagrangian transform similarly as the original Langrangian transformed under global transformations.

The covariant derivative cannot be constructed without introducing new fields, because the terms containing partial derivatives of the gauge group parameters in the gauge transformations of a partial derivative (2.28) do not cancel each other in any combination of the partial derivatives $\partial_\mu u$.

Decomposing the covariant derivative with respect to the generators P_μ and $M_{\mu\nu}$

As usual, the gauge transformation operator (2.27) can be decomposed with respect to the generators of the gauge symmetry group, given in the field representation (2.17), where the operator acts. Thus, we are led to choose the following Ansatz for the covariant derivative

$$\nabla_\mu = \partial_\mu + A_\mu(x) \ , \tag{2.31}$$

where the gauge fields $A_\mu(x)$ can be decomposed with respect to the generators of the representation on which the covariant derivative operates. The multipliers of the generators are the gauge fields which we introduce in the gauge theory of the Poincaré symmetry

$$A_\mu(x) = -iA_\mu{}^\nu(x)P_\nu + \frac{i}{2}A_\mu{}^{\nu\rho}(x)M_{\nu\rho} \ . \tag{2.32}$$

The purpose of the gauge fields $A_\mu{}^\nu$ and $A_\mu{}^{\nu\rho} = -A_\mu{}^{\rho\nu}$ is to act as compensating fields for the local translations and Lorentz transformations, respectively. In 4-dimensional Minkowski spacetime, four gauge fields are needed to compensate each dimension of the local Poincaré symmetry: $4\times 4 = 16$ gauge fields $A_\mu{}^\nu$ for local translations and $6\times 4 = 24$ gauge fields $A_\mu{}^{\nu\rho}$ for local Lorentz rotations.

This decomposition is important in algebraic calculations, because it enables us to directly use the algebra of generators (2.2). This is particularly useful in perturbative calculations.

We obtain the local gauge transformations of the gauge fields A_μ by substituting the Ansatz (2.31) into the covariance condition (2.30) and by solving it:

$$\begin{aligned}\delta A_\mu(x) = A'_\mu(x) - A_\mu(x) &= [\Theta, \nabla_\mu] \\ &= \omega_\mu{}^\nu\nabla_\nu + [\Theta,\partial_\mu] + [\Theta, A_\mu] \ .\end{aligned} \tag{2.33}$$

The crucially important first term in the right-hand side of (2.33) has its origin in the fact that the covariant derivative ∇_μ is a covariant vector and therefore the generators $\Sigma_{\mu\nu}$ act on it as a linear transformation — more precisely we have to use the $(0,1)$-tensor representation of the generators $\Sigma_{\mu\nu}$ (2.10) when we operate on the covariant derivative. Gauge transformation laws for the individual gauge fields are found by substituting the decomposition (2.32) and the variation

$$\delta A_\mu = -i\delta A_\mu{}^\nu P_\nu + \frac{i}{2}\delta A_\mu{}^{\nu\rho} M_{\nu\rho} \tag{2.34}$$

into the gauge transformation law (2.33). Evaluation of the commutators in the right-hand side of (2.33) — especially $[\Theta, A_\mu]$ — is a lengthy calculation, since even the generators (2.17) and the gauge fields do not commute. By grouping the terms of the result we obtain the required decomposition (2.34) which gives us the gauge transformation laws:

$$\begin{aligned}
\delta A_\mu{}^\nu &= -\big(\varepsilon^\rho + \omega^{\rho\sigma} x_\sigma\big)\partial_\rho A_\mu{}^\nu + \omega_\mu{}^\rho A_\rho{}^\nu + \omega^\nu{}_\rho A_\mu{}^\rho + \varepsilon_\rho A_\mu{}^{\rho\nu} && \text{(2.35a)}\\
&\quad + \partial_\mu \varepsilon^\nu + \big(A_\mu{}^\rho + A_\mu{}^{\rho\sigma} x_\sigma\big)\partial_\rho \varepsilon^\nu \ , \\
\delta A_\mu{}^{\nu\rho} &= -\big(\varepsilon^\sigma + \omega^{\sigma\tau} x_\tau\big)\partial_\sigma A_\mu{}^{\nu\rho} + \omega_\mu{}^\sigma A_\sigma{}^{\nu\rho} + \omega^\nu{}_\sigma A_\mu{}^{\sigma\rho} + \omega^\rho{}_\sigma A_\mu{}^{\nu\sigma} && \text{(2.35b)}\\
&\quad + \partial_\mu \omega^{\nu\rho} + \big(A_\mu{}^\sigma + A_\mu{}^{\sigma\tau} x_\tau\big)\partial_\sigma \omega^{\nu\rho} \ .
\end{aligned}$$

These gauge transformations are quite complicated and they also greatly differ from the corresponding gauge transformation formulae found both in non-Abelian gauge theories [29] and in the gauge theory of the translation symmetry [34, 42]. It should also be noted that the gauge fields associated with translations and Lorentz rotations are strongly involved through the gauge transformations (2.35). For these reasons we do not further pursue this decomposition of the gauge fields.

Decomposing the covariant derivative with respect to ∂_α and $\Sigma_{\mu\nu}$

For our purposes, a more useful form for the covariant derivative is obtained by introducing a matrix of effective gauge fields,

$$e_\mu{}^\alpha = \delta_\mu^\alpha + A_\mu{}^\alpha + A_\mu{}^{\alpha\nu} x_\nu \ , \tag{2.36}$$

as replacements for the gauge fields $A_\mu{}^\alpha$. This enables us to write the covariant derivative in the form

$$\nabla_\mu = d_\mu + \mathcal{A}_\mu \tag{2.37}$$

where we have defined

$$d_\mu = e_\mu{}^\alpha \partial_\alpha \ , \quad \mathcal{A}_\mu = \frac{i}{2} A_\mu{}^{\nu\rho}\Sigma_{\nu\rho} \ . \tag{2.38}$$

The first term d_μ of the covariant derivative (2.37) is a linear combination of partial derivatives and the second term $\mathcal{A}_\mu$ operates on the fields as a linear transformation. In each point of spacetime the gauge fields $e_\mu{}^\alpha$ are used to cancel the first term in the gauge transformations of partial derivatives (2.28) and the gauge fields $A_\mu{}^{\nu\rho}$ are used to do the same for the fourth term of the transformations. This form of the covariant derivative is perhaps the most natural decomposition for the field representations of the Poincaré symmetry (2.17), where both P_μ and $M_{\mu\nu}$ contain partial derivatives.

We obtain the local gauge transformations of the gauge fields by substituting the covariant derivative (2.37) into the covariance condition (2.30) and by solving the equation

$$\begin{aligned} \delta e_\mu{}^\alpha \partial_\alpha + \frac{i}{2}\delta A_\mu{}^{\nu\rho}\Sigma_{\nu\rho} &= [\Theta, \nabla_\mu] \\ &= \omega_\mu{}^\nu \nabla_\nu + [\Theta, d_\mu] + [\Theta, \mathcal{A}_\mu] \ . \end{aligned} \tag{2.39}$$

The evaluation of the right-hand side of (2.39) gives us the local gauge transformations of the gauge fields

$$\delta e_\mu{}^\alpha = -\big(\varepsilon^\beta + \omega^{\beta\nu} x_\nu\big)\partial_\beta e_\mu{}^\alpha + \omega_\mu{}^\nu e_\nu{}^\alpha + e_\mu{}^\beta \partial_\beta \big(\varepsilon^\alpha + \omega^{\alpha\nu} x_\nu\big) \tag{2.40a}$$

$$\delta A_\mu{}^{\nu\rho} = -\big(\varepsilon^\beta + \omega^{\beta\sigma} x_\sigma\big)\partial_\beta A_\mu{}^{\nu\rho} + \omega_\mu{}^\sigma A_\sigma{}^{\nu\rho} + \omega^\nu{}_\sigma A_\mu{}^{\sigma\rho} + \omega^\rho{}_\sigma A_\mu{}^{\nu\sigma} + e_\mu{}^\alpha \partial_\alpha \omega^{\nu\rho} \ . \tag{2.40b}$$

Alternatively, these transformations can be derived from the definition of the gauge fields $e_\mu{}^\alpha$ (2.36) and from the gauge transformation laws (2.35). The gauge transformations of the gauge fields $e_\mu{}^\alpha$ do not involve the gauge fields $A_\mu{}^{\nu\rho}$, which will turn out to be a most useful improvement compared to the complicated gauge transformations of $A_\mu{}^\nu$. This form of the gauge fields also has the most direct geometrical interpretation. Hence this will be our primary choice of the covariant derivative and of the gauge fields — unless otherwise stated, "covariant derivative" refers to (2.37).

Finally, it should be pointed out that the derivative operator d_μ and the fields $e_\mu{}^\alpha$ are the covariant derivative and the gauge fields of the gauge theory of the translation symmetry, respectively.

Expanding the covariant derivative in two factors

Yet another form of the covariant derivative can be obtained by factorizing (2.37) into two factors

$$\nabla_\mu = e_\mu{}^\alpha D_\alpha = e_\mu{}^\alpha \big(\partial_\alpha + B_\alpha\big) = e_\mu{}^\alpha \left(\partial_\alpha + \frac{i}{2} B_\alpha{}^{\mu\nu}\Sigma_{\mu\nu}\right) \ , \tag{2.41}$$

where we have defined the gauge fields

$$B_\alpha = \big(e^{-1}\big)^\mu{}_\alpha \mathcal{A}_\mu \quad \Leftrightarrow \quad B_\alpha{}^{\mu\nu} = \big(e^{-1}\big)^\rho{}_\alpha A_\rho{}^{\mu\nu} \tag{2.42}$$

and replaced $\mathcal{A}_\mu$ by them.[13] The operator D_α has the same form as the covariant derivative has in non-Abelian gauge theories and the full covariant derivative (2.41) is a linear combination of these operators at each point. The gauge fields $B_\alpha{}^{\mu\nu}$ have an interesting geometrical intepretation, as will be seen shortly.

Alternatively, the covariant derivative (2.41) can be introduced in two steps: First by introducing "the incomplete covariant derivative" D_α and then by adding the $e_\mu{}^\alpha$ factor to complete it. Similarly as in non-Abelian gauge theories, the D_α factor cancels the fourth term — the nonlinear term which contains partial derivatives of the transformation parameters $\omega^{\nu\rho}$ — of the gauge transformation (2.28). Then

[13] We are implicitly requiring that the gauge field matrix $e_\mu{}^\alpha$ is invertible: $\det e \neq 0$.

the ${e_\mu}^\alpha$ factor cancels the linear term, which contains partial derivatives of the transformation parameters, of the same transformation. This is similar to what Kibble did in [30], though not in the internal-like symmetry setting presented here.

Behaviour of the gauge fields under the local gauge transformations is again obtained from the covariance condition (2.30). For that we can solve the equation

$$\begin{aligned} \delta {e_\mu}^\alpha D_\alpha + {e_\mu}^\alpha \delta B_\alpha &= [\Theta, \nabla_\mu] \\ &= {\omega_\mu}^\nu \nabla_\nu + \big[\Theta, {e_\mu}^\alpha\big] D_\alpha + {e_\mu}^\alpha \big[\Theta, D_\alpha\big] \ . \end{aligned} \tag{2.43}$$

By evaluating the commutators in the right-hand side of (2.43) and by regrouping the result we obtain the gauge transformation laws

$$\delta {e_\mu}^\alpha = -\big(\varepsilon^\beta + \omega^{\beta\nu} x_\nu\big) \partial_\beta {e_\mu}^\alpha + {\omega_\mu}^\nu {e_\nu}^\alpha + {e_\mu}^\beta \partial_\beta \big(\varepsilon^\alpha + \omega^{\alpha\nu} x_\nu\big) \ , \tag{2.44a}$$

$$\begin{aligned} \delta {B_\alpha}^{\mu\nu} = &-\big(\varepsilon^\beta + \omega^{\beta\rho} x_\rho\big) \partial_\beta {B_\alpha}^{\mu\nu} + {\omega_\alpha}^\beta {B_\beta}^{\mu\nu} + {\omega^\mu}_\rho {B_\alpha}^{\rho\nu} + {\omega^\nu}_\rho {B_\alpha}^{\mu\rho} \\ &- \partial_\alpha \big(\varepsilon^\beta + \omega^{\beta\rho} x_\rho\big) {B_\beta}^{\mu\nu} + \partial_\alpha \omega^{\mu\nu} \ . \end{aligned} \tag{2.44b}$$

These gauge transformations can of course be deduced from the definition of the gauge fields ${B_\alpha}^{\mu\nu}$ (2.42) and from the gauge transformation laws (2.40). Notice that now even the gauge transformations of the gauge fields ${B_\alpha}^{\mu\nu}$ (2.44b) do not involve the gauge fields ${e_\mu}^\alpha$ — the inverse of this is still true as it was before in (2.40a). Thus the gauge fields ${\varepsilon_\mu}^\alpha$ and ${B_\alpha}^{\mu\nu}$ are quite independent at this point. These two type of gauge fields, however, do depend on each other through the equations of motion, which we will derive later.

2.4.3 Lagrangian with minimal coupling to the gauge fields

We follow the well-known convention and choose the minimal coupling of matter fields to the gauge fields. This is achieved by replacing the partial derivatives in the Langrangian L_{M} by the covariant derivatives (2.37). So, we define a new Lagrangian

$$\tilde{L}_{\mathrm{M}}\big(u_i, \partial_\mu u_i, {e_\mu}^\alpha, {A_\alpha}^{\mu\nu}\big) = L_{\mathrm{M}}(u_i, \nabla_\mu u_i) \ . \tag{2.45}$$

This Lagrangian transforms under local Poincaré gauge transformations in the same way as the Lagrangian L_{M} transforms under global Poincaré transformations

$$\delta \tilde{L}_{\mathrm{M}} = -\left(\varepsilon^\mu + \omega^{\mu\nu} x_\nu\right) \partial_\mu \tilde{L}_{\mathrm{M}} \ .$$

The right-hand side of this transformation is not a pure divergence anymore, because of the coordinate dependence of the transformation parameters. We can repair this issue by multiplying the Lagrangian with a factor $\mathcal{E}$ which has the needed gauge transformation behaviour:

$$\begin{aligned} \delta(\mathcal{E}\tilde{L}_{\mathrm{M}}) = \delta\mathcal{E}\tilde{L}_{\mathrm{M}} + \mathcal{E}\delta\tilde{L}_{\mathrm{M}} &= -\partial_\mu \Big(\big(\varepsilon^\mu + \omega^{\mu\nu} x_\nu\big)\mathcal{E}\tilde{L}_{\mathrm{M}}\Big) \ , \\ \Rightarrow \ \delta\mathcal{E} = -\partial_\mu\big(\varepsilon^\mu + \omega^{\mu\nu} x_\nu\big)\mathcal{E} &- \big(\varepsilon^\mu + \omega^{\mu\nu} x_\nu\big)\partial_\mu \mathcal{E} \ . \end{aligned} \tag{2.46}$$

This behaviour implies that the factor $\mathcal{E}$ has to be constructed from the gauge fields. The simplest choice that has the required gauge transformation characteristics is

$$\mathcal{E} = \det\big(e^{-1}\big) = (\det e)^{-1} = \exp(-\operatorname{Tr}\, \ln e) \ , \tag{2.47}$$

where $\left(e^{-1}\right)$ is the inverse of the matrix $e \equiv ({e_\mu}^\alpha)$. The gauge transformations of (2.47) can be confirmed to fulfill (2.46) by a small calculation:

$$\begin{aligned}\delta \det\left(e^{-1}\right) &= -\det\left(e^{-1}\right)\cdot\left(e^{-1}\right)^\mu{}_\alpha \delta {e_\mu}^\alpha \\ &= -\partial_\mu\left(\varepsilon^\mu + \omega^{\mu\nu}x_\nu\right)\det\left(e^{-1}\right) - \left(\varepsilon^\mu + \omega^{\mu\nu}x_\nu\right)\partial_\mu \det\left(e^{-1}\right) \ ,\end{aligned}$$

where

$$\partial_\mu \det\left(e^{-1}\right) = -\det\left(e^{-1}\right)\cdot\left(e^{-1}\right)^\rho{}_\alpha \partial_\mu {e_\rho}^\alpha \ .$$

According to (2.46) the infinitesimal gauge transformations of the Lagrangian $\mathcal{E}\tilde{L}_\mathrm{M}$ are pure divergences, which vanish under integration over spacetime. Thus, the Lagrangian that provides a gauge invariant action and that minimally couples the matter fields to the gauge fields is

$$\mathcal{L}_\mathrm{M}\left(u_i, \partial_\mu u_i, {e_\mu}^\alpha, {A_\mu}^{\nu\rho}\right) = \det\left(e^{-1}\right)L_\mathrm{M}(u_i, \nabla_\mu u_i) \ . \tag{2.48}$$

2.4.4 Field strength and free Lagrangian of the gauge fields

In order to complete the action of the theory we need to construct a gauge invariant action of the free gauge fields, which defines physics completely in the absence of matter. Since there is an infinite number of such actions we follow the common practice in gauge theories by choosing the simplest nontrivial action.

The best way to construct a gauge invariant action of the free gauge fields is to seek for tensor fields which are covariant under the gauge transformations and to contract such tensors in order to obtain scalars, whose behaviour under the gauge transformations is identical with the matter Lagrangian (2.45). Since the commutator of two covariant derivatives is by definition gauge covariant, it can be used to define covariant tensor fields, especially one called the field strength.[14] Let us calculate the commutator of two covariant derivatives:

$$\begin{aligned}[\nabla_\mu, \nabla_\nu] &= [d_\mu + \mathcal{A}_\mu, d_\nu + \mathcal{A}_\nu] \qquad (2.49)\\ &= [d_\mu, d_\nu] - \left({A_{\mu\nu}}^\rho - {A_{\nu\mu}}^\rho\right)d_\rho + d_\mu\mathcal{A}_\nu - d_\nu\mathcal{A}_\mu + [\mathcal{A}_\mu, \mathcal{A}_\nu] \\ &= \frac{i}{2}R^{\rho\sigma}{}_{\mu\nu}\Sigma_{\rho\sigma} - T^\rho{}_{\mu\nu}\nabla_\rho \ ,\end{aligned}$$

where we have defined the tensor fields

$$R^{\rho\sigma}{}_{\mu\nu} = d_\mu {A_\nu}^{\rho\sigma} - d_\nu {A_\mu}^{\rho\sigma} - {A_\mu}^\rho{}_\tau {A_\nu}^{\tau\sigma} + {A_\nu}^\rho{}_\tau {A_\mu}^{\tau\sigma} - {C_{\mu\nu}}^\tau {A_\tau}^{\rho\sigma} \ , \tag{2.50a}$$

$$T^\rho{}_{\mu\nu} = {A_{\mu\nu}}^\rho - {A_{\nu\mu}}^\rho - {C_{\mu\nu}}^\rho \ . \tag{2.50b}$$

R and T are covariant under Poincaré gauge transformations and R is the field strength tensor we are seeking. C is the covariant field strength tensor of the local translation group,

$$[d_\mu, d_\nu] = {C_{\mu\nu}}^\rho d_\rho \ , \quad {C_{\mu\nu}}^\rho = \left({e_\mu}^\alpha\partial_\alpha {e_\nu}^\beta - {e_\nu}^\alpha\partial_\alpha {e_\mu}^\beta\right)\left(e^{-1}\right)^\rho{}_\beta \ , \tag{2.51}$$

[14] An altervative way to obtain these tensors is to use geometric reasoning. This is analogous to the argumentation used in Riemannian geometry to find the curvature tensor and the torsion tensor.

though it is not covariant under Poincaré gauge transformations. The second term in the middle form of the commutator (2.49) is a result of the fact that the covariant derivative ∇_μ is a covariant vector and therefore $\mathcal{A}_\mu$ operates on it as a local linear transformation — recall that we have to use the $(0,1)$-tensor representation of the generators $\Sigma_{\mu\nu}$ when operating on the covariant derivative.

We have stated that the field strength R is a tensor field, but we have not explicitly written down how such an object transforms under the Poincaré gauge transformations. For completeness and as an important example, we give the gauge transformations of the field strength

$$\begin{aligned}\delta R^{\rho\sigma}{}_{\mu\nu} = -\big(\varepsilon^\tau + \omega^{\tau\upsilon}x_\upsilon\big)\partial_\tau R^{\rho\sigma}{}_{\mu\nu} + \omega^\rho{}_\tau R^{\tau\sigma}{}_{\mu\nu} + \omega^\sigma{}_\tau R^{\rho\tau}{}_{\mu\nu} \\ + \omega_\mu{}^\tau R^{\rho\sigma}{}_{\tau\nu} + \omega_\nu{}^\tau R^{\rho\sigma}{}_{\mu\tau}\ . \quad (2.52)\end{aligned}$$

In other words, the field strength $R^{\rho\sigma}{}_{\mu\nu}$ is a $(2,2)$-tensor field representation of the local Poincaré gauge group, given by (2.17) and (2.11).

The simplest free Lagrangian is the one that has the lowest order in terms of the gauge fields. A constant Λ has zero order and it is trivially gauge invariant. The simplest nontrivial scalar combination of gauge fields can be obtained by contracting the field strength tensor

$$\mathcal{R} = R^{\mu\nu}{}_{\mu\nu}\ .$$

We cast aside all higher order terms like $T^\rho{}_{\mu\nu}T_\rho{}^{\mu\nu}$, $\mathcal{R}^2$, $R^{\mu\mu}{}_{\rho\sigma}R^{\rho\sigma}{}_{\mu\nu}$ etc. By multiplying the chosen scalars with the factor (2.47), we will obtain the free Lagrangian of the gauge fields (taking the scalars in the lowest order according to the priciple of correspondence),

$$\mathcal{L}_{\mathrm{G}}(e_\mu{}^\alpha, \partial_\beta e_\mu{}^\alpha, A_\mu{}^{\nu\rho}, \partial_\alpha A_\mu{}^{\nu\rho}) = -\frac{1}{2\kappa}\det\big(e^{-1}\big)(\mathcal{R} + 2\Lambda)\ . \quad (2.53)$$

The constant κ in (2.53) is evaluated experimentally through comparison with GR.

2.4.5 Action of the gauge theory and the equations of motion

The complete action of the gauge theory of the Poincaré symmetry is obtained by combining the Lagrangian of the matter fields (2.48) — includes the minimal coupling to the gauge fields — and the Lagrangian of the free gauge fields (2.53):

$$\begin{aligned}S\big[u_i, e_\mu{}^\alpha, A_\alpha{}^{\mu\nu}\big] &= \int \mathrm{d}^4x\,(\mathcal{L}_{\mathrm{G}} + \mathcal{L}_{\mathrm{M}}) \quad (2.54)\\ &= \int \mathrm{d}x^4 \det\big(e^{-1}\big)\left(-\frac{1}{2\kappa}(\mathcal{R} + 2\Lambda) + L_{\mathrm{M}}(u_i, \nabla_\mu u_i)\right)\ .\end{aligned}$$

The equations of motion for the Poincaré gauge theory are obtained in the usual way, i.e. by varying the action with respect to the gauge fields $e_\mu{}^\alpha$ and $A_\mu{}^{\nu\rho}$ and with respect to the matter fields u_i. Since we are describing gravitation by gauge fields, the gravitational dynamics is encoded in the equations of motion for the gauge fields. They are written as:[15]

$$\frac{1}{2}\det\left(e^{-1}\right)\left(-e_{\nu}{}^{\alpha}\frac{\delta\mathcal{R}}{\delta e_{\mu}{}^{\alpha}}+\delta^{\mu}_{\nu}(\mathcal{R}+2\Lambda)\right)=\kappa\det\left(e^{-1}\right)\mathcal{T}^{\mu}_{\ \nu}\ , \tag{2.55a}$$

$$\det\left(e^{-1}\right)\left(T^{\rho}{}_{\mu\nu}+\delta^{\rho}_{\mu}T^{\sigma}{}_{\nu\sigma}-\delta^{\rho}_{\nu}T^{\sigma}{}_{\mu\sigma}\right)=\kappa\det\left(e^{-1}\right)\mathcal{S}^{\rho}{}_{\mu\nu}\ , \tag{2.55b}$$

where we have defined the *energy-momentum tensor* and the *spin density tensor* for the matter fields as

$$\mathcal{T}^{\mu}_{\ \nu}=-e_{\nu}{}^{\alpha}\frac{1}{\det\left(e^{-1}\right)}\frac{\partial\mathcal{L}_{\mathrm{M}}}{\partial e_{\mu}{}^{\alpha}}=-\frac{\partial L_{\mathrm{M}}(u,\nabla_{\rho}u)}{\partial(\nabla_{\mu}u)}d_{\nu}u+\delta^{\mu}_{\nu}L_{\mathrm{M}}(u,\nabla_{\rho}u)\ , \tag{2.56a}$$

$$\mathcal{S}^{\rho}{}_{\mu\nu}=-\frac{2}{\det\left(e^{-1}\right)}\frac{\partial\mathcal{L}_{\mathrm{M}}}{\partial A_{\rho}{}^{\mu\nu}}=-\frac{\partial L_{\mathrm{M}}(u,\nabla_{\sigma}u)}{\partial(\nabla_{\rho}u)}i\Sigma_{\mu\nu}u\ , \tag{2.56b}$$

respectively.

2.5 Geometrical interpretation and comparison with General Relativity

In this section we establish the relation between the gauge fields of the gauge theory of the Poincaré symmetry and the geometry of spacetime. This enables us to interpret the fundamental gauge fields as gravitational potential fields and to give the gauge theory a complete geometrical re-interpretation, which turns out to be the *Einstein-Cartan theory of gravitation* (ECT). Finally, we compare the theory with GR.

2.5.1 Geometry associated with the Poincaré gauge fields

So far we have insisted that we were developing the gauge theory of the Poincaré symmetry in the flat Minkowski spacetime background. In order to compare the Poincaré gauge theory with measurements and with the theory of GR, we are going to give the theory a geometrical interpretation. The geometrical interpretation of gravitation is very convenient, since gravitation has only been measured at macroscopic ($\gtrsim$ 1 mm) and astronomical scales. The primary reference of geometrical concepts for this section is [51].

We intepret that the spacetime is not necessarily flat, but instead we have just been considering physics in orthonormal *non-coordinate bases* of the spacetime manifold. We interpret that the gauge field matrix $e_{\mu}{}^{\alpha}$ is the *vierbein* system of the spacetime manifold.[16] The orthonormal basis vectors for the tangent spaces of the spacetime manifold are

$$\hat{e}_{\mu}=e_{\mu}{}^{\alpha}\partial_{\alpha}=d_{\mu}$$

[15] We do not write the variation $\frac{\delta\mathcal{R}}{\delta e_{\mu}{}^{\alpha}}$ explicitly, because it is a lengthy expression and we are not going to use it.

[16] This is enabled by the invertibility and by the correct transformation properties of $e_{\mu}{}^{\alpha}$.

and the basis one-forms for the cotangent spaces are

$$\hat{\theta}^\mu = e^\mu{}_\alpha \mathrm{d}x^\alpha \ ,$$

where $\{\partial_\alpha\}$ and $\{\mathrm{d}x^\alpha\}$ are the coordinate bases for the tangent and cotangent spaces of the spacetime manifold and we have conventionally denoted the inverse of the vierbein: $e^\mu{}_\alpha \equiv \left(e^{-1}\right)^\mu{}_\alpha$. In spacetime indices, the first half of the Greek alphabet $\alpha, \beta, \ldots$ refers to the coordinate bases and the latter half $\mu, \nu, \ldots$ refers to the orthonormal non-coordinate bases — we have anticipated this notation in the previous sections. Tensor components transform linearly between these bases at each point: $V^\alpha = e_\mu{}^\alpha V^\mu$, $V_\alpha = e^\mu{}_\alpha V_\mu$ and so forth.

The *metric tensor* of the spacetime manifold is defined by the orthonormality of the basis vectors $\hat{e}_\mu$ with respect to the metric g in all points of the spacetime,

$$g(\hat{e}_\mu, \hat{e}_\nu) = g_{\mu\nu} \equiv \eta_{\mu\nu} \ ,$$

where $\eta_{\mu\nu}$ is the Minkowski metric. Hence, in the coordinate bases the metric has the components

$$g_{\alpha\beta} = \eta_{\mu\nu} e^\mu{}_\alpha e^\nu{}_\beta \ . \tag{2.57}$$

The signature of the metric is the same as the signature of the Minkowski metric. Thus the spacetime is a *Lorentzian manifold.* According to (2.57) the metric itself can be seen as an effective gauge field, a dynamical variable. Hence the spacetime is not anymore flat.

The rest of the gauge fields are interpreted to be the *connection coefficents* given in orthonormal non-coordinate bases

$$\Gamma^\rho{}_{\mu\nu} = -A_\mu{}^\rho{}_\nu \ . \tag{2.58}$$

The antisymmetry of the gauge fields $A_\rho{}^{\mu\nu}$ with respect to the last two indices implies

$$\Gamma^\mu{}_\rho{}^\nu = -A_\rho{}^{\mu\nu} = A_\rho{}^{\nu\mu} = -\Gamma^\nu{}_\rho{}^\mu \ , \tag{2.59}$$

which ensures the *metric compatibility* of the connection associated with the gauge fields.

The covariant derivative of the Poincaré gauge theory is replaced by the covariant derivative of a Lorentzian manifold

$$\nabla_\mu = e_\mu{}^\alpha \left(\partial_\alpha - \frac{i}{2} \Gamma^\nu{}_\alpha{}^\rho \Sigma_{\nu\rho} \right) \ , \tag{2.60}$$

where the connection

$$\Gamma^\nu{}_{\alpha\rho} = e^\sigma{}_\alpha \Gamma^\nu{}_{\sigma\rho} \tag{2.61}$$

is called the *spin connection.* The name refers to the fact that one is compelled to use the covariant derivative (2.60) for all other representations of the Lorentz symmetry except for the tensor representations, which also accept the covariant derivative with respect to the coordinate bases, and spinors are of course the most important non-tensorial representations. Notice that the gauge fields (2.42) have a direct interpretation as the spin connection (2.61).

In a Lorentzian spacetime manifold we can perform both local Lorentz transformations with respect to the non-coordinate bases $(\mu, \nu, \ldots)$ and general coordinate transformations with respect to the coordinate bases $(\alpha, \beta, \ldots)$, without affecting the physics of the system. The local Lorentz transformations are the local frame rotations which leave the metric tensor invariant, $\Lambda^{\rho}{}_{\mu}\Lambda^{\sigma}{}_{\nu}\eta_{\rho\sigma} = \eta_{\mu\nu} \Rightarrow \Lambda^{\mu}{}_{\nu} \in SO(1,3)$, in each point of the manifold. Invariance under the general coordinate transformations realizes the fact that we are free to choose any coordinate system for the spacetime manifold. The question is, how do the local Poincaré gauge transformations translate to these two kinds of symmetry transformations? We demonstrate this by considering the transformation behaviour of the mixed basis object $e_{\mu}{}^{\alpha}$, first as a vierbein and then as a gauge field. Under infinitesimal local Lorentz transformations and under general infinitesimal coordinate transformations

$$x^{\alpha} \longrightarrow x'^{\alpha} = x^{\alpha} + \epsilon^{\alpha}(x)$$

the vierbeins transform as

$$\begin{aligned} e_{\mu}{}^{\alpha}(x) \longrightarrow e'_{\mu}{}^{\alpha}(x') &= \Lambda^{\nu}{}_{\mu}(x)\, e_{\nu}{}^{\beta}(x)\, \frac{\partial x'^{\alpha}}{\partial x^{\beta}} \\ &= \big(\delta^{\nu}_{\mu} + \omega_{\mu}{}^{\nu}(x)\big) e_{\nu}{}^{\beta}(x) \big(\delta^{\alpha}_{\beta} + \partial_{\beta}\epsilon^{\alpha}(x)\big) \\ &= e_{\mu}{}^{\alpha}(x) + \omega_{\mu}{}^{\nu}(x) e_{\nu}{}^{\alpha}(x) + e_{\mu}{}^{\beta}(x)\partial_{\beta}\epsilon^{\alpha}(x)\ . \end{aligned} \tag{2.62}$$

We can cast this to a form of an internal-like transformation by expanding the transformed vierbein around x: $e'_{\mu}{}^{\alpha}(x') = e'_{\mu}{}^{\alpha}(x) + \epsilon^{\beta}\partial_{\beta}e_{\mu}{}^{\alpha}(x) + \mathcal{O}\left(\epsilon^2\right)$. This gives us the infinitesimal transformations of the vierbeins at x,

$$\delta e_{\mu}{}^{\alpha} = e'_{\mu}{}^{\alpha} - e_{\mu}{}^{\alpha} = -\epsilon^{\beta}\partial_{\beta}e_{\mu}{}^{\alpha} + \omega_{\mu}{}^{\nu}e_{\nu}{}^{\alpha} + e_{\mu}{}^{\beta}\partial_{\beta}\epsilon^{\alpha}\ . \tag{2.63}$$

If we now in the local Poincaré gauge transformations of the gauge fields $e_{\mu}{}^{\alpha}$ (2.40a) identify the parameter fields

$$\epsilon^{\alpha}(x) \equiv \varepsilon^{\alpha}(x) + \omega^{\alpha\beta}(x)x_{\beta}\ , \tag{2.64}$$

we see that the gauge transformations are identical with the transformations of the vierbeins (2.63). Thus the gauge invariance of the gauge theory of the Poincaré symmetry can be re-interpreted as covariance under the group of general infinitesimal coordinate transformations and as invariance under the group of local Lorentz frame rotations.

By inserting the connection coefficients (2.58) to the expressions for R and T tensor fields of the Poincaré gauge theory (2.50), we find out that apart from the sign of R they are identical with the *Riemann curvature tensor* R_{Γ} and the *torsion tensor* T_{Γ} of a pseudo-Riemannian manifold with a metric connection, given in an orthonormal basis $\{\hat{e}_{\mu}\}$,

$$R_{\Gamma}{}^{\rho}{}_{\sigma\mu\nu} = -R^{\rho}{}_{\sigma\mu\nu} = \hat{e}_{\mu}(\Gamma^{\rho}{}_{\nu\sigma}) - \hat{e}_{\nu}(\Gamma^{\rho}{}_{\mu\sigma}) + \Gamma^{\rho}{}_{\mu\tau}\Gamma^{\tau}{}_{\nu\sigma} - \Gamma^{\rho}{}_{\nu\tau}\Gamma^{\tau}{}_{\mu\sigma} - c_{\mu\nu}{}^{\tau}\Gamma^{\rho}{}_{\tau\sigma}\ , \tag{2.65a}$$

$$T_{\Gamma}{}^{\rho}{}_{\mu\nu} = T_{\mu\nu}{}^{\rho} = \Gamma^{\rho}{}_{\mu\nu} - \Gamma^{\rho}{}_{\nu\mu} - c_{\mu\nu}{}^{\rho}\ , \tag{2.65b}$$

where the anholonomy coefficients $c_{\mu\nu}{}^{\rho}$ originate from the Lie algera of basis vectors,

$$[\hat{e}_\mu, \hat{e}_\nu] = c_{\mu\nu}{}^{\rho}\hat{e}_\rho \ .$$

By introducing the *connection one-form*

$$\omega^{\rho}{}_{\sigma} = \Gamma^{\rho}{}_{\nu\sigma}\hat{\theta}^{\nu} \tag{2.66}$$

and the *torsion two-form* and the *curvature two-form*

$$T^{\rho} = \frac{1}{2}T_{\Gamma}{}^{\rho}{}_{\mu\nu}\hat{\theta}^{\mu}\wedge\hat{\theta}^{\nu} \ , \quad R^{\rho}{}_{\sigma} = \frac{1}{2}R_{\Gamma}{}^{\rho}{}_{\sigma\mu\nu}\hat{\theta}^{\mu}\wedge\hat{\theta}^{\nu} \tag{2.67}$$

of the Cartan formalism we can express the identities (2.65) as the *Cartan's structure equations*,

$$\mathrm{d}\hat{\theta}^{\rho} + \omega^{\rho}{}_{\sigma}\wedge\hat{\theta}^{\sigma} = T^{\rho} \ , \tag{2.68a}$$

$$\mathrm{d}\omega^{\rho}{}_{\sigma} + \omega^{\rho}{}_{\tau}\wedge\omega^{\tau}{}_{\sigma} = R^{\rho}{}_{\sigma} \ . \tag{2.68b}$$

The *Bianchi identities* can be obtained by taking exterior derivative of each of Cartan's structure equations (recall $\mathrm{d}^2 = 0$),

$$\mathrm{d}T^{\rho} + \omega^{\rho}{}_{\sigma}\wedge T^{\sigma} = R^{\rho}{}_{\sigma}\wedge\hat{\theta}^{\sigma} \ , \tag{2.69a}$$

$$\mathrm{d}R^{\rho}{}_{\sigma} + \omega^{\rho}{}_{\tau}\wedge R^{\tau}{}_{\sigma} - R^{\rho}{}_{\tau}\wedge\omega^{\tau}{}_{\sigma} = 0 \ . \tag{2.69b}$$

The connection coefficients with respect to the coordinate bases are obtained from the definition of the connection coefficients with respect to the non-coordinate bases

$$A_{\mu}{}^{\rho}{}_{\nu}\hat{e}_\rho = \Gamma^{\rho}{}_{\mu\nu}\hat{e}_\rho \equiv \nabla_\mu\hat{e}_\nu = e_{\mu}{}^{\alpha}\left(\partial_\alpha e_{\nu}{}^{\gamma} + \Gamma^{\gamma}{}_{\alpha\beta}e_{\nu}{}^{\beta}\right)e^{\rho}{}_{\gamma}\hat{e}_\rho \ ,$$

$$\Rightarrow \ \Gamma^{\gamma}{}_{\alpha\beta} = e^{\mu}{}_{\alpha}e^{\nu}{}_{\beta}e_{\rho}{}^{\gamma}A_{\mu}{}^{\rho}{}_{\nu} - e^{\nu}{}_{\beta}\partial_\alpha e_{\nu}{}^{\gamma} \ . \tag{2.70}$$

This enables us to write the curvature and the torsion tensors with respect to the coordinate bases, which is the form familiar from elementary GR,

$$R_{\Gamma}{}^{\gamma}{}_{\delta\alpha\beta} = \partial_\alpha\Gamma^{\gamma}{}_{\beta\delta} - \partial_\beta\Gamma^{\gamma}{}_{\alpha\delta} + \Gamma^{\gamma}{}_{\alpha\zeta}\Gamma^{\zeta}{}_{\beta\delta} - \Gamma^{\gamma}{}_{\beta\zeta}\Gamma^{\zeta}{}_{\alpha\delta} \ , \tag{2.71a}$$

$$T_{\Gamma}{}^{\gamma}{}_{\alpha\beta} = \Gamma^{\gamma}{}_{\alpha\beta} - \Gamma^{\gamma}{}_{\beta\alpha} \ . \tag{2.71b}$$

Lastly we emphasize that the complementary concept of the Poincaré symmetry as an external symmetry (2.13), can equally well be generalized to a local gauge symmetry and that the resulting gauge theory is complementary to the gauge theory of the Poincaré symmetry we constructed in the section 2.4. In the complementary approach the local gauge transformations are:

$$x^{\alpha} \longrightarrow x'^{\alpha} = x^{\alpha} + \epsilon^{\alpha}(x) \ , \tag{2.72a}$$

$$u(x) \longrightarrow u'(x') = u(x) - \frac{i}{2}\omega^{\mu\nu}\Sigma_{\mu\nu}u(x) \ . \tag{2.72b}$$

This approach has a clear advantage over the present approach when it comes to the geometrical interpretation of the invariance under the Poincaré gauge transformations. The gauge transformations (2.72a) and (2.72b) translate directly to the general infinitesimal coordinate transformations and to the local Lorentz frame rotations, respectively. No "internalization" of the transformations (2.62) is needed to provide the correspondence. Indeed, this is the way the gauge theory of the Poincaré symmetry has most often been considered [30],[33],[37]. The theory is called the Einstein-Cartan-Sciama-Kibble theory of gravitation.

2.5.2 Gravitational action and the equations of motion

We can re-interpret the action of the free gauge fields S_{G} as a gravitational action of the spacetime by rewriting the Lagrangian of the free gauge fields (2.53) in terms of the metric (2.57) and the connection coefficients (2.70) with respect to the coordinate bases,

$$S_{\mathrm{G}}[g,\Gamma] = \frac{1}{2\kappa}\int \mathrm{d}^4x\,\sqrt{-g}\left(\mathcal{R}_{\Gamma} - 2\Lambda\right)\ , \tag{2.73}$$

where the determinant of the metric is denoted as $g \equiv \det(g_{\alpha\beta})$ and we have defined the *scalar curvature* and the *Ricci tensor* as

$$\mathcal{R}_{\Gamma} = g^{\alpha\beta}R_{\Gamma\alpha\beta}\ , \quad R_{\Gamma\alpha\beta} = R_{\Gamma}{}^{\gamma}{}_{\alpha\gamma\beta}\ , \tag{2.74}$$

respectively. We set $\kappa \equiv 8\pi G$, where G is the *gravitational constant*, in order to match with experimental data. This is the well-known *Palatini action* — spiced with the *cosmological constant* Λ — which can be deduced from ECT.[17] The independent variables of the action are the metric and the connection coefficients with respect to the coordinate bases.

The equations of motion for the action (2.73) can be written [30, 31]

$$R_{\Gamma\alpha\beta} - \frac{1}{2}g_{\alpha\beta}\mathcal{R}_{\Gamma} + g_{\alpha\beta}\Lambda = \kappa\mathcal{T}_{\alpha\beta}\ , \tag{2.75a}$$

$$T_{\Gamma}{}^{\gamma}{}_{\alpha\beta} + \delta^{\gamma}_{\alpha}T_{\Gamma}{}^{\delta}{}_{\beta\delta} - \delta^{\gamma}_{\beta}T_{\Gamma}{}^{\delta}{}_{\alpha\delta} = \kappa\mathcal{S}^{\gamma}{}_{\alpha\beta}\ , \tag{2.75b}$$

where the energy-momentum tensor

$$\mathcal{T}_{\alpha\beta} = -\frac{2}{\sqrt{-g}}\frac{\delta S_{\mathrm{M}}}{\delta g^{\alpha\beta}} \tag{2.76}$$

and the spin density tensor

$$\mathcal{S}^{\gamma}{}_{\alpha\beta} = -\frac{2}{\sqrt{-g}}\frac{\delta S_{\mathrm{M}}}{\delta \Gamma^{\alpha}{}_{\gamma}{}^{\beta}} \tag{2.77}$$

are comparable to the corresponding tensors of the Poincaré gauge theory (2.56). The first equations of motion (2.75a) are the *Einstein equations* of GR, which connect the curvature of spacetime and the energy-momentum tensor of matter and radiation, but with the difference that now the torsion does not vanish and therefore the Ricci tensor and the energy-momentum tensor are not symmetric. The second equations of motion (2.75b) connect the torsion of spacetime and the spin density tensor of matter and radiation linearly, and they can be solved for the torsion

$$T_{\Gamma}{}^{\gamma}{}_{\alpha\beta} = \kappa\left(\mathcal{S}^{\gamma}{}_{\alpha\beta} + \frac{1}{2}\delta^{\gamma}_{\alpha}\mathcal{S}^{\delta}{}_{\beta\delta} - \frac{1}{2}\delta^{\gamma}_{\beta}\mathcal{S}^{\delta}{}_{\alpha\delta}\right)\ . \tag{2.78}$$

In the gauge theory approach the non-vanishing torsion and the spin density tensor are necessary consequences of the local Poincaré gauge symmetry.

The Bianchi identities (2.69) can be used to construct conservation equations for ECT. Equivalently the conservation equations can be derived from the gauge theory of gravitation we have constructed. The covariant divergence of the energy-momentum tensor does not vanish as it does in GR.

[17] For a recent review on the Einstein-Cartan theory and for references, see [52].

2.5.3 Comparison with General Relativity

We have shown that the gauge theory of the Poincaré symmetry we have constructed is equivalent with ECT, which was originally developed by Élie Cartan in the 1920s — before spin was discovered — and later rediscovered by Sciama and Kibble.

The difference between ECT and GR is that in ECT the torsion of spacetime is not set to zero and the spin of matter and radiation is explicitly recognized. In ECT the torsion of spacetime and the spin density are directly proportional in each point of spacetime (2.78). In the absence of spin, the torsion vanishes, the connection becomes a *Levi-Civita connection* and so ECT reduces to GR. This also means that the torsion cannot propagate, in other words "torsion waves" do not exist. There is no difference between ECT and GR in empty space.

We can eliminate the torsion of spacetime in the equations of motion of ECT (2.75) by separating the connection to a Levi-Civita connection and to a torsion dependent part,

$$\Gamma^{\gamma}{}_{\alpha\beta} = \overset{\circ}{\Gamma}{}^{\gamma}{}_{\alpha\beta} + \frac{1}{2}\left(T^{\gamma}{}_{\alpha\beta} + T_{\alpha}{}^{\gamma}{}_{\beta} + T_{\beta}{}^{\gamma}{}_{\alpha}\right) ,$$

and by replacing the torsion with the spin density (2.78). This way, we can replace the equations of motion (2.75) with the Einstein equation

$$\overset{\circ}{R}_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} \overset{\circ}{\mathcal{R}} + g_{\alpha\beta}\Lambda = \kappa \mathcal{T}^{\text{eff}}_{\alpha\beta} , \tag{2.79}$$

where the circle accent refers to a Levi-Civita connection depending only on the metric and the effective energy-momentum tensor is of the form

$$\mathcal{T}^{\text{eff}}_{\alpha\beta} = \mathcal{T}_{\alpha\beta} + \frac{1}{2}\overset{\circ}{\nabla}_{\gamma}\left(\mathcal{S}_{\alpha\beta}{}^{\gamma} + \mathcal{S}_{\beta}{}^{\gamma}{}_{\alpha} + \mathcal{S}_{\alpha}{}^{\gamma}{}_{\beta}\right) + \kappa \mathcal{S}^2 .$$

From the effective energy-momentum tensor it can be calculated that the contribution of the spin density becomes equally important with the energy-momentum, when the density of matter is $10^{47}\,\text{g/cm}^3$ for electrons or $10^{54}\,\text{g/cm}^3$ for protons [37]. These densities are so high that they can only be encountered in the early universe and in black holes, but they are still much smaller than the Planck density $m_\text{P}/l_\text{P}^3 \sim 10^{94}\,\text{g/cm}^3$ at which the quantum gravitational effects are believed to dominate. Thus the predictions of ECT and GR are identical in most circumstances.

Since all experiments devised to test GR so far only involve relatively low densities, all experimental data that support GR also support ECT. Thus, ECT is a viable theory of gravitation. In the light of today's knowledge, the elegant incorporation of spin in ECT is very welcome. It may also turn out that ECT is a better limit of a yet unknown quantum theory of gravitation than GR.

Chapter 3

Noncommutative field theory and twisted Poincaré symmetry

In this chapter we first review the concepts and tools needed to define field theories on noncommutative spacetimes. Then we discuss the twisted Poincaré symmetry, which provides the concept of special relativity for noncommutative field theories.

3.1 Noncommutative spacetime

The type of noncommutativity of spacetime coordinates discussed in this study is defined by the commutation relations

$$\left[\hat{x}^\mu, \hat{x}^\nu\right] = i\theta^{\mu\nu} \ , \tag{3.1}$$

where $\theta^{\mu\nu}$ is a real *constant* antisymmetric matrix. This is the form of noncommutativity encountered in the Seiberg-Witten low-energy limit of certain string theories [24], as we explained in the section 1.1. Since this observation, noncommutative quantum and gauge field theories have been under intense research (for reviews, see [53, 54]). We assume the matrix $\theta^{\mu\nu}$ is invertible, which implies that the spacetime has to be even-dimensional. Since we are primarily interested in the usual four-dimensional spacetime, this is not a problem.

In a more general case, $\theta^{\mu\nu}$ could be any kind of function of spacetime. For a comprehensive treatment of noncommutative spaces and their application to mathematical physics, see [1, 2, 3, 4, 55].

3.2 Weyl quantization and the Moyal $\star$-product

The concepts presented in this section enable us to represent noncommutative C^*-algebras of quantum operators on the algebra of commutative functions on an ordinary phase space. In the coordinate and momentum space representations of a noncommutative field theory, we can utilize most of the tools and methods of the ordinary commutative QFT.

3.2.1 Weyl operators and symbols

Hermann Weyl invented an elegant method of canonical quantization that utilizes the symmetrical operator ordering [56], which is perfectly suited for describing operators on noncommutative spacetime.

We consider a unital commutative algebra $\mathcal{A}$ of complex-valued functions on D-dimensional Minkowskian (or Euclidean) space, with the usual point-wise product of functions

$$fg(x) = f(x)g(x) = g(x)f(x) = gf(x) \ .$$

We assume that the functions belong to an appropriate Schwartz space with sufficiently rapid decrease at infinity, so that each function can be represented by its Fourier transform

$$\tilde{f}(k) = \int \mathrm{d}^D x\, f(x) e^{-ik_\mu x^\mu} \ . \tag{3.2}$$

The *Weyl operator* that corresponds to the function $f(x)$ is defined by

$$\hat{W}[f] = \int \frac{\mathrm{d}^D k}{(2\pi)^D}\, \tilde{f}(k) e^{ik_\mu \hat{x}^\mu} \ , \tag{3.3}$$

where the coordinate operators $\hat{x}^\mu$ satisfy the noncommutativity relations (3.1). The function $f(x)$ itself is a *Weyl symbol*. When a Weyl symbol is real-valued, the corresponding Weyl operator is Hermitian. We can further define the Hermitian operator

$$\hat{\Delta}(x) = \int \frac{\mathrm{d}^D k}{(2\pi)^D}\, e^{-ik_\mu \hat{x}^\mu} e^{ik_\nu x^\nu} \ , \tag{3.4}$$

which enables us to write the relation between the Weyl symbol $f(x)$ and its Weyl operator (3.3) in the explicit form

$$\hat{W}[f] = \int \mathrm{d}^D x\, f(x) \hat{\Delta}(x) \ . \tag{3.5}$$

Thus the Weyl symbol $f(x)$ can be interpreted to be the coordinate space representation of the Weyl operator $\hat{W}[f]$. The Weyl operators (3.5) generated by the noncommutative coordinates (3.1) for $\mathcal{A}$ constitute a unital noncommutative algebra $\hat{\mathcal{A}}$.

We can calculate the products of operators by using the Baker-Campbell-Hausdorff formula for exponential functions

$$e^{ik_\mu \hat{x}^\mu} e^{ik'_\nu \hat{x}^\nu} = e^{i(k+k')_\mu \hat{x}^\mu} e^{-\frac{i}{2}\theta^{\mu\nu} k_\mu k'_\nu}$$

and by inserting the operator relation (3.5) for $f(z) = e^{i(k+k')_\mu z^\mu}$:

$$e^{i(k+k')_\mu \hat{x}^\mu} = \int \mathrm{d}^D z\, e^{i(k+k')_\mu z^\mu} \hat{\Delta}(z) \ .$$

This gives us the products of the operators $\hat{\Delta}(x)$ as

$$\hat{\Delta}(x)\hat{\Delta}(y) = \iint \frac{\mathrm{d}^D k}{(2\pi)^D} \frac{\mathrm{d}^D k'}{(2\pi)^D} \int \mathrm{d}^D z\, e^{i(k+k')_\mu z^\mu} \hat{\Delta}(z) e^{-\frac{i}{2}\theta^{\mu\nu} k_\mu k'_\nu} e^{-ik_\mu x^\mu} e^{-ik_\nu y^\nu} \ . \tag{3.6}$$

Since the matrix $\theta^{\mu\nu}$ is invertible, we can perform the Gaussian momentum integrals in (3.6) and reach the result

$$\hat{\Delta}(x)\hat{\Delta}(y) = \frac{1}{\pi^D\,|\det\theta|}\int \mathrm{d}^D z\,\hat{\Delta}(z)e^{-i2(\theta^{-1})_{\mu\nu}(x-z)^\mu(y-z)^\nu}\ . \tag{3.7}$$

Derivatives of the noncommutative operators can be introduced through the standard operator commutation relations

$$\left[\hat{\partial}_\mu, \hat{x}^\nu\right] = \delta_\mu^\nu\ , \quad \left[\hat{\partial}_\mu, \hat{\partial}_\nu\right] = 0\ . \tag{3.8}$$

From (3.8) we can deduce the following important results for the derivative and for the trace of Weyl operators

$$\left[\hat{\partial}_\mu, \hat{W}[f]\right] = \hat{W}[\partial_\mu f(x)]\ , \tag{3.9}$$

$$\operatorname{Tr}\hat{W}[f] = \int \mathrm{d}^D x\, f(x) \quad ;\ \operatorname{Tr}\hat{\Delta}(x) = 1\ . \tag{3.10}$$

The relation between a Weyl operator and its Weyl symbol (3.5) can be reversed as:

$$f(x) = \operatorname{Tr}\left(\hat{W}[f]\hat{\Delta}(x)\right)\ . \tag{3.11}$$

3.2.2 The ⋆-product

The ⋆-product of two functions (Weyl symbols) is defined to be the representation of the product of their Weyl operators on the commutative algebra of functions

$$\hat{W}[f\star g] = \hat{W}[f]\hat{W}[g]\ . \tag{3.12}$$

The ⋆-product can be written explicitly by using the result (3.7)

$$\begin{aligned} f(x)\star g(x) &= f(x)\exp\left(\frac{i}{2}\overleftarrow{\partial}_\mu\theta^{\mu\nu}\overrightarrow{\partial}_\nu\right)g(x) \qquad (3.13)\\ &= f(x)g(x) + \sum_{n=1}^{\infty}\frac{1}{n!}\left(\frac{i}{2}\right)^n \theta^{\mu_1\nu_1}\cdots\theta^{\mu_n\nu_n}\partial_{\mu_1}\cdots\partial_{\mu_n}f(x)\partial_{\nu_1}\cdots\partial_{\nu_n}g(x)\ , \end{aligned}$$

where $\theta^{\mu\nu}$ is the constant matrix in (3.1). The nonlocality of the ⋆-product is obvious in (3.13). The value of $f(x)\star g(x)$ receives contributions not only from the point x, but also from an area around it, whose shape and size is defined by the constant $\theta^{\mu\nu}$. More precisely, if the fields $f(x)$ and $g(x)$ vanish outside a small region of size $\delta \ll \sqrt{\|\theta\|}$, then $f(x)\star g(x)$ is nonvanishing in a much larger region of size $\|\theta\|/\delta$ [57]. Physically this means that a high energy process can have immediate effects over a large — potentially infinite — region of spacetime. The ⋆-product has been named the Moyal product or the Groenewold–Moyal product after its inventors [58, 59].

The ⋆-product can be generalized for multiple functions at possibly different points

$$f_1(x_1)\star\cdots\star f_n(x_n) = \prod_{1\le a<b\le n}\exp\left(\frac{i}{2}\theta^{\mu\nu}\frac{\partial}{\partial x_a^\mu}\frac{\partial}{\partial x_b^\nu}\right)f_1(x_1)\cdots f_n(x_n)\ . \tag{3.14}$$

The noncommutative algebra of Weyl operators $\hat{\mathcal{A}}$ can be represented on the algebra of ordinary complex-valued functions by using the associative and noncommutative $\star$-product. Particularly, the commutator of Weyl operators is represented on the algebra of functions by the *Moyal bracket*

$$[f(x), g(x)]_\star = f(x) \star g(x) - g(x) \star f(x) \; . \tag{3.15}$$

The commutation relations of coordinates can be easily calculated to be of the same form as the corresponding operator relations (3.1)

$$[x^\mu, x^\nu]_\star = i\theta^{\mu\nu} \; , \tag{3.16}$$

since the $\star$-product of coordinates has the simple form

$$x^\mu \star x^\nu = x^\mu x^\nu + \frac{i}{2}\theta^{\mu\nu} = x^\nu x^\mu - \frac{i}{2}\theta^{\nu\mu} \; .$$

Indeed, the noncommutative algebra of functions $\mathcal{A}_\theta$, obtained by replacing the point-wise product of $\mathcal{A}$ with the noncommutative $\star$-product (3.13), is isomorphic to the algebra of Weyl operators $\hat{\mathcal{A}}$ generated by the noncommutative coordinate operators (3.1). This relation is often called the *Weyl-Moyal correspondence.*

The trace of the product of Weyl operators is represented by the integral of the $\star$-product of functions

$$\mathrm{Tr}\left(\hat{W}[f_1]\cdots\hat{W}[f_n]\right) = \int \mathrm{d}^D x \, f_1(x) \star \cdots \star f_n(x) \; . \tag{3.17}$$

Due to the invariance of the trace under cyclic permutations of the Weyl operators, also the integral of the $\star$-product of functions is invariant under cyclic permutations of functions. This implies, in particular,

$$\int \mathrm{d}^D x \, f(x) \star g(x) = \int \mathrm{d}^D x \, f(x) g(x) \; . \tag{3.18}$$

3.3 The twisted Poincaré symmetry of noncommutative spacetime

3.3.1 Breaking of the Lorentz symmetry

The defining commutation relation of the noncommutative spacetime (3.1) is clearly not covariant under Lorentz transformations, because the left-hand side of the relation is a tensor and the right hand-side is a constant. The noncommutative spacetime does not posses the Lorentz symmetry. Hence, Lorentz invariant theories are not supported. The noncommutative spacetime, however, is symmetric under translations.

In four dimensions we can always choose a reference frame where the θ-matrix in the commutation relations of coordinates (3.1) takes the block-diagonal form

$$\theta^{\mu\nu} = \begin{pmatrix} 0 & \vartheta_1 & 0 & 0 \\ -\vartheta_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \vartheta_2 \\ 0 & 0 & -\vartheta_2 & 0 \end{pmatrix} \; , \tag{3.19}$$

where we can see that for arbitary values ϑ_i the largest subgroup of the Lorentz group that conserves the commutation relations of coordinates is $SO(1,1) \times SO(2)$ [60, 61]. Since both of the group factors are one-dimensional Abelian rotation groups, we only have one-dimensional irreducible representations. This is a serious problem, because the quantum and gauge field theories of high energy physics are vitally dependent on the richer representation content of the Poincaré algebra — discussed in the section 2.2 — including spinor, vector and Lorentz tensor fields. Especially, the lack of representations with the spin posed a major threat to the usability of noncommutative field theories. The solution to the problems arising from the breaking of the Lorentz symmetry is the twisted Poincaré symmetry, a new kind of "quantum symmetry", which we will discuss next.

The Lorentz invariance has been verified with high accuracy. In spite of much effort, no sign of violation of the Lorentz invariance has been observed (for a review and references, see [62]). The noncommutativity scale Λ_{NC} is defined by

$$\theta^{\mu\nu} = \frac{1}{\Lambda_{\mathrm{NC}}^2}\,\epsilon_{\mu\nu}\ , \tag{3.20}$$

where $\epsilon_{\mu\nu} \sim 1$ is assumed. A low energy bound on Λ_{NC} was obtained from clock-comparison precison tests in [63] to be

$$\Lambda_{\mathrm{NC}} \gtrsim 10\,\mathrm{TeV} \sim (10^{-20}\,\mathrm{m})^{-1}\ . \tag{3.21}$$

A high energy bound on the noncommutativity scale was obtained in [64] by analysing high energy $2 \to 2$ processes

$$\Lambda_{\mathrm{NC}} \gtrsim 1\,\mathrm{TeV}\ . \tag{3.22}$$

These are only few important examples on the phenomenological research done around noncommutative physics. If the Lorentz invariance is indeed broken at high energies (low distances), the twisted Poincaré symmetry may well turn out to be a very important concept for Planck scale physics. It may as well turn out that the Lorentz symmetry is truly an exact symmetry, in which case we will have to find a more elaborate way to implement the nonlocality, and possibly the noncommutativity, of Planck scale physics.

3.3.2 Mathematics of twisted Hopf algebras

Before diving into the twisted Poincaré symmetry, we shall discuss the mathematical concepts needed to grasp the idea of this new symmetry. These concepts enable us to generalize the Poincaré symmetry of ordinary field theories to the noncommutative setting. For a further introduction on the subject and for a complete treatment of *quantum groups*, see the monographs [65, 66].

Universal enveloping algebras

Universal enveloping algebras are nearly as common as Lie algebras in physics, but we often take them for granted or do not even think about them. The universal enveloping algebra of a Lie algebra is the most general *unital associative algebra* into which the Lie algebra can be embedded.

Let us consider a Lie algebra generated by T_i, $i = 1, 2, \ldots, n$,

$$[T_i, T_j] = ic_{ijk}T_k \ .$$

The Lie algebra could for instance be the algebra of angular momentum operators J_i, a cornerstone of quantum mechanics. We do not usually see the Lie bracket $[T_i, T_j]$ as a non-associative product of generators T_i, but instead as a commutator

$$[T_i, T_j] \equiv T_iT_j - T_jT_i = ic_{ijk}T_k \ , \tag{3.23}$$

where the associative product is T_iT_j. Thus, we have embedded the Lie algebra into its universal enveloping algebra that consists of the polynomials in the generators T_i modulo the commutation relations (3.23) and of the unit element $\mathbf{1}$. The basis of the universal enveloping algebra can be chosen to consists of $\mathbf{1}$ and of the fully symmetrized products of the generators

$$T_{(i_1}T_{i_2}\cdots T_{i_n)} \ , \ n \in \mathbb{N} \ .$$

Since the universal enveloping of a Lie algebra fully captures the structure of the Lie algebra, the representation theory of the common generators are identical for the two algebras. In the universal enveloping of a Lie algebra we can define such polynomial operators as the quadratic Casimir operators, which can be used to classify the representations of the Lie algebra (for an example, see the discussion on the Poincaré algebra in section 2.2).

Every Lie algebra has a universal enveloping algebra, which is uniquely determined up to a unique algebra isomorphism by the Lie algebra. This property of "universality" is the reason why enveloping algebras of Lie algebras are called universal. The associativity of universal enveloping algebras enables the introduction of interesting additional structures and that is what makes universal enveloping algebras so useful for us.

Hopf algebras and their twists

Unital associative algebras have a natural hidden Hopf algebra structure. A unital associative algebra can be extended to a *bialgebra* by introducing a compatible *coalgebra* structure.[1] A bialgebra can be extended to a Hopf algebra by introducing an automorphism called the *antipode* (or coinverse) that is compatible with the bialgebra. The antipode is related to the fact that every element of a group has its inverse in the group. Indeed, Hopf algebras can be understood as generalizations of groups, since they can be used to describe groups and also more general concepts like quantum groups.

In this study we are interested in the structure of the universal enveloping algebra of a Lie algebra $\mathcal{U}$ as a Hopf algebra and in the function representations of $\mathcal{U}$ on spacetimes. The universal enveloping algebra $\mathcal{U}$ consists of a vector space V over the field $\mathbb{C}$ and of the multiplication and unit linear maps

$$\begin{aligned} m &: V \otimes V \to V \ , \\ \eta &: \mathbb{C} \to V \ , \end{aligned}$$

[1] It should be stressed that this coalgebra is not a dual of the algebra structure. A bialgebra is both an algebra and a coalgebra.

respectively.[2] Explicitly the multiplication is usually written

$$m(X \otimes Y) = XY \quad ; \ X, Y \in \mathcal{U} \ .$$

The multiplication m is *associative*

$$m \circ (\mathbf{1} \otimes m) = m \circ (m \otimes \mathbf{1})$$

and the unit map η implies the existence of a unit element $\mathbf{1}$ in V

$$m \circ (\mathbf{1} \otimes \eta) = m \circ (\eta \otimes \mathbf{1}) = \mathrm{id}\ (\text{identity map}) \ \Rightarrow \ \eta(\alpha) = \alpha \mathbf{1} \ .$$

The bialgebra structure for $\mathcal{U}$ is constructed by introducing the *coproduct* and *counit* homomorphisms

$$\Delta : V \to V \otimes V \ ,$$
$$\varepsilon : V \to \mathbb{C} \ ,$$

respectively. The coproduct Δ is *coassociative*,

$$(\mathrm{id} \otimes \Delta)\Delta = (\Delta \otimes \mathrm{id})\Delta \ ,$$

and the counit ε satisfies

$$(\mathrm{id} \otimes \varepsilon) \circ \Delta = (\varepsilon \otimes \mathrm{id}) \circ \Delta \ .$$

We complete the Hopf algebra by introducing the antipode, an antihomomorphism that is compatible with the bialgebra structure [3]

$$S : V \to V \ , \ m \circ (S \otimes \mathbf{1}) \circ \Delta = m \circ (\mathbf{1} \otimes S) \circ \Delta = \eta \circ \varepsilon \ .$$

The natural hidden Hopf algebra structure of $\mathcal{U}$ is defined by

$$\Delta_0(X) = X \otimes \mathbf{1} + \mathbf{1} \otimes X \ , \qquad \Delta_0(\mathbf{1}) = \mathbf{1} \otimes \mathbf{1} \ , \tag{3.24}$$
$$\varepsilon(X) = 0 \ , \qquad \varepsilon(\mathbf{1}) = 1 \ , \tag{3.25}$$
$$S(X) = -X \ , \qquad S(\mathbf{1}) = \mathbf{1} \ , \tag{3.26}$$

for all $X \in V - \{\mathbf{1}\}$. It is easy to see that these maps satisfy all the above requirements. The Hopf algebra $\mathcal{U}$ is noncommutative, but cocommutative due to the symmetry of the coproduct (3.24).

We can deform a cocommutative Hopf algebra like $\mathcal{U}$ to a noncocommutative one by introducing a twist element

$$\mathcal{F} \in \mathcal{U} \otimes \mathcal{U}$$

and by redefinig the coproduct of the Hopf algebra with a similarity transformation

$$\Delta_0(X) \longrightarrow \Delta_t(X) = \mathcal{F} \Delta_0(X) \mathcal{F}^{-1} \ , \tag{3.27}$$

[2] Generally the field $\mathbb{C}$ could instead be any field.

[3] Recall that a homomorphism h satisfies $h(xy) = h(x)h(y)$ and that an antihomomorphism a satisfies $a(xy) = a(y)a(x)$.

in other words by twisting the coproduct of $\mathcal{U}$. In order to preserve the Hopf algebra structure, the twist element has to satisfy the twist conditions

$$\mathcal{F}_{12}(\Delta_0 \otimes \mathrm{id})\mathcal{F} = \mathcal{F}_{23}(\mathrm{id} \otimes \Delta_0)\mathcal{F} \ , \tag{3.28a}$$
$$(\varepsilon \otimes \mathrm{id})\mathcal{F} = \mathbf{1} = (\mathrm{id} \otimes \varepsilon)\mathcal{F} \ , \tag{3.28b}$$

where $\mathcal{F}_{12} = \mathcal{F} \otimes \mathbf{1}$ and $\mathcal{F}_{23} = \mathbf{1} \otimes \mathcal{F}$. We denote the twist deformed $\mathcal{U}$ by $\mathcal{U}_t$. The twist element does not affect the multiplication m of the algebra $\mathcal{U}_t$ and therefore the commutation relations (3.23) among the generators of $\mathcal{U}$ are preserved. This means that *the representation content of $\mathcal{U}_t$ is identical with that of $\mathcal{U}$*. What is affected by the twist, is the action of $\mathcal{U}_t$ onto the tensor products of its representations, i.e. the Leibniz rule.

Let $\mathcal{A}$ be a *commutative* associative algebra of functions — consistent with the coproduct Δ_0 — that holds a representation of $\mathcal{U}$. The commutative multiplication on $\mathcal{A}$ is defined by

$$m^{\mathcal{A}} : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A} \ , \quad fg = m^{\mathcal{A}}(f \otimes g) \ . \tag{3.29}$$

We denote the action of $X \in \mathcal{U}$ on $f \in \mathcal{A}$ by $X \triangleright f$. An element of the universal enveloping algebra $X \in \mathcal{U}$ acts on a product of representations $f, g \in \mathcal{A}$ by the standard Leibniz rule [4]

$$X \triangleright (fg) = m^{\mathcal{A}}\big(\Delta_0(X) \triangleright (f \otimes g)\big) = (X \triangleright f)g + f(X \triangleright g) \ . \tag{3.30}$$

When the coproduct of $\mathcal{U}$ is twisted, while preserving the action of $\mathcal{U}$ on the elements of $\mathcal{A}$, the multiplication map of the algebra holding the representations has to be redefined to

$$m^{\mathcal{A}} \longrightarrow m_t^{\mathcal{A}} = m^{\mathcal{A}} \circ \mathcal{F}^{-1} \ , \tag{3.31}$$

where $\mathcal{F}^{-1}$ is the inverse of the twist element $\mathcal{F}$ acting on $\mathcal{A} \otimes \mathcal{A}$. The noncommutative algebra $\mathcal{A}_t$ obtained by replacing the multiplication (3.29) with the deformed multiplication

$$m_t^{\mathcal{A}} : \mathcal{A}_t \otimes \mathcal{A}_t \to \mathcal{A}_t \ , \quad f \star g = m_t^{\mathcal{A}}(f \otimes g) = m^{\mathcal{A}}\big(\mathcal{F}^{-1} \triangleright (f \otimes g)\big) \ , \tag{3.32}$$

which is consistent with the twisted coproduct of $\mathcal{U}_t$ (3.27), holds the representation of the twisted universal enveloping algebra $\mathcal{U}_t$. The action $\triangleright_t$ of the twisted universal enveloping algebra $\mathcal{U}_t$ on the representation $f \in \mathcal{A}_t$ is not affected by the twisting

$$X \triangleright_t f = X \triangleright f \ . \tag{3.33}$$

However, the action of $X \in \mathcal{U}_t$ on the product of representations $f, g \in \mathcal{A}_t$ is altered by the twist and it is given by the deformed Leibniz rule

$$X \triangleright_t (f \star g) = m_t^{\mathcal{A}}\big(\Delta_t(X) \triangleright_t (f \otimes g)\big) \ . \tag{3.34}$$

Lastly we note that the above argument can be reversed: If we deform an algebra $\mathcal{A}$ holding a representation of $\mathcal{U}$ by redefining the multiplication as in (3.32), then we have to consistently twist the coproduct of $\mathcal{U}$ by (3.27).

[4] The action of $X \otimes Y \in \mathcal{U} \otimes \mathcal{U}$ on $f \otimes g \in \mathcal{A}_1 \otimes \mathcal{A}_2$ is defined: $(X \otimes Y) \triangleright (f \otimes g) = (X \triangleright f) \otimes (Y \triangleright g)$.

Example of a twist Consider a Lie group G — a smooth manifold — and a set of vector fields $\{X_i\}_{i=1}^m$ on G that commute with each other. The element

$$\mathcal{F} = e^{\sigma^{ij} X_i \otimes X_i} \equiv \mathbf{1} \otimes \mathbf{1} + \sum_{n=1}^{\infty} \frac{1}{n!} (\sigma^{ij} X_i \otimes X_i)^n \ \in \mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g}) \ , \tag{3.35}$$

where σ^{ij} is an arbitary constant matrix and $\mathfrak{g}$ is the Lie algebra of G, satisfies the twist conditions (3.28)

$$\mathcal{F}_{12}(\Delta_0 \otimes \mathrm{id})\mathcal{F} = e^{\sigma^{ij}(X_i \otimes X_j \otimes \mathbf{1} + X_i \otimes \mathbf{1} \otimes X_j + \mathbf{1} \otimes X_i \otimes X_j)} = \mathcal{F}_{23}(\mathrm{id} \otimes \Delta_0)\mathcal{F} \tag{3.36}$$

and the second one is trivially true due to the counit (3.25) and $X_i \neq \mathbf{1}$. Therefore (3.35) is a twist for $\mathcal{U}(\mathfrak{g})$.

3.3.3 Twisting the Poincaré symmetry

For some time the problem of insufficient representation content of noncommutative field theories, due to the breaking of the Lorentz symmetry, was often ignored and studies on noncommutative quantum field theories were made by using the full representation content of the Poincaré algebra. These studies include the important discussions on unitarity [67] and causality [68],[69] and on noncommutative counterparts of some specific models like QED [70] and SM [71]. One could try to justify this approach by considering the spacetime noncommutativity as a pertubation of commutative QFT, but this would be far too limited, since it rules out all non-pertubative considerations.

An elegant solution to the problem was discovered by M. Chaichian et al. [43, 44] in the form of a twisted Poincaré symmetry. They introduced a twist deformation of the universal enveloping algebra $\mathcal{U}(\mathcal{P})$ of the Poincaré algebra $\mathcal{P}$ that provides a new symmetry that is respected by the noncommutative theory obtained by Weyl quantization on the noncommutative spacetime (3.1). Since the twist deformation does not alter the multiplication in $\mathcal{U}(\mathcal{P})$, the commutation relations among its generators (2.2) are preserved. Thus *the representation content of the twisted algebra $\mathcal{U}_t(\mathcal{P})$ is the same as the representation content of the usual Poincaré algebra.* This legitimates the usage of the familiar representations of the Poincaré symmetry in the context of noncommutative field theories.

The Poincaré algebra $\mathcal{P}$ (2.2) has a commutative subalgebra of translation generators P_μ that can be used to construct the Abelian twist element

$$\mathcal{F} = e^{\frac{i}{2}\theta^{\mu\nu} P_\mu \otimes P_\nu} \ , \tag{3.37}$$

where $\theta^{\mu\nu}$ is a real *constant* antisymmetric matrix. This twist element is a special case of the twist (3.35), so it clearly satisfies the twist conditions (3.28) and thus it can be used to consistently twist the coproduct (3.24) of the Hopf algebra $\mathcal{U}(\mathcal{P})$. Explicitly, the twisted coproduct (3.27) for $X \in \mathcal{P}$ is written

$$\Delta_t(X) = e^{\frac{i}{2}\theta^{\mu\nu} P_\mu \otimes P_\nu} (X \otimes \mathbf{1} + \mathbf{1} \otimes X) e^{-\frac{i}{2}\theta^{\mu\nu} P_\mu \otimes P_\nu} \ .$$

The coproduct of the translation generators P_μ is not affected by the twist (3.37) due to commutativity of translations (2.2a),

$$\Delta_t(P_\mu) = \Delta_0(P_\mu) = P_\mu \otimes \mathbf{1} + \mathbf{1} \otimes P_\mu \ . \tag{3.38}$$

The coproduct of the Lorentz generators $M_{\mu\nu}$ is altered by the twist, because of the non-vanishing commutation relations (2.2b). The coproduct

$$\Delta_t(M_{\mu\nu}) = \mathrm{Ad}\, e^{\frac{i}{2}\theta^{\rho\sigma} P_\rho \otimes P_\sigma} \Delta_0(M_{\mu\nu})$$

can be evaluted by using (2.2b) and the operator formula

$$\mathrm{Ad}\, e^B C = e^B C e^{-B} = \sum_{n=0}^{\infty} \frac{1}{n!} \underbrace{[B,[B,\dots[}_{n} B, C]] = \sum_{n=0}^{\infty} \frac{(\mathrm{Ad}\, B)^n}{n!} C \ .$$

The result is

$$\begin{aligned}
\Delta_t(M_{\mu\nu}) &= \Delta_0(M_{\mu\nu}) + \frac{i}{2}\theta^{\rho\sigma}\big([P_\rho, M_{\mu\nu}] \otimes P_\sigma + P_\rho \otimes [P_\sigma, M_{\mu\nu}]\big) \\
&= M_{\mu\nu} \otimes \mathbf{1} + \mathbf{1} \otimes M_{\mu\nu} - \frac{1}{2}\theta^{\rho\sigma}\big((\eta_{\rho\mu}P_\nu - \eta_{\rho\nu}P_\mu) \otimes P_\sigma + P_\rho \otimes (\eta_{\sigma\mu}P_\nu - \eta_{\sigma\nu}P_\mu)\big).
\end{aligned} \tag{3.39}$$

Let us next consider the commutative algebra $\mathcal{A}$ of smooth complex-valued functions on Minkowski spacetime, with the usual commutative point-wise multiplication

$$m\big(f(x) \otimes g(x)\big) = f(x)g(x) \ . \tag{3.40}$$

In $\mathcal{A}$ the representation of $\mathcal{U}(\mathcal{P})$ is generated by the standard realization of the Poincaré algebra (2.4)

$$P_\mu \triangleright f(x) = i\partial_\mu f(x) \tag{3.41a}$$

$$M_{\mu\nu} \triangleright f(x) = i(x_\mu \partial_\nu - x_\nu \partial_\mu) f(x) \ , \tag{3.41b}$$

that acts on the coordinates in the standard way: $\partial_\mu x^\nu = \delta^\nu_\mu$. $\mathcal{U}(\mathcal{P})$ acts on the product of representations (3.40) through the standard Leibniz rule (3.30) defined by the symmetric coproduct (3.24).

When $\mathcal{U}((\mathcal{P})$ is twisted with the twist element (3.37), the multiplication of its reprentations on $\mathcal{A}$ has to be redefined according to (3.32). The noncommutative algebra $\mathcal{A}_t$ of functions that holds the representation of the *twisted Poincaré algebra* $\mathcal{U}_t(\mathcal{P})$,[5] has the noncommutative multiplication rule

$$\begin{aligned}
m_t\big(f(x) \otimes g(x)\big) &= m\Big(\mathcal{F}^{-1} \triangleright \big(f(x) \otimes g(x)\big)\Big) \\
&= m\Big(e^{-\frac{i}{2}\theta^{\mu\nu} P_\mu \otimes P_\nu} \triangleright \big(f(x) \otimes g(x)\big)\Big) \\
&= m\Big(e^{\frac{i}{2}\theta^{\mu\nu} \partial_\mu \otimes \partial_\nu} \big(f(x) \otimes g(x)\big)\Big) \\
&= f(x) \star g(x) \ .
\end{aligned} \tag{3.42}$$

[5]A more verbose name for $\mathcal{U}_t(\mathcal{P})$ would be *the twist deformed universal enveloping algebra of the Poincaré algebra.*

The comparison of the Moyal $\star$-product (3.13) and of the multiplication m_t on $\mathcal{A}_t$ (3.42) reveals they are in fact the same noncommutative product of functions. Thus the algebra $\mathcal{A}_t$ holding the representation of $\mathcal{U}_t(\mathcal{P})$ is indeed the same algebra of functions $\mathcal{A}_\theta$ that was found to be isomorphic to the algebra of Weyl operators $\hat{\mathcal{A}}$ generated by the noncommutative coordinate operators (3.1) (see subsection 3.2.2), $\mathcal{A}_t = \mathcal{A}_\theta \cong \hat{\mathcal{A}}$. As an example, we calculate the coordinate commutation relations on $\mathcal{A}_t$,

$$[x^\mu, x^\nu]_\star = m_t(x^\mu \otimes x^\nu) - m_t(x^\nu \otimes x^\mu) = \frac{i}{2}(\theta^{\mu\nu} - \theta^{\nu\mu}) = i\theta^{\mu\nu} \ , \tag{3.43}$$

which are of course identical with the Moyal-brackets of coordinates (3.16). Now it is clear that building quantum field theory through Weyl quantization with noncommutative coordinate operators is equivalent to the procedure of twisting the Poincaré algebra and redefining the multiplication of its representations.

According to (3.33), the action of the twisted Poincaré algebra $\mathcal{U}_t(\mathcal{P})$ on its representation in $\mathcal{A}_t$ is not affected by the twist, but the action on the product of representations is given by the deformed Leibniz rule (3.34). Let us calculate the Leibniz rules for the generators P_μ, $M_{\mu\nu}$ of $\mathcal{U}_t(\mathcal{P})$ by using their coproducts (3.38) and (3.39). Since the coproduct of P_μ is not deformed, it has the standard Leibniz rule

$$P_\mu \triangleright_t \big(f(x) \star g(x)\big) = \big(P_\mu \triangleright_t f(x)\big) \star g(x) + f(x) \star \big(P_\mu \triangleright_t g(x)\big) \ . \tag{3.44}$$

For the generators $M_{\mu\nu}$ we obtain the deformed Leibniz rule

$$\begin{aligned} M_{\mu\nu} \triangleright_t \big(f(x) \star g(x)\big) = {} & M_{\mu\nu} \triangleright_t f(x) \star g(x) + f(x) \star M_{\mu\nu} \triangleright_t g(x) \\ & - \frac{1}{2}\theta^{\rho\sigma}\big((\eta_{\rho\mu}P_\nu - \eta_{\rho\nu}P_\mu) \triangleright_t f(x) \star P_\sigma \triangleright_t g(x) \\ & + P_\rho \triangleright_t f(x) \star (\eta_{\sigma\mu}P_\nu - \eta_{\sigma\mu}P_\nu) \triangleright_t g(x)\big) \ . \end{aligned} \tag{3.45}$$

Next we show that a noncommutative quantum field theory built through the Weyl-Moyal correspondence is invariant under the twisted Poincaré algebra. We do this by first checking that the commutation relations of coordinates (3.43) — equivalently (3.1) — are invariant under the twisted Poincaré transformations and then by showing that functions on the noncommutative spacetime are covariant under the twisted Poincaré algebra.

Invariance of $[x^\mu, x^\nu]_\star$ under translations is obvious due to the invariance under ordinary translations and the standard Leibniz rule (3.44). According to (3.45) and (3.41), $M_{\mu\nu}$'s action on the $\star$-product of coordinates is

$$\begin{aligned} M_{\mu\nu} \triangleright_t (x^\rho \star x^\sigma) = {} & i(\eta_{\mu\tau}\delta^\rho_\nu - \eta_{\nu\tau}\delta^\rho_\mu)x^\tau \star x^\sigma + i(\eta_{\mu\tau}\delta^\sigma_\nu - \eta_{\nu\tau}\delta^\sigma_\mu)x^\rho \star x^\tau \\ & + \frac{1}{2}\theta^{\tau\upsilon}\big((\eta_{\mu\tau}\delta^\rho_\nu - \eta_{\nu\tau}\delta^\rho_\mu)\delta^\sigma_\upsilon - (\eta_{\mu\tau}\delta^\sigma_\nu - \eta_{\nu\tau}\delta^\sigma_\mu)\delta^\rho_\upsilon\big) \ , \end{aligned} \tag{3.46}$$

where the last term is antisymmetric in $\rho \leftrightarrow \sigma$. Thus the coordinate commutators are invariant under the action of $M_{\mu\nu}$

$$\begin{aligned} M_{\mu\nu} \triangleright_t [x^\rho, x^\sigma]_\star &= i\big(\eta_{\mu\tau}\delta^\rho_\nu - \eta_{\nu\tau}\delta^\rho_\mu\big)\big([x^\tau, x^\sigma]_\star - i\theta^{\tau\sigma}\big) \\ &\quad - i\big(\eta_{\mu\tau}\delta^\sigma_\nu - \eta_{\nu\tau}\delta^\sigma_\mu\big)\big([x^\tau, x^\rho]_\star - i\theta^{\tau\rho}\big) \\ &= 0 \end{aligned}$$

i.e. invariant under twisted Lorentz transformations. This is consistent with the $\theta^{\mu\nu}$ being an invariant antisymmetric tensor

$$P_\mu \triangleright_t \theta^{\mu\nu} = 0 \ , \quad M_{\mu\nu} \triangleright_t \theta^{\mu\nu} = 0 \ .$$

So, the commutation relations (3.43) are invariant under twisted Poincaré transformations. Note that (3.46) also implies the twisted-Poincaré invariance of the Minkowski length $x^2 \equiv x_\mu \star x^\mu = x_\mu x^\mu$

$$M_{\mu\nu} \triangleright_t x^2 = \eta_{\rho\sigma} M_{\mu\nu} \triangleright_t (x^\rho \star x^\sigma) = 0 \ .$$

Because of the invariant commutation relations (3.43)

$$[x^\mu, x^\nu]_\star = 2x^{[\mu} \star x^{\nu]} = 0 \ ,$$

every tensorial object of the form

$$x^{\mu_1} \star x^{\mu_2} \star \cdots \star x^{\mu_n}$$

can be written as a sum of symmetrized tensors

$$x^{(\mu_1} \star x^{\mu_2} \star \cdots \star x^{\mu_m)}$$

with equal or lower ranks $m \leq n$, which means that the basis of the representation algebra $\mathcal{A}_t$ is symmetric. Hence, it is sufficient to show that symmetrized tensors on $\mathcal{A}_t$ are covariant under $\mathcal{U}_t(\mathcal{P})$. Consider the transformation behaviour of the symmetric rank two tensor

$$f_\star^{\rho\sigma} = x^{(\rho} \star x^{\sigma)} \equiv \frac{1}{2}\big(x^\rho \star x^\sigma + x^\sigma \star x^\rho\big) \ .$$

The twisted Lorentz transformations of this tensor are directly obtained from (3.46)

$$M_{\mu\nu} \triangleright_t f_\star^{\rho\sigma} = i(\eta_{\mu\tau}\delta_\nu^\rho - \eta_{\nu\tau}\delta_\mu^\rho) f_\star^{\tau\sigma} + i(\eta_{\mu\tau}\delta_\nu^\sigma - \eta_{\nu\tau}\delta_\mu^\sigma) f_\star^{\rho\tau} \ .$$

This is the same tensor transformation rule obtained for the corresponding symmetric tensor $f^{\mu\nu} = x^{(\mu}x^{\nu)} = x^\mu x^\nu$ in the commutative theory,

$$M_{\mu\nu} \triangleright f^{\rho\sigma} = i(\eta_{\mu\tau}\delta_\nu^\rho - \eta_{\nu\tau}\delta_\mu^\rho) f^{\tau\sigma} + i(\eta_{\mu\tau}\delta_\nu^\sigma - \eta_{\nu\tau}\delta_\mu^\sigma) f^{\rho\tau} \ .$$

Thus, $f_\star^{\rho\sigma}$ is twisted-Poincaré covariant. Generalization to arbitary symmetric tensors $x^{(\mu_1} \star x^{\mu_2} \star \cdots \star x^{\mu_n)}$ follows by induction.

This implies that the noncommutative quantum field theories built through Weyl quantization and Moyal $\star$-product possess the twisted Poincaré symmetry. Finally we can conclude that, *if in commutative theories, relativistic invariance means invariance under the Poincaré transformations, then in noncommutative theories relativistic invariance means invariance under the twisted Poincaré transformations.* This also enables us to adopt the point of view, where the noncommutativity of coordinates (3.1) is an implication of the twisted Poincaré symmetry of spacetime.

The generalization of the above discussion for all physically interesting representations of the Poincaré algebra has not yet been achieved. Because the Poincaré algebra is twisted, the definition of the concept of field is more involved on noncommutative space than on commutative space. In the commutative setting, Minkowski space is realized as the quotient of the Poincaré group by the Lorentz group, and a classical field is a section of a vector bundle induced by some representation of the Lorentz group. However, the universal enveloping algebra of the Lorentz Lie algebra is not a Hopf subalgebra of the twisted Poincaré algebra, so that there exists no noncommutative analogue of the Minkowski space. For a proper discussion on this problem, see [72].

3.4 Basic results in noncommutative QFT

In this section we briefly mention some important general aspects and results in noncommutative QFT. The subjects discussed are perturbation theory and UV/IR mixing, unitarity, causality and the spin-statistics relation.

3.4.1 Example: Noncommutative scalar field $\lambda\phi^4$-theory

Noncommutative action

As a concrete example, let us consider the QFT of the massive real-valued scalar field ϕ with the $\lambda\phi^4$-interaction defined on the noncommutative version of four-dimensional Minkowski spacetime.[6] The action of the theory is written in terms of the Weyl operator $\hat{W}[\phi]$ associated with the field $\phi(x)$ (Weyl symbol)

$$S[\phi] = \mathrm{Tr}\left(\frac{1}{2}\left[\hat{\partial}_\mu, \hat{W}[\phi]\right]^2 - \frac{m^2}{2}\hat{W}[\phi]^2 - \frac{\lambda}{4!}\hat{W}[\phi]^4\right) . \tag{3.47}$$

The path integral measure is chosen to be the usual Feynman measure for $\phi(x)$. We can write the action as an integral over spacetime by using the formulae (3.9) and (3.17). By using the property (3.17), we can replace the single $\star$-products with the ordinary point-wise product under the spacetime integral. So we have

$$S[\phi] = \int \mathrm{d}^4x \left(\frac{1}{2}\partial_\mu\phi(x)\partial^\mu\phi(x) - \frac{m^2}{2}\phi(x)^2 - \frac{\lambda}{4!}\phi(x)\star\phi(x)\star\phi(x)\star\phi(x)\right) . \tag{3.48}$$

As we can see from the action, the free theory of the field ϕ is identical with the corresponding commutative theory. Only the higher-than-second-order interaction term with multiple $\star$-products separates the action (3.48) from the more familiar action of commutative fields, but this "small" difference has some remarkable implications.

Noncommutative pertubation theory

The fact that the free noncommutative scalar field theory is identical with ordinary commutative theory means especially that the bare Feynman propagator for the

[6] For the commutative counterpart of this theory, see for example the books [49, 73].

field ϕ is the same as in the commutative case

$$D_{\mathrm{F}}(x-y) = \int \frac{\mathrm{d}^D k}{(2\pi)^D}\, i\Delta_{\mathrm{F}}(k) e^{-ik\cdot(x-y)} \quad ; \; i\Delta_{\mathrm{F}}(k) \equiv \frac{i}{k^2 - m^2 + i\epsilon} \; .$$

The new features arise from the interaction vertex of four fields. The interaction can be written

$$\mathrm{Tr}\Big(\hat{W}[\phi]^4\Big) = \int \left[\prod_{i=1}^{4} \frac{\mathrm{d}^D k_i}{(2\pi)^D}\, \tilde{\phi}_a(k_i)\right] (2\pi)^D \delta^D\left(\sum_{i=1}^{4} k_i\right) V(k_1,k_2,k_3,k_4) \; ,$$

where the momentum space interaction vertex is the phase factor

$$V(k_1,k_2,k_3,k_4) = \prod_{1\le i<j\le 4} e^{-\frac{i}{2}k_i\times k_j} = e^{-\frac{i}{2}\sum_{i<j} k_i\times k_j} \; , \tag{3.49}$$

and the antisymmetric product of momenta is denoted by

$$k_i \times k_j = k_{i\mu}\theta^{\mu\nu}k_{j\nu} \; .$$

Hence, instead of the ordinary momentum space Feynman rule $-i\lambda$, we now have the Feynman rule $-i\lambda V(k_1,k_2,k_3,k_4)$ for the $\lambda\phi^4$-interaction vertex. The phase factor (3.49) describes the nonlocality of the interaction and the fact that the order of the fields in a vertex is significant due to the noncommutativity of the fields.

The phase factor of Feynman graphs and crossing propagators The total phase factor received by a Feynman graph due to the spacetime noncommutativity depends crucially on the order the lines in the graph and on whether or not the lines in the graph cross over each other. The following results for the phase factor are presented in [74] and reviewed in [57].

Consider a connected Feynman graph with n external lines, which are cyclically labelled with momenta $k_1, k_2, \ldots, k_n$. If the lines in the Feynman graph do not cross over each other — i.e. we have a planar graph —, the graph receives the phase factor

$$V_{\text{planar}}(k_1,\ldots,k_n) = \prod_{1\le i<j\le n} e^{-\frac{i}{2}k_i\times k_j} = e^{-\frac{i}{2}\sum_{i<j} k_i\times k_j} \; ,$$

which is completely independent on the internal structure of the Feynman graph. If the lines in the Feynman graph do cross over each other — i.e. we have a nonplanar graph —, the graph receives the phase factor

$$\begin{aligned} V_{\text{nonplanar}}(k_1,\ldots,k_n) &= V_{\text{planar}}(k_1,\ldots,k_n)\prod_{i,j} e^{-\frac{i}{2}C_{ij}k_i\times k_j} \\ &= e^{-\frac{i}{2}\sum_{i<j} k_i\times k_j - \frac{i}{2}\sum_{i,j} C_{ij}k_i\times k_j} \; , \end{aligned} \tag{3.50}$$

where C_{ij} is the intersection matrix which counts how many times the line k_i (external or internal) crosses over the line k_j. If the line k_i crosses over the line k_j that is moving to the left (right), then C_{ij} is $+1$ (-1). If the lines do not cross, C_{ij} is 0. Since now also the internal lines are taken into a count, the phase factor depends on the internal structure of the nonplanar Feynman graph. Especially in the nonplanar case the $\theta^{\mu\nu}$ dependence of the phase factor can be complex. In the nonplanar case, the rapidly oscillating phase factor can provide an effective ultraviolet (UV) cutoff $\Lambda_{\text{eff}} \sim \|\theta\|^{-\frac{1}{2}}$ that makes an otherwise ultraviolet divergent graph converge. In planar graphs the phase factor does not improve the UV behaviour of the graph.

3.4.2 UV/IR mixing

The most novel feature of noncommutative pertubation theory is the interesting mixing of perturbative high energy and low energy dynamics, named the UV/IR mixing. It was first discovered in noncommutative scalar field theory [57] and similar results were also independently reached in noncommutative quantum electrodynamics [70].

The UV/IR mixing of noncommutative field theories originates from the additional phase factor of nonplanar Feynman graphs — like (3.50) in the above example — when conventional methods are used to regulate the graphs. In [57] it was discovered that in the noncommutative Euclidean ϕ^4-theory, the effective UV cutoff Λ_{eff} arising from the one loop nonplanar graph in mass renormalization is of the form

$$\Lambda_{\text{eff}}^2 = \frac{1}{\Lambda^{-2} + p \circ p} \quad ; \; p \circ p \equiv p_\mu (\theta^{\mu\nu})^2 p_\nu \ ,$$

where the parameter Λ regulates the integrals of the one loop graphs and p is the external momentum. In the limit $\Lambda \to \infty$ the effective UV cutoff is finite

$$\Lambda_{\text{eff}}^2 = \frac{1}{p \circ p}$$

and so is the nonplanar graph. If we now take $p \to 0$ (or $\theta \to 0$) the nonplanar graph diverges quadratically along with the effective UV cutoff $\Lambda_{\text{eff}} \to \infty$ in a way that is not encountered in the commutative theory. Although this is an IR divergence ($p \to 0$), it is the result of high energy virtual processes. If the UV ($\Lambda \to \infty$) and the IR ($p \to 0$) limits are taken in the opposite order, we have $\Lambda_{\text{eff}} = \Lambda \to \infty$ and so the one loop graph is divergent in the same way as in the commutative theory. This kind of noncommutativity of UV and IR limits is not found in commutative field theories.

3.4.3 Unitarity and causality

Unitarity and causality are essential ingriedients of local QFT. It is hard to imagine a well-defined physical field theory without them and therefore we will reject theories that are not both unitary and causal.

The noncommutativity of time and space $\theta^{0i} \neq 0$ violates both the unitarity [67] and the causality [68] of noncommutative field theories.[7] Noncommutative field theories with space-space noncommutativity $\theta^{0i} = 0$ have been concluded to be both

[7] Theories with noncommutative time and space coordinates $\theta^{0i} \neq 0$ have many names. They are called “time-space noncommutative” or “space/time noncommutative” or “space-time noncommutative” or “time-like noncommutative”.

unitary and causal [67, 68] and the noncommutative field theories obtained as low energy limits of string theory [24] are precisely of this type.

In the further study of unitarity [75], it was discovered that only unitary noncommutative field theories can be obtained as decoupled field theory limits of string theory. Non-unitary field theories cannot be obtained from string theory because for them the massive open string modes do not decouple. It was also found that field theories with light-like noncommutativity $\theta^{\mu\nu}\theta_{\mu\nu} = 0$ (i.e. $\theta^{0i} = -\theta^{1i}$) are unitary after all.

In theories with time-space noncommutativity $\theta^{0i} \neq 0$, causality is violated on both micro- and macroscopic levels [68, 60]. Noncommutativity of the time coordinate makes physical events nonlocal with respect to time, which means that the value of a field at given time t depends on both the value of the field before and after t. Therefore the effect, "the value of the field right now", can precede the cause, "the value of the field in future". However, nonlocality in space coordinates does not lead to acausal effects [68].

For a physical interpretation of the violation of unitarity in theories with time-space noncommutativity and for a quantitative discussion of unitarity and causality in such theories, see [60].

For these reasons, the space-space noncommutative field theories ($\theta^{0i} = 0$) — and posssibly the light-like theories ($\theta^{\mu\nu}\theta_{\mu\nu} = 0$) — are considered to be the only valid choices for physical models.

3.4.4 Spin-statistics relation and CPT theorem

The spin-statistics relation and the CPT theorem are fundamental results in local QFT and therefore we would very much like to preserve these results in the noncommutative theory. Fortunately, it has been convincingly argued that both the spin-statistics relation and the CPT theorem stand steady in noncommutative QFT.

Pauli's spin-statistics relation, stating that integer spin particles have bosonic (symmetric) statistics and half-integer spin particles have fermionic (antisymmetric) statistics, was proved to hold in noncommutative QFT [69] by using the Lagrangian formalism according to Pauli and revisited in [76] after the discovery of the twisted Poincaré symmetry. The same result was reached through axiomatic approach in [61, 77, 78]. The only remaining question, does the spin-statistics relation hold in light-like noncommutative $\theta^{\mu\nu}\theta_{\mu\nu} = 0$ theories, received a positive answer in [79], where it was shown that the twisted Poincaré symmetry does not experience nontrivial statistics, regardless of the form of the noncommutativity.

The CPT theorem has been proved to hold in noncommutative field theory, in spite of the nonlocality and the violation of the Lorentz symmetry, even though the individual charge conjugation (C) and time reversal (T) symmetries, and also the parity (P) in some cases, are broken [80]. In [69], the CPT theorem of noncommutative field theory was proved within the general Hamiltonian framework. This was followed by the proof through the axiomatic approach [61, 77].

Chapter 4

Noncommutative gauge theory

Gauge theories are vitally important when building realistical physical models. So, in order to get any real results out of the noncommutative field theory, the notion of gauge symmetry had to be generalized to the noncommutative setting. Since gauge symmetries are essentially *local*, generalizing them to the nonlocal noncommutative spacetime is highly nontrivial.

There are two methods to construct gauge field theories in noncommutative spacetime. First uses the Seiberg-Witten map, obtained from string theory [24], which maps a noncommutative gauge theory to a commutative gauge theory. In the second, one uses a noncommutative generalization of a gauge group and the $\star$-product to construct a gauge theory in the framework of noncommutative field theory. Both methods have been further developed and they offer some flexibility in their approaches. We will first present a short introduction to the idea of the Seiberg-Witten maps and to a noncommutative SM based on them, and then move to discuss the field theoretical approach.

In this chapter the noncommutative spacetime is considered to be of the same type as in the chapter 3, i.e. to have noncommutative coordinates that satisfy (3.1).

For future use the Moyal $\star$-product and the Moyal bracket (see section 3.2.2) are naturally generalized for the algebra of matrix-valued functions $M_{n\times n} \otimes \mathcal{A}_\theta$

$$\big(f(x) \star g(y)\big)_{ij} = f(x)_{ik} \star g(y)_{kj} \ . \tag{4.1}$$

The Hermitean conjugation for the algebra $M_{n\times n} \otimes \mathcal{A}_\theta$ can be defined by the usual Hermitean conjugation of matrices $\big(f(x)^\dagger\big)_{ij} = \big(f(x)_{ji}\big)^*$ and by the definition stating how the $\star$-product behaves under the operation

$$\big(f(x) \star g(x)\big)^\dagger = g(x)^\dagger \star f(x)^\dagger \ . \tag{4.2}$$

4.1 The Seiberg-Witten map method

When one in the open string theory in a constant antisymmetric background field, with string end points constrained on D-branes, performs quantization by using the Pauli-Villars or the point-splitting reqularization, one obtains a commutative or a noncommutative gauge theory, respectively. The Seiberg-Witten maps provide a correspondence between these two gauge theories, which should be equivalent, since a well-defined quantum theory does not depend on the regularization technique.

The method has been extended to enable the use of any commutative gauge group in [81, 82, 83, 84] (and in some earlier works referenced in these work.). In this approach it is argued that, because most of the gauge theories on noncommutative spaces cannot be formulated with Lie algebra valued infinitesimal gauge transformations, the infinitesimal gauge transformations should instead be taken to be enveloping algebra valued. The idea is to bypass the difficulties in constructing noncommutative gauge groups — we will discuss these problems later — by letting the generators of gauge transformations and the gauge fields to take values in the universal enveloping of the gauge algebra. The main problem with this approach is that enveloping algebras are infinite dimensional, which means that naively the numbers of both gauge transformation parameters and the gauge fields are infinite. The gauge transformation parameters and the gauge fields can, however, be defined to be functions of the corresponding Lie algebra valued objects — the functions being obtained through the Seiberg-Witten maps —, so that their numbers are the same as in the corresponding commutative gauge theories.

Let us consider the noncommutative gauge theory of a non-Abelian gauge algebra, say the algebra $\mathfrak{su}(n)$, with the matter fields $\hat{\psi}$ and the gauge fields $\hat{A}_\mu$. The infinitesimal local gauge transformations are

$$\hat{\delta}_{\hat{\lambda}}\hat{\psi} = i\rho_\psi(\hat{\lambda}) \star \hat{\psi} \ , \tag{4.3}$$

$$\hat{\delta}_{\hat{\lambda}}\hat{A}_\mu = \partial_\mu\hat{\lambda} + i\big[\hat{\lambda}, \hat{A}_\mu\big]_\star \ , \tag{4.4}$$

where the noncommutative infinitesimal gauge transformation parameter $\hat{\lambda}$ is valued in the universal enveloping of the gauge algebra $\mathcal{U}\big(\mathfrak{su}(n)\big)$ and ρ_ψ is the matter representation of $\mathcal{U}\big(\mathfrak{su}(n)\big)$. It should be noted that there is no gauge symmetry group, since this gauge symmetry is only defined for infinitesimal gauge transformations.[1] Generally, the gauge transformation parameter $\hat{\lambda}$ cannot be Lie algebra valued, because the commutator of two Lie algebra valued parameters $\hat{\lambda} = \hat{\lambda}_i T_i$ and $\hat{\sigma} = \hat{\sigma}_i T_i$ does not close in the Lie algebra

$$[\hat{\lambda}, \hat{\sigma}] = \frac{1}{2}\{\hat{\lambda}_i, \hat{\sigma}_j\}_\star \underbrace{[T_i, T_j]}_{ic_{ijk}T_k} + \frac{1}{2}\underbrace{[\hat{\lambda}_i, \hat{\sigma}_j]_\star}_{\neq 0}\{T_i, T_j\} \ . \tag{4.5}$$

Therefore, we have to use fields and gauge transformations that are $\mathcal{U}\big(\mathfrak{su}(n)\big)$-valued. The gauge fields $\hat{A}_\mu$ have to be in the adjoint representation. The gauge covariant derivative and the field strength are introduced by

$$\hat{D}_\mu\hat{\psi} = \partial_\mu\hat{\psi} - i\rho_\psi(\hat{A}_\mu) \star \hat{\psi} \ , \tag{4.6}$$

$$\hat{F}_{\mu\nu} = \partial_\mu\hat{A}_\nu - \partial_\nu\hat{A}_\mu - i\big[\hat{A}_\mu, \hat{A}_\nu\big]_\star \ , \tag{4.7}$$

with the gauge transformations

$$\hat{\delta}_{\hat{\lambda}}\hat{D}_\mu\hat{\psi} = i\rho_\psi(\hat{\lambda}) \star \hat{D}_\mu\hat{\psi} \ , \tag{4.8}$$

$$\hat{\delta}_{\hat{\lambda}}\hat{F}_{\mu\nu} = i\big[\hat{\lambda}, \hat{F}_{\mu\nu}\big]_\star \ . \tag{4.9}$$

[1] For a $\mathcal{U}(L)$ there is nothing like the exponential map that maps a Lie algebra L to a Lie group.

The gauge invariant action for the gauge fields is defined by

$$S[\hat{A}, \partial\hat{A}] = -\frac{1}{4}\int \mathrm{d}^D x\ \mathrm{Tr}\big(\hat{F}_{\mu\nu} \star \hat{F}^{\mu\nu}\big) \tag{4.10}$$

and the action for the matter fields is constructed by using the covariant derivative. For example, the action of a noncommutative fermion is written as

$$S[\hat{\psi}, \partial\hat{\psi}, \hat{A}] = \int \mathrm{d}^D x\, \bar{\hat{\psi}} \star (\gamma^\mu D_\mu - m)\hat{\psi}\ . \tag{4.11}$$

The corresponding concepts for the commutative $\mathfrak{su}(n)$ gauge theory are defined similarly, the differences being the ordinary point-wise product and the Lie algebra valued fields and gauge transformation parameters. We denote the commutative concepts without the hats: ψ, A_μ, λ etc.

Since the gauge invariance of the commutative gauge theory should map to the gauge invariance of the noncommutative gauge theory, the gauge transformations in the latter theory are induced by the transformations of the former theory:

$$\hat{A}_\mu[A] + \hat{\delta}_{\hat{\lambda}[\lambda,A]}\hat{A}_\mu[A] = \hat{A}_\mu[A + \delta_\lambda A]\ , \tag{4.12}$$

$$\hat{\psi}[\psi, A] + \hat{\delta}_{\hat{\lambda}[\lambda,A]}\hat{\psi}[\psi, A] = \hat{\psi}[\psi + \delta_\lambda\psi, A + \delta_\lambda A]\ . \tag{4.13}$$

In other words, if the commutative fields A_μ and ψ are related to the fields A_μ^Λ and ψ^Λ through the gauge transformation $\Lambda = e^{i\lambda}$ generated by λ, then the noncommutative fields $\hat{A}_\mu[A]$ and $\hat{\psi}[\psi, A]$ are related to the fields $\hat{A}_\mu[A^\Lambda]$ and $\hat{\psi}[\psi^\Lambda, A^\Lambda]$ through the gauge transformation $\hat{\Lambda} = e^{i\hat{\lambda}[\lambda,A]}$ generated by $\hat{\lambda}[\lambda, A]$. These gauge equivalence relations can be solved pertubatively in θ in order to obtain the Seiberg-Witten maps. In the leading order in θ they can be written

$$\hat{A}_\mu[A] = A_\mu + \frac{1}{4}\theta^{\nu\rho}\{A_\rho, \partial_\nu A_\mu + F_{\nu\mu}\} + \mathcal{O}\left(\theta^2\right)\ , \tag{4.14}$$

$$\hat{\psi}[\psi, A] = \psi + \frac{1}{2}\theta^{\mu\nu}\rho_\psi(A_\nu)\partial_\mu\psi + \frac{i}{8}\theta^{\mu\nu}\big[\rho_\psi(A_\mu), \rho_\psi(A_\nu)\big]\psi + \mathcal{O}\left(\theta^2\right)\ , \tag{4.15}$$

$$\hat{\lambda}[\lambda, A] = \lambda + \frac{1}{4}\theta^{\mu\nu}\{A_\nu, \partial_\mu\lambda\} + \mathcal{O}\left(\theta^2\right)\ . \tag{4.16}$$

In [85] this approach was used to construct a noncommutative SM based on the standard gauge algebra $\mathfrak{su}(3) \times \mathfrak{su}(2) \times \mathfrak{u}(1)$. The noncommutative fields and gauge parameters of the model are expressed as towers built upon the commutative fields and gauge parameters that transform under the standard SM gauge group $SU(3)_\mathrm{C} \times SU(2)_\mathrm{L} \times U(1)$. The result is a minimal noncommutative extension of SM that has the same number of fields and free coupling constants as SM, and that gives SM as the zeroth order of the θ expansion. The model introduces a new kind of mixing or unification of the interactions, since all the gauge fields are combined to a single master field and its Seiberg-Witten map (4.14) is nonlinear. In the lowest order in θ the gauge bosons of the group factors do decouple, but because the quarks carry both $SU(3)_\mathrm{C}$ and $SU(2)_\mathrm{L} \times U(1)$ charges, there are vertices where two quarks couple to both the $SU(3)_\mathrm{C}$ and the $U(1)$ gauge boson. This implies parity violation in QCD. There are also new vertices in the pure gauge sector, namely vertices with five and six gauge bosons for the gauge groups $SU(3)_{\mathrm{C}}$ and $SU(2)_{\mathrm{L}}$. The $U(1)$ gauge bosons do not self-interact nor do they couple to neutral particles, like the Higgs boson. This means that neutral and charged currents are affected by the noncommutativity. The effects may be observable particularly in the neutral decays of heavy particles like the b and t quarks, and in the CP-violation phase extracted from the Cabibbo-Kobayashi-Maskawa matrix.

4.2 The field theoretical method

4.2.1 Simple noncommutative gauge groups

In this section we consider local noncommutative gauge groups constructed directly by replacing the Lie bracket with the Moyal bracket (3.15). There are two main problems in the construction of noncommutative local groups. One is the fact that the $\star$-product usually destroys the closure condition of local groups [86, 83, 87]. Second are the restrictions on the representation content of noncommutative local groups [70, 88].

Let us consider a local Lie group generated by the Lie algebra of generators T_i

$$[T_i, T_j] = ic_{ijk}T_k \ . \tag{4.17}$$

The local group elements $e^{if(x)}$ are parameterized by the Lie algebra valued functions

$$f(x) = f_i(x)T_i \ . \tag{4.18}$$

As we already saw while discussing the Seiberg-Witten map method, generally, two Lie algebra valued parameters f and g do not close in the Lie algebra under the generalized Moyal bracket

$$[f,g]_\star = \frac{i}{2}c_{ijk}\{f_i, g_j\}_\star T_k + \frac{1}{2}[f_i, g_j]_\star\{T_i, T_j\} \ , \tag{4.19}$$

because of the $\star$-product. This is the reason why there are no minimal noncommutative extensions for most groups, e.g. $SU(n)$, $SO(n)$, $O(n)$, $Sp(n)$ and $USp(n)$, that we are interested in. The group $U(n)$ is a "fortunate" exception. In the defining representation it is parameterized by anti-Hermitean complex-valued $n\times n$ matrices if and the Moyal bracket of two of these parameters is also anti-Hermitean ($f^\dagger = f$)

$$\big([if, ig]_\star\big)^\dagger = [(ig)^\dagger, (if)^\dagger]_\star = [-ig, -if]_\star = -[if, ig]_\star \ . \tag{4.20}$$

This implies that we can construct a minimal noncommutative extension of $U(n)$, which we name $U_\star(n)$. We choose the standard normalization $\mathrm{Tr}(T_iT_j) = \frac{1}{2}\delta_{ij}$ for the basis of the generator algebra $\mathfrak{u}_\star(n)$. In order to cover all Hermitean $n\times n$ matrices we have to include the unit matrix $T_0 = \frac{1}{\sqrt{2n}}\mathbf{1}_{n\times n}$ to the basis of $\mathfrak{u}_\star(n)$. Now the elements of $\mathfrak{u}_\star(n)$,

$$if(x) = i\sum_{i=0}^{n^2-1} f_i(x)T_i \ , \tag{4.21}$$

can be arbitary anti-Hermitean $n \times n$ matrices and the Moyal bracket of two of them is still anti-Hermitean (4.20). The elements of the group $U_\star(n)$ are star-unitary operators defined by

$$(e\star)^{if(x)} = \sum_{k=0}^{\infty} \frac{i^k}{k!} \underbrace{f(x) \star \cdots \star f(x)}_{k} \ . \tag{4.22}$$

By star-unitarity we mean that for $U \in U_\star(n)$,

$$U^\dagger = U^{-1} \quad \text{and} \quad U^{-1} \star U = U \star U^{-1} = \mathbf{1} \ . \tag{4.23}$$

The adjoint of the group element (4.22) is easy to obtain,

$$\left((e\star)^{if(x)}\right)^\dagger = (e\star)^{(if(x))^\dagger} = (e\star)^{-if(x)} \ .$$

This construction does not work for $SU(n)$ because the zero trace condition of $\mathfrak{su}(n)$ is not preserved by the Moyal bracket.

From the above construction it is evident that only the $n \times n$ irreducible representation is possible for $U_\star(n)$. If the dimension of an irreducible matrix representation is $N \geq n$, the algebra $\mathfrak{u}_\star(n)$ for this representation closes in $\mathfrak{u}_\star(N)$, i.e. in Hermitean $N \times N$ matrices. Thus, if $N > n$, the representation is reducible and it cannot be used as a basis for the $\mathfrak{u}_\star(n)$ gauge fields.

The action of the algebra $\mathfrak{u}_\star(n)$ on its modules (representations) in $\mathcal{A}_\theta$ is realized through the $\star$-product and the primitive coproduct (3.24). Concretely, for the left $\mathfrak{u}_\star(n)$ module we have

$$\begin{aligned} f \triangleright \psi_1 &= f \star \psi_1 \ , \\ f \triangleright (\psi_1 \star \psi_2) &= (f \triangleright \psi_1) \star \psi_2 + \psi_1 \star (f \triangleright \psi_2) \ , \end{aligned} \tag{4.24}$$

for $f \in \mathfrak{u}_\star(n)$ and $\psi_i \in \mathcal{A}_\theta$.

4.2.2 Noncommutative $U_\star(n)$ gauge theory

The gauge fields

The $U_\star(n)$ gauge theory is based on the $\mathfrak{u}_\star(n)$-valued gauge fields

$$A_\mu = \sum_{i=0}^{n^2-1} A^i_\mu(x) T_i \ . \tag{4.25}$$

We emphasize that for $U_\star(n)$ the gauge fields are necessarily in the $n \times n$ matrix form. The gauge transformations of the gauge fields

$$A_\mu \longrightarrow A'_\mu = U \star A_\mu \star U^{-1} + \frac{i}{g} U \star \partial_\mu U^{-1} \tag{4.26}$$

are generated by the elements $U \in U_\star(n)$ in the adjoint representation

$$U = (e\star)^{i\lambda} \ , \ \lambda \in \mathfrak{u}_\star(n) \ . \tag{4.27}$$

The field strength is defined by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\mu A_\mu - ig[A_\mu, A_\nu]_\star \tag{4.28}$$

and it is easy to show that it transforms in the adjoint representation of $U_\star(n)$

$$F_{\mu\nu} \longrightarrow F'_{\mu\nu} = U \star F_{\mu\nu} \star U^{-1} \ , \tag{4.29}$$

i.e. the field strength is covariant under the gauge transformations. The infinitesimal gauge transformations are written in the familiar form

$$\delta A_\mu = \partial_\mu \lambda + i[\lambda, A_\mu]_\star \ , \tag{4.30}$$
$$\delta F_{\mu\nu} = i[\lambda, F_{\mu\nu}]_\star \ . \tag{4.31}$$

Combined integration over spacetime and matrix trace provides the trace operation for the noncommutative gauge fields (see the section 3.2). The gauge invariant action for the gauge fields can be defined by

$$S_{\mathrm{G}}[A, \partial A] = -\frac{1}{4} \int \mathrm{d}^D x \ \mathrm{Tr}\big(F_{\mu\nu} \star F^{\mu\nu}\big) \ . \tag{4.32}$$

Even the $U_\star(1)$ gauge theory — unlike the commutative $U(1)$ theory — is self-interacting due to the Moyal bracket of the gauge fields in the field strength (4.28).

The matter fields

Since only the $n \times n$ representation is available for $U_\star(n)$, the matter fields have to live either in the fundamental representation ψ or in the anti-fundamental representation χ or in the adjoint representation ϕ. The gauge transformations for these matter field repesentations are

$$\psi \longrightarrow \psi' = U \star \psi \ , \tag{4.33a}$$
$$\chi \longrightarrow \chi' = \chi \star U^{-1} \ , \tag{4.33b}$$
$$\phi \longrightarrow \phi' = U \star \phi \star U^{-1} \ , \tag{4.33c}$$

where U is given by (4.27). The corresponding covariant derivatives for the matter field representations are given by

$$D_\mu \psi = \partial_\mu \psi - igA_\mu \star \psi \ , \tag{4.34a}$$
$$D_\mu \chi = \partial_\mu \chi + ig\chi \star A_\mu \ , \tag{4.34b}$$
$$D_\mu \phi = \partial_\mu \phi - ig[A_\mu, \phi]_\star \ , \tag{4.34c}$$

where g is the gauge coupling constant. The gauge invariant action for the matter fields is constructed as usual by using the covariant derivatives instead of the partial derivatives. As an example, the noncommutative extension of the action for the fermionic Dirac field ψ with the mass m is written as

$$\begin{aligned} S_{\mathrm{M}}[\bar\psi, \psi, \partial\psi, A] &= \int \mathrm{d}^D x \, \bar\psi \star (i\gamma^\mu D_\mu - m)\psi \\ &= \int \mathrm{d}^D x \, \big(\bar\psi \star i\gamma^\mu \partial_\mu \psi - m\bar\psi \star \psi + g\bar\psi \star \gamma^\mu A_\mu \star \psi\big) \ . \end{aligned} \tag{4.35}$$

Adding the action of the gauge fields gives us the action of the $U_\star(n)$ gauge theory

$$\begin{aligned} S &= S_{\mathrm{G}} + S_{\mathrm{M}} \qquad (4.36)\\ &= \int \mathrm{d}^D x \left(-\frac{1}{4}\operatorname{Tr}\big(F_{\mu\nu} \star F^{\mu\nu}\big) + \bar{\psi} \star i\gamma^\mu \partial_\mu \psi - m\bar{\psi} \star \psi + g\bar{\psi} \star \gamma^\mu A_\mu \star \psi \right) . \end{aligned}$$

The result (3.18) can be used in the terms with a single $\star$-product.

The restrictions on the representations of the matter fields (4.33),(4.34) imply that the $U_\star(1)$ gauge theory exhibits *charge quantization* [70]. There are only three possible $U_\star(1)$ charges for the matter fields: $+1$ for ψ, -1 for χ and zero for ϕ. Though the ϕ field has no charge, all the matter fields have a dipole moment [89, 90]. This charge quantization is problematic, because in a noncommutative version of SM we should be able to have particles with both integer and fractional hypercharges: $-1, -2$ for electrons and neutrinos, and $\frac{1}{3}, \frac{4}{3}, -\frac{2}{3}$ for quarks and 1 for Higgs. A similar, though less problematic, charge quantization appears in ordinary non-Abelian semi-simple gauge theories, where the charge for a matter field is fixed by choosing the representation for the field.

The no-go theorem In physical models we need to consider gauge groups with several simple factors. Let G_1 and G_2 be two local gauge groups. The gauge group $G = G_1 \times G_2$ is defined by the relations

$$\begin{aligned} g = g_1 \times g_2 \ , \quad h = h_1 \times h_2 \ , \quad g, h \in G \ , \quad g_i, h_i \in G_i \ , \\ g \cdot h = (g_1 \times g_2) \cdot (h_1 \times h_2) \equiv (g_1 \cdot h_1) \times (g_2 \cdot h_2) \ , \end{aligned} \tag{4.37}$$

where $\cdot$ is the appropriate group multiplication for each group. If we now take the groups to be the noncommutative ones, $G_1 = U_\star(n)$ and $G_2 = U_\star(m)$, we see that because of the $\star$-product we cannot re-arrange the elements of the subgroups as in (4.37). Therefore the matter fields cannot be in the fundamental representation of both $U_\star(n)$ and $U_\star(m)$. However, there is one possibility left. The matter field Ψ can be in the fundamental representation of one group, say $U_\star(n)$, and in the anti-fundamental representation of the other group

$$\Psi \longrightarrow \Psi' = U \star \Psi \star V^{-1} \quad ; \ U \in U_\star(n), \ V \in U_\star(m) \ . \tag{4.38}$$

In the general case the gauge group consists of N factors $G = \prod_{i=1}^N U_\star(n_i)$. The matter fields can at most be charged under two of the $U_\star(n_i)$ factors and they have to be singlets under the rest of them. This is a strong constraint on the possible models.

4.2.3 Noncommutative standard model

Since the $SU(n)$ factors in the gauge group $G_{\mathrm{SM}} \equiv SU(3)_{\mathrm{C}} \times SU(2)_{\mathrm{L}} \times U(1)_{\mathrm{Y}}$ of the usual SM do not have simple noncommutative extensions, the SM cannot be generalized to noncommutative spacetime just by replacing the gauge group with a noncommutative one. Instead, we have to use a higher rank gauge group that has G_{SM} as its subgroup. At low energies, we should of course recover the usual SM.

The minimal noncommutative gauge group for the noncommutative standard model (NCSM) is $G_{\rm SM} \equiv U_\star(3) \times U_\star(2) \times U_\star(1)$, which contains $G_{\rm SM}$ and has two additional $U(1)$ factors. Thus the minimal gauge group of NCSM introduces two extra $U(1)$ gauge fields compared to the usual gauge group of SM. In order to recover SM at low energies, the two extra gauge bosons should be considerably heavier than the massive SM gauge bosons, the weak gauge bosons $m_Z \sim m_W \sim 100\,\text{GeV}$.

The first problem in constructing NCSM is the quantization of $U_\star(1)$ charges, as we already mentioned. The restrictions placed by the no-go theorem on the representations of matter fields are another challenge for building NCSM. Since SM quarks are charged under all the three factors of the gauge group $G_{\rm SM}$, the no-go theorem has to be evaded somehow. Both of these problems can be solved by spontaneously breaking the NCSM gauge group $G_{\rm NCSM}$ down to $G_{\rm SM}$.

In [71] NCSM was constructed by using the gauge group $G_{\rm NCSM}$. The two extra $U(1)$ symmeties were reduced through the Higgs mechanism, which needs to be ran twice. The two fields used for this reduction were named the Higgsac fields — in order to distinguish them from the usual Higgs doublet that is used for spontaneous breaking of the electroweak symmetry, giving the masses to the SM particles — and they were taken to be in the trace-$U(1)$ part representation of $U(n)$. Unfortunately these Higgsac fields are not representations of the noncommutative gauge group $U_\star(n)$. Thus the symmetry breaking mechanism of [71] is not gauge invariant and therefore not spontaneous, which leads to the violation of unitarity. The model also contains gauge anomalies related to the extra trace-$U(1)$ factors in $G_{\rm NCSM}$. In [91], the gauge group reduction $U_\star(n) \to SU(n)$ mechanism was redefined in a gauge invariant way by using half infinite Wilson lines, providing a spontaneous breaking of noncommutative $U_\star(n)$ gauge theories down to $SU(n)$ gauge theories. The noncommutative SM built with the new Higgsac fields is free of gauge anomalies, when a pair of $U_\star(2)$ doublet lepton fields is introduced to cancel the chiral anomalies arising from the extra gauge fields [91].

As anticipated, NCSM has several new features beyond the usual SM. The most important achievement of the model is that the no-go theorem together with the Higgsac mechanism uniquely defines the weak hypercharges (consequently, also the electric charges) of all the SM particles to their observed values. The hypercharge is the linear combination of the generators of the trace-$U(1)$ subgroups of the $U_\star(n)$ factors of $G_{\rm NCSM}$

$$Y = -\frac{2}{3}T^0_{U_\star(3)} - T^0_{U_\star(2)} - 2T^0_{U_\star(1)}\ . \tag{4.39}$$

Since in QED the values of electric charges have to be introduced by hand, this reduces the number of free parameters needed in NCSM compared to SM.[2]

NCSM contains two new massive gauge bosons named W^0 and G^0. These gauge bosons contribute to the Z gauge boson mass eigenstate, although the contribution is suppressed by $\left(\frac{m_Z}{m_{W^0}}\right)^2$. A lower bound on the masses of the new gauge bosons can be found by comparing the quantum corrections on the parameter

$$\rho \equiv \left(\frac{m_Z}{m_W}\right)^2 \cos^2\theta^0_W$$

[2]The noncommutativity parameters $\theta^{\mu\nu}$ naturally increase the number of parameters.

to the precision experiments used to test the same parameter in SM. The result is

$$m_{W^0}, m_{G^0} \gtrsim 25 m_Z \approx 2.3\,\mathrm{TeV}\ . \tag{4.40}$$

Below these high energies the gauge bosons decouple from the particle spectrum and the same may be the case for the extra anomaly cancelling leptons.

The neutrino has a dipole moment interaction vertex with the photon, just like all other fermions in NCSM. This interaction can be used to calculate a lower bound on the noncommutativity scale,

$$\Lambda_{\mathrm{NC}} \gtrsim 1\,\mathrm{TeV}\ , \tag{4.41}$$

based on astrophysical observations. This is consistent with the lower bounds mentioned in the subsection 3.3.1.

NCSM also exhibits the inherent CP violation of noncommutative QFT in both lepton and quark sectors.

Lastly we note that so far the NCSM has primarily been considered at the classical level and in the leading order in the noncommutativity parameter θ. Hence, all the features and the final faith of the model are still open questions. However, the Higgsac mechanism in one form or another will most probably continue to play an important role in the noncommutative gauge theories based on the gauge groups $U_\star(n)$.

4.2.4 Noncommutative orthogonal and sympletic gauge groups

The constraint to use only the $U_\star(n)$ gauge groups for noncommutative gauge theories is so restricting that the search for alternatives has received attention. We have already seen that enveloping algebra valued gauge transformations can be used, but how about Lie algebra valued, are we really stuck with $\mathfrak{u}_\star(n)$ for good?

Noncommutative orthogonal and sympletic gauge symmetry groups can be formulated [86, 87]. These approaches are based on elaborating the Hermitean conjugation (transposition) of the gauge algebra. Indeed, if the multiplication of an algebra of matrix-valued functions is redefined, surely we can also try to redefine the Hermitean conjugation in order to preserve the closure condition under the Moyal bracket (4.1). Particularly the noncommutative local orthogonal groups may turn out to be important when trying to construct a gauge theory of the Lorentz rotation group.

$\mathfrak{o}_\star(n)$ gauge algebra

For a concrete example, let us consider the algebra $\mathfrak{o}(n)$ of the orthogonal gauge group $O(n)$. It consists of antisymmetric $n\times n$ matrices $\{\, a \in M_{n\times n} \otimes \mathcal{A} \mid a^t = -a \,\}$, where $\mathcal{A}$ is a commutative algebra of functions and t is the ordinary matrix transposition that does not act on the coordinate degrees of freedom; $(a^t)_{ij} = a_{ji}$.

When we move to the noncommutative space (1.3), the point-wise matrix multiplication of $\mathfrak{o}(n)$ is replaced with the noncommutative $\star$-multiplication of matrices (4.1). Now the functions are necessarily complex-valued due to the $\star$-product (3.13).

In order to construct the noncommutative algebra $\mathfrak{o}_\star(n)$, we need to find a transposition operator(s) T that is consistent with the $\star$-product (4.1) and the form(s) of the matrix-valued functions in the algebra. We need

$$a(x)^T = -a(x) \ , \quad \big(a(x) \star b(x)\big)^T = b(x)^T \star a(x)^T \ , \tag{4.42}$$

for all $a(x), b(x) \in \mathfrak{o}_\star(n)$. This is sufficient to provide the closure under the generalized Moyal bracket

$$\big(\big[a(x), b(x)\big]_\star\big)^T = \big[a(x), b(x)\big]_\star \ .$$

The transposition operator T can be defined as a combination of the matrix transposition t and of the transposition operator τ acting on the coordinates in the element functions of the matrices $a_{ij}(x)$. The full transposition T is given by

$$\big(a(x)^T\big)_{ij} = (a^t)_{ij}(x)^\tau = a_{ji}(x)^\tau \ . \tag{4.43}$$

For simplicity, we consider explicitly only two coordinates (x^1, x^2). It was discovered in [87] that there are at least two τ operators we can use to construct $\mathfrak{o}_\star(n)$. They $(\tau_{1,2})$ are defined by

$$f(x^1, x^2)^{\tau_1} = f(x^1, -x^2) \ , \tag{4.44}$$

$$f(x^1, x^2)^{\tau_2} = f(x^2, x^1) \ , \tag{4.45}$$

for all $f(x) \in \mathcal{A}_\theta$. According to (4.42) the τ operators have to satisfy

$$(f \star g)(x)^{\tau_i} \equiv \big(f(x) \star g(x)\big)^{\tau_i} = g(x)^{\tau_i} \star f(x)^{\tau_i} \ , \ i = 1, 2 \ . \tag{4.46}$$

Indeed, it is fairly easy to check that this is true for the noncommutative algebra $\mathcal{A}_\theta$ generated by the commutators (1.3). As a consistency check we calculate the $\tau_{1,2}$ transpositions of the coordinate commutators

$$\begin{aligned}
\big([x^1, x^2]_\star\big)^{\tau_1} &= (x^2)^{\tau_1} \star (x^1)^{\tau_1} - (x^1)^{\tau_1} \star (x^2)^{\tau_1} = (-x^2) \star x^1 - x^1 \star (-x^2) \\
&= -x^2 \star x^1 + x^1 \star x^2 = [x^1, x^2]_\star = i\theta^{12} = (i\theta^{12})^{\tau_1} \ , \\
\big([x^1, x^2]_\star\big)^{\tau_2} &= (x^2)^{\tau_2} \star (x^1)^{\tau_2} - (x^1)^{\tau_2} \star (x^2)^{\tau_2} = x^1 \star x^2 - x^2 \star x^1 \\
&= [x^1, x^2]_\star = i\theta^{12} = (i\theta^{12})^{\tau_2} \ .
\end{aligned}$$

Now we can write down the requirements for the elements of the two algebras $\mathfrak{o}_\star(n)_{\tau_1}$ and $\mathfrak{o}_\star(n)_{\tau_2}$:

$$\mathfrak{o}_\star(n)_{\tau_1} : \quad a_{ji}(x^1, -x^2) = -a_{ij}(x^1, x^2) \ , \tag{4.47a}$$

$$\mathfrak{o}_\star(n)_{\tau_2} : \quad a_{ji}(x^2, x^1) = -a_{ij}(x^1, x^2) \ , \tag{4.47b}$$

where $1 \leq i, j \leq n$. Notice that the diagonal elements $a_{ii}(x)$ do not vanish in the noncommutative case and therefore we have n more degrees of freedom than in the commutative case, the total of $n(n+1)/2$ independent components for $u_\star(n)$. Hence, we also have the non-trivial $\mathfrak{o}_\star(1)$ algebra. The functions of the forms (4.47) close under the Moyal bracket and they can be used to formulate gauge transformations and gauge fields for the $O_\star(n)$ gauge theories, although the process is a bit more involved than in the $U_\star(n)$ case.

Chapter 5

Noncommutative gravitation

The formulation of a quantum theory of gravitation has been an important goal for theoretical physicists since the birth of quantum mechanics. The consistent introduction of quantum effects to the classical theories of gravitation has proven to be extremely difficult. In spite of the great progress made in string theory during the last few decades, quantum gravitation is still out of our reach.

Noncommutative gravitation could offer an alternative way to formulate a theory of gravitation that is compatible with quantum mechanics and that is able to capture the expected nonlocality of the Planck scale physics. In this chapter we further discuss what is known about noncommutative gravitation and how the existing problems could be solved. Our emphasis is on the gauge theory point of view, since we believe that noncommutative gravitation should be formulated as a gauge theory.

5.1 Overview

In this section we present a compact overview of recent approaches to noncommutative gravitation. A review on noncommutative gravitation with emphasis on its string theoretical origin can be found in [92].

Since the discovery of noncommuattive geometry in string theory [24], the idea of noncommutative gravitation has inspired many. Several different approaches to the noncommutative deformation of the classical theories of gravitation — especially Einstein's GR and the gauge theory of the Poincaré symmetry — have been tried. These have mainly been based on the canonical noncommutative structure (1.3) and the Moyal $\star$-product (3.13). Generically, such noncommutative deformations of gravitation result into a complexification of the connection and of the metric tensor as well as of the local Lorentz invariance.

One of the first of these recents approaches to noncommutative gravitation [93] used the simple prescription: in GR, replace the point-wise product of functions with the Moyal $\star$-product and use complex-valued fields when necessary, resulting into a complex symmetric metric on a complex manifold and the local complex Lorentz gauge symmetry $CSO(1,3)$ instead of the local Lorentz symmetry $SO(1,3)$. This was followed by the treatment of gravitation as the gauge theory of the unitary group $U(1, D-1)$ [94, 95], which can be generalized to the noncommutative spacetime (see section 4.2). After it was shown that the special orthogonal and sympletic groups can be used to build noncommutative gauge theories [86, 83] through the Seiberg-Witten maps [24], a theory of noncommutative gravitation was derived by gauging the $SO(4,1)$ symmetry and by reducing it to the inhomogenius Lorentz gauge symmetry $ISO(3,1)$ [96].[1]

In addition to the violation of the Lorentz invariance, the canonical noncommutativity of spacetime coordinates (1.3) also breaks the covariance under the general coordinate transformations. There is, however, a subgroup of volume-preserving coordinate transformations that acts covariantly on the canonical noncommutative spacetime. This residual symmetry together with the gauge algebra $\mathfrak{so}(3,1)$ has been used to build a theory of noncommutative gravitation [99].

A problem with most of the approaches to noncommutative gravitation is that they are physically ad hoc, because they are not based on any general symmetrical or dynamical principle. An attempt to amend this issue was made in [27, 28], where noncommutative gravitation was derived by twisting the Hopf algebra structure of the infinitesimal diffeomorphisms of a smooth spacetime manifold, so that the twisted infinitesimal diffeomorphisms act covariantly on the corresponding noncommutative spacetime. In short the idea is: In GR, the covariance under general coordinate transformations is implemented through the covariance under diffeomorphisms that are geneated by vector fields forming a Lie algebra, whose universal enveloping algebra has the natural Hopf algebra structure (see subsection 3.3.2). The diffeomorphism Hopf algebra is deformed by using the Abelian twist element (3.37), with the hope of obtaining general coordinate transformations on the noncommutative spacetime. However, since the deforming of the diffeomorphisms is made in a frame-dependent way, the twisted diffeomorphisms cannot be the correct general coordinate transformations. Moreover, it turned out that the gravitational dynamics obtained from string theory in the Seiberg-Witten limit is much richer than that provided by the $\star$-product [26], particularly in the mentioned works [27, 28]. This suggests that the simple $\star$-product (3.13) is not able to fully codify the gravitational dynamics on noncommutative spacetimes.

Other Riemannian geometries for noncommutative spacetimes have also been developed [100, 101, 102] and they are also based on generalizations of the familiar concepts of metric and curvature. The latest of these [102] is the most important one for this work, not least because it is where the idea of the possibility to derive noncommutative gravitation as a gauge theory of the twisted Poincaré symmetry was mentioned for the first time.[2] In [102], a Riemannian geometry on noncommutative n-dimensional surfaces is constructed through concrete examples, beginning from 2-dimensional noncommutative surfaces embedded in flat 3-dimensional noncommutative space — the simplest nontrivial case — and by building up from there, finally generalizing to n-dimensional noncommutative surfaces. General coordinate transformations are introduced as gauge transformations on the underlying noncommutative associative algebra of functions, mapping it to another non-trivially isomorphic algebra. It is hoped that this approach will eventually lead us to the

[1]Recently this approach was used to derive deformed Schwarzschild and Reissner-Nordström solutions in noncommutative spacetime [97, 98].

[2]In fact, we have not found any other mentioning on this idea.

correct geometrical concepts for noncommutative gravitation.

5.2 Challenges of noncommutative gravitation

In this section we further discuss the main challenges that need to be met, in order to begin to understand noncommutative gravitation.

5.2.1 Covariance under general coordinate transformations

Covariance under general coordinate transformations is an essential ingredient of GR and ECT, and theories alike. Such theories can be formulated similarly in any reference frame. Hence, they are called frame-independent, in contrast to the frame-dependent theories like special relativity and SM.

The noncommutative quantum and gauge field theories discussed in chapters 3 and 4 are also frame-dependent. By choosing the frame-dependent twist element

$$\mathcal{F} = e^{-\frac{i}{2}\theta^{\mu\nu}\partial_\mu \otimes \partial_\nu} , \tag{5.1}$$

where $\theta^{\mu\nu}$ is a *constant* antisymetric matrix, we have defined the $\star$-product of the representation algebra to the frame-dependent form (3.42) for good. The deformations of GR that have been based on the frame-dependent $\star$-product are physically inconsistent, because in them the $\star$-product does not transform properly under general coordinate transformations. The deformation of the general coordinate transformations has to be done in a frame-independent way, so that the $\star$-product transforms covariantly under them. Giving the success of twisting Hopf algebras so far, it is expected that the twisting may be a right approach to achieve this. We "just" have to find a consistent way to deform both the algebra of functions on spacetime and the general coordinate transformations.

5.2.2 Local Lorentz invariance

In addition to the frame-independence, the second essential feature of GR is the local Lorentz invariance, i.e. the local $SO(1,3)$ gauge symmetry. As we have already discussed in chapter 4, noncommutative gauge theory of the Lorentz algebra can be constructed through the extended Seiberg-Witten map [81, 82, 83, 84, 86]. The fact that the Lorentz symmetry is twisted in noncommutative spacetime complicates the formulation of the local Lorentz invariance. At present, we are lacking a *symmetry principle* that would guide us when formulating deformed symmetries in noncomutative spacetime.

5.3 Gauge theory of the twisted Poincaré symmetry

5.3.1 Twisted gauge theories — Twist as a symmetry principle

Since the discovery of the twisted Poincaré symmetry of noncommutative quantum and gauge field theories, the role of the twisting procedure in high energy physics has been under research and discussion [103, 104, 105, 106]. The essential question is: does the twisting of symmetries itself provide a *symmetry principle* for building noncommutative field theories? Giving the fact that the twisted Poincaré symmetry can be used as the principle of special relativity when constructing frame-dependent noncommutative field theories, it is expected that a generalized twisting procedure could provide a symmetry principle that would enable the construction of consistent noncommutative field theories with different kinds of gauge symmetries.

Covariance in twisted gauge theories

Here we present a general observation on the covariance in twisted gauge theories.

Let us consider a gauge theory of the gauge symmetry G on a noncommutative spacetime that is twist deformed with the twist element

$$\tilde{\mathcal{F}} = e^{\frac{i}{2}\theta^{\mu\nu}\mathcal{D}_\mu\otimes\mathcal{D}_\nu+\mathcal{O}(\theta^2)} \ , \tag{5.2}$$

where $\mathcal{D}_\mu$s are operators acting on the representations of G (e.g. partial or covariant derivatives etc.). Let R_1 and R_2 be two representations of G and ϕ_1, ϕ_2 be fields transforming in the representations

$$\phi_1 \longrightarrow \phi_1' = U_1\phi_1 \ , \quad U_1 \in R_1 \ , \tag{5.3a}$$
$$\phi_2 \longrightarrow \phi_2' = U_2\phi_2 \ , \quad U_2 \in R_2 \ . \tag{5.3b}$$

According to (3.32), the noncommutative $\star$-product of the fields is defined by

$$\phi_1 \star \phi_2 = m\big(\tilde{\mathcal{F}}^{-1}(\phi_1\otimes\phi_2)\big) \ . \tag{5.4}$$

Notice that the $\star$-product is not present in the gauge transformations (5.3), because a twisted gauge algebra acts on its representations in the same way as the original gauge algebra acts on its representations before the twisting. Assuming $\phi_1 \star \phi_2$ transforms in the product representation $R_1 \otimes R_2$,

$$(\phi_1\star\phi_2)' \equiv m\big(\tilde{\mathcal{F}}'^{-1}(\phi_1'\otimes\phi_2')\big) = m\big((U_1\otimes U_2)\tilde{\mathcal{F}}^{-1}(\phi_1\otimes\phi_2)\big) \ , \tag{5.5}$$

the twist element (5.2) has to transform as

$$\tilde{\mathcal{F}}^{-1} \longrightarrow \tilde{\mathcal{F}}'^{-1} = (U_1\otimes U_2)\tilde{\mathcal{F}}^{-1}(U_1^{-1}\otimes U_2^{-1}) = e^{\frac{i}{2}\theta^{\mu\nu}U_1\mathcal{D}_\mu U_1^{-1}\otimes U_2\mathcal{D}_\nu U_2^{-1}+\mathcal{O}(\theta^2)} \ . \tag{5.6}$$

In order to obtain a gauge covariant $\star$-product, we have to demand that the inverse of the twist is covariant under the gauge transformations. According to (5.6), this can be true if and only if the operators $\mathcal{D}_\mu$ are covariant under the gauge transformations, i.e.

$$\mathcal{D}_\mu \longrightarrow \mathcal{D}'_\mu = U_i \mathcal{D}_\mu U_i^{-1} \ , \ i = 1, 2 \ , \tag{5.7}$$

and of course the same applies to the rest of the terms $\mathcal{O}(\theta^2)$ in the twist (5.2).

The crucial question that we will soon discuss is: Are such gauge covariant twist elements valid twists, i.e. do they satisfy the twist conditions (3.28)?

Internal gauge symmetries

An attemp to build twisted gauge theories of internal gauge symmetries was made in [103, 104]. The idea is to take the *combination* of the Poincaré algebra $\mathcal{P}$ and of the Lie algebra $\mathcal{G}$ of an internal gauge symmetry and to twist the coproduct of the universal enveloping algebra for the combined algebra with the Abelian twist element (3.37) of the twisted Poincaré symmetry. Soon this approach was shown to contradict the very idea of a gauge symmetry [105] and such twisted gauge theories were shown be incompatible with the twisted Poincaré symmetry [106] (see also [26]). Next we review the results of the latter works.

Let us consider the Lie algebra $\mathcal{G}$ of an internal gauge symmetry, generated by T_i, $i = 1, 2, \ldots, n$,

$$[T_i, T_j] = i c_{ijk} T_k \ . \tag{5.8}$$

The local gauge transformations are generated by the Hermitean functions

$$\alpha(x) = \alpha_i(x) T_i \ , \tag{5.9}$$

which do not commute with the generators (2.17) of the global Poincaré algebra (2.2). Thus, we can extend the Poincaré algebra by semidirect product with $\mathcal{G}$ and twist the universal enveloping algebra $\mathcal{U}(\mathcal{P} \ltimes \mathcal{G})$ of the combination. We choose the Abelian twist element

$$\mathcal{F} = e^{\frac{i}{2}\theta^{\mu\nu} P_\mu \otimes P_\nu} \tag{5.10}$$

and follow the twisting procedure presented in section 3.3. Since the elements of the twisted algebra $\mathcal{U}_t(\mathcal{P} \ltimes \mathcal{G})$ act on the noncommutative algebra of functions $\mathcal{A}_\theta$ in the same way as the algebra $\mathcal{U}(\mathcal{P} \ltimes \mathcal{G})$ acts on the commutative algebra of functions on Minkowski spacetime (3.33), the infinitesimal gauge transformations for a pair of fields on $\mathcal{A}_\theta$ are defined by

$$\delta_\alpha \phi_k(x) = i\alpha(x) \phi_k(x) \ , \ k = 1, 2 \ . \tag{5.11}$$

The coproduct of the algebra of infitesimal gauge transformations δ_α is twisted as in (3.27)

$$\Delta_0(\delta_\alpha) \to \Delta_t(\delta_\alpha) := \mathcal{F} \Delta_0(\delta_\alpha) \mathcal{F}^{-1} \quad ; \ \Delta_0(\delta_\alpha) = \delta_\alpha \otimes \mathbf{1} + \mathbf{1} \otimes \delta_\alpha \tag{5.12}$$

and the action of the gauge transformation δ_α on the $\star$-product of fields is defined by (3.34). Thus the gauge transformation of the quadratic term $\phi_1(x) \star \phi_2(x)$ in a Lagrangian is written

$$\begin{aligned} \delta_\alpha\big(\phi_1 \star \phi_2\big) &= m_t\big(\Delta_t(\delta_\alpha)(\phi_1 \otimes \phi_2)\big) \\ &= m\big(\mathcal{F}^{-1} \mathcal{F} \Delta_0(\delta_\alpha) \mathcal{F}^{-1} (\phi_1 \otimes \phi_2)\big) \\ &= m\big((\delta_\alpha \otimes \mathbf{1} + \mathbf{1} \otimes \delta_\alpha) e^{\frac{i}{2}\theta^{\mu\nu} \partial_\mu \otimes \partial_\nu} (\phi_1 \otimes \phi_2)\big) \ . \end{aligned} \tag{5.13}$$

In [103, 104], the result of the gauge transformation (5.13) is alleged to be

$$\delta_\alpha\big(\phi_1(x)\star\phi_2(x)\big) = i\alpha_i(x)\big[\big(T_i\phi_1(x)\big)\star\phi_2(x) + \phi_1(x)\star\big(T_i\phi_2(x)\big)\big]\ , \tag{5.14}$$

but from (5.13) we can see that this can be true only if the partial derivatives of the fields $\phi_k(x)$ $(k = 1, 2)$ transform under the gauge transformations in the same representation as the field itself

$$\delta_\alpha\big((-i)^m P_{\mu_1}\cdots P_{\mu_m}\phi_k(x)\big) = \delta_\alpha\big(\partial_{\mu_1}\cdots\partial_{\mu_m}\phi_k(x)\big) = i\alpha(x)\big(\partial_{\mu_1}\cdots\partial_{\mu_m}\phi_k(x)\big)\ . \tag{5.15}$$

If this would be the case, there would be no reason to introduce the gauge fields and the covariant derivatives, because the partial derivatives would already be covariant under the gauge transformations. Thus, this approach contradicts the very essence of gauge symmetries.

The form of the gauge transformations (5.14) is very desirable, because it would enable any gauge group to close on the noncommutative spacetime, so that we would not have to worry about the no-go theorem. Therefore, we can try to repair this problem by replacing the twist element (5.10) with the gauge covariant non-Abelian twist element

$$\tilde{\mathcal{F}} = e^{-\frac{i}{2}\theta^{\mu\nu}D_\mu\otimes D_\nu+\mathcal{O}(\theta^2)}\ , \tag{5.16}$$

where the covariant derivative is defined by

$$D_\mu = -i(P_\mu + A^i_\mu T_i) = \partial_\mu - iA_\mu \tag{5.17}$$

and the gauge fields $A_\mu(x) = A^i_\mu(x)T_i$ are in the adjoint representation of the gauge algebra

$$\delta_\alpha A_\mu(x) = i\big[\alpha(x), A_\mu(x)\big] + \partial_\mu\alpha(x)\ . \tag{5.18}$$

According to (3.32), the noncommutative multiplication of the algebra of functions that is holding the representations of the twisted algebra $\mathcal{U}_t(\mathcal{P}\ltimes\mathcal{G})$ is defined by (we denote the product by $\bigstar$)

$$\phi_1(x)\bigstar\phi_2(x) := m\Big(\tilde{\mathcal{F}}^{-1}\big(\phi_1(x)\otimes\phi_2(x)\big)\Big)\ . \tag{5.19}$$

Since D_μs are covariant under the gauge transformations

$$\delta_\alpha\big(D_{\mu_1}\cdots D_{\mu_m}\phi_k(x)\big) = i\alpha(x)\big(D_{\mu_1}\cdots D_{\mu_m}\phi_k(x)\big)\ , \tag{5.20}$$

the gauge transformations of the $\bigstar$-product (5.19) can be of the form (5.14) without contradicting the idea of gauge symmetries. The twist element (5.16), however, has a fatal flaw: $\tilde{\mathcal{F}}$ does not satisfy the first twist condition in (3.28), i.e. the following equation is not true

$$\tilde{\mathcal{F}}_{12}(\Delta_0\otimes\mathrm{id})\tilde{\mathcal{F}} = \tilde{\mathcal{F}}_{23}(\mathrm{id}\otimes\Delta_0)\tilde{\mathcal{F}}\ . \tag{5.21}$$

The twist condition (5.21) is violated already in the second order in θ. The terms responsible for this are of the forms $\theta\theta D\otimes D\otimes DD$, $\theta\theta D\otimes DD\otimes D$ and $\theta\theta DD\otimes D\otimes D$, where the two indices of the DD factor come from both θs, so that the alternative covariant first order terms $\theta^{\mu\nu}F_{\mu\nu}\otimes\mathbf{1}$ and $\theta^{\mu\nu}\mathbf{1}\otimes F_{\mu\nu}$ cannot cancel the terms of these forms. The possible second order terms of the covariant $\mathcal{O}(\theta^2)$ in (5.16) can cancel some of the terms, but not all of them. In [106], it was also verified that even the most general covariant twist element — a general invertible function — cannot be made to satisfy the twist condition (3.28). Thus, *a non-Abelian covariant generalization of the twist* (5.10) *does not exist.* This implies that the noncommutative $\bigstar$-multiplication for the representations of $\mathcal{U}_t(\mathcal{P} \ltimes \mathcal{G})$ cannot be associative.

We can conclude that the *external* Poincaré symmetry and the *internal* gauge symmetry cannot be unified under a common twist. Hence, only the $\star$-gauge symmetries discussed in chapter 4, where the action of the gauge algebra on fields is deformed $\delta_\alpha = i\alpha(x)\phi(x) \rightarrow i\alpha(x) \star \phi(x)$ and the coproduct of δ_α is left untouched, can currently be used to construct noncommutative gauge theories.

5.3.2 Gauging the twisted Poincaré symmetry

The local Poincaré gauge symmetry is an *external* gauge symmetry. Through geometrical interpretation the Poincaré gauge symmetry translates to the covariance under general coordinate transformations and to the local Lorentz symmetry (see the section 2.5). This "duality" of the Poincaré gauge symmetry is both a problem and a possibility, since we just saw that an internal gauge symmetry cannot be twisted together with the Poincaré symmetry. What about gauging the twisted Poincaré symmetry itself. We intend to find out whether the gauge theory of the Poincaré symmetry on noncommutative spacetime can be formulated by means of a twist.

We could take the direct naive approach and try to construct a noncommutative gauge theory of the twisted Poincaré symmetry by using the Abelian twist (3.37) and by replacing the point-wise product of functions with the Moyal $\star$-product in the classical theory constructed in chapter 2. The result would, however, be an inconsistent frame-dependent theory — in many ways similar to those already developed — that would certainly not be a plausible theory of gravitation. We would not be able to give any meaningful geometrical interpretation to a theory of this type.

Since the global Poincaré symmetry is twisted with the Abelian twist (3.37) in the case of the flat noncommutative spacetime, also the generalized local Poincaré gauge symmetry on noncommutative spacetime should be a quantum symmetry. A natural way to generalize the local Poincaré gauge symmetry into the noncommutative setting is to consider it as a twisted gauge symmetry, so that the global twisted Poincaré symmetry is obtained in the limit of vanishing gauge fields. When the global twisted Poincaré symmetry is generalized to a local gauge symmetry, we have to introduce the gauge fields in order to compensate the non-covariance of the partial derivatives, similarly as we did in the commutative case in section 2.4. Instead of the partial derivatives we have to use the covariant derivatives (2.31) or equivalently (2.37),

$$\nabla_\mu = d_\mu + \mathcal{A}_\mu = -i\Big(e_\mu{}^\alpha P_\alpha - \frac{1}{2} A_\mu{}^{\nu\rho} \Sigma_{\nu\rho}\Big) , \tag{5.22}$$

where the $\Sigma_{\nu\rho}$s generate a finite-dimensional representation of the Lorentz algebra.

The only difference compared to the covariant derivative of an internal gauge symmetry (5.17) are the vierbein gauge fields ${e_\mu}^\alpha$ multiplying the P_μs in (5.22). $\mathcal{A}_\mu$s are the gauge fields of the internal Lorentz rotations. In order to obtain a theory that is covariantly deformed under the local Poincaré gauge transformations, according to (5.6) and (5.7), the frame-dependent P_μs have to be replaced with the covariant derivatives $i\nabla_\mu$ in the Abelian twist element (5.16). The covariant non-Abelian twist element is of the form

$$\tilde{\mathcal{F}} = e^{-\frac{i}{2}\theta^{\mu\nu}\nabla_\mu\otimes\nabla_\nu+\mathcal{O}\left(\theta^2\right)} \ , \tag{5.23}$$

where $\mathcal{O}\left(\theta^2\right)$ stands for the possible additional covariant terms in higher orders of the noncommutativity parameter $\theta^{\mu\nu}$. Because of the similar forms of the covariant derivatives (5.22) and (5.17) and of the twist elements (5.23) and (5.16), the basic algebraic reasoning presented in [106] holds also for the twist element (5.23) proposed here. The gauge fields $\mathcal{A}_\mu$ alone in ∇_μ will violate the twist condition (5.21) and the rest of gauge fields ${e_\mu}^\alpha$ are not able to rescue the twist condition. The fact that there are now two second rank (field strength) tensors (2.50) does not help us to satisfy the twist condition.

We present the core argumentation of [106] applied to the present case. First we consider the twist element (5.23) with only the first order term in θ in the exponent. The second order terms in θ that do not cancel in the twist condition (5.21) are, in the left-hand side

$$\begin{aligned}\frac{1}{2}\left(-\frac{i}{2}\right)^2\theta^{\mu\nu}\theta^{\rho\sigma}\big(&2\nabla_\mu\nabla_\rho\otimes\nabla_\nu\otimes\nabla_\sigma+2\nabla_\mu\otimes\nabla_\nu\nabla_\rho\otimes\nabla_\sigma\\&+\nabla_\mu\otimes\nabla_\rho\otimes\nabla_\nu\nabla_\sigma+\nabla_\rho\otimes\nabla_\mu\otimes\nabla_\nu\nabla_\sigma\big)\end{aligned} \tag{5.24}$$

and in the right-hand side

$$\begin{aligned}\frac{1}{2}\left(-\frac{i}{2}\right)^2\theta^{\mu\nu}\theta^{\rho\sigma}\big(&2\nabla_\rho\otimes\nabla_\mu\nabla_\sigma\otimes\nabla_\nu+2\nabla_\rho\otimes\nabla_\mu\otimes\nabla_\nu\nabla_\sigma\\&+\nabla_\mu\nabla_\rho\otimes\nabla_\nu\otimes\nabla_\sigma+\nabla_\mu\nabla_\rho\otimes\nabla_\sigma\otimes\nabla_\nu\big)\ .\end{aligned} \tag{5.25}$$

These terms cannot be cancelled by terms that have second rank tensors

$$R^{\rho\sigma}{}_{\mu\nu}\Sigma_{\rho\sigma}\ , \quad T^\rho{}_{\mu\nu}\nabla_\rho \tag{5.26}$$

in them, because the two indices for such tensors come from the same $\theta^{\mu\nu}$, unlike for the $\nabla\nabla$ factors in (5.24) and (5.25). This is why such terms were not included in twist element (5.23) in the first place. The possible second order terms in (5.23) have the forms

$$\theta^{\mu\nu}\theta^{\rho\sigma}\,1\otimes\nabla_\mu\nabla_\nu\nabla_\rho\nabla_\sigma\ , \qquad \theta^{\mu\nu}\theta^{\rho\sigma}\,\nabla_\mu\nabla_\nu\nabla_\rho\nabla_\sigma\otimes 1\ , \tag{5.27}$$

$$\theta^{\mu\nu}\theta^{\rho\sigma}\,\nabla_\mu\otimes\nabla_\nu\nabla_\rho\nabla_\sigma\ , \qquad \theta^{\mu\nu}\theta^{\rho\sigma}\,\nabla_\mu\nabla_\nu\nabla_\rho\otimes\nabla_\sigma\ , \tag{5.28}$$

$$\theta^{\mu\nu}\theta^{\rho\sigma}\,\nabla_\mu\nabla_\nu\otimes\nabla_\rho\nabla_\sigma\ , \tag{5.29}$$

with all the permutations of indices of the covariant derivatives — although the antisymmetry of θ greatly reduces the number of independent permutations. In the twist condition (5.21), these second orders terms can never cancel all the terms in (5.24) and (5.25). Therefore, the twist condition (5.21) is necessarily violated in the second order in θ.

It is well known that the gauging of the translation symmetry leads to the Einstein-Hilbert Lagrangian and to the covariance under general coordinate transformations (see [34, 42]). Hence, it is interresting to see whether the gauge theory of the *external* translation symmetry group $\mathcal{T}_4$ can be consistently defined together with the twisted Poincaré symmetry. The covariant derivative for the local translations is

$$d_\mu = -ie_\mu{}^\alpha P_\alpha \ . \tag{5.30}$$

In fact, this is also the covariant derivative of the Poincaré gauge symmetry for one-dimensional representations, for which the covariant derivative (5.22) should reduce to (5.30), where the gauge fields $e_\mu{}^\alpha$ now contain contributions also from the local Lorentz transformations. Since the covariant derivatives of the translation group do not commute (2.51), the covariant twist element

$$\tilde{\mathcal{F}} = e^{-\frac{i}{2}\theta^{\mu\nu} d_\mu \otimes d_\nu + \mathcal{O}\left(\theta^2\right)} = e^{\frac{i}{2}\theta^{\mu\nu} e_\mu{}^\alpha P_\alpha \otimes e_\nu{}^\beta P_\beta + \mathcal{O}\left(\theta^2\right)} \tag{5.31}$$

cannot be of the Abelian type (3.35) that is known to be a twist. Because of this and the high level of arbitarity in the gauge fields $e_\mu{}^\alpha$ of the covariant derivative (5.30), we face the similar algebraic problems with the twist element (5.31) as we did with the twist element of the full Poincaré symmetry (5.23). It is difficult to see how (5.31) could satisfy the twist condition (5.21) any better than (5.23), even though the twist candidate (5.31) is now much simpler. Thus, it is not only the internal Lorentz rotation symmetry that breaks the validity of the non-Abelian Poincaré gauge covariant twist element (5.23). The external gauge symmetry associated with the general coordinate transformations is evenly problematic.

Thus the Poincaré gauge covariant non-Abelian twist element (5.23) is not a twist and the $\bigstar$-product (5.19) defined by it is not associative. We can conclude that *the twisted Poincaré symmetry cannot be gauged by generalizing the Abelian twist* (5.10) *to a covariant non-Abelian twist* (5.23), *nor by introducing a more general covariant twist element.*

It should be mentioned that from the mathematical point of view, we could try to deform the action of the twisted Poincaré algebra on its representations, instead of generalizing the twist element, but it seems unlikely that such an approach could solve the problems related to the frame-dependent twist element (5.1).

It is suggested in [106] that supersymmetric gauge theories may be formulated by means of a twist, but such considerations are beyond the scope of this work.

Chapter 6

Conclusions

We have discussed the essential role of the Poincaré symmetry in relativistic field theories and we have explained how the local Poincaré symmetry produces the classical gauge theory of gravitation, equivalent to the Einstein-Cartan theory of gravitation. Next we discussed the properties of canonical noncommutative spaces and the formulation of quantum and gauge field theories on noncommutative spacetimes. We emphasized the twisted Poincaré symmetry that is respected by these noncommutative field theories and we discussed the underlying mathematics of this quantum symmetry. Then we discussed the main problems in formulating a theory of gravitation on noncommutative spacetimes, emphasizing the need to formulate the concept of general coordinate transformations in the noncommutative setting. Finally we addressed twisted gauge symmetries, especially the possibility to generalize the global twisted Poincaré symmetry to a local gauge symmetry. Based on the current understanding of noncommutative gauge theories and twisted symmetries, we concluded that both the internal and the external twisted gauge symmetries in noncommutative field theory are very problematic. We showed that the formulation of a twisted Poincaré gauge symmetry cannot be made by means of a gauge covariant twist. This suggests that in noncommutative spacetimes the gravitational interaction does not arise from a twisted Poincaré gauge symmetry in the same way as it arises from the Poincaré gauge symmetry in commutative spacetimes. Thus more elaborate approaches to noncommutative gravitation will have to be invented in order to understand the role of the twisted Poincaré symmetry with respect to the gravitational interaction on noncommutative spacetimes.

Lastly, a final word about the significance of spacetime noncommutativity is in place. Spacetime noncommutativity can be seen as the next major conceptual step we may have to take in order to begin to resolve the incompatibility of general relativity and quantum mechanics. Noncommutative field theories certainly will not be the ultimate theory of physics. They, however, offer a nice test ground for studying noncommutativity, nonlocality and all the fascinating things related to these. They may also well be the theories that will offer first solid predictions on physics at the Planck scale.

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