Equilibrium Bounded Bargaining

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Abstract

We study one-sided offers bargaining game g where players cannot commit finalizing the trade. The game never ends but implements terms of trade iff there is stage k from which on players agree on these terms. Otherwise, no-trade results. We show that equilibria induced by g are equivalent to *equilibrium bounded bargaining schemes* that are *robust against bound extensions*.

Game g has many sequential equilibria. We give the seller a leading role in equilibrium selection. The seller is allowed to swift from one equilibrium to another iff this is dynamically consistent. We show that the set of admissible equilibria, or stable set, is unique. Under typical conditions, equilibrium in a stable set allocates the good to the buyer with price equal to the buyer's least valuation in the *belief closed set* of valuations. Thus higher order uncertainty does not counterbalance the situation in favor of the seller.

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1 Introduction

Bargaining is about commitment. The aim of this paper is to study one-sided offers bargaining under the logical benchmark assumption that players do not have *any* commitment power. To achieve this goal, a new approach is developed. General beliefs are allowed.[1]

By the Coase conjecture,[2] if a seller of a good is unable to commit to her price offers in a one-sided offers bargaining situation, then trade takes place

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[1] Weakening the informational assumptions is an apt research topic in bargaining and game theory. Recent examples include Yildiz (2002) and Bergemann-Morris (2003).

[2] The conjecture is established by Coase (1972) in the context of durable good monopolist.

without delay with the price equal to the least possible reservation valuation of the buyer. Gul *et.el.* (1986) and Fudenberg *et.el.* (1986) confirm this conjecture when seller's beliefs of the buyer's valuations are common knowledge, and buyer's least possible valuation is bounded away from seller's valuation.[3] The result is sensitive to the informational assumptions. Feinberg and Skrzypacz (2002) show that with one additional level of uncertainty an equilibrium may emerge, where the seller achieves surplus above the least possible valuation of the buyer, and delays occur.

In the standard model[4] one is typically interested in the limit case where the cost of waiting goes to zero. However, even in the limit, the model relies on a commitment assumption: Once the buyer accepts seller's offer, trade takes place and payoffs materialize. Thus the seller *can* commit not to change her offer once the offer is accepted. This paper argues that without external commitment devices, it is difficult for the seller to take advantage of her private information.[5]

A problem with no-commitment is that any game that ends up implementing an outcome at some finite terminal history contains some commitment power - an implemented outcome cannot be *non*-implemented even if wanted by the players. How should one model bargaining without any commitment? Muthoo (1990, 1994) studies bargaining games when the players cannot commit not to retract their offers. However, in his model they *can* commit to implement trade when they do not retract an accepted offer. Our aim is to analyse a games without any commitment.

A standard position in noncooperative game theory is that any feature of a social situation that affects player's behavior should be clearly spelled out through the *extensive game form*[6] that represents the situation (see Rubinstein, 1991). Ideally, if an extensive form reflects no-commitment, then it should allow players to change their past actions if they wish to. Formally this desideratum can be met by game forms that are structurally stationary in a sense that *players' moves do not have any influence on the continuation extensive form.* If this is true, then past actions are never physically restrictive. Note that the standard model does not meet the desideratum since after buyer's acceptance the continuation game is different from the continuation game after rejection.

The first contribution of the paper is to come up with an extensive game form that meets the desideratum. We analyse the following sequential moves game form g: At each odd stage, the seller makes an offer and at each even stage

[3] More precisely, this is true in the "Gap" case and only in the limit. Once a delay between consecutive offers becomes small, the price of the good approaches buyer's lowest possible valuation. In the "No Gap" case a stationarity restrictions is needed to obtain the same result (see Ausubel and Deneckere 1989a,b).

[4] In the standard model, the seller makes offers which are either rejected or accepted by the buyer. If an offer is rejected, the seller makes a new offer. If an offer is accepted, it is implemented. The buyer knows his reservation valuation whereas the seller does not. Seller's valuation is common knowledge.

[5] Feinberg-Skrzypacz (2002) use infinitely mixed strategies in their equilibria whereas we only allow finitely mixed strategies. See the end of Section 5 for discussion on this.

[6] A game form specifies the order of moves and the physical consequences of all possible combinations of the moves. Together with preferences and information structure, the game form constitutes a game. See e.g. Osborne-Rubinstein (1995).

the buyer either accepts or rejects. The game never ends. If there is a stage from which onwards the seller always makes offer m, for some $m \in \mathsf{R}$, and the buyer always accepts, then payoffs equal to trade with transfer m become associated to the players (at the infinity). Otherwise, zero payoffs become associated to the players.

Although payoffs do not ever materialize with g in finite time, the order of moves is well defined and each (infinite) strategy vector is associated with a physical outcome. Thus, together with belief structure p, pair (g, p) constitutes a proper extensive form game. As players' strategic options in the continuation game are independent of their past actions, also the desideratum above is met. Extensive form g does not reflect any commitment power on the part of the players.

We allow all beliefs over a *finite* set of possible signals. Our first observation is that whenever players' *pure* or *finitely mixed* strategies are common knowledge, there exists a stage from which onwards it is common knowledge which outcome is to be implemented. Thus, in equilibrium where strategies *are* common knowledge the game unavoidably reveals the outcome that will be implemented. Hence, if information regarding the implemented outcome is disperse at the beginning of the game, then there has to be a player who *reveals* the eventual outcome. Much of the consecutive analysis of this paper is about showing that such player typically does not exist.

Game (g, p) hosts many pure strategy (or finitely mixed) sequential equilibria. The second contribution of the paper is to develop an equilibrium selection argument that is motivated by the idea that the seller plays a leading role in choosing the equilibrium.[7] Equilibrium selection takes place as a function of beliefs. The central feature of the equilibrium selection rule is *dynamic consistency.* Namely, in consent with the no-commitment assumption, we assume that the seller cannot commit not to switch to a *new* more profitable equilibrium whenever such becomes feasible. Of course, in any *admissable* equilibrium this should not be the case. Hence we need to answer which equilibrium selection rules are admissible, i.e. dynamically consistent.

Admissible equilibria are characterized in the language of vonNeumann-Morgenstern stable sets. First we establish a dominance relation - which we call *upsetting* - on the set of equilibria. Roughly, an equilibrium is upset by another if it is common knowledge that the seller strictly prefers the latter at some positive probability information node of the former equilibrium, given the posterior beliefs induced at this node. If the seller is empowered to move from one equilibrium to another, then an upset equilibrium should be deemed unstable. The problem is that there typically does not exist any non-upset equilibrium. However, dynamic consistency does not require admissible equilibria to be non-upset but only *stable* in the following sense. Any stable equilibrium cannot be upset by another stable equilibrium (internal stability), and any unstable equilibrium must be upset by a stable one (external stability).[8] The set

[7] This is in line with the standard mechanism design approach that allows the designer to choose her most desired equilibrium of the mechanism.

[8] Since upsetting is defined with respect to the beliefs, a stable equilibrium selection rule has much Markovian flavor.

of stable equilibria that meets the two stability criteria, or a stable set, is the object of our study.

vNM stable set has not gone without recognition in the noncooperative game theory. Two immediately related papers are Blume and Sobel (1997) and Vartiainen (2001).[9] The first one introduces refinement of a cheap talk game that uses stable set as the solution device. Their aim is to remove equilibria that are not robust against the sender wanting to continue communication. The motivation of their dominance relation and upsetting are similar, even if they differ formally. The second paper uses upsetting based stable set to refine equilibria in an auction design game where the seller is unable to commit not to change the rules of the designed auction. All these papers are based on the idea that stable set reflects sequential rationality, given the behavior induced by the dominance relation at hand. A general taxonomy of consistency approaches, relating them back to stable set, is offered by Greenberg (1990).

First we apply the solution to the binary reservation valuations' case, which is special and transparent. Assume that buyer's valuations are derived from the set $\{\theta^0, \theta^1\}$ such that $0 < \theta^0 < \theta^1$. Information of a player is represented by a private signal. Signal of the buyer determines his valuation in the set $\{\theta^0, \theta^1\}$. Players' signals can be correlated in arbitrary way.

We construct a set of equilibria G^* where each equilibrium sells the good to the buyer with price equal to his least possible valuation in the belief closed set of valuations.[10] A belief closed set can be larger than the support of seller's belief over buyer's reservation valuations, and hence the lowest valuation lower than the one in the support of seller's beliefs (of buyer's valuations). Our main finding here is that the constructed set of equilibria is a stable set, and that the stable set is unique up to payoff irrelevant moves. Thus, the seller is forced to sell the good with a low price if there is k such that (1) the buyer knows that the buyer is willing to pay high price, (2) the buyer knows (1), (3) the seller knows (2),..., (k) player i does *not* know (k-1).

For example, under the informational assumptions of Feinberg-Scrzypacz, the seller is always forced to sell the good with price equal to the low valuation. Thus in our framework the seller *cannot* take advantage his private information.

In the more general case buyer's valuations are derived from a finite set $\{\theta^0, ..., \theta^K\}$ such that $0 < \theta^0 < \theta^1 < \cdots < \theta^K$. Also there we establish that a stable set exists, and it is unique up to payoff irrelevant moves. The general structure of the stable set is difficult to describe. In particular, G^* fails to be stable. However, in cases where information structure is sufficiently well behaved, "monotonic", we can say more. Consider the following case, familiar from the Global Games literature. Assume the seller obtains a noisy signal $\theta^{\tilde{k}}$, where $\tilde{k} = k + \varepsilon$, and ε is a noise term with support $\{0, 1\}$ for $k = 0, ...,$ of buyer's valuation. If $\tilde{k}$ is not commonly known, i.e. not observable by the buyer, then the buyer knows the signal of the buyer is at least θ^{k-1}. However, θ^{k-1} and $\theta^{\tilde{k}}$ do not coincide and therefore players cannot ever agree that seller's reservation

[9] Others include Asheim (1992), Asheim-Nilssen (1996), and Kahn-Mookherjee (1995).

[10] A belief closed set of types satisfies the property that it is common knowledge that all player's types belong to this set.

valuation is above θ^0. This is the least possible valuation in the belief closed set of valuations of the seller. In any stable equilibrium the seller is always forced to sell the good to the buyer with price θ^0.[11]

Extensive form games with general beliefs are typically difficult to analyze. Due to leeway provided by the disequilibrium beliefs, the set of equilibria easily becomes unmanageable. Hence, refining equilibria through restrictions on disequilibrium beliefs has become a huge research project. However, even if theories abound, no commonly agreed principle how to restrict disequilibrium beliefs exists.[12] An attractive feature of our equilibrium selection criterion - that also drives the uniqueness results - is that it does not rely on to any restrictions on disequilibrium beliefs (except by the assumption that they are common). Stable set imposes conditions only on the *on*-the-equilibrium occurrences.

Admittedly, the assumption that an outcome is never implemented is cumbersome, and needs to be scrutinized. The third and most interesting contribution of the paper is to associate equilibria of game form g to concrete bargaining schemes that always implement trade in finite time. We use the following procedure to generate such scheme. Take any strategy σ on g, and choose an initial bound K (an even integer). 1: Given prior p, on the play path of (σ, p) implement outcome after stage K based on the most recent offer by the seller and rejection/acceptance of the buyer. 2: After any zero-probability move, identify (any) new p' and K', and repeat Step 1 with respect to them. By repeatedly applying Steps 1 and 2 one comes up with an extensive game form and a strategy that is a truncated version of σ. If the induced strategy constitutes a sequential equilibrium of the induced game, then the equilibrium play automatically ends in finite time. Such scheme, a combination of a strategy and game, is called as *equilibrium bounded bargaining scheme.*[13]

A virtue of the concept of equilibrium bounded bargaining scheme is that it allows us to analyze the effects of removing physical commitment devices. Namely, if the equilibrium bounded bargaining scheme is not dependent on the commitment aid provided by the bargaining bound, then the equilibrium should survive any extension of it. Equilibrium bounded bargaining schemes that meet such property are said to be *robust against bound extensions* (or simply robust equilibrium bounded bargaining scheme). We show that the set of equilibrium outcome functions induced by g and outcome functions induced by robust equilibrium bounded bargaining schemes are equivalent. Any element of the former corresponds with some element of the latter and vice versa. Thus, g can be can be thought of as a device that only admits equilibrium outcome functions that do not reflect any commitment power.

Finally, we focus on equilibrium strategies rather than outcome functions

[11] *If* $\tilde{k}$ is commonly known, then the Coase conjecture follows: in any stable equilibrium the seller is sells with price $\theta^{\tilde{k}}$, which is the least reservation value that players commonly agree is possible.

[12] In fact, many signaling theories are subject to the criticism that the assumption they made cannot be consistent with them being common knowledge among players. For discussion on this so called Stiglitz-critique, see e.g. Mailath *et.al* (1993).

[13] The concept is heavily motivated by Ray and Vohra (1997), who analyse equilibrium binding coalitional agreements.

induced by them. We show that if one starts from robust equilibrium bounded bargaining schemes, and derives equilibria A^{R} of g that correspond these schemes, then the stable set G^* belongs to A^{R}. This implies that G^* is the *unique stable set* on A^{R} as well. Hence, in terms of the stable set, nothing is added by focusing on the grand set of equilibria of g rather than robust equilibrium bounded bargaining schemes. E.g. in the binary valuations case, any robust equilibrium bounded bargaining scheme allocates the good to the buyer with price θ^0 unless it is common knowledge that the buyer has higher valuation.

The paper is organized as follows: Section 2 defines the informational conditions, establishes the game form, and develops the solution concept. Section 3 deals with the binary valuations case, and Section 4 the many-valuations case. Section 5 introduces the concept of equilibrium bounded bargaining scheme, and establishes the equivalence result. The final section closes the paper with dscussion.

2 Framework

2.1 Information

There is a seller and a buyer bargaining over an indivisible object. Normalize seller's reservation value, which is not subject to uncertainty, to 0. The buyer *privately* knows his strictly positive reservation valuation θ on Θ. Let $T = T_{\mathsf{s}} \times T_{\mathsf{b}}$ be a *finite* set of types. Let function $\theta : T_{\mathsf{b}} \to \Theta$ describe how buyer's reservation valuation is associated to his type. Given set D of types t, denote the least possible seller's reservation valuation by $\underline{\theta}(D) = \min\{\theta(t_{\mathsf{b}}) : (t_{\mathsf{b}}, t_{\mathsf{s}}) \in D\}$.

Denote by Δ the of all probability distributions on T, representing players' beliefs in all states. Each t_{i}'s beliefs are then represented by a conditional distribution $p(\cdot : t_{\mathsf{i}})$ on $T_{-\mathsf{i}}$. Given $p \in \Delta$, a *belief closed* subset $BC = BC_{\mathsf{s}} \times BC_{\mathsf{b}}$ of T satisfies the property that whenever $t \in BC$, then both players know their types are in BC, they know that they know their types are in BC, and so forth. Thus BC can be brought to players' knowledge without affecting their beliefs. A belief closed subset is *smallest* if it holds true that the only belief closed subset it contains is the set itself.

More formally, let B_{s} and B_{b} be the knowledge operators on T identifying the largest event that t_{s} and t_{b} associate positive probability under (common) p:[14]

$$\begin{aligned} B_{\mathsf{b}}(p, t_{\mathsf{s}}) &= S(p(\cdot : t_{\mathsf{s}})), \\ B_{\mathsf{s}}(p, t_{\mathsf{b}}) &= S(p(\cdot : t_{\mathsf{b}})). \end{aligned}$$

Similarly, $B_{\mathsf{i}}(p, D_{-\mathsf{i}}) = \cup_{t_{-\mathsf{i}} \in D_{-\mathsf{i}}} B_{\mathsf{i}}(p, t_{-\mathsf{i}})$, for $i = s, b$. Construct inductively

$$\begin{aligned} B^{0}_{\mathsf{s}}(p,t) \times B^{0}_{\mathsf{b}}(p,t) &= B_{\mathsf{s}}(p, t_{\mathsf{b}}) \times B_{\mathsf{b}}(p, t_{\mathsf{s}}), \quad \text{and} \\ B^{\mathsf{n}}_{\mathsf{s}}(p,t) \times B^{\mathsf{n}}_{\mathsf{b}}(p,t) &= B_{\mathsf{s}}(p, B^{\mathsf{n}-1}_{\mathsf{b}}(p,t)) \times B_{\mathsf{b}}(p, B^{\mathsf{n}-1}_{\mathsf{s}}(p,t)) \quad \text{for } n = 1, \ldots . \end{aligned}$$

Note that $[B^{\mathsf{n}}_{\mathsf{s}}(p,t) \times B^{\mathsf{n}}_{\mathsf{b}}(p,t)] \subseteq [B^{\mathsf{n}+1}_{\mathsf{s}}(p,t) \times B^{\mathsf{n}+1}_{\mathsf{b}}(p,t)] \subseteq T$. Since T is finite $B^{\infty}_{\mathsf{s}}(p,t) \times B^{\infty}_{\mathsf{b}}(p,t)$ well defined. Denote by $BC(p,t)$ the smallest belief closed

[14]Support of p is a smallest closed set Y such that $p(Y) = 1$.

set that contains t, given p. Then

$$BC(p,t) = B_s^\infty(p,t) \times B_b^\infty(p,t).$$

Note that $D = BC(p,t)$ for $t \in D = D_s \times D_b$ if and only if $B_i(p, B_{-i}(p, D_i))) = D_i$ for all $i = s, b$. On the other hand, $B_i(p, B_{-i}(p, D_i))) = D_i$ implies $B_{-i}(p, B_i(p, D_{-i}))) = D_{-i}$ for D_{-i} such that $D_{-i} = B_{-i}(p, D_i)$. For such D, then, $D = BC(p,t)$ for $t \in D = D_s \times D_b$. Alternatively, $BC \subset T$ is a belief closed set if and only if $t, t' \in BC$ implies there is chain $t^1, ..., t^k$, $k \geq 1$, such that, for some $i, j \in \{b, s\}$

$$p(t_i^1, t_{-i}^1)p(t_i^2, t_{-i}^1)p(t_i^2, t_{-i}^2) \cdots p(t_j^n, t_{-j}^{n-1})p(t_j^n, t_{-j}^n) > 0.$$

Finally, if D is a belief closed set then so is $T \setminus D$.

2.2 The Model

Let $(a, m) \in \{0, 1\} \times \mathsf{R}$ describe an outcome of a bargaining procedure where $a = 1$ if and only if trade takes place, and $m \in \mathsf{R}$ is a transfer from the buyer to the seller. With outcome (a, m) the payoff associated to the buyer with valuation θ is $u(a, m, \theta) = a\theta - m$ and the payoff associated to the seller is $v(a, m) = m$.

2.2.1 The Extensive Form

A standard position in noncooperative game theory is that any element of a social situation that may affect player's behavior should be clearly spelled out through the extensive game form that represents the strategic situation. The game form should specify who moves when and what are the consequences of all possible combinations of the moves. Thus non-commitment assumption, if it makes sense, should also be describable by an extensive game form. Such extensive form should be consistent with the idea that players cannot commit to their past actions.

We take as a *desideratum* that a game form containing no commitment power *exhibits structural stationarity in a sense that player's choice after any history does not affect the continuation extensive game form.* That is, players' past choices do not affect what they can achieve in the future. Note that a game that meets this desideratum cannot have finite terminal histories - if it had, then the player's move at the node preceding terminal history would have a dramatic effect on the continuation extensive game. This in turn implies that the game at hand cannot contain discounting - if it had, then all terminal histories would generate zero payoff.

Consider the following extensive game form g: Players move in a sequential order, and there is no finite terminal history (any terminal history infinite). At odd stages the seller makes an offer (a, m) and at even stages the buyer either accepts or rejects. If there is stage k from which onwards the seller always offers (a, m) and the buyer always accepts, then payoffs $u(a, m, \theta)$ and $v(a, m)$ are associated to the players. In all other cases, payoffs $u(0, 0, \theta) = 0$ and $v(0, 0) = 0$ are associated to the players.

As the continuation game form is independent of the past history, game form g satisfies the desideratum stated above.[15] Further motivation for the game is provided in Section 5, where it is shown that equilibria induced by g are coincide with equilibria of bargaining schemes that implement outcomes in bounded time but that are robust against bound extensions.

Pure strategies σ_{b} and σ_{s} of the buyer and the seller define an action for each player after any finite history.[16] Strategy $\sigma = (\sigma_{\mathsf{s}}, \sigma_{\mathsf{b}})$ then determines players' payoffs uniquely. Denote by h a typical history of g. Let p reflect players' prior beliefs. Given strategy σ, denote the beliefs associated to history h by $p(\sigma, h)$. Abusing the language slightly, pair (g, p) constitutes a *bargaining game*.[17]

If there is a stage from which onwards the seller always offers (a, m) and the offer is always accepted by the buyer, then we say players *agree* on (a, m) from that stage onwards. Note that the game allows players agree also on not implementing trade. If the seller offers $(1, m)$, then she is said to offer price m.

There is no finite history at where players would know which outcome will be implemented *unless* they know one anothers' strategies in the continuation game. On the other hand, *if* players's strategies are common knowledge they do become to know after finite history which outcome will be implemented, as the next Lemma establishes.

Lemma 1 *Fix p. Suppose players' strategies σ are common knowledge. Then there is stage K such that from K onwards it is common knowledge which outcome will be implemented.*

Proof. Let outcome function $f : T \to \{0, 1\} \times \mathsf{R}$ associate players to the outcomes according to their common knowledge strategies σ. Take any $(a, m) \neq (0, 0)$. Denote by $k(t)$ the stage from which onwards $t \in f^{-1}(a, m) \subseteq T$ always agree on (a, m) under σ. By the construction of the game, and the definition of f, such $k(t)$ exists. Since T is finite, $k = \max\{k(t) : t \in f^{-1}(a, m)\}$ exists. Now all types in $f^{-1}(a, m) \subset T$ agree on implementing (a, m) from k onwards. Since f^{-1} is a partition of T, and since T is finite, there is the highest k' such that all types $\cup_{(\mathsf{a},\mathsf{m}) \neq (0,0)} f^{-1}(a, m)$ agree on what outcome they will implement from k' onwards. Thus, if there is a deviation from a consent at some stage higher than k', then it must be by types in $f^{-1}(0, 0)$. Let $K \geq k'$ be highest stage where some of the types in $f^{-1}(0, 0)$ deviate from the consent of types not in $f^{-1}(0, 0)$. Then from K onwards the outcome that is to be implemented is common knowledge. ■

[15] Vartiainen (2003) argues that game form g, or an game form that is formally equivalent to it, is the *only* game form that meets a larger set of desiderata for commitment-free game forms.

[16] We focus on pure strategies. However, all our results would remain unchanged with *finitely* mixed strategies. Feinberg-Skrzypacz (2002) explicitly use infinite mixing when constructing an equilibrium that vielates the Coase conjecture. In our framework infinite mixing, which is not an unproblematic concept, is a necessary condition to restore their result. Infinite mixing is further discussed in the end of Section 5.

[17] For notational simplicity, we refrain from including out-of-equilibrium beliefs into the description of sequential equilibria. Implictly, of course, they are well defined.

Lemma 1 is based on a very simple idea. If strategies are common knowledge, and type sets finite, it must be common knowledge among players when the they will at the most start agreeing on any $(a,m) \neq (0,0)$, if they are about to implement it at the infinity. Thus, such maximal such stage is reached, they also know that if none of these cases has materialized, then they are about to implement $(0,0)$. Note that we could allow players use strategies that are mixed in *finitely* many stages without affecting Lemma 1.

Lemma 1 plays an important role in the analysis. The following Corollary, which is employed later in the paper, is a direct consequence of Lemma 1.

Corollary 1 *Suppose players' strategies σ are common knowledge. Then there is the least stage $k \geq 0$ such that from k onwards it is always common knowledge whether outcome (a,m) is implemented or not.*

Proof. By Lemma 1, there is K after which it is common knowledge whether outcome (a,m) is implemented or not, along with any equilibrium history. Ask whether this holds for stage $K-1$, too. If not, we are done. If yes, move to stage $K-2$. Working backwards on the cardinality of the stages, one finds the least stage k after which it is common knowledge whether outcome (a,m) is implemented or not. ■

Suppose the buyer knows, or thinks he knows, the seller's strategy σ_{S}, as is the case in sequential equilibria. Since the buyer always possesses the right to force $(0,0)$ before any other outcome is materialized, his expectation of the equilibrium outcome must always generate him nonnegative payoff. Such property of the game is called *ex post individual rationality,* or EXPIR for short.

We maintain the hypothesis that equilibrium strategies are common knowledge. Given strategy σ and beliefs p, we say history h is on the *off-the-equilibrium* path if it is common knowledge that it cannot be reached with positive probability. *On-the-equilibrium* histories constitute the complement of the off-the-equilibrium histories. Deviation by a player from the equilibrium does not necessarily lead the game to an off-the-equilibrium history. On-the-equilibrium beliefs are derived according to Bayes' rule. The following consistency assumption restricts beliefs also at the off-the-equilibrium histories.

Assumption Beliefs are common at any off-the-equilibrium history.

That is, players coordinate their beliefs even after zero probability occurrences.

2.2.2 Stable Set

The game supports many equilibria.[18] Which to focus? We assume that the seller has a leading role in equilibrium selection but, in consent with the noncommitment idea, she cannot commit not to switch to a more profitable equilibrium if such becomes available, rather than stick to an originally chosen

[18] Muthoo (1994) (see also Muthoo 1999) shows in a model where an offer can be retracted (but nonretracted offer not) how to build up a continuum of equilibria. All of his equilibria are supported by our game, too.

equilibrium. Of course, in any admissible equilibrium deviations cannot take place. A natural restriction for admissible equilibrium would then be that no more profitable equilibria should become available once the players update their beliefs along the equilibrium play. But the problem is that there may not be any equilibrium that is seller-optimal in the class equilibria as information is being revealed.

We now develop an equilibrium selection argument that allows the seller to switch to another equilibrium if and only if such swift is "dynamically consistent". Equilibria that are non-upset according to this criterion are then deemed admissible, or *stable*.

Given beliefs p, sequential equilibrium (SE) (σ, p) is upset by equilibrium (σ', p') if it is common knowledge that the seller prefers the latter equilibrium at some point on the equilibrium path of first one. Thus, the commitment problem induces a relation on equilibria (σ, p) and (σ', p'). To model such relation more formally, let A be the graph of the equilibrium correspondence.

$$\mathsf{A} = \{(\sigma, p) : \sigma \text{ is SE, } p \in \Delta\}.$$

Then the the seller's commitment problem spans a partial order on A as follows:

Definition 1 *Equilibrium (σ, p) is* upset *by equilibrium (σ', p) if $p' = p(\sigma, h)$ for some terminal history h on-the-equilibrium path of σ, and it is common knowledge that the seller strictly prefers (σ', p') over (σ, p) at h.*

Any upset equilibrium is destabilized by the upsetting equilibrium. But this may mot be the end of the story. What if the upsetting equilibrium is itself upset by a third equilibrium, this by fourth, and so on? Which equilibria can the seller commit herself to? We solve the problem by imposing a dynamic consistency constraint on the equilibrium selection rule: any stable equilibrium should not be upset by another stable equilibrium, and any unstable equilibrium should be upset by some stable equilibrium.

Definition 2 *Set $\mathsf{G} \subset \mathsf{A}$ is* stable *if:*

1. *(Internal stability) No element of G is upset by an element in G.*
2. *(External stability) Every element not in G is upset by an element in G.*

The aim of the rest of the paper is to characterize properties of stable sets. The terms external and internal stability are drawn from the literature on vonNeumann-Morgenstern stable standards of behavior.[19] In the current set up, external and internal stabilities describe Markovian decision making that is dynamically consistent in the following sense.

To gain some intuition, interpret the equilibrium selection problem itself as a game where the seller is the only mover, and where the buyer accommodates to the declaration of an equilibrium as long as the seller commit to it. Think

[19] Cf. von Neumann-Morgenstern (1944). For comprehensive application of the stable set apparatus, see Greenberg (1991).

an equilibrium as a machine that takes as input a prior distribution and as an output the final posterior distribution along the equilibrium path.[20] The seller must now come up with a plan tells which equilibria she can commit to even if she entitled to implement a new equilibrium once a posterior of the implemented equilibrium has been materialized. Which equilibria can the seller commit to? First, divide the set of equilibria to those she can commit to and those she cannot. A direct implication of sequential rationality is that given that she can commit to equilibrium A in the first group there *cannot* be equilibrium B that she can commit to and that she prefers over A, given the posterior generated by A. On the other hand, if she cannot commit to equilibrium A, then there *must* be equilibrium B that she can commit to and that she prefers over A, given the posterior generated by A. Thus external and external stabilities can be viewed as a *necessary* condition for dynamically consistent decision making under non-commitment.

Note that as upsetting takes place only on-the-equilibrium path, stability does not impose any restriction on players off-the-equilibrium beliefs. For any $p \in \Delta$, the belief closed set $BC(p,t)$ and hence the least possible valuation $\underline{\theta}(BC(p,t))$ is well defined and common knowledge in $BC(p,t)$, for all $t \in S(p)$.

3 Binary Valuations

It is instructive to first characterize stable set in the binary valuations case. Let $\Theta = \{\theta^0, \theta^1\}$ with $0 < \theta^0 < \theta^1$. First we solve the existence question by constructing a particular stable set. Then we go on to show that there cannot be any other stable sets.

For any p, construct the following simple equilibrium strategy σ^{p}. Given $t \in S(p)$, the seller always offers price $\underline{\theta}(BC(p,t))$, the buyer always accepts offer $\underline{\theta}(BC(p,t))$ and rejects any other offer. Off-the-equilibrium actions do not affect priors. If players adhere these instructions, then no information is ever revealed and trade with price $\underline{\theta}(BC(p,t))$ is implemented.

It is clear that (σ^{p}, p) constitutes a sequential equilibrium. There is no profitable deviation, one-shoot or infinite, for either player.[21]

Construct set G^* as follows:

$$\mathsf{G}^* = \{(\sigma, p) : \text{not upset by } (\sigma^{\mathsf{q}}, q), q \in \Delta\}.$$

That is, G^* comprises all equilibria (σ, p) that are not upset by price offers $\underline{\theta}(BC(q,t))$, where posterior q is derived along the equilibrium path of (σ, p). Note that (σ^{p}, p) is not upset by (σ^{p}, p), and thus $(\sigma^{\mathsf{p}}, p) \in \mathsf{G}^*$, for all $p \in \Delta$.

Lemma 2 $(\sigma, p) \in \mathsf{G}^*$ *only if it is common knowledge along the equilibrium path that $\underline{\theta}(BC(p,t))$ will become implemented, for all $t \in S(p)$.*

[20] Thus we assume that the equilibrium decision rule is dependent only on the current beliefs. Hence the decision problem has much Markovian flavor.

[21] In the current set up, one-shoot-deviation principle may not be sufficient condition for sequential equilibrium.

Proof. Take $(\sigma, p) \in \mathsf{G}^*$. Since (σ, p) is not upset by any (σ^{q}, q), (σ, p) implements trade with probability one with price at least $\underline{\theta}(BC(p,t))$, for all $t \in S(p)$. Since trade always takes place, and since type $t_{\mathsf{b}} \in \theta^{-1}(\underline{\theta}(BC(p,t)))$ can at most get zero payoff, t_{b} only trades with price $m = \underline{\theta}(BC(p,t))$.

Suppose that the Lemma does not hold. Then, since Θ is a binary set, (σ, p) implements price strictly higher than $\underline{\theta}(BC(p,t)) = \theta^0$ under some $t' \in BC(p,t)$. Since equilibrium strategies are common knowledge, identify set H of on-the equilibrium histories at which it becomes common knowledge that price higher than θ^0 is to be implemented. Pick $\bar{h} \in H$ that is the longest of all histories in H. By Corollary 1, such longest history exists. Let $\bar{p}$ be the posterior belief at $\bar{h}$, before price higher than θ^0 becomes common knowledge, and p' a typical posterior belief immediately after $\bar{h}$, when price higher than θ^0 is common knowledge. Since after $\bar{h}$ it is common knowledge whether price θ^0 or higher is to be implemented, it follows from EXPIR that $\underline{\theta}(BC(p',t)) = \theta^1$. Since (σ, p) is not upset by $(\sigma^{\mathsf{p}'}, p')$, the implemented price is θ^1. Thus, after $\bar{h}$, either price θ^0 or θ^1 is common knowledge.

First we argue that the player who moves at $\bar{h}$ cannot be the buyer. Suppose, to the contrary, that the buyer is the mover. Let $D_{\mathsf{b}} \subset B_{\mathsf{b}}(\bar{p}, t)$ constitute the set of types of the buyer whose choice at $\bar{h}$ make price θ^0 common knowledge. After p' is induced by $t'_{\mathsf{s}} \in B_{\mathsf{b}}(\bar{p}, t) \setminus D_{\mathsf{b}}$, it becomes common knowledge that price θ^1 will be implemented. By EXPIR, type's $t'_{\mathsf{s}} \in B_{\mathsf{b}}(\bar{p}, t) \setminus D_{\mathsf{b}}$ valuation is θ^1. Thus, by choosing her equilibrium action, type $t'_{\mathsf{b}} \in B_{\mathsf{b}}(\bar{p}, t) \setminus D_{\mathsf{b}}$ gets zero payoff. By imitating $t_{\mathsf{b}} \in D_{\mathsf{b}}$ type t'_{b} induces payoff $\theta^1 - \theta^0$ when any t_{s} such that $\bar{p}(t'_{\mathsf{b}}, t_{\mathsf{s}})\bar{p}(t_{\mathsf{b}}, t_{\mathsf{s}}) > 0$ materializes. When such t_{s} does not materialize, t'_{b} still guarantees zero payoff by EXPIR.

Since after $\bar{h}$ it will be common knowledge whether trade with price θ^0 will be implemented, it must be that $D_{\mathsf{b}} = B_{\mathsf{b}}(\bar{p}, B_{\mathsf{s}}(\bar{p}, D_{\mathsf{b}})))$, as otherwise some type $t'_{\mathsf{b}} \in B_{\mathsf{b}}(\bar{p}, B_{\mathsf{s}}(\bar{p}, D_{\mathsf{b}}))) \setminus D_{\mathsf{b}}$ would strictly benefit from imitating type in D_{b}. But this implies that $D_{\mathsf{b}} = BC_{\mathsf{b}}(\bar{p}, t)$ for all $t \in D_{\mathsf{b}}$. Consequently, also $S_{\mathsf{b}}(\bar{p}) \setminus D_{\mathsf{b}} = BC_{\mathsf{b}}(\bar{p}, t)$, for all $t_{\mathsf{b}} \in S_{\mathsf{b}}(\bar{p}) \setminus D_{\mathsf{s}}$. Because of this, it must be common knowledge *at* $\bar{h}$ that types in D_{b} implement price θ^0 and in $S_{\mathsf{b}}(\bar{p}) \setminus D_{\mathsf{s}}$ implement price θ^1. A contradiction.

The player who moves at $\bar{h}$ must be the seller. Let $D_{\mathsf{s}} \subset B_{\mathsf{s}}(\bar{p})$ constitute the set of types of the seller whose choice at $\bar{h}$ make price θ^0 common knowledge. After p' is induced by $t_{\mathsf{s}} \in B_{\mathsf{s}}(\bar{p}) \setminus D_{\mathsf{s}}$, it becomes common knowledge that price θ^1 will be implemented. By choosing her equilibrium action, type $t_{\mathsf{s}} \in D_{\mathsf{s}}$ gets payoff θ^0. By imitating $t'_{\mathsf{s}} \in B_{\mathsf{s}}(\bar{p}) \setminus D_{\mathsf{b}}$, type t_{s} induces payoff θ^1 when any t_{b} such that $\bar{p}(t_{\mathsf{b}}, t'_{\mathsf{s}})\bar{p}(t_{\mathsf{b}}, t_{\mathsf{s}}) > 0$ materializes. When such t_{b} does not materialize, buyer's choice after $\bar{h}$ reveals publicly that the seller has falsely imitated t_{s}. Let p'' reflect the common belief under such out-of-equilibrium information set. Since any $(\sigma', p'') \in \mathsf{G}^*$ is not upset by any (σ^{q}, q), (σ', p'') implements trade with price at least θ^0. Thus, t_{s} guarantees θ^0 payoff, even after imitating t'_{s}.

Since no information is revealed after $\bar{h}$, it must be that $D_{\mathsf{s}} = B_{\mathsf{s}}(\bar{p}, B_{\mathsf{b}}(\bar{p}, D_{\mathsf{s}})))$. Otherwise some type $t_{\mathsf{s}} \in D_{\mathsf{s}}$ would strictly benefit from imitating type in $B_{\mathsf{s}}(\bar{p}, B_{\mathsf{b}}(\bar{p}, D_{\mathsf{s}}))) \setminus D_{\mathsf{s}}$. But this implies that $D_{\mathsf{s}} = BC_{\mathsf{s}}(\bar{p}, t)$ for all $t \in D_{\mathsf{s}}$. Consequently, also $S_{\mathsf{s}}(\bar{p}) \setminus D_{\mathsf{s}} = BC_{\mathsf{s}}(\bar{p}, t)$, for all $t_{\mathsf{s}} \in S_{\mathsf{s}}(\bar{p}) \setminus D_{\mathsf{s}}$. Because of this, it must be common knowledge at $\bar{h}$ that types in $S_{\mathsf{s}}(\bar{p}) \setminus D_{\mathsf{s}}$ have valuation θ^1. Since (σ, p) is not upset at $\bar{h}$, it is also common knowledge that types in $S_\mathsf{S}(\bar{p}) \setminus D_\mathsf{S}$ implement price θ^1. Thus it is common knowledge at $\bar{h}$ which outcome will become implemented, to the contrary of the assumption. ■

The proof of the Lemma 2 is roughly as follows: By Lemma 1, there is a finite set of histories on the equilibrium path where it *becomes* common knowledge that trade with price θ^0 will not be implemented. Thus, one can also identify the longest history $\bar{h}$ after which this becomes common knowledge. After $\bar{h}$ but *not* before it is then common knowledge whether price θ^0 or higher will be implemented. Since the equilibrium is not upset by any simple equilibrium and since any equilibrium meets the EXPIR constraint, it follows that any implemented price is coincides with θ^0 or θ^1. In the former case the least possible valuation of the buyer is θ^0 and in the latter θ^1. The rest of the proof argues that neither the buyer nor seller can commit to reveal which is the case: in the latter case the seller is tempted to mimic the former, and in the former the buyer is tempted to mimic the latter. Thus information cannot be revealed credibly by neither party. This implies that the seller is forced to sell with price θ^0.[22]

Theorem 1 G^* *is a stable set.*

Proof. Since $(\sigma^\mathsf{p}, p) \in \mathsf{G}^*$, for all $p \in \Delta$, external stability is met. For internal stability, note that $(\sigma, p) \in \mathsf{G}^*$ implies by Lemma 2 that it is common knowledge at all decision nodes along the equilibrium path that $\underline{\theta}(B(p,t))$ will be implemented. But then $(\sigma', p') \in \mathsf{G}^*$ implements $\underline{\theta}(B(p,t))$ if $\underline{\theta}(B(p,t)) = \underline{\theta}(B(p',t))$. Thus (σ', p') cannot upset (σ, p). ■

Since $(\sigma^\mathsf{p}, p) \in \mathsf{G}^*$, the only nonobvious part of the proof is internal stability. But this is implies by Lemma 2 which says that any two stable equilibria must be equivalent in terms of outcome expectations, and hence they cannot upset one another. Thus G^* meets both criteria.

No we turn to the question of uniqueness. First we argue that if G is a stable set, then $\mathsf{G} \subseteq \mathsf{G}^*$.

Lemma 3 *Let* G *be a stable set. If* $(\sigma, q) \in \mathsf{G}$*, then* (σ, q) *is not upset by* (σ^p, p)*, for any* p*.*

Proof. Suppose (σ, q) is upset by (σ^p, p). By internal stability, $(\sigma^\mathsf{p}, p) \notin \mathsf{G}^*$. By external stability, there is $(\sigma', p) \in \mathsf{G}^*$ that upsets (σ^p, p). But then (σ', p) also upsets (σ, q), a contradiction. ■

By Lemma 3, $(\sigma, q) \in \mathsf{G}$ implies $(\sigma, q) \in \mathsf{G}^*$. Now we prove the other direction: if G and G^* are stable sets, then $\mathsf{G}^* \subseteq \mathsf{G}$.

Lemma 4 *Let* G *be a stable set. If* (σ, q) *is not upset by* (σ^p, p)*, for any* p*, then* $(\sigma, q) \in \mathsf{G}$*.*

[22] It should be noted that sequentiality of bargaining is crucial for the result. Because of sequentiality, the problem becomes dominance solvable à la Rubinstein (1989).

Proof. Suppose (σ, q) is not upset by (σ^p, p), for any p, but $(\sigma, q) \in \mathsf{G}^* \backslash \mathsf{G}$. By external stability, there is $(\sigma, p) \in \mathsf{G}$ that upsets (σ, q). By Lemma 3, $\mathsf{G} \subseteq \mathsf{G}^*$, and hence $(\sigma, p) \in \mathsf{G}^*$. But then $(\sigma, q) \in \mathsf{G}^*$ and $(\sigma, p) \in \mathsf{G}^*$ violate internal stability, a contradiction ■

By Lemma 4, $(\sigma, q) \in \mathsf{G}^*$ implies $(\sigma, q) \in \mathsf{G}$. Collecting the Lemmata, a uniqueness result follows.

Theorem 2 *If* G *is a stable set, then* $\mathsf{G} = \mathsf{G}^*$.

Thus, the good is always traded with price θ^0 if (1) the seller knows the buyer's reservation valuation, (2) the buyer knows the seller knows (1), (3) the seller knows (2), ..., (k) player i does *not* know (k-1). We become to show that even if the seller is privately informed about the buyer's valuation, she cannot commit not sell the good to the buyer *if* she cannot commit to her offers. Our conclusion is that the seller can take advantage of her private information only if she simultaneously possesses some commitment power.

4 General Finite Case

The previous analysis concentrates on the binary reservation valuations case. The restriction is unappealing but, unfortunately, necessary for G^* to be stable. Nevertheless, we argue that unqueness result remains valid in more general set up.

Assume the set of buyers valuations is $\{\theta^0, ..., \theta^K\}$ such that $0 < \theta^0 < \cdots < \theta^K$. Again, let T be a finite set and let us focus on pure strategy equilibria. First we develop an algorithm that generates a stable set, and proves that the stable set is unique.

Theorem 3 *Stable set exists and is unique.*

Proof. Construct a following algorithm:

$$
\begin{aligned}
\mathsf{G}^0 &= \left\{(\sigma, p) : \text{not upset by any } (\sigma, q) \in \mathsf{A}\right\}, \\
\mathsf{B}^1 &= \left\{(\sigma, p) : \text{upset by some } (\sigma, q) \in \mathsf{G}^0\right\}, \\
\mathsf{G}^1 &= \left\{(\sigma, p) : \text{not upset by any } (\sigma, q) \in \mathsf{A} \setminus \mathsf{B}^1\right\}, \\
&\vdots \\
\mathsf{B}^k &= \left\{(\sigma, p) : \text{upset by some } (\sigma, q) \in \mathsf{G}^{k-1}\right\}, \\
\mathsf{G}^k &= \left\{(\sigma, p) : \text{not upset by any } (\sigma, q) \in \mathsf{A} \setminus \mathsf{B}^k\right\}, \\
&\vdots
\end{aligned}
$$

It is clear that $\mathsf{G}^0 = \emptyset$ (take the class of degenerate distributions $p(t) = 1$ and let σ choose maximal EXPIR outcome). Note that $\mathsf{B}^k \subseteq \mathsf{B}^{k+1}$ and $\mathsf{G}^k \subseteq \mathsf{G}^{k+1}$ for all k. Thus if $\cup_{k=1}^{\infty}[\mathsf{G}^k \cup \mathsf{B}^k] = \mathsf{A}$, then $\overline{\mathsf{G}} = \mathsf{G}^\infty$ exists and is the unique stable set.

Suppose $(\sigma, p) \in \mathsf{A} \setminus \cup_{k=1}^{\infty}(\mathsf{G}^k \cup \mathsf{B}^k)$. If

$$(\sigma, p) \notin \mathsf{G}^k \cup \mathsf{B}^k \text{ for all } k = 0, ...,$$

then, by de Morgan's law,

(σ, p) is upset by some $(\sigma, q) \in [\mathsf{A} \setminus \mathsf{B}^k] \setminus \mathsf{G}^{k-1} = \mathsf{A} \setminus [\mathsf{G}^{k-1} \cup \mathsf{B}^k]$, for all $k = 1, ...$.

Since $[\mathsf{G}^{k-1} \cup \mathsf{B}^k] \subset [\mathsf{G}^k \cup \mathsf{B}^{k+1}]$ for all k, we have

$$(\sigma, p) \text{ is upset by some } (\sigma, q) \in \mathsf{A} \setminus \cup_{k=1}^{\infty}[\mathsf{G}^{k-1} \cup \mathsf{B}^k].$$

Applying the same argument inductively to the upsetting equilibria, we can, then, construct a sequence of equilibria $\{(\sigma^k, p^k)\}_{k=0}^{\infty}$ such that $(\sigma^k, q^k) \in \mathsf{A} \setminus \cup_{k'=1}^{\infty}[\mathsf{G}^{k'-1} \cup \mathsf{B}^{k'}]$ for all k, and such that (σ^{k+1}, p^{k+1}) upsets (σ^k, p^k). Of all such sequences, pick a one that meets the following additional property:

$$(\sigma^{k+1}, p^{k+1}) = (\sigma^{p^k}, p^k) \text{ if } (\sigma^k, p^k) \neq (\sigma^{p^k}, p^k) \in \mathsf{A} \setminus \cup_{k=1}^{\infty}[\mathsf{G}^{k-1} \cup \mathsf{B}^k].$$

Since T is finite and strategies are pure, there is L and $\tilde{p}$ such that $p^l = \tilde{p}$ for all $l > L$, and such that $(\sigma^{\tilde{p}}, \tilde{p}) \in \cup_{k=1}^{\infty}[\mathsf{G}^{k-1} \cup \mathsf{B}^k] = [\cup_{k=0}^{\infty}\mathsf{G}^k] \cup [\cup_{k=1}^{\infty}\mathsf{B}^k]$. Since prior $\tilde{p}$ does not change, it must be that under σ^{l+1} players agree to implement price higher than under σ^l. By EXPIR, $(\sigma^{\tilde{p}}, \tilde{p})$ upsets $(\sigma^l, \tilde{p})$ for all $l > L$. Since $(\sigma^l, \tilde{p}) \notin [\cup_{k=0}^{\infty}\mathsf{G}^k] \cup [\cup_{k=1}^{\infty}\mathsf{B}^k]$, it follows that $(\sigma^{\tilde{p}}, \tilde{p}) \in \cup_{k=1}^{\infty}\mathsf{B}^k$. Thus, since $(\sigma^{\tilde{p}}, \tilde{p})$ is a constant equilibrium, $(\sigma^{\tilde{p}}, \tilde{p})$ is upset by some $(\sigma', \tilde{p}) \in \mathsf{G}^k$, for some k. But then $(\sigma', \tilde{p}) \in \mathsf{G}^k$ also upsets $(\sigma^l, \tilde{p}) \notin \mathsf{B}^{k+1}$, a contradiction. ■

The algorithm starts from those equilibria that are inherently stable, namely those that are not upset by any other equilibria. For example, if $\theta(BC(p,t)) = \{\theta^0\}$, then it is common knowledge that the buyer's reservation valuation is θ^0 and, given EXPIR, (σ^p, p) cannot by upset by any (σ, p). Thus $(\sigma^p, p) \in \mathsf{G}^0$ belongs to the stable set. The inductive step is to identify set B^1 whose all elements are upset by an element in G^0. These cannot belong to the stable set. The second inductive step is identify set G^1 whose all elements are not upset by an element of the complement of B^0. These in turn must belong to the stable set; they can only be upset by an element outside stable set. Then we continue like this, adding on the one hand to the stable set elements that certainly belong there and, on the other, to the "unstable" set those elements that cannot belong to the stable set. Finally, we show that that the union of the evolving stable and unstable set exhausts all the equilibria. This implies that a stable set exists and that it is unique. Of course, in the binary valuations case, the unique stable set is G^*.

In the general case, the structure of the stable set is difficult to describe. However, we can show that any stable equilibrium is efficient.

Theorem 4 *Let* G *be a stable set. If* $(\sigma, q) \in \mathsf{G}$, *then* (σ, q) *implements trade with probability one.*

Proof. If (σ, q) does not implement trade with probability one, then there is stage from which onwards it is common knowledge that $(0,0)$ will be implemented. Let p reflect beliefs at this stage. Then (σ^{p}, p) upsets (σ, q). But by Lemma 3 this leads to contradiction. ■

To see why G^* fails to constitute a stable set in the multiple valuations case, consider the following example.

Example 1 *Let* $\Theta = \{\theta^0, \theta^1, \theta^2\}$ *such that* $\theta^1 = (\theta^0 + \theta^2)/2$, *and let* $T = \Theta \times \{t^0, t^1\}$. *Find* p *such that* $p(\theta^0, t^0) = p(\theta^1, t^1) = 1/3$, $p(\theta^2, t^0) = p(\theta^2, t^1) = 1/6$. *Construct seller's strategy* σ_{s} *where* t^0 *and* t^1 *first offer* θ^1.

- *If the first offer is accepted, then* t^0 *and* t^1 *always offer* θ^1.
- *If the first offer is not accepted, then* t^0 *always offers* θ^0, *and* t^1 *always offers* θ^2.

Construct buyer's strategy σ_{b} *where* θ^0 *always accepts any offer at most* θ^0, *and* θ^1 *and* θ^2 *always accept any offer at most* θ^1. *If the seller observes a zero probability action by the seller, then she believes the deviator is* θ^2.

To see why the constructed strategy constitutes an equilibrium, observe that no one shoot nor infinite deviation can improve any player's position. In particular, neither θ^1 nor θ^2 benefit from downgrading their valuations as this would lead in the first case to the eventual no trade, and in the second to trade with price θ^1. We now argue that existence of equilibrium (σ, p) implies G^* cannot be a stable set. To verify that (σ, p) constitutes an element of G^*, note that if price θ' is to implemented, then $\theta(BC(p', t)) = \theta'$ for any second period equilibrium posterior. Thus (σ, p) is not upset by any $(\sigma^{\mathsf{p}'}, p')$. On the other hand, the same applies to (σ^{p}, p), which trades with price $\theta(BC(p, t)) = \theta^0$, for all $t \in S(p)$. But now (σ^{p}, p) *is* upset by (σ, p) implying that G^* violates internal stability.[23]

Belief Monotonicity However, under certain restriction on priors, we can say more. Next we argue that the outcome of the bargaining procedure is uniquely determined whenever the set-up contains some degree of monotonicity. The following condition captures a monotonicity restriction the is sufficient for uniqueness:

Definition 3 *(Belief monotonicity) If* $\theta(t_{\mathsf{b}}) < \theta(t'_{\mathsf{b}}) < \theta(t''_{\mathsf{b}})$ *and* $p(t_{\mathsf{b}}, t_{\mathsf{s}})p(t''_{\mathsf{b}}, t_{\mathsf{s}}) > 0$, *then* $p(t'_{\mathsf{b}}, t_{\mathsf{s}}) > 0$, *for all* $t_{\mathsf{b}}, t'_{\mathsf{b}}, t''_{\mathsf{b}} \in BC_{\mathsf{b}}(p, t)$, *and for all* $t_{\mathsf{s}} \in BC_{\mathsf{s}}(p, t)$.

[23] Note that equilibrium presented in the Example can also be supported as an equilibrium of the standard one-sided offers bargaining game when the discount factor approaches unity. Thus with more than two valuations, the Feinberg-Skrzypacz result can be obtained with *without* appealing to infinitely mixed strategies that do not implement an outcome within a bound.

That is, if the seller sees two valuations of the buyer possible, and there is a third valuation that cannot be ruled out on the common knowledge grounds that lies in between the two valuations, then the seller conceives also the middle valuation possible. Thus the property guarantees that there are no "holes" in the beliefs of the seller.

Theorem 5 *Let* G *be stable set and let* p *satisfy beliefs monotonicity. Then* $(\sigma, p) \in \mathsf{G}$ *if and only if* σ *always sells the good with price* $BC(p,t)$*.*

Proof. W.l.o.g., assume $BC(p,t) = T$, for all t. Denote a typical element of $\theta^{-1}(\theta^{\mathsf{k}})$ by $t_{\mathsf{b}}^{\mathsf{k}}$, for $k = 0, ..., K$. Since p satisfies belief monotonicity, if there is $t_{\mathsf{b}}^{0}, t_{\mathsf{s}}$ such that $p(t_{\mathsf{b}}^{0}, t_{\mathsf{s}})p(t_{\mathsf{b}}^{\mathsf{k}}, t_{\mathsf{s}}) > 0$ for $k > 2$, then $p(t_{\mathsf{b}}^{1}, t_{\mathsf{s}}) > 0$ for some t_{b}^{1}. Suppose the latter does not hold. Then $p(t_{\mathsf{b}}^{0}, t_{\mathsf{s}})p(t_{\mathsf{b}}^{\mathsf{k}}, t_{\mathsf{s}}) = 0$ for $k > 2$ for all $(t_{\mathsf{b}}^{0}, t_{\mathsf{s}}) \in \theta^{-1}(\theta^{0}) \times T_{\mathsf{s}}$. Then $B_{\mathsf{b}}(p, B_{\mathsf{s}}(p, t_{\mathsf{b}}^{0})) \subseteq \theta^{-1}(\theta^{0})$, for all $t_{\mathsf{b}}^{0} \in \theta^{-1}(\theta^{0})$ and, consequently, $B_{\mathsf{b}}(p, B_{\mathsf{s}}(p, \theta^{-1}(\theta^{0}))) = BC_{\mathsf{b}}(p, t_{\mathsf{b}}^{0})$. Then $T_{\mathsf{b}} = \theta^{-1}(\theta^{0})$.

By EXPIR it follows that t_{b}^{0} trades with price at most θ^{0}. Suppose t_{b}^{0} trades with price lower than θ^{0} under some types. By Corollary 1, this fact will become common knowledge after some finite history. Let q be the posterior at such history. Since (σ, q) is upset by (σ^{q}, q), it follows that $(\sigma^{\mathsf{q}}, q) \notin \mathsf{G}$. Thus there is $(\sigma', q) \in \mathsf{G}$ that upsets (σ^{q}, q). But then (σ', q) also upsets (σ, q) and, a fortiori, (σ, p), a contradiction. Thus t_{b}^{0} trades with price θ^{0}, and the Lemma is established.

Suppose now that $p(t_{\mathsf{b}}^{1}, t_{\mathsf{s}}) > 0$ for some t_{b}^{1}. We claim that type t_{b}^{1} trades with price θ^{0} as well. Suppose not. Fix equilibrium σ, and call the play path played with positive probability in the equilibrium under t_{b}^{0} by $t_{\mathsf{b}}^{0}-$equilibrium path. Let $\sigma_{\mathsf{b}}^{\mathsf{k}}$ be buyer $t_{\mathsf{b}}^{\mathsf{k}}$'s and $\sigma_{\mathsf{s}}^{\mathsf{k}}$ be seller $t_{\mathsf{s}}^{\mathsf{k}}$'s equilibrium strategy. Then $\sigma_{\mathsf{b}}^{\mathsf{k}}$ and $\sigma_{\mathsf{s}}^{\mathsf{k}}$ specify for each history an action. By imitating σ_{b}^{0} on the $t_{\mathsf{b}}^{0}-$equilibrium path, buyer t_{b}^{1} induces payoff $\theta^{1} - \theta^{0}$ when t_{s} materializes. When such t_{s} does not materialize, t_{b}^{1} still guarantees zero payoff, by EXPIR. Thus t_{b}^{1}'s equilibrium strategy must generate him positive payoff.

If there is a history on the $t_{\mathsf{b}}^{0}-$equilibrium path at which σ_{b}^{0} and σ_{b}^{1} do not coincide, then it becomes common knowledge that buyer's type is at least θ^{1}. Let p' denote the posterior belief at such history. Suppose (σ, p') is upset by $(\sigma^{\mathsf{p}'}, p')$. By internal stability, $(\sigma^{\mathsf{p}'}, p') \notin \mathsf{G}^{*}$. By external stability, there is $(\sigma', p') \in \mathsf{G}^{*}$ that upsets $(\sigma^{\mathsf{p}'}, p')$. But then (σ', p') also upsets (σ, p), a contradiction. Since (σ, p') is not upset by $(\sigma^{\mathsf{p}'}, p')$, the implemented price is $\underline{\theta}(BC(p', t)) \geq \theta^{1}$, for all $t \in S(p')$. But this implies that t_{b}^{1}'s equilibrium strategy generates him zero payoff. Thus σ_{b}^{1} coincides with σ_{b}^{0} on the $t_{\mathsf{b}}^{0}-$equilibrium path. Thus if σ_{b}^{1} and σ_{b}^{0} do not coincide, then it must be out of the $t_{\mathsf{b}}^{0}-$equilibrium path. This implies that the player who moves the game out of the $t_{\mathsf{b}}^{0}-$equilibrium path is the seller.

We now argue that this is not possible. Since p satisfies belief monotonicity, there is t_{s}^{0} such that $p(t_{\mathsf{b}}^{0}, t_{\mathsf{s}}^{0})p(t_{\mathsf{b}}^{1}, t_{\mathsf{s}}^{0}) > 0$. If there is t_{s} that moves the game out of the $t_{\mathsf{b}}^{0}-$equilibrium path when t_{b}^{1} is present but not when t_{b}^{0} is present, it must be the case that t_{s} can condition his action on the realization of buyer's type in $\{t_{\mathsf{b}}^{0}, t_{\mathsf{b}}^{1}\}$. This is possible only if $t_{\mathsf{b}}^{0} \notin B_{\mathsf{b}}(p, t_{\mathsf{s}})$ and $t_{\mathsf{b}}^{1} \in B_{\mathsf{b}}(p, t_{\mathsf{s}})$. Thus $t_{\mathsf{s}}^{0} \neq t_{\mathsf{s}}$. By imitating t_{s}'s strategy on the $t_{\mathsf{b}}^{0}-$equilibrium path, seller t_{s}^{0} induces payoff θ^1 when t_b^1 materializes. When t_b^0 materializes, σ_b^1 and σ_b^0 do not coincide at the out of t_b^0-equilibrium path, and hence it is publicly revealed that the seller has falsely imitated t_s. Let p'' reflect the common belief under such out-of-equilibrium information set. As above, any $(\sigma', p'') \in G$ implements trade with price at least θ^0. Thus, t_s^0 guarantees payoff θ^0, even when t_b^0 materializes. This implies that the player who moves the game out of the t_b^0-equilibrium path cannot be the seller. Thus t_b^1 trades with price θ^0.

Finally, use the argument inductively to prove that t_b^k trades with price θ^{k-1}, if $t_b^0, ..., t_b^{k-1}$ trade with price θ^0. Then any t_b^k, $k = 0, ..., K$ trades with price θ^0. ■

The simplest example is the standard one where the seller does not have any private information, that is T_s is single valued. Then, under the common prior assumption, $p(t_b, t_s) > 0$ for all $(t_b, t_s) \in T$ and belief monotonicity is automatically met. Thus, by Theorem 5, we get the standard Coase conjecture outcome:[24]

Corollary 2 *If seller's beliefs are common knowledge, then trade takes place with price θ^0 in any equilibrium belonging to the stable set.*

Another scenario where monotonicity automatically binds is where it is common knowledge that the seller receives a private and noisy signal about buyer's valuation. This structure is familiar from the Global Games literature.

Suppose there are nonnegative numbers l and h such that $l + h > 0$, and let the belief structure p satisfy

$$\begin{aligned} T &= \{\theta^0, ..., \theta^K\} \times \{t^0, ..., t^K\}, \qquad (1) \\ \theta(\theta^k) &= \theta^k \text{ and } \prod_{k'=\max\{k-l,0\}}^{\min\{k+h,K\}} p(\theta^{k'}, t^k) > 0, \text{ for all } k = 0, ..., K. \end{aligned}$$

That is, the seller's only knows his payoff relevant type and the buyer privately receives a noisy signal about buyer's valuation. Structure of the signal common knowledge. By assumption, this structure satisfies belief monotonicity: Take $\theta < \theta' < \theta''$ and t_s. If $p(\theta, t_s)p(\theta'', t_s) > 0$, then by (1) also $p(\theta, t_s)p(\theta', t_s)p(\theta'', t_s) > 0$. Thus $p(\theta', t_s) > 0$.

More interesting is that as the buyer only gets to know his valuation, and he updates his beliefs over the seller's signal in an imperfect manner. By construction, therefore,

$$\begin{aligned} \text{either } \prod_{k=0}^{K-1} p(\theta^{k+1}, t^k)p(\theta^k, t^k) &> 0, \\ \text{or } \prod_{k=1}^{K} p(\theta^{k-1}, t^k)p(\theta^k, t^k) &> 0. \end{aligned}$$

[24] Excepts that there may be costless delays.

Thus, $BC(p,t) = T$ and hence $\theta^0 = \underline{\theta}(BC(p,t))$ for all $t \in T$. Since this structure satisfies belief monotonicity, we have by Theorem 5 that trade always takes place with price θ^0.

Corollary 3 *If the buyer only knows his valuation and the seller receives noisy private signal about the buyer's valuation, as defined in 1, then trade takes place with price θ^0 in any equilibrium equilibrium belonging to the stable set.*

Thus, even if the seller *knows* the buyer has valuation of at least $\theta^k > \theta^0$, she is forced to sell with price above θ^0.

5 Equilibrium Bounded Bargaining Schemes

An obvious conceptual problem with game form g is that it never actually implements trade. One could argue that reasonable bargaining theory should not allow bargaining to continue without a bound. In this section we provide further justification for the construction of g by showing that equilibrium outcome functions under g and only those survive if one focuses on *equilibrium bounded bargaining procedures* that are robust against bound extensions. That is, those bargaining procedures that in equilibrium are bounded to end in finite time, but which do not crucially depend on the commitment aid provided by the bound.

By Lemma 1 we know that when players strategies are common knowledge it will sooner or later be common knowledge which outcome is eventually implemented, even though implementation does not ever take place. Could not, then, players agree on executing the trade immediately when the outcome is common knowledge rather than at the infinity?

Let us now formalize this idea. Assume that the seller and the buyer want to implement trade once the outcome becomes common knowledge. Thus, before the game (g,p) starts, they have to *simultaneously* agree on the stage of the game when they implement the trade conditional on realized messages, *and* on their communication strategies (through offers and acceptances/rejections) up to the implementation stage. Agreed communication must constitute an equilibrium. Thus out-of-equilibrium consequence must be defined as well. When players observe an out-of-equilibrium move, they agree on a *new* strategy-implementation plan given their (common) out-of-equilibrium beliefs. And after an out-of-equilibrium move from this, they agree on a still new strategy-implementation plan given their beliefs, and so forth. Thus the system of subgame equilibria conditional on prior beliefs has to be designed such a way that (i) no player ever deviates, and (ii) with any prior an outcome is implemented in finite time. Of course, principles (i) and (ii) has to be applied to deviation from the equilibrium path following an out-of-equilibrium move, and so forth.

To tailor such *equilibrium bounded bargaining scheme,* take bargaining strategy σ of extensive form g. Starting from history h, denote the positive probability play-path under σ and prior p by (σ,p,h). Let p^0 be the initial beliefs and h^0 the initial history of g. Construct game form g^B inductively according to the following procedure:

Step 0. Take an even integer K^0, a *bound,* and choose $(p, h, K) = (p^0, h^0, K^0)$. Go to Step 1.

Step 1. Truncate g immediately after the K^{th} stage on the play-path (σ, p, h) by implementing (a, m) if it is the most recent offer and it been accepted. Otherwise implement $(0, 0)$.

Step 2. After deviation from the play-path (σ, p, h) to history h' at or before the K^{th} stage, choose new prior p' and even integer K'. Choose $(p, h, K) = (p', h', K')$ and go to step 1.

Thus, given σ, we construct game tree g^{B} by imposing a uniform finite bound on g on-the-play-path (σ, p^0, h^0), on-the-play-path of (σ, p', h') where where p' is any off-the-play-path belief associated to history h' that follows a deviation from (σ, p^0, h^0), and so forth. Continuing inductively, we simultaneously come up with a system of beliefs and a system of finite bounds.

The system of bounds defines a game tree that is a truncated version of g. Any finite terminal history of the tree associates to an outcome. By letting any *infinite* sequence of actions implement outcome $(0, 0)$,[25] we come up with a proper game form which we call g^{B}. The corresponding truncated version of strategy σ, defined on g^{B}, is denoted by σ^{B}. Thus σ and σ^{B} coincide on g^{B}.

Given prior p, call triple $(g^{\mathsf{B}}, \sigma^{\mathsf{B}}, p)$ as a *bargaining scheme.*[26] We say that this bargaining scheme is *equilibrium bounded* if (σ^{B}, p) also constitutes a sequential equilibrium of g^{B}. If bargaining scheme $(g^{\mathsf{B}}, \sigma^{\mathsf{B}}, p)$ is equilibrium bounded, then equilibrium (σ^{B}, p) of game g^{B} implements an outcome in finite time. Abusing the language, call K as the *initial bound* and σ_{K} as the *initial equilibrium* of $(g^{\mathsf{B}}, \sigma^{\mathsf{B}}, p)$.

The concept of equilibrium bounded bargaining scheme allows us to evaluate the importance of the commitment assumption, that is, existence of bounds. If the equilibrium (σ^{B}, p) is not dependent on the commitment power provided by bounds K, then one should be able to *extend* them without affecting the equilibrium behavior. Our goal is to characterize such bounded equilibria that are robust against bound extensions.

Suppose $(g^{\mathsf{B}}, \sigma^{\mathsf{B}}, p)$ is an equilibrium bounded bargaining scheme. Take $L \in \{0, 2, 4, ...\}$. We say that bargaining scheme (g', σ', p) is an L*-extension* of $(g^{\mathsf{B}}, \sigma^{\mathsf{B}}, p)$ if (i) g' is derived from g^{B} by extending *all* bounds for L stages, (ii) σ' and σ^{B} coincide within g^{B}, and (iii) on the play-path from any bound, say K, onwards, σ repeats the actions taken by σ^{B} at stages $K - 1, K$. Thus (g', σ', p) and $(g^{\mathsf{B}}, \sigma^{\mathsf{B}}, p)$ coincide whenever the former is defined, and they induce the same outcomes. If $(g^{\mathsf{B}}, \sigma^{\mathsf{B}}, p)$ can be $L-$extended, then extending the bound does not affect the equilibrium in a meaningful way.

Definition 4 *Equilibrium bounded bargaining scheme* $(g^{\mathsf{B}}, \sigma^{\mathsf{B}}, p)$ *is* robust *against bound extensions if there is an equilibrium bounded bargaining scheme that is an* $L-$*extension of* $(g^{\mathsf{B}}, \sigma^{\mathsf{B}}, p)$, *for any even integer* L.

[25] Infinite strategies never materialize in equilibrium.

[26] As before, we drop disequilibrium beliefs from the description of an equilibrium bounded scheme. Implicitly, such beliefs are correctly defined.

That is, if $(g^{\mathsf{B}}, \sigma^{\mathsf{B}}, p)$ is robust against bound extensions, then the length of a bounded the one-sided-offers bargaining game can be extended without affecting players behavior nor the resulting outcomes. Robustness against bound extensions implicates ability to *commit* to a particular behavior even if physical commitment devices are removed.

Denote by $f(\cdot : g, \sigma) : S(p) \to A \times \mathsf{R}$ the outcome function induced by game g and strategy σ, given prior p.

Definition 5 *Triple (g, σ, p), where σ constitutes an equilibrium of g under p, is* outcome equivalent *to an equilibrium bounded bargaining scheme $(g^{\mathsf{B}}, \sigma^{\mathsf{B}}, p)$ if $f(\cdot : g^{\mathsf{B}}, \sigma^{\mathsf{B}}) = f(\cdot : g, \sigma)$. It is an* extension equivalent *if it meets the criteria of $\infty-$extension of $(g^{\mathsf{B}}, \sigma^{\mathsf{B}}, p)$.*

If (g, σ, p) and $(g^{\mathsf{B}}, \sigma^{\mathsf{B}}, p)$ are outcome equivalent, then they induce the same outcomes given the data of types. If (g, σ, p) and $(g^{\mathsf{B}}, \sigma^{\mathsf{B}}, p)$ are extension equivalent, then they are not only outcome equivalent but the strategies also agree on the truncated game g^{B}, and no information is revealed within (g, σ, p) beyond the bounds imposed by g^{B}.

Before establishing the theorems, call equilibrium (σ, p) of g *regular* if players agree on outcome $(0, 0)$ rather than never agree on any other outcome whenever they end up implementing outcome $(0, 0)$. The assumption of regularity is strategically immaterial. For any equilibrium (σ', p) there exists a regular equilibrium (σ, p) that coincides with (σ', p) before stage, say, K when $(0, 0)$ becomes common knowledge (by Lemma 1, such K exists), and afterwards agree on $(0, 0)$. Agreeing on $(0, 0)$ can be supported as equilibrium from K onwards by punishing any deviant with the same punishment as in the original strategy.

Theorem 6 *The set of outcome functions induced by equilibria σ of g under p is equivalent with the set of outcome functions induced by equilibrium bounded bargaining schemes $(g^{\mathsf{B}}, \sigma^{\mathsf{B}}, p)$ that are robust against bound extensions.*

Proof. "Only if": Suppose σ constitutes an equilibrium of g under p. W.l.o.g. assume that σ is regular. First we derive an equilibrium bounded bargaining scheme $(g^{\mathsf{B}}, \sigma^{\mathsf{B}}, p)$ that is outcome equivalent to (g, σ, p). Given initial history h and equilibrium (σ, p), identify an even stage K from which onwards the implemented outcome is common knowledge under (σ, p, h). Similarly, to any off-the-equilibrium history h' and beliefs p' reachable from (σ, p), identify stage K' from which onwards the implemented outcome is common knowledge under (σ, p', h'). By Lemma 1, such stages exist for any prior p'. Working inductively we come up with strategy σ^{B} and, simultaneously, game form g^{B}.

Since σ is regular, $(g^{\mathsf{B}}, \sigma^{\mathsf{B}}, p)$ and (g, σ, p) are outcome equivalent. Thus it suffices to show that $(g^{\mathsf{B}}, \sigma^{\mathsf{B}}, p)$ is robust against bound extensions. Construct an $L-$extension (g', σ', p) of $(g^{\mathsf{B}}, \sigma^{\mathsf{B}}, p)$, for any $L = 0, 2, 4, \ldots$. By construction, no new information is revealed from the bounds onwards under (g, σ, p). Hence, between any old and new bound in the equilibrium path, one shoot deviation from (g', σ', p) is not profitable if deviation from $(g^{\mathsf{B}}, \sigma^{\mathsf{B}}, p)$ is not profitable. Thus, since $(g^{\mathsf{B}}, \sigma^{\mathsf{B}}, p)$ constitutes an equilibrium bounded bargaining scheme, so does (g', σ', p).

"If": Suppose that $(g^{\mathsf{B}}, \sigma^{\mathsf{B}}, p)$ is a robust equilibrium bounded bargaining scheme. Construct strategy σ that coincides with σ^{B} on g^{B}, and from any bound, say K, onwards, σ repeats the actions taken by σ^{B} at stages $K-1, K$. By construction, (g, σ, p) is outcome equivalent to any equilibrium bounded bargaining scheme that is an extension of $(g^{\mathsf{B}}, \sigma^{\mathsf{B}}, p)$. We need to show that σ constitutes an equilibrium of g under p. Suppose not. There are two cases:

(i) A player deviates from σ after K. Suppose a deviation takes place at stage $K' > K$. Since (g, σ, p) is equivalent to equilibrium bounded bargaining scheme (g', σ', p) that is a $(K'-K)-$extension of $(g^{\mathsf{B}}, \sigma^{\mathsf{B}}, p)$, the expected payoffs from deviation are the same as from deviating from (g', σ', p). This contradicts the assumption that (g', σ', p) is an equilibrium bounded bargaining scheme.

(ii) A player deviates from σ at or before K. The expected payoffs from deviating from σ at or before K are the same as from deviating from $(g^{\mathsf{B}}, \sigma^{\mathsf{B}}, p)$ at or before K. This contradicts the assumption that $(g^{\mathsf{B}}, \sigma^{\mathsf{B}}, p)$ is an equilibrium bounded bargaining scheme. ■

That is, looking at equilibria of g under p is outcome equivalent to looking at the robust equilibrium bounded bargaining schemes under p. Thus, game (g, p) can be seen as a refinement vehicle: focusing on equilibria of (g, p) forces us to drop out those equilibrium bounded bargaining schemes that are crucially dependent on the bound. Therefore, the infinite version of the game refines all equilibria of a finite game that are not robust against increasing the number of stages. Under such interpretation, we are focusing on situations where players agree on a scheme to exchange information until a stage where they can commit not change information any more, and then they execute the trade.

Note that any equilibrium σ of game (g, p) that *always induces trade* is also regular. Thus by the "only if" part of the proof of Theorem 6, any equilibrium σ that always induces trade satisfies the property that there is an outcome equivalent equilibrium σ^{B} of a robust equilibrium bounded bargaining scheme $(g^{\mathsf{B}}, \sigma^{\mathsf{B}}, p)$ that has the additional property that σ coincides with σ^{B} on g^{B}, and agrees on the implemented outcome beyond the bounds imposed by g^{B}.

Theorem 7 *If equilibrium σ of game (g, p) induces trade with probability one, then (g, σ, p) is extension equivalent to an equilibrium bounded bargaining scheme $(g^{\mathsf{B}}, \sigma^{\mathsf{B}}, p)$ that is robust against bound extensions.*

Proof. Construct an equilibrium bounded bargaining scheme $(g^{\mathsf{B}}, \sigma^{\mathsf{B}}, p)$ from (g, σ, p) as in the only if -part of the proof of Theorem 6. By construction, (g, σ, p) is extension equivalent to an equilibrium bounded bargaining scheme $(g^{\mathsf{B}}, \sigma^{\mathsf{B}}, p)$. Since no new information is revealed from the bounds onwards under (g, σ, p), we have that $(g^{\mathsf{B}}, \sigma^{\mathsf{B}}, p)$ is robust against bound extensions. ■

That is, if σ always induces trade, then the truncated version of σ constitutes an equilibrium of a bounded game that implements trade in finite time, just as the conjectured in the beginning of this section. Since any equilibria in the stable set always induces trade, it follows that stable equilibria can also be implemented by equilibrium bounded schemes that are robust against bound extensions.

Let A^{R} be the subgraph of A that consists of equilibria of g that are extension equivalent to robust equilibrium bounded bargaining schemes. Since the stable set defined w.r.t. A is contained in A^{R}, it follows that this set also meets the internal and external stabilities w.r.t. A^{R}. Moreover, since equilibrium σ^{p} is regular, we have $(\sigma^{\mathsf{p}}, p) \in \mathsf{A}^{\mathsf{R}}$ for all p. By this, the algorithm used in the proof of Theorem 3 can be used here, too. Combining these, we get the following important corollary.

Corollary 4 *The unique stable set defined on* A *is the unique stable set defined on* A^{R}.

A Note on Mixed Strategies Let us now allow players use mixed strategies in game (g, p). By definition, equilibrium bounded bargaining schemes may use randomized strategies only if they randomize on finitely many periods. Thus if one takes as the starting point set A^{R}, then infinite randomization is automatically ruled out. This implies that Lemma 1 and, *a fortiori*, Theorems 2 and 3 are in force even if finite mixing is allowed.

Feinberg and Skrzypacz (2002) build up equilibria in the binary valuations case that generate the seller profit above θ^0 even if it is not common knowledge that seller's valuation is θ^1. Their equilibrium uses infinitely mixed strategies. By Theorem 2, such equilibria cannot be generated in our setting, where only finite mixing is permitted. Thus, our theory is sensitive to infinite mixing.

Infinite mixing is impossible to combine with players becoming to agree on an implemented outcome in finite time. Because of this, equilibria using infinite mixing cannot be replicated by equilibrium bounded bargaining schemes. One either has to allow the possibility that the outcome is not implemented within a bound, or one has to assume that infinite mixing does not take place. Which is more reasonable?

To fix the ideas, let us assume that a reasonable theory does not allow boundless bargaining. This restriction not only removes infinitely mixed strategies, but also questions the validity of game form g. But the aim of Theorems 6 and Corollary 4 is to justify the use of g as a *shorthand* description of robust equilibrium bounded bargaining schemes rather than expression of a reasonable bargaining situation *per se*. Since any equilibrium bounded bargaining scheme implements all outcomes in finite time, the restriction above is avoided by the equilibria of game form g. To conclude, the restriction to not allow bargaining to go on forever permits game form g but does not permit infinite mixing.

6 Conclusions

The key contribution of this paper is methodological. We have modeled commitment-free bargaining reduces to cheap talk where offers and acceptances convey information only in equilibrium, and trade may be implemented only when no further information is communicated. Thus, even without external commitment devices one can design a bargaining game in such a way that (i) the equilibrium play of the game always implemented trade within certain time bound, (ii) the equilibrium play is unaffected by any extension of the bound. A pair of game and its equilibrium satisfying these properties is called an *equilibrium bounded bargaining scheme* that is *robust against bound extensions.*

In more concrete terms, one may think the process as one where the players first agree on a bounded communication protocol, i.e. who says what and when, and on the implementation procedure as a function of the exchanged messages. Since players lack any commitment power, the communication protocol has to be designed in such a way that no-one wants to continue communication once the bound is reached. Since players are rational, the planned communication must consitute an equilibrium within the protocol. After agreeing on the rules and the equilibrium, they play the game. Thus players *agree* to implement an outcome in equilibrium rather than are bounded to by external forces.

We show that robust equilibrium bounded bargaining schemes can be characterized in a parsimonious way by bargaining game g that never ends but associates players with payoffs (a, m) if players agree on (a, m) starting from some stage. Thus focusing on equilibria of g is without loss of generality.

Like many infinite horizon games, g hosts many equilibria. We argue that the selection criterion that allows the seller to possess a leading role in equilibrium selection can be compactly represented by a vNM stable set once one imposes a dynamic consistency criterion on the selection rule. With finite type space, a stable set is unique, and stable equilibria in the stable set have in the binary valuations case a very simple and intuitive characterization: stable equilibrium always sells the good to buyer with the least possible valuation in the *belief close set* of buyer's reservation valuations. In the many valuations case the same is true when beliefs satisfy certain (weak) monotonicity condition. Thus, when seller's beliefs are common knowledge, the result coincides with the standard result in the Gap case. However, it questions the validity of an argument that with private information about buyers valuation the seller can counterbalance the situation in favor of her.

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