Bimatrix games in general equilibrium

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Abstract

We embed bimatrix games in a random matching model where the players play the game only after they have been paired. Before playing the game they may make side payments whose magnitude depends on how many agents meet. It turns out that the overall equilibrium features the Pareto-efficient Nash equilibrium of the bimatrix game.

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1. Introduction

A two-player bimatrix game is a very partial equilibrium type situation. The players are fixed, their pay-offs are fixed, and in a sense they are forced to play the game, as they do not have any explicitly modelled outside option. Of course, one can think that there is some implicit outside option and the utility from that is less than any of the payoffs in the game.

There are typically a multiplicity of equilibria in these games, and the literature on refinements presents arguments that favour some of these but there is no general understanding about which equilibria one can expect the players to co-ordinate to. To get some more insight into this one can allow the players more choices. This amounts to endogenising the game's payoffs to some extent. An example of this line of reasoning is Jackson and Wilkie (2002) where the players may, prior to playing the game, make commitments for side payments that depend on the chosen actions. Jackson and Wilkie show that this possibility does change the way the players play the game but it does not guarantee efficiency nor result in a unique equilibrium; quite to the contrary efficient equilibria may cease to exist altogether.

While Jackson and Wilkie retain the partial equilibrium framework we approach the situation from another angle. We embed the two-person games in a general equilibrium framework. There are many potential players who meet randomly, and a player gets to play the game once he has found a partner. There is also a possibility of sidepayments which have a dual role. First, they determine who gets to play with whom, and secondly they co-ordinate the players to a certain equilibrium of the bimatrix game.

It turns out that competition, i.e. the threat that some other player gets to play with one's potential partner, guarantees that generically the (general) equilibrium is such that in the bimatrix game the players play the efficient equilibrium. We also get an endogenous outside option for each player since they can always leave their current partner and start looking for a new one.

We think that the usefulness of embedding partial equilibrium models, be they games or some other situation, in a general equilibrium framework is strongly conveyed in our analysis. It shows that competitive price formation may get rid of the multiplicity of equilibria easily. It is also a useful way of checking the robustness of partial equilibrium models. Similar technique is used by Blouin (2003), Janssen and Karamychev (2002) and Inderst and Müller (2002).

Price formation, i.e. the sidepayments, is in an important role in our approach, and the meeting technology we use makes it particularly simple. We think that the column players contact the row players randomly. This is the so-called urn-ball model, and it is very convenient because it makes multiple meetings possible. A particular row player may find that no column players contacted him. In this case he just waits till next period to meet new column players. He may also find that exactly one column player contacted him. In this case we postulate that the column player makes a take-it-or-leave-it offer to the row player. This means that the column player proposes a sum of money (positive or negative) to the row player who either accepts or rejects. In the former case they proceed to play the bimatrix game. In the latter case they separate and look for new partners next period. The row player may also find that two or more column players contacted him. In this case the column players engage in an auction for the right to play the bimatrix game with the row player. A justification for these price formation rules can be found in Halko, Kultti and Niinimäki (2002).

The rest of this article is organised as follows. In section 2 we present the model, and in section 3 the analysis. In section 4 we show that generically the efficient equilibrium is the only reasonable one in any *nxm* bimatrix game. In section 5 we present conclusions.

2. The Model

Let us first fix notation and terminology. We study an infinite horizon discrete time setting with R row players and C column players. The players have to form pairs to play a bimatrix game, and they meet randomly. The meetings are governed by an urn-ball meeting process where the row players are urns and the column players are balls. The number of column players a row player meets is a binomial random variable, and the probability that a particular row player meets exactly $k \le C$ column players is $\binom{C}{k}\left(\frac{1}{R}\right)^k\left(\frac{R-1}{R}\right)^{C-k}$. As binomials are awkward to deal with we assume that both R and C are large so that we can approximate the binomial by a Poisson distribution with rate $q = \frac{C}{R}$. Then the probability that a row player meets exactly k column players is given by $e^{-q}\frac{q^k}{k!}$. All the agents discount future by the same factor $d \in (0,1)$. We assume that agents that play the bimatrix game exit the economy and are replaced by identical agents so that the number of row and column players remains the same, and then we focus on a steady state equilibrium.

Upon meeting, but before playing the bimatrix game the agents may agree on monetary transfers, and their strategic situations are as follows: If a column player meets a row player, and no other column players meet this row player, the column player may propose any side payment to the row player. The row player either accepts or rejects. Rejection causes the pair to separate. If several column players meet a row player the column players can make any bids, the highest bidder wins and pays the second highest bid. Again the row player either accepts or rejects, and rejection dissolves the match. If there are several equally good offers the row player selects from them randomly.

In case the row player accepts an offer or a bid of a column player the column player first make the sidepayment, and then plays the bimatrix game with the row player.

3. The Analysis

Consider an n x m bimatrix game and two of its equilibria with payoffs (a,b) and (A,B) where the row player’s payoff is given by the first co-ordinate. Assume that the players play an equilibrium that results in payoffs (a,b) in a pairwise meeting where the side payment is p, and in payoffs (A,B) in a multiple meeting where the side payment is q. The system of equations that determines their expected utilities is

$$\begin{cases} V_r = d[e^{-q}V_r + e^{-q}q \cdot V_r + (1-e^{-q} - e^{-q}q)(A+q)] \\ V_c = d[e^{-q}(b-p) + (1-e^{-q})V_c)] \\ a + p = V_r \\ B - q = V_c \end{cases} \qquad (1)$$

The expected utilities are evaluated at the end of a period, and so everything on the right hand side is discounted. The first equation is the row player's expected utility. If he meets no-one he gets his expected utility as he just waits till next period to meet column players. If he meets exactly one column player he gets his reservation value, i.e. his expected utility, since the column player's optimal offer is such that the row player is indifferent between accepting and rejecting. If the row player meets many column players he gets side payment q as well as the payoff A from the bimatrix game.

If the column player is the only one to meet the row player he gets payoff a from the bimatrix game, and he also has to pay his offer p. If two or more column players meet the row player an auction ensues, and in equilibrium they get their reservation utility. The third equation states that the optimal offer of a column player is such that the row player is indifferent between accepting and rejecting. The fourth equation states that in an auction the side payment goes up until the column players are indifferent between paying the side payment q and waiting till next period.

The resulting utilities and prices are then

$$\begin{cases} V_r = \dfrac{d(1-e^{-q} - qe^{-q})}{(1-d)(1-dqe^{-q})}\left[(A+B)(1-d+de^{-q}) - (a+b)de^{-q}\right] \\ V_c = \dfrac{de^{-q}}{(1-d)(1-dqe^{-q})}\left[-(A+B)d(1-e^{-q} - qe^{-q}) + (a+b)(1-de^{-q} - dqe^{-q})\right] \\ p = \dfrac{1}{(1-d)(1-dqe^{-q})}\begin{Bmatrix} (A+B)d(1-e^{-q} - qe^{-q})(1-d+de^{-q}) \\ -a(1-d+de^{-q})(1-de^{-q} - dqe^{-q}) \\ -bd^2e^{-q}(1-e^{-q} - qe^{-q}) \end{Bmatrix} \\ q = \dfrac{1}{(1-d)(1-dqe^{-q})}\begin{Bmatrix} Ad^2e^{-q}(1-e^{-q} - qe^{-q}) \\ +B(1-d+de^{-q})(1-de^{-q} - dqe^{-q}) \\ -(a+b)de^{-q}(1-de^{-q} - dqe^{-q}) \end{Bmatrix} \end{cases} \qquad (2)$$

It is remarkable that the players' utilities depend only on the sum of the payoffs of the equilibria of the bimatrix game, not on the row and column players' individual payoffs. In the general equilibrium framework only the aggregate payoff matters as it is possible to transfer utility. This also hints why the only reasonable equilibrium is such that the Pareto-efficient equilibrium of the bimatrix game is played.

For equations (2) to depict an equilibrium it must be the case that $V_r \geq 0$ and $V_c \geq 0$. These inequalities are equivalent to

$$\frac{1-d+de^{-q}}{de^{-q}} \geq \frac{a+b}{A+B} \geq \frac{d\left(1-e^{-q}-qe^{-q}\right)}{1-de^{-q}-dqe^{-q}} \tag{3}$$

When the discount factor approaches unity, which is the interesting case, the upper and lower bounds for $\frac{a+b}{A+B}$ also approach unity. This means that the sum of the payoffs in the equilibria of the bimatrix game has to be practically the same if the players are to play two different equilibria. Generically there are no such equilibria, and for this reason we focus only on such equilibria of the whole game where a particular equilibrium of the bimatrix game is played.[1] Substituting $A=a$ and $B=b$ in (3) the players' expected utilities in such an equilibrium are

$$\begin{cases} V_r = \dfrac{d(e^q-1-q)}{e^q-dq}(a+b) \\ V_c = \dfrac{d}{e^q-dq}(a+b) \end{cases} \tag{4}$$

4. Eliminating Inefficient Equilibria

We see that any of the equilibria, pure or mixed, of a bimatrix game can be supported in a general equilibrium framework. Some of them are, however, not very good predictions because they are easy to break. Breaking an equilibrium is based on a forward induction argument which is nothing new. Unlike in a partial equilibrium setting burning money or offering side payments (p. 113 Osborne and Rubinstein, 1994), in our general equilibrium setting it is very natural. This in mind we offer the following definition of an equilibrium.

Definition. The equilibrium consists of prices p and q and a bimatrix game Nash equilibrium (s_1, s_2) where p is the side payment offered by a column player when he is the only one to meet a row player, and q is the winning bid in an auction that takes place when two or more column players meet a row player. Further, there is no transfer $\tilde{p}$ that a column player could offer instead of p such that a row player could unambiguously interpret it as a suggestion to play another equilibrium that would make both parties better-off than (s_1, s_2).

The definition above generically isolates the Pareto-efficient equilibrium of the bimatrix game because, if some other equilibrium played there exists a transfer $\tilde{p}$ such the following condition is satisfied: There exists and equilibrium of the bimatrix game such that if the row player and the column player co-ordinate to it both fare better, given $\tilde{p}$, than in the current equilibrium.

[1] It is possible that the equilibrium to be played in the bimatrix game depends on the number of column players that meet a row player. This is also not feasible for large enough discount factors.

Proposition 1.
In the generic nxm game, if the equilibrium to be played is not Pareto efficient, there always exists a side payment $\tilde{p}$ so that given $\tilde{p}$ the only reasonable equilibrium is the efficient one.

Proof. Assume that the players play an equilibrium with payoffs *(c,d)* while there exists an equilibrium with payoffs *(a,b)* such that $a+b>c+d$. We prove the proposition by breaking the inefficient equilibrium in the easiest case possible, i.e. when one column player meets one row player and the column player makes the offer of $\tilde{p}$ with the intention to move to an equilibrium with payoffs (a,b). We have two constraints for the side payment offer, namely the individual rationality constraint and the unambiguousness constraint. Now the IR constraints for the column player and the row player are

$$\begin{cases} b-\tilde{p}>d-p \\ a+\tilde{p}>c+p=V_r \end{cases}. \tag{5}$$

From the upper equation we get

$$b-d+\frac{d(1-e^{-q}-qe^{-q})}{1-dqe^{-q}}d-\frac{1-d+de^{-q}}{1-dqe^{-q}}c>\tilde{p},$$

which yields

$$b-\frac{1-d+de^{-q}}{1-dqe^{-q}}(c+d)>\tilde{p}. \tag{6}$$

From the lower equation in (5) we get

$$\tilde{p}>V_r-a=\frac{d(1-e^{-q}-qe^{-q})}{1-dqe^{-q}}(c+d)-a. \tag{7}$$

Combining (6) and (7) gives

$$b-\frac{1-d+de^{-q}}{1-dqe^{-q}}(c+d)>\frac{d(1-e^{-q}-qe^{-q})}{1-dqe^{-q}}(c+d)-a \tag{8}$$

which amounts to

$$a+b>c+d \qquad (9).$$

But (9) holds by assumption so the IR constraint is satisfied. The unambiguousness constraint requires that in any other equilibrium, one player has a lower utility with the side payment $\tilde{p}$ than in the equilibrium with payoffs *(c,d)*. We assume that there is another equilibrium with payoffs (e,f) for which it holds that a+b>e+f. Then we must show that for $\tilde{p}$ satisfying (6) and (7) one of the following is also satisfied

$$f - \tilde{p} < d - p \tag{10}$$

or

$$e + \tilde{p} < V_r\,. \tag{11}$$

From (10) we get

$$\tilde{p} > -d + p + f = -d + \frac{d(1 - e^{-q} - qe^{-q})}{1 - dqe^{-q}} d - \frac{1 - d + de^{-q}}{1 - dqe^{-q}} c + f \tag{12}$$

which is equivalent to

$$\tilde{p} > f - \frac{1 - d + de^{-q}}{1 - dqe^{-q}} (c + d) \tag{13}$$

Expression (11) gives

$$\tilde{p} < V_r - e = \frac{d(1 - e^{-q} - qe^{-q})}{1 - dqe^{-q}} (c + d) - e \tag{14}$$

which is equivalent to

$$\tilde{p} < \frac{d(1 - e^{-q} - qe^{-q})}{1 - dqe^{-q}} (c + d) - e \tag{15}$$

Since a+b>e+f, it must be that either a>e, or b>f. Assume that a>e: Since by (7) we have

$$\tilde{p} > \frac{d(1 - e^{-q} - qe^{-q})}{1 - dqe^{-q}} (c + d) - a \tag{16}$$

and by the assumption a>e the RHS is less than

$$\frac{d(1 - e^{-q} - qe^{-q})}{1 - dqe^{-q}} (c + d) - e \tag{17}$$

so that a $\tilde{p}$ that satisfies (7) and (15) can be found.

Assume next that b>f: By (6) we have

$$b - \frac{1 - d + de^{-q}}{1 - dqe^{-q}} (c + d) > \tilde{p} \tag{18}$$

and by the assumption b>f the LHS is greater than

$$f - \frac{1-d+de^{-q}}{1-dqe^{-q}}(c+d) \qquad (19)$$

so that a $\tilde{p}$ that satisfies (6) and (13) can be found.

It is also easily shown that the offer cannot be mistaken for playing the equilibrium with payoffs *(c,d)*, i.e. for a "do nothing" approach. By substituting c and d for e and f, correspondingly, we can see that with the side payment $\tilde{p}$ one of the players (or both) is worse-off. We thus conclude that if the inefficient equilibrium is to be played it can be broken by means of this side payment. Q.E.D.

5. Conclusions

Embedding two-player matrix games in a general equilibrium framework allows quite a sharp prediction about the way the game is played. In general the players co-ordinate to the Pareto-efficient equilibrium of the bimatrix game. This is accomplished by transfers that determine the agents who end up playing the game, as well as co-ordinate to a particular equilibrium. On top of this we attain an endogenous reservation value, or the value of the outside option of continuing search, that behaves very nicely. One can immediately see that letting the number of column players approach zero means that they reap practically all the available surplus. The opposite happens when the number of row players approaches zero. In this sense the utilities of the players reflect the demand and supply conditions.

The only case when it is not possible to escape an inefficient equilibrium is the case where there are Pareto-efficient equilibria with exactly the same payoffs to the players. This is illuminated in the following co-ordination game

		B	
		L	R
S	T	2,2	0,0
	B	0,0	2,2

Figure 1. A symmetric co-ordination game.

If the players happen to play the mixed strategy equilibrium there is no way they can use out-of-equilibrium-transfers to deviate to one of the Pareto-efficient equilibria as there is no way to tell them apart.

The idea of endogenising the pay-offs of the players by considering games in a general equilibrium framework may be a useful practice when the games are projects where two parties are needed, and where it is not possible to make complete contracts. Then the players' utilities depend on how other players behave because this determines the value of their outside option, and this helps to co-ordinate on a particular equilibrium.

References.

Blouin M., 2003, Equilibrium in a decentralized market with adverse selection, Econ Theory 22, 2, 245-262.

Halko M-L., K. Kultti and J-P. Niinimäki, 2002, Evolutionary stability of trading rules in an urn-ball matching model, Department of Economics, University of Helsinki, Discussion Papers No. 541.

Inderst R. and H. Müller, 2002, Competitive search markets for durable goods, Econ Theory 19, 3, 599-622.

Jackson M. and S. Wilkie, 2002, Endogenous games and mechanisms: Side payments among players, manuscript.

Janssen M. and V. Karamychev, 2002, Cycles and multiple equilibria in the market for durable lemons, Econ Theory 20, 3, 579-601.

Osborne M. and A. Rubinstein, 1994, A course in game theory. MIT Press.