p(x+y)=p(x)+p(y)-p(x)p(y)

Klaus Kultti*
Department of Economics, University of Helsinki,
e-mail: klaus.kultti@helsinki.fi

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Abstract

I show that if x and y are regarded as investments in R&D and p is the probability of discovery and if p satisfies p(x+y)=p(x)+p(y)-p(x)p(y) then p is practically uniquely determined.

1 Introduction

In modelling economic phenomena the following problem comes recurrently up. An agent invest in some activity which may be research and development, education, job search or the agent may invest into keeping things secret. The returns to the investment are not deterministic but typically it is assumed that the higher the investment the higher the probability of success. Now there is an infinite number of ways to pick up a probability measure that is increasing in the investment.

In some cases one can expect more structure from the probability measure. For instance, in R&D, which is the example I use for the rest of the note, one may think that R&D activity consist of choosing different paths of research, and each of the paths either leads to a success or to a failure. Then the probability function p should satisfy the following equation

$$p(x+y) = p(x) + (1 - p(x))\, p(y) \tag{1}$$

The interpretation is that it does not matter whether a firm, say, invests amount $x + y$ to R&D or whether it first invests x and then only in case this investment

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results in a failure invest amount y more. Another way to look at (1) is that it does not matter to the firm whether it divides its R&D activity into many small sequential activities or puts up just one large activity. I show here that this structure already pins down the form of the probability function almost uniquely. The point here is to show how easily a form of a function may be deduced if one is able to formulate a functional equation of which (1) is an example.

Before going through the analysis I want to point out that here the focus is the probability function not the objective function of a firm. Dividing R&D into small pieces may still be preferable depending on the form of the cost function. For instance, ff the cost function were linear investing $x+y$ to R&D would cost $x+y$ while first investing x and only in case of failure incurring the additional investment y would result in expected costs $x+(1-p(x))\,y$. Both investment strategies would lead to the same probability of success while the latter one would result in lower costs. Thus, not dividing the investment into small pieces must rely on some kind of non-convexities in the cost function.

2 Analysis

A more convenient form to (1) is

$$p(x+y) = p(x) + p(y) - p(x)p(y) \tag{2}$$

Insert $y = x$ into (2) to get

$$p(2x) = 2p(x) - (p(x))^2 = -(1-p(x))^2 + 1 \tag{3}$$

In a similar way

$$\begin{aligned} p(3x) \quad = \quad & p(2x+x) = p(2x) + p(x) - p(2x)p(x) = 3p(x) - 3\,(p(x))^2 + (p(x))^3 = \\ & -(1-p(x))^3 + 1 \end{aligned} \tag{4}$$

where in the second equality (2) is used and in the third equality (3) is used. By induction it is straightforward to show that

$$p(nx) = -(1-p(x))^n + 1 \tag{5}$$

Notice that by (2) we get

$$p(x) = p\left(2\frac{x}{2}\right) = -\left(1 - p\left(\frac{x}{2}\right)\right)^2 + 1 \leq 1 \tag{6}$$

Next let us determine the value of p at rational arguments. First we get from (5) inserting $x = 1$

$$p(n) = -(1-p(1))^n + 1 \tag{7}$$

On the other hand inserting $x = \frac{m}{n}$ into (5) we get

$$p(m) = -\left(1 - p\left(\frac{m}{n}\right)\right)^n + 1 \tag{8}$$

Evaluating (7) at $n = m$, equating the right hand sides of (7) and (8), and solving for $p\left(\frac{m}{n}\right)$ yields

$$p\left(\frac{m}{n}\right) = -\left(1 - p(1)\right)^{m/n} + 1 \tag{9}$$

By (6) $p(x)$ is at most one. Let us assume that $p(1) = 1$. Then it follows from (2) that

$$p(x) = p\left(1 + (x-1\right) = p(1) + p(x-1)\left(1 - p(1)\right) = 1 \tag{10}$$

for all $x \geq 1$. Take $x \in [0, 1)$, and let n be an integer such that $nx \geq 1$. Now $p(nx) = p\left(1 + (nx - 1)\right) = p(1) + p(nx-1) - p(1)p(nx-1) = -\left(1 - p(x)\right)^n + 1$ where the second equality comes from (2) and the third equality comes from (5). By (10) this yields $-\left(1 - p(x)\right)^n = 0$, and thus $p(x) = 1$ for all $x \geq 0$.

If $p(x) \neq 1$ we can define

$$c \equiv \log\left(1 - p(1)\right) \tag{11}$$

Inserting this into (9) yields

$$p\left(\frac{m}{n}\right) = -e^{c\frac{m}{n}} + 1 \tag{12}$$

Next we assume that p is continuous, and as its value is known on a dense set or reals (rationals) we immediately get that for all $x \in R_+$ (12) implies that

$$p(x) = 1 - e^{cx} \tag{13}$$

In particular, (13) implies that $p(0) = 0$. Notice that in (11) $c < 0$, and defining $\gamma = -c$ (13) can be expressed in the form

$$p(x) = 1 - e^{-\gamma x} \tag{14}$$

Thus, we get that the only continuous solutions to (1) for $x \in R_+$ are $p(x) = 1$ and $p(x) = 1 - e^{-\gamma x}$, $\gamma > 0$, where the latter one is the relevant solution if we are after a non-trivial probability function. It is very pleasing that the probability function is of this form as it is easy to apply, and behaves very nicely; for instance, it is infinitely many times continuously differentiable.

It should be noted that the same result can be derived using much weaker conditions, and that an almost identical derivation, as well as much more material can be found in Aczel (1966).

References

Aczel J. 1966 Lectures on functional equations and their applications. Academic Press: New York and London.