Learning with Heterogeneous Expectations in an Evolutionary World

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Abstract

This paper studies a game theoretic model where agents choose between two updating rules to predict a future endogenous variable. Agents rationally choose between these predictors based on relative performance. Conditions for evolutionary stability and stability under learning are found for the Nash solutions and corresponding parameter equilibria. Stability conditions are contingent upon parameter values and the initial distribution of heterogeneity. However, when the cost of using the more advanced updating rule is sufficiently large, all agents will asymptotically use the more parsimonious, or Minimum State Variable (MSV), updating rule.

Key Words: Adaptive Learning; Evolutionary Dynamics; Heterogeneous Expectations; Multiple Equilibria; Rational Expectations.

JEL Classification: C62, D84, E37

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1 Introduction

Expectations continue to play a key role in macroeconomic research. Since its introduction by Muth [25] and Lucas [21], [22], the Rational Expectations Hypothesis (REH) has been the dominant paradigm in expectations formation. According to the REH, agents form expectations using the mathematical expectations operator conditioned upon available information. In modelling, economists usually assume that agents posses perfect knowledge of the market equilibrium equations and use these to form their expectations.

Two main objections to the REH come from the literature on bounded rationality. The first is that it may be unreasonable to assume that agents have perfect knowledge of the market equilibrium. The literature has suggested that we allow agents to form expectations from less sophisticated schemes as in Bray and Savin [6], Evans and Honkapohja [11], and Hommes and Sorger [19]. The second objection to the REH is that with heterogeneous expectations, economic outcomes depend upon expectations of all participants. Hence, agents using rational expectations not only possess perfect knowledge of the market equilibrium, but they also posses perfect knowledge of the beliefs of all agents in the economy. The literature has also discussed expectation formation schemes with heterogeneity, e.g. in Evans and Honkapohja [10], Evans, Honkapohja, and Marimon [12], Honkapohja and Mitra [20], Giannitsarou [16], and Guse [17].

Recent work has also considered a different approach to expectation formation by including predictor choice as an economic decision.[1] In Evans and Ramey [13] agents choose whether or not to use a costly algorithm to update beliefs every period. This is later extended in Evans and Ramey [14] where they allow agents to pay a resource cost for the privilege to use a mechanism that directly calculates expectations. Sethi and Franke [27] consider a model where agents have the choice between using a costless adaptive expectations rule or using rational expectations which incurs a cost. In this paper, predictor decision is dictated via an evolutionary process. Brock and Hommes [7] use an approach they call the Adaptively Rational Equilibrium Dynamics (A.R.E.D.) to examine predictor decision. They consider a cobweb model with a finite set of predictors that each incur a cost for use. In each period, agents choose a predictor based on the performance of each predictor in the previous period. They conclude that when the set of predictors are a stable predictor, rational expectations, and an unstable

[1] The works of Arthur [1], De Grauwe, DeWachter, and Embrechts [9], and Sethi [26] present numerical results for this type of research.

predictor, naive expectations, the dynamics of the system may not settle down to an equilibrium. However, this instability result disappears when the set of predictors available increases. Branch [4] examines Brock and Hommes' model and finds that the set of predictors available affects the local stability properties of the system.

Previous literature on predictor choice has ignored two topics that should be examined. First, the models studied have a unique equilibrium. Multiple equilibria has been examined extensively in many papers including Azariadis [2], Cass and Shell [8], Azariadis and Guesnerie [3], and Farmer [15]. One could question how a model with multiple equilibria would act in an environment such as the A.R.E.D discussed in Brock and Hommes [7]. Second, the models examined in the literature are purely deterministic. Random shocks always occur to the economy due to unpredictable events such as changes in weather. The bounded rationality literature examines stochastic models with boundedly rational predictors. Here, multiple equilibria is a well discussed topic (e.g. Evans and Honkapohja [11]). With heterogeneity, multiple equilibria, and adaptive learning with more than one available updating rule, a model can be constructed where agents not only learn the equilibrium but also learn the quality of each predictor.

The purpose of this paper is to incorporate predictor choice into a model with heterogenous expectations, multiple equilibria, and adaptive learning. In the previous adaptive learning literature, the assumed level of heterogeneity has been exogenous and in this sense, ad hoc. An ad hoc level of heterogeneity may produce a result where many agents are using an obviously inefficient predictor due to the form of the parameter equilibrium. Agents who notice this efficiency disparity could form better expectations by switching to using the most efficient predictor. In this manner, agents not only learn the parameters of the model, but they also learn the best way to learn these parameters. Therefore, in this paper, heterogeneity will now be determined endogenously from a type of "social learning."

Marimon and McGratton [23] show that there is an isomorphism between adaptive learning and evolutionary learning. As heterogeneity is expressed as a proportion of agents in this paper, it is only natural to use evolutionary learning as the mechanism for social learning in the dynamic system. The speed of the two learning mechanisms will differ as social learning should be slower than parameter learning. The concept of "fast-slow" learning where agents learn the parameters much faster than the quality of the predictors creates an easy way to analytically determine stability of the dynamic system.

The technique used in this paper will provide a tool for a test of robustness of learnability of equilibria under homogeneous expectations. A rational expectations equilibrium (REE) is commonly considered to be relevant if it is stable under learning, or expectationally stable (E-stable). However, some models produce multiple equilibria where many solutions may be E-stable. A natural question would be: If more than one solution (predictor) is available to the agents of an economic model and agents can choose a predictor based on past performance, what solution, or solutions, would be used by these agents?[2] Furthermore, would the resulting (stable) Nash solution, under the predictor choice model, involve homogeneous or heterogeneous expectations? Conditions for stability under predictor choice and learning may be more strict than E-stability conditions for homogeneous expectations.

This technique may also provide for further available solutions when heterogeneity is considered in an economic model with forward expectations. As suggested by the above paragraph, the predictor choice model may result in a Nash solution where heterogeneous expectations exist asymptotically. Even when only one REE is E-stable under homogeneous expectations, this may not guarantee that this solution is the only stable solution under heterogeneity and predictor choice. Therefore, further problems may arise if there exists multiple Nash solutions under heterogeneous expectations and predictor choice that are stable under learning and evolutionary behavior.

This paper presents a self-referential linear stochastic model with the possibility of multiple equilibria. Guse [17] discusses the stability results under learning of such a model when agents have different perceptions of the true equilibrium. In this paper, the model in Guse [17] is expressed as a game where agents benefit from using the most efficient predictor of the economy. I examine the stability properties of the equilibria in the game under RE and under least squares learning. When the model is expressed as a game with predictors, only some Nash equilibria are shown to be evolutionary stable when disturbed by mutant populations. Furthermore, only some Nash equilibria are evolutionary stable with a corresponding learnable parameter equilibrium. Stability conditions are found for both homogenous and heterogeneous expectations. The central conclusion is that homogeneity and heterogeneity results depend on the initial level of heterogeneity and parameter values in the model. The stability path dependence disappears when the cost of using the expensive predictor is sufficiently high and leads to all agents asymptotically using the more parsimonious, or Minimum State Variable (MSV), updating

[2] One could consider sunspot equilibria as well, but this paper only considers non-sunspot equilibria as potential predictor choices.

rule.

2 The Model and E-stability

In this paper, I use a self-referential linear stochastic macroeconomic model with the possibility of multiple REE, as presented in Taylor [28] and discussed in the learning literature, e.g. Evans and Honkapohja [11] and Heinemann [18]. It is a linear stochastic model with real balance effects consisting of four parts: aggregate demand, aggregate supply, money demand, and a fixed money supply. The reduced form is as follows:

$$y_t = \alpha + \beta_0 E^*_{t-1} y_t + \beta_1 E^*_{t-1} y_{t+1} + v_t \tag{1}$$

where E^* denotes a not necessarily rational expectation and v_t is a linear combination of stochastic shocks where $v_t \sim N\left(0, \sigma^2\right)$. Although it may be any variable that is affected by expectations, think of the variable y_t to be the inflation rate at time t. Assume that agents have the choice of using one of two predictors:

$$\begin{aligned} PLM_1 &: \quad y_t = a_1 + v_t \\ PLM_2 &: \quad y_t = a_2 + b_2 y_{t-1} + v_t. \end{aligned}$$

where agents recursively estimate the coefficients of their PLM to form expectations. If a proportion of μ agents uses PLM_1, then the actual law of motion (ALM) is:

$$\begin{aligned} y_t &= \alpha + \mu a_1(\beta_0 + \beta_1 + \beta_1(1-\mu)b_2) + (1-\mu)a_2(\beta_0 + \beta_1 + \beta_1(1-\mu)b_2) \\ &\quad + [(1-\mu)b_2(\beta_0 + (1-\mu)\,\beta_1 b_2)]y_{t-1} + v_t \end{aligned} \tag{2}$$

The above system defines a mapping from the PLM to the ALM as follows:

$$\begin{aligned} T(\phi) &= T\begin{pmatrix} a_1 \\ a_2 \\ b_2 \end{pmatrix} \\ &= \begin{pmatrix} \frac{\alpha+\mu a_1(\beta_0+\beta_1+\beta_1(1-\mu)b_2)+(1-\mu)a_2(\beta_0+\beta_1+\beta_1(1-\mu)b_2)}{1-(1-\mu)b_2(\beta_0+(1-\mu)\beta_1 b_2)} \\ \alpha+\mu a_1(\beta_0+\beta_1+\beta_1(1-\mu)b_2)+(1-\mu)a_2(\beta_0+\beta_1+\beta_1(1-\mu)b_2) \\ (1-\mu)b_2(\beta_0+(1-\mu)\,\beta_1 b_2) \end{pmatrix} \end{aligned} \tag{3}$$

The resulting equilibria are expressed as :

$$\begin{aligned} a_1 &= \frac{\alpha}{1-b_2-\mu\lambda-(1-\mu)\,(1-b_2)\,\lambda}, \\ a_2 &= \frac{(1-b_2)\,\alpha}{1-b_2-\mu\lambda-(1-\mu)\,(1-b_2)\,\lambda}, \text{and} \\ b_2 &= \frac{1-(1-\mu)\beta_0}{(1-\mu)^2\beta_1} \end{aligned} \tag{4}$$

or

$$\begin{aligned} a_1 &= \frac{\alpha}{1-\beta_0-\beta_1}, \\ a_2 &= \frac{\alpha}{1-\beta_0-\beta_1}, \text{ and} \\ b_2 &= 0 \end{aligned} \tag{5}$$

where:

$$\lambda = 1+\beta_1+\mu\beta_0.$$

Equilibrium (4) is referred to the AR(1) mixed expectations equilibria (MEE).[3] In this equilibrium, the proportion of agents using PLM_1 are underparameterizing the model when they are forming their expectations. Therefore, prediction errors for PLM_1 will tend to be larger, on average, in this equilibrium. Equilibrium (5) is referred to as the minimum state variable (MSV) MEE. In this equilibrium, prediction errors will be the same for each PLM as they produce the same prediction. Although the

[3] Branch and McGough [5] refer to such an equilibrium as the Heterogeneous Expectations Equilibrium (HEE). The MEE includes the REE when $\mu = 0$ or $\mu = 1$.

equilibria expectations in the MSV solution are homogenous, it will be considered heterogeneous expectations since two predictors are used to form expectations. The equilibria are referred to as "mixed" because they are generated from more than one predictor.

Under the two MEE's, economic agents have a great deal of knowledge of the economy. It is common to ask whether these equilibria are robust when agents form expectations using less sophisticated schemes than RE. Suppose that the agents act like econometricians and construct forecasts using their econometric model that they update every period when new information becomes available. The condition for an equilibrium to be (locally) stable under such a learning rule is known as Expectational Stability (E-stability):

Definition 1: *E-stability is the condition of local asymptotic stability of* $\bar{\phi}$ *under the differential equation*[4]

$$\frac{d\phi}{d\tau} = T(\phi) - \phi,$$

where T *is the mapping from the perceived law of motion,* ϕ*, to the implied actual law of motion,* $T(\phi)$ *and* τ *denotes "notional" or "artificial" time.*

It is commonly known that, under least squares learning, an E-stable equilibrium is learnable. Learnability of an equilibrium may be regarded as a necessary condition for the relevance of that equilibrium. Guse [17] presents the E-stability conditions for a fixed proportion of heterogeneity, μ, in the following proposition:

Proposition 1: *E-stability conditions for the above linear stochastic model with heterogeneous expectations.*

1. *All MSV MEE in the set*

$$ES_1 = \left\{ (\beta_0, \beta_1) \,|\beta_0 < \left(\frac{1}{1-\mu} \right), \beta_0 + \beta_1 < 1 \right\}$$

are E-stable. All MSV MEE outside of this set are E-unstable.

[4] In the homogenous expectations case, $\phi = (a_1)$ if all agents use PLM_1 and $\phi = \begin{pmatrix} a_2 \\ b_2 \end{pmatrix}$ if they use PLM_2.

2. All AR(1) MEE in the set

$$ES_2 = \left\{ (\beta_0, \beta_1) \,|\, \frac{1}{1-\mu} < \beta_0 < \frac{1}{1-\mu} - (1-\mu)\,\beta_1, \beta_1 < 0 \right\}$$

are E-stable. All AR(1) MEE outside of this set are E-unstable.

For the Taylor [28] real balance model, the parameter restrictions are $\beta_1 = -\beta_0$ and $\beta_0 \neq 0$. Therefore, either solution, MSV or AR(1) may be E-stable under heterogeneous expectations, depending on the parameter values of the model. E-stability of the two solutions may change when the level of heterogeneity, μ, is allowed to change. When μ is allowed to change, Guse [17] presents the condition for E-stability for any μ:[5]

Proposition 2: *Let*

$$A = \{(\beta_0, \beta_1) \,|\, \beta_0 < 1, \beta_0 + \beta_1 < 1\}$$

and

$$S = \left\{ (\beta_0, \beta_1) \,|\, \beta_0 > 1, \beta_1 < -\frac{1}{4}\beta_0^2 \right\}.$$

If $(\beta_0, \beta_1) \in A \cup S$, *then for each* $\mu \in [0,1]$ *exactly one of the two MEE is E-stable.*

Within this set, if μ changes for some reason, the other MEE may become E-stable, but there is no μ such that both solutions are E-unstable. The set A is where only the MSV equilibrium is E-stable for all $\mu \in [0,1]$. The set S is the set where the two equilibria exchange E-stability at $\mu = 1 - \frac{1}{\beta_0}$. When determining stability of the system with both social learning and parameter learning, I will assume that $(\beta_0, \beta_1) \in A \cup S$. I will focus on the set S where stability of each equilibrium is determined by the level of heterogeneity, μ, however, I will also discuss stability properties when $(\beta_0, \beta_1) \in A$.

3 Evolutionary Stability

Next, focus on the AR(1) and MSV processes and ignore E-stability for the time being. Suppose that the agents have the ability to change their PLM if they believe that the other PLM is doing a better job at predicting the economic variable. Agents observe the Mean Squared Error of the predictor at

[5] This proposition is a combination of two propositions presented in Guse [17].

each time period and decide whether to continue using their current predictor or switch to using the other predictor. Assume the following utility function for an agent using predictor i:

$$U_i = \frac{1}{MSE_i} - \text{ cost of using predictor } i$$

To solve for the Nash equilibria of the game and simplify the math, I will use the MSE of each predictor that corresponds to the MEE of the current value of μ. The MSE's are written as MSE_1 and MSE_2 and can be found in the appendix. Assume that the cost of using the AR(1) process is greater or equal to the cost of using the MSV process, so, without loss of generality, the cost of using the MSV process will be normalized to zero.[6]

3.1 The Model Expressed as a Game with a Continuum of Players

Assume that there are many agents where each agent's decision does not affect the state of the economy. For simplicity, assume that there is a continuum of players. Let $([0,1], \mathcal{B})$ be the underlying space where $[0,1]$ is the player set and $\mathcal{B}$ is the σ-algebra of Borel subsets of $[0,1]$. Let $S_i = \{PLM_1, PLM_2\}$ be the set of strategies for each player. In this artificial game, assume that the agents will know the equilibrium values based on their predictor choice.

Suppose that each player receives a payoff from choosing either strategy in the following manner:

$$\begin{aligned} v_i(s_i, \mu) &= \frac{1}{MSE_1} = U_1 \text{ if } s_i = PLM_1 \\ &= \frac{1}{MSE_2} - k = U_2 \text{ if } s_i = PLM_2 \end{aligned}$$

where $k \geq 0$ is the cost of using the AR(1) predictor and the population state at time t is:

$$x_t = (\mu_t, 1 - \mu_t).$$

The associated population payoff average is then

$$v(\mu, \mu) = \mu * v_1 + (1 - \mu) * v_2 = \frac{\mu}{MSE_1} + (1 - \mu)\left(\frac{1}{MSE_2} - k\right).$$

[6] This can be done since agents will only consider differences in utility and not the differences in the estimated parameter values.

This utility function defines the utility an agent receives for playing a mixed strategy of μ, given that everyone else plays an average of μ. In the framework of the paper, we are not concerned with mixed strategies, but the average payoff with agents using one of two pure strategies must be considered. The above function will be used for this purpose.

3.1.1 The Nash equilibria of the game

Next, I solve for the Nash equilibria for the above game. Since there are two possible solutions, MSV and AR(1), I solve for the Nash equilibria associated with each solution.

For the MSV solution, it turns out that $\forall \mu \in [0,1]$, $MSE_1 = MSE_2 = \sigma^2$. If $k = 0$, agents are indifferent in which PLM they use, so the Nash equilibria consists of $\mathcal{B}$, the set of all possible combinations of heterogeneous expectations. If $0 < k \leq (\sigma^2)^{-1}$, the Nash equilibrium is $\mu = 1$ where all of the agents choose to use PLM_1.[7]

There are several Nash equilibria for the AR(1) solution. This is solved for $v_1 = v_2$ for $\mu \in (0,1)$, $v_1 > v_2$ for $\mu = 1$, and $v_2 > v_1$ for $\mu = 0$. This is where no agents will want to deviate from their current PLM. If $k = 0$, then there are two Nash equilibria which are

$$\begin{aligned} \mu &= 0 \\ \mu &= 1 - \frac{1}{\beta_0}. \end{aligned}$$

If $0 < k \leq k_1$, then the three Nash equilibria are:

$$\begin{aligned} \mu &= 0 \\ \mu &= \mu_2 \\ \mu &= \mu_3. \end{aligned}$$

[7] An upper bound for the cost parameter is not necessary since the Nash equilibria are the same. The case where the cost is higher than the upper bound is an uninteresting case since the utility for an agent using PLM_2 is always negative.

where

$$\begin{aligned}
\mu_1 &= 1 + \frac{\beta_0 + \sqrt{\beta_0^2 + 4\beta_1\sqrt{k\sigma^2}}}{2\beta_1\sqrt{k\sigma^2}} \\
\mu_2 &= 1 + \frac{\beta_0 - \sqrt{\beta_0^2 + 4\beta_1\sqrt{k\sigma^2}}}{2\beta_1\sqrt{k\sigma^2}} \\
\mu_3 &= 1 - \frac{\beta_0 - \sqrt{\beta_0^2 - 4\beta_1\sqrt{k\sigma^2}}}{2\beta_1\sqrt{k\sigma^2}} \\
k_1 &= \frac{(1-\beta_0)^2}{\beta_1^2\sigma^2} \\
k_2 &= \frac{\beta_0^4}{16\beta_1^2\sigma^2}
\end{aligned}$$

If $k_1 < k < k_2$ and $\beta_0 > 2$, then there are three Nash equilibria, these are

$$\begin{aligned}
\mu &= \mu_1 \\
\mu &= \mu_2 \\
\mu &= \mu_3.
\end{aligned}$$

If $k_1 < k \leq k_2$ and $1 < \beta_0 \leq 2$, then the Nash equilibria are:

$$\begin{aligned}
\mu &= \mu_2 \\
\mu &= \mu_3.
\end{aligned}$$

When $k = k_2$ and $\beta_0 > 2$, then the two Nash equilibria are:

$$\begin{aligned}
\mu &= \mu_1 = \mu_2 = 1 - \frac{2}{\beta_0} \\
\mu &= \mu_3.
\end{aligned}$$

Finally, when $k > k_2$, there is only one Nash equilibrium, which is

$$\mu = \mu_3.$$

3.1.2 Evolutionary Stability of the Nash Equilibria

Evolutionary game theory considers a pure or mixed strategy and determines whether this strategy is stable when the population is disturbed by some "mutant strategy." The game described above does not allow mixed strategies, but the population average of those choosing PLM_1, μ, will be considered a "mixed strategy." To determine stability in the context of the above game, consider all "mixed" and pure equilibria for the population and determine if it will be beneficial for some agents to switch from using their current PLM to using the other PLM. If some "mutant" population strategy is allowed to enter, will the given (equilibrium) population strategy be stable to this given mutant if agents are allowed to switch strategies? Following Weibull [29], Evolutionary Stability is defined as follows:

Definition 2: $x \in \Delta$ *is an evolutionary stable strategy (ESS)*[8] *if for every strategy* $y \neq x$, there exists some $\bar{\varepsilon}_y \in (0,1)$ such that

$$v\left(x, \varepsilon y + (1-\varepsilon)\, x\right) > v\left(y, \varepsilon y + (1-\varepsilon)\, x\right) \tag{6}$$

holds for all $\varepsilon \in (0, \bar{\varepsilon}_y)$.

Under an evolutionary stable strategy, if a small proportion of agents "mutate" from using one predictor to the other predictor, then they will not receive more utility than before the mutation. Furthermore, no other agents will wish to follow the "mutants." When there exists a selection criterion for the population, the population will tend to move to the evolutionary stable strategy.

A best response function can be drawn to present evolutionary stability. $x \in \Delta$ is evolutionary stable if:[9]

$$v\left(s_1, \varepsilon y + (1-\varepsilon)\, x\right) - v\left(s_2, \varepsilon y + (1-\varepsilon)\, x\right) \lesseqgtr 0 \text{ if } y \gtreqless x$$

The only potential strategies that can be evolutionary stable are the Nash equilibria. The set of ESS will thus be a subset of the Nash equilibria. Formally, $\Delta^{ESS} \subset \Delta^{NE}$.

Consider the Nash equilibria for the MSV solution to the above model set in the game. For the MSV solution where $k = 0$, if μ_0 is allowed to change, there is equality for equation (6) for any $\mu_0 \in [0,1]$. Therefore, all the Nash equilibria in this case fail to be evolutionary stable strategies. This results from the fact that utility from each updating rule is the same for all $\mu \in [0,1]$. I will assume that $k > 0$ for

[8] Δ denotes the set of potential strategies. In this particular continuous framework, $x = (\mu_t, 1-\mu_t) \in [0,1]^2$.

[9] Note that $\mu \in [0,1]$, so for pure strategies, we only have to increase or decrease μ depending on which strategy we are considering.

the MSV MEE to ignore this uninteresting result. Next, consider the Nash equilibrium for the MSV solution when $0 < k < \frac{1}{\sigma^2}$. When a small proportion of agents use PLM_2, inequality (6) always holds. Therefore, the Nash equilibrium of $\mu = 1$ for the MSV solution where $0 < k < \frac{1}{\sigma^2}$ is an evolutionary stable strategy.

Next, consider the Nash equilibria for the AR(1) solution. Figures 1-3 depict the best response functions used to determine evolutionary stability. Figure 1 shows the best response function when $0 \leq k \leq k_1$. The three Nash equilibria are $\mu = 0$, $\mu = \mu_2$, and $\mu = \mu_3$ where $\mu_2 \leq \bar{\mu} \leq \mu_3$. The first and third of these Nash equilibria are ESS's, but the second solution is not an ESS.[10] Figure 2 presents the best response function when $k_1 < k \leq k_2$ and $\beta_0 > 2$. The Nash equilibria are $\mu = \mu_1$, $\mu = \mu_2$ $\mu = \mu_3$. The first and third of these three Nash equilibria are ESS, while the second solution is not.[11] One can also use figure 3 to observe the case where $k_1 < k \leq k_2$ and $1 < \beta_0 < 2$ by moving the origin between μ_1 and μ_2. Here, only μ_3 is ESS since $\mu_1 < 0$. Finally, when $k > k_2$, the Nash Equilibrium is $\mu = \mu_3$. This solution is ESS as shown in figure 3.

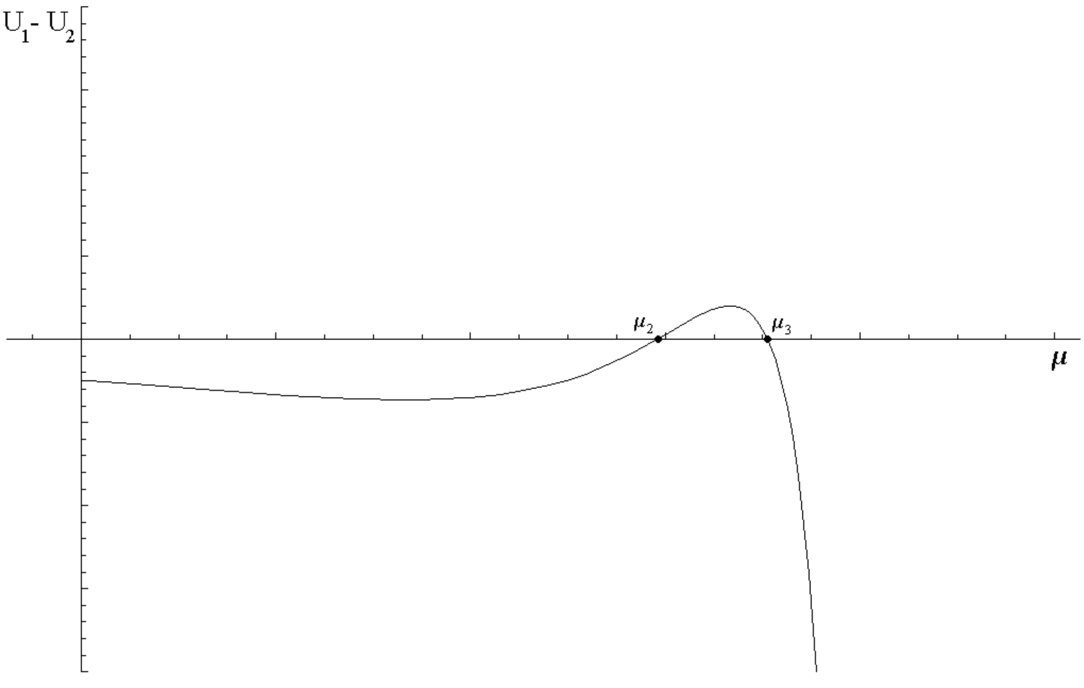

FIGURE 1. Best Response Function for AR(1) Solution when $0 \leq k \leq k_1$.

[10]When $k = 0$, there are only two Nash solutions, $\mu = 0$ and $\mu = \mu_2 = \mu_3 = \bar{\mu}$. In this case, $\mu = 0$ is the ESS.
[11]When $k = k_2$, there are only two Nash solutions, $\mu = \mu_1 = \mu_2$ and $\mu = \mu_3$. In this case, $\mu = \mu_3$ is the ESS.

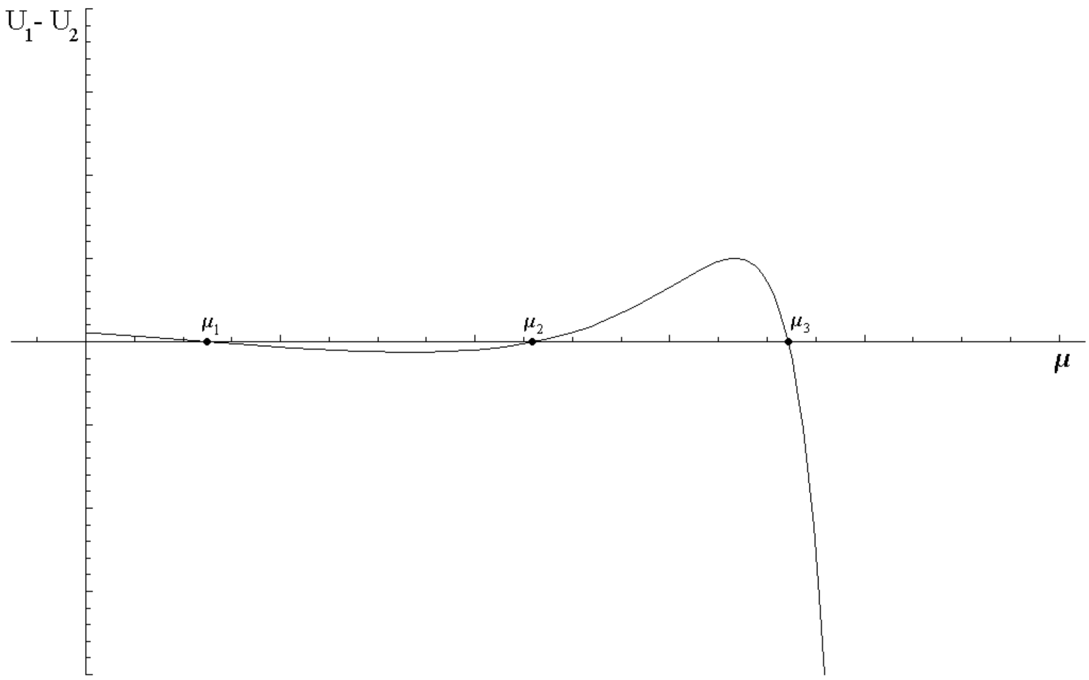

FIGURE 2. Best Response Function for AR(1) Solution when $2 < \beta_0 < 4$ and $k_1 < k < k_2$.

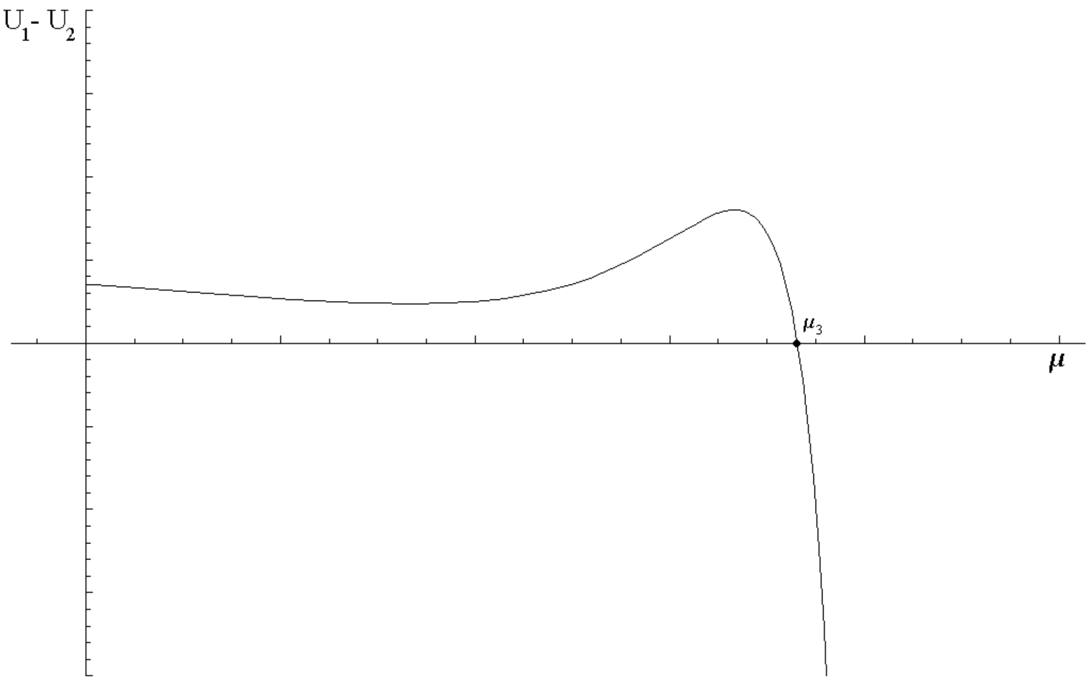

FIGURE 3. Best Response Function for AR(1) Solution when $k > k_2$.

These results bring forward a natural question, “Are there any evolutionary stable Nash equilibria with E-stable MEE's?” The most interesting candidates are the Nash MEE defined with $0 < \mu < 1$.

The following proposition states that these candidates can be evolutionary stable with E-stable solutions as long as the cost parameter is contained in a specific range of values.

Proposition 3: *If* $k_1 < k < k_2$ and $2 < \beta_0 < 4$*, then there does exists a Nash equilibrium with* $\mu \in (0, 1)$ *that is evolutionary stable with an E-stable MEE. For any* $k < k_1$ *or* $k > k_2$*, there does not exist a Nash equilibrium with* $\mu \in (0, 1)$ *that is evolutionary stable with an E-stable MEE.*

4 The Replicator Dynamics and Evolutionary E-stability

Since I have defined utility functions for both of the estimators, I now define the replicator dynamics for μ. There are two elements of evolutionary game theory: a mutant mechanism which provides variety and a selection criterion that favors one variety over another. The replicator dynamics provides the role of selection. Following Weibull [29], define the replicator dynamics as follows:

$$\mu_t = \left(\frac{\zeta + U_1}{\zeta + \mu_{t-1} * U_1 + (1 - \mu_{t-1}) * U_2} \right) * \mu_{t-1} \tag{7}$$

The replicator dynamics directs the population to use the parameter updating rule that awards a higher utility at time $t - 1$. Brock and Hommes [7] and others assumed that the role of selection was dictated by a multinomial logit law of motion. With a multinomial logit, convergence to a single predictor is not necessarily attainable unless the expected value of utility from each predictor, except one, is equal to zero. The replicator dynamics will provide a tool to produce the possibility of convergence to homogeneous expectations due to the exponential nature of the replicator dynamics. However, this does not guarantee convergence to a single predictor as it may be that $U_1 = U_2$ for some μ_{t-1}. The ζ variable can be used for two purposes: first, it will determine the speed of convergence for real time dynamics, and second, it can be used to make sure that both the numerator and denominator of the replicator dynamics are positive. Since μ_t is allowed to vary through time, it must be that each PLM is a reasonable choice within the model. Therefore, assume that the AR(1) solution is stationary for all $\mu \in \left[0, 1 - \beta_0^{-1}\right)$, or $(\beta_0, \beta_1) \in S \cup A$ as defined above.

4.1 Fast-Slow Dynamics

For mathematical purposes, I will assume that the MSE that agents observe will be the MSE corresponding to the MEE for the current value of μ. This will be from a process of fast parameter learning dynamics with slow replicator dynamics, or "fast-slow" learning.[12] The agents will fully learn the MEE corresponding to the current value of μ before each period when μ is updated by the replicator dynamics. Therefore, the speed of the learning is infinitely faster than the speed of the replicator dynamics. I make this assumption in order to theoretically evaluate for evolutionary E-stability defined below.[13]

4.2 Evolutionary E-stability

Now, I will examine when a Nash solution is stable given β_0, β_1, α, and a cost parameter, k. Here, I introduce a concept I will call evolutionary E-stability:

Definition 3: *Assume that the model is updated using fast parameter learning dynamics with slow replicator dynamics. An MEE or REE, $\phi^*(\mu^*)$, is Evolutionary E-stable, under the defined game above, if for all $\mu \in [0,1]$* sufficiently close to μ^* *(1) $\mu_t \to \mu^*$ under the replicator dynamics and (2) $\phi(\mu_t)$ is E-stable for all μ_t.*

Here, $\phi(\mu)$ refers to an E-stable MEE that is determined by the level of heterogeneity, μ, and $\phi^*(\mu^*)$ is the MEE determined by the Nash solution of μ^*. Under evolutionary E-stability, if a mutation occurs to change the level of heterogeneity, then the system will return to the evolutionary E-stable MEE or REE. Furthermore, at each μ in the neighborhood of μ^*, the corresponding MEE is E-stable. Like E-stability, this is a local condition, but unlike E-stability, the boundary of attractive may be determined under the replicator dynamics for each Nash solution.

[12] One could also investigate slow-fast dynamics, but here I focus on fast-slow dynamics because of its appealing theoretical results.

[13] When agents compute the MSE as $MSE_{i,t} = MSE_{i,t-1} + t^{-1}((y_t - z'_{t-1}\phi_{i,t})^2 - MSE_{i,t-1})$, simulations show that the results are comparable to the results below. Where $i = 1$ or 2 corresponding to PLM's 1 and 2. Simulations are not included as they do not provide any additional conclusions.

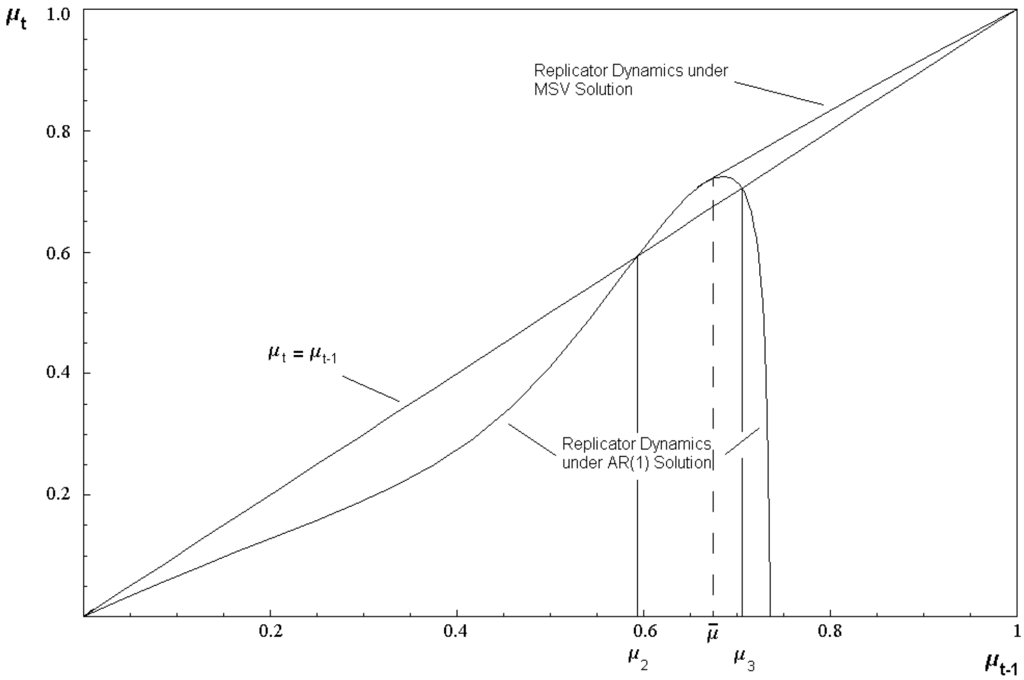

FIGURE 4. The Replicator Dynamics for an AR(1) Evolutionary E-stable REE.

4.3 AR(1) Evolutionary E-stability

First, consider the AR(1) REE where all agents use the AR(1) predictor. Figure 4 presents the replicator dynamics for the AR(1) REE when there is a deviation to $\mu_0 < 1 - \beta_0^{-1} = \bar{\mu}$. If the population begins at a μ to the left of the intersection point of $\mu_t = \mu_2$ of the replicator dynamics, then the replicator dynamics will direct the entire population to using the AR(1) predictor. If μ is to the right of this intersection point, then the replicator dynamics will direct the agents away from using the AR(1) predictor. This result will be explained later with MSV Dominance. The following proposition presents the conditions for stability for the AR(1) REE under the replicator dynamics:

Proposition 4: *Under fast-slow dynamics, the AR(1) REE is stable under the replicator dynamics for all*

$$0 \leq \mu_0 < \mu_2$$

if

$$0 \leq k < k_1.$$

Note that this stability result is path dependent. It must be that the initial level of heterogeneity must be contained in the above limits stated in the proposition. Evolutionary E-stability conditions for the AR(1) REE come from the previous proposition.

Corollary 1: *The AR(1) REE is Evolutionary E-stable for*

$$0 \leq k < k_1.$$

4.4 MSV Evolutionary E-stability

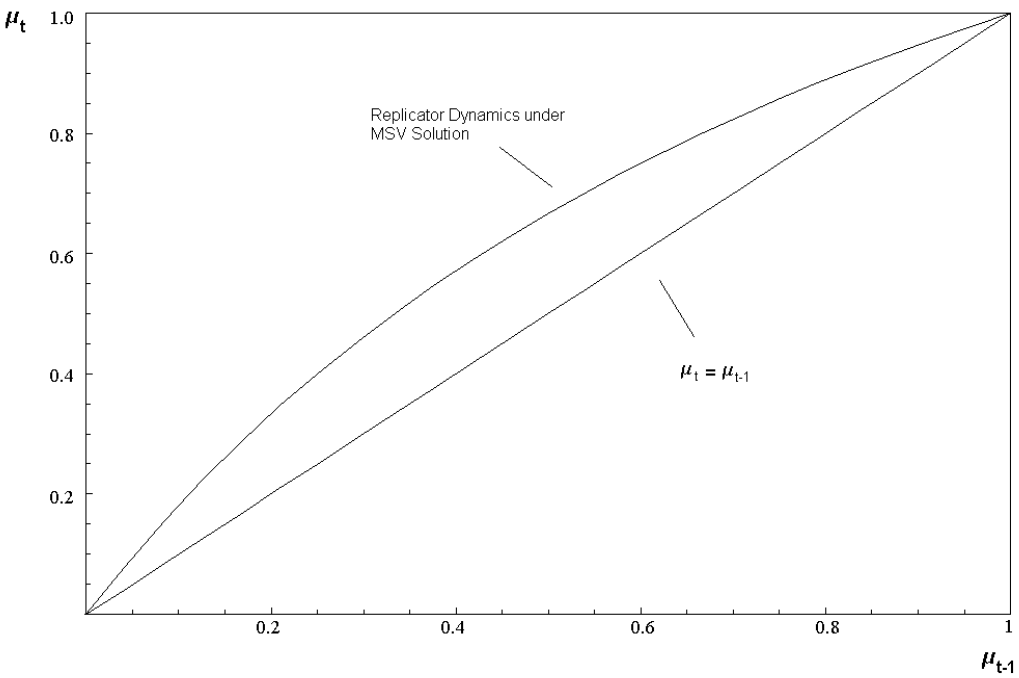

FIGURE 5. The Replicator Dynamics for an MSV Evolutionary E-stable REE.

Next, I examine when the MSV REE is evolutionary E-stable. It turns out that it can not be Evolutionary E-stable when the cost of using the second predictor is zero since both of the MSE's are equal to σ^2. The replicator dynamics in this case would be $\mu_{t+1} = \mu_t$. Since the MSV predictor is easier to use, suppose that there is a preference of using the MSV predictor so that $k > 0$. Figure 5 shows the replicator dynamics for MSV evolutionary E-stability which gives us the following proposition:

Proposition 5: *Under Fast-slow dynamics, the MSV REE is Evolutionary E-stable if* $k > 0$. *Furthermore, the solution is always stable under the replicator dynamics for all*

$$\begin{aligned} 1 - \frac{1}{\beta_0} &< \mu_0 \leq 1 \text{ if } \beta_0 > 1 \\ 0 &< \mu_0 \leq 1 \text{ if } \beta_0 < 1 \end{aligned}$$

if $k > 0$.

Figure 5 shows the replicator dynamics for the case where $\beta_0 < 1$, i.e. where $(\beta_0, \beta_1) \in A$. For $(\beta_0, \beta_1) \in S$, one can look at figures 4 and 6 to the right of $\mu = \bar{\mu} = 1 - \frac{1}{\beta_0}$. When the MSV MEE is E-stable, both updating rules provide the same MSE. Therefore, as long as $k > 0$, the replicator dynamics will direct the population to all use the MSV predictor. As the cost of using the AR(1) predictor, k, increases, the replicator dynamics becomes more bowed out from the line $\mu_t = \mu_{t-1}$. This will create a result of MSV dominance which is discussed below.

4.5 Evolutionary E-stability for the AR(1) MEE

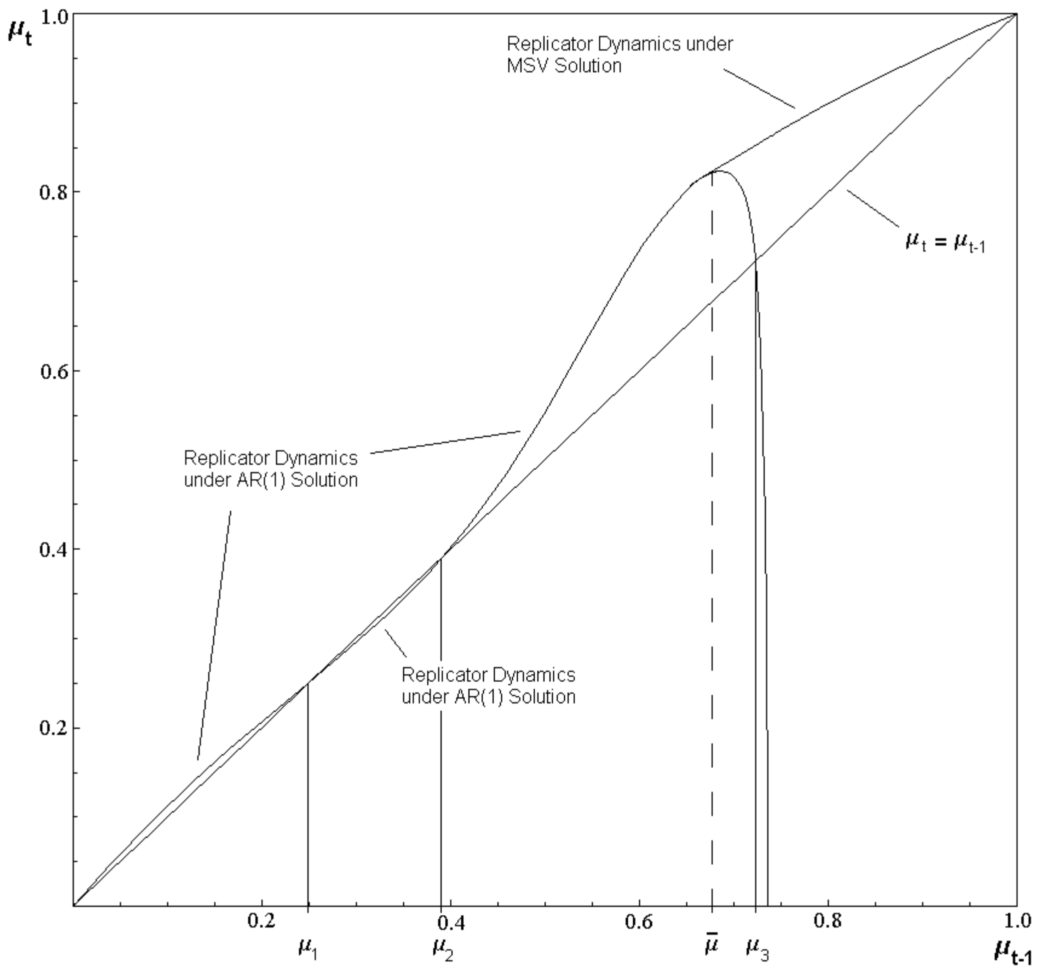

FIGURE 6. The Replicator Dynamics in the Case of an Evolutionary E-stable MEE.

Finally, for some $k > 0$, it is possible for an MEE to be simultaneously evolutionary stable and E-stable. Figure 6 shows the replicator dynamics when the evolutionary stable AR(1) MEE exists. Here, the stable Nash solution is μ_1. As long as $0 < \mu_0 < \mu_2$, the replicator dynamics will direct the population to using a mix of predictors described by μ_1. If $\mu_0 > \mu_2$, then the replicator dynamics will direct the population away from using the AR(1) predictor. The result here will be discussed later in MSV dominance. The following proposition states the conditions for the existence of an evolutionary E-stable MEE.

Proposition 6: *Under fast-slow dynamics, the AR(1) MEE, μ_1, is always stable under the replicator dynamics when*

$$2 < \beta_0 < 4$$

and

$$0 < \mu_0 < \mu_2$$

if

$$k \in (k_1, k_2) .$$

These stability conditions for the replicator dynamics give the condition for Evolutionary E-stability of the AR(1) MEE, μ_1.

Corollary 2: *The AR(1) MEE, μ_1, is Evolutionary E-stable for*

$$k \in (k_1, k_2) .$$

The fact that the agents form expectations of the variable in time $t+1$ produces this result of an evolutionary E-stable MEE. The agents that are using the AR(1) PLM are looking in the future also take in account what they predicted for the variable at time t. The agents using the MSV PLM are predicting the variable in time $t+1$ not taking into account that the process is AR(1) at time t. This leads to an MSE that is increasing when μ is small and is decreasing when μ is larger. The increasing function is a result of the process of not updating the $t+1$ variable. The MSE then becomes decreasing as more agents use the MSV predictor, so the AR(1) solution appears more like the MSV solution. The equilibria then exchange stability when the solution for b_2 in PLM_2 reaches $b_2 = 0$.

4.6 MSV Dominance

There is one more question to answer. From propositions 4 and 6, if k is large enough, then the AR(1) REE, is not evolutionary E-stable and there is no evolutionary E-stable AR(1) MEE. What happens in this case? It turns out that the solution converges to the MSV REE and all agents switch to using the MSV predictor. I refer to this phenomenon as minimum state variable dominance.

Definition 4: *Minimum state variable (MSV) dominance is said to occur if a model begins at an AR(1) E-*

stable MEE and converges to an MSV E-stable REE with homogenous expectations under the replicator dynamics.

MSV dominance is shown in figures 4 and 6. If $\mu_2 < \mu_0 < 1 - \frac{1}{\beta_0} = \bar{\mu}$, the AR(1) MEE is E-stable and the MSV MEE is not, however, the replicator dynamics are directing the population away from using the AR(1) updating rule. As more agents use the MSV updating rule, the AR(1) MEE solution becomes more like the MSV MEE. In fact, when $\mu = \bar{\mu}$, both solutions are the same where[14]

$$\begin{aligned} a_1 &= a_2 = \frac{\alpha}{1 - \beta_0 - \beta_1} \\ b_2 &= 0. \end{aligned}$$

At $\mu > \bar{\mu}$, the AR(1) MEE is no longer E-stable, however, the MSV MEE is E-stable. Now the population is in the area of MSV evolutionary E-stability and the replicator dynamics will continue to direct all agents to use the MSV predictor. Therefore, the relevant branch for the replicator dynamics in figures 4 and 6 is the one corresponding to the MSV solution. The E-stable MEE was initially the AR(1) solution, but due to the replicator dynamics, all agents asymptotically switched to the MSV updating rule which corresponds to the new E-stable MEE. The following proposition gives the conditions for MSV dominance:

Proposition 7: *If*

$$0 < k \leq k_2$$

and

$$\mu_2 < \mu_0 < 1 - \frac{1}{\beta_0} = \bar{\mu},$$

and the MEE is an E-stable AR(1) solution, then MSV dominance will occur. If

$$k > k_2$$

and

$$0 < \mu_0 < 1 - \frac{1}{\beta_0}$$

[14] The solution is, however, not E-stable. Simulations suggest this does not present a problem as long as b_2 is sufficiently near zero when $\mu = \bar{\mu}$.

and the MEE is an E-stable AR(1) solution, then MSV dominance will always occur.

4.7 Global Stability, MSV Dominance, and Path Dependence

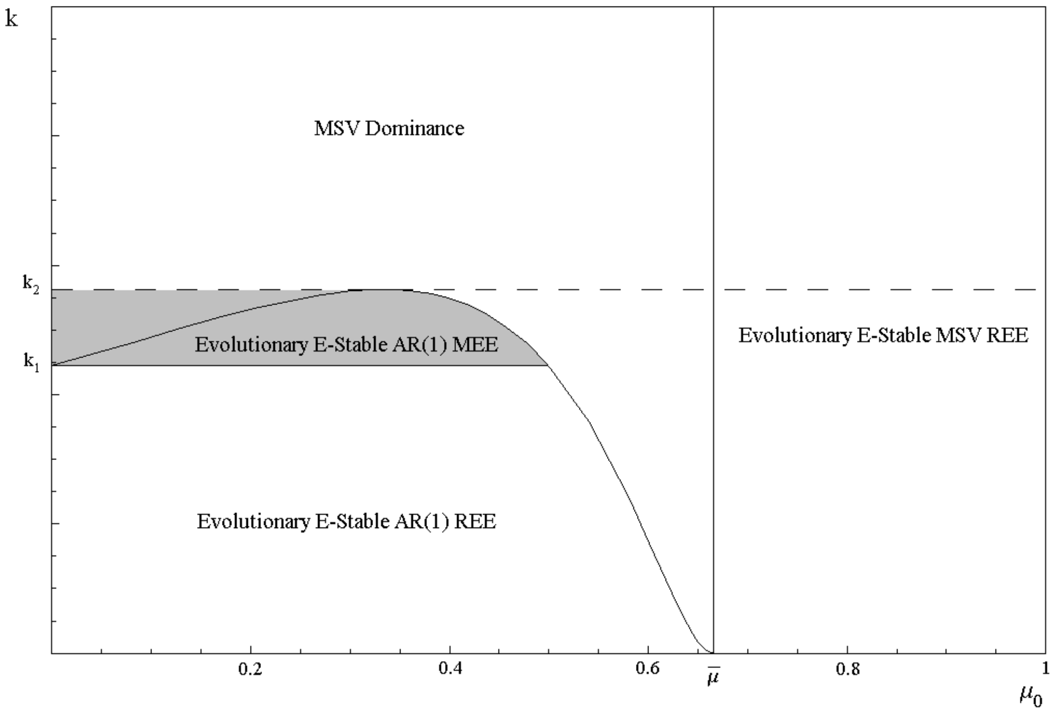

FIGURE 7. Evolutionary E-stability Conditions

In the previous section, the system converged to the Nash equilibrium was dependent upon the initial population level, μ_0. Figure 7 presents the Nash solution for every corresponding μ_0 and k for $2 < \beta_0 < 4$. The curve in the figure represents μ_1 and μ_2 for the corresponding cost, k. Recall that these Nash solutions did not exist for some (β_0, β_1, k). This curve will shift to the left as β_0 decreases and disappear when $\beta_0 < 1$, the case where the MSV REE is evolutionary E-stable for all $\mu_0 \in [0, 1]$.

When μ_0 is to the left of this curve, the resulting Nash solution of the system is $\mu = 0$ for $k < k_1$ and $\mu = \mu_1$ for $k_1 < k < k_2$. Therefore, the corresponding parameter equilibria are the AR(1) REE and the AR(1) MEE respectively. When μ_0 is to the right of this curve, the resulting Nash solution is $\mu = 1$ with the corresponding MSV parameter equilibrium. The resulting Nash solutions are path dependent when the cost of using the AR(1) predictor is below k_2. However, when $k > k_2$, for all $\mu \in (0, 1]$, the model is MSV dominant. Therefore, path dependence of the Nash solution no longer exists when the cost of the AR(1) predictor is sufficiently high.

MSV dominance provides an interesting result to consider the relevance of the two solutions of the above model. McCallum [24] shows that if a model is "well formulated",[15] parameters are such that expectations do not have implausible discontinuities, then the MSV solution should be E-stable for a specific self-referential linear stochastic model with an endogenous lag term. However, in the above model, a endogenous lag term does not exist. Following McCallum (2002), for homogeneous expectations,

$$E_{t-1}y_t = \frac{\alpha}{1-\beta_0} + \frac{1}{1-\beta_0}\beta_1 E_{t-1}y_{t+1}.$$

One may believe that for the model to be well formulated, it must be that $\beta_0 < 1$. However, under the only REE solution of $\left(\frac{\alpha}{1-\beta_0-\beta_1}, 0\right) = \left(\frac{\alpha}{1-\beta_1}, 0\right)$, when $\beta_0 = 1$, it follows that

$$E_{t-1}y_t = E_{t-1}y_{t+1} = \frac{\alpha}{1-\beta_0-\beta_1}$$

so that expectations here are well defined.[16] For this model, a reasonable condition for the model to be well formulated is that $\beta_0 + \beta_1 < 1$. Note that for every possible $(\beta_0, \beta_1) \in A \cup S$, it follows that $\beta_0 + \beta_1 < 1$.[17] Therefore, the model discussed in this paper is well formulated, and under both homogenous and heterogeneous expectations, the MSV solution is E-unstable for some $(\beta_0, \beta_1) \in A \cup S$. With MSV dominance, even the existence of a single agent who believes that the law of motion is MSV, will provide a result of asymptotic homogeneity of the MSV predictor provided the cost parameter is large enough. This result provides a reasonable situation where the MSV solution may be the relevant solution even when it is not initially learnable.

5 Conclusion

This paper introduces the use of evolutionary learning to further evaluate REE under learning. Furthermore, it investigates the possibility of more equilibria defined under heterogeneous expectations. Evolutionary and adaptive learning are combined so agents not only learn the parameter values of a

[15] My objective in discussing this somewhat controversial term is not to defend its use, but to show that a "well formulated model" does not guarantee that the MSV solution is E-stable.

[16] Neither solution is E-stable, so one must assume that agents believe that the model is such that $\beta_0 \neq 1$ for homogeneous expectations and that $\beta_0 \neq \frac{1}{1-\mu}$ for heterogeneous expectations. This assumption is common in the literature.

[17] In fact, the condition for the model to be stationary for every E-stable AR(1) solution is a stronger condition than the condition for a well formulated model.

perceived equilibrium, they also learn the "best" equilibrium to learn.

The paper investigates a well discussed model with the possibility of multiple equilibria and shows that each solution may be stable under the combined evolutionary-adaptive learning dynamics. In these equilibria, one of the two predictors is always superior to the other, so the superior predictor is used by all agents. I also discover that there is a possibility of convergence to an equilibrium where both predictors have the same quality of prediction. It turns out that the for a large enough cost of using the AR(1) predictor, the less parsimonious predictor becomes the unambiguously preferred predictor. This results in a global convergence of the minimum state variable (MSV) as long as at least one individual initially believes the model to be of this form. This result suggests that the MSV solution, of the above model, may be the universally relevant solution even when it is not initially learnable.

I believe that this type of modelling may be useful in macroeconomic literature such as monetary policy and business cycles. The combination of adaptive learning and evolutionary learning produces interesting dynamics that can be studied in some economic models. In one application, the parameter values of a model may change over time due to exogenous or endogenous shocks. As a result, there may not be convergence to a single equilibrium as found in this paper, but instead there may be a non-converging dynamic system. This is a topic of current research.

6 Appendix

6.1 Proofs

Proof of Proposition 3

For $k \notin \left(\frac{(1-\beta_0)^2}{\beta_1^2\sigma^2}, \frac{\beta_0^4}{16\beta_1^2\sigma^2}\right)$ *the only evolutionary stable Nash MEE with* $\mu \in (0,1)$ *is the AR(1) Nash solution* $\mu = \mu_3$. *For this value of* μ, *we find that* $\beta_0 < \frac{1}{1-\mu}$. *The AR(1) solution is not E-stable at this value. For* $k \in \left(\frac{(1-\beta_0)^2}{\beta_1^2\sigma^2}, \frac{\beta_0^4}{16\beta_1^2\sigma^2}\right)$ *and* $2 < \beta_0 < 4$, *there exists two evolutionary stable Nash equilibria with* $\mu \in (0,1)$. *These two solutions are* $\mu = \mu_1$ *and* $\mu = \mu_3$. *We find that*

$$\beta_0 < \frac{1}{1-\mu_3},$$

so this solution is not E-stable. We also find that

$$\beta_0 > \frac{1}{1-\mu_1},$$

so this solution is evolutionary stable with an E-stable MEE. ■

Proof of Proposition 4

When $0 \le k < \frac{(1-\beta_o)^2}{\beta_1^2\sigma^2}$*, solutions to the replicator dynamics are:*

$$\begin{aligned}
\mu_t &= 0 \\
\mu_t &= 1 + \frac{\beta_0 - \sqrt{\beta_0^2 + 4\beta_1\sqrt{k\sigma^2}}}{2\beta_1\sqrt{k\sigma^2}} = \mu_2 \\
\mu_t &= 1 - \frac{\beta_0 - \sqrt{\beta_0^2 - 4\beta_1\sqrt{k\sigma^2}}}{2\beta_1\sqrt{k\sigma^2}} = \mu_3 \\
\mu_t &= 1
\end{aligned}$$

For μ_1 *and* μ_2*, we must see when these solutions are between zero and one. For* $k = 0$*, it turns out that*

$$\mu_2 = \mu_3 = 1 - \frac{1}{\beta_0},$$

so here both solutions are between zero and one. For $k = \infty$*, we see that*

$$\mu_3 = 1$$

and μ_1 *is an imaginary number. Therefore,* $\mu_3 \in \left(1 - \frac{1}{\beta_0}, 1\right)$ *for all* $k > 0$*, and* $\mu \notin [0,1]$ *for some* $k > 0$*. However, it turns out that* $\mu_2 \in [0,1]$ *when* $0 \le k < \frac{(1-\beta_0)^2}{\beta_1^2\sigma^2}$*. As k increases, we see that* μ_2 *becomes smaller and* μ_3 *becomes larger. The four solutions of the replicator dynamics are as follows:*

$$0 < \mu_2 \le \mu_3 < 1 \text{ for } 0 \le k < \frac{(1-\beta_0)^2}{\beta_1^2\sigma^2}$$

In the above region for k, the slope of the replicator dynamics,

$$0 < \frac{\partial\mu_t}{\partial\mu_{t-1}}|_{\mu=0} < 1$$

and

$$\frac{\partial \mu_t}{\partial \mu_{t-1}}|_{\mu=\mu_2} > 1.$$

So if $\mu_0 < \mu_2$, then the system will converge to $\mu = 0$ and if $\mu_0 > \mu_2$, the system will diverge away from $\mu = 0$. Therefore the replicator dynamics are stable under the above conditions. ■

Proof of Corollary 1

Proposition 1 shows the stability properties for the replicator dynamics. For the AR(1) solution to be E-stable, it must be that:

$$\begin{aligned} \frac{1}{1-\mu} &< \beta_0 < \frac{1}{1-\mu} - (1-\mu)\beta_1 \\ \beta_1 &< 0. \end{aligned}$$

We have assumed that $(\beta_0, \beta_1) \in S \cup A$, so that

$$\beta_0 < \frac{1}{1-\mu} - (1-\mu)\beta_1$$

and

$$\beta_1 < 0.$$

Also note that $\mu_2 \leq 1 - \beta_0^{-1}$, so for any μ_0 sufficiently close to $\mu = 0$, it must be that $\beta_0 > \frac{1}{1-\mu_0}$. Therefore, the AR(1) REE is evolutionary E-stable for

$$0 \leq k < \frac{(1-\beta_0)^2}{\beta_1^2 \sigma^2}.$$

■

Proof of Proposition 5

The only solution to the replicator dynamics under the MSV MEE is

$$\mu = 1.$$

It can be shown that

$$\frac{\partial \mu_t}{\partial \mu_{t-1}}|_{\mu=1} < 1,$$

so the replicator dynamics are stable here. Also, as long as

$$\mu_0 > 1 - \frac{1}{\beta_0},$$

the MEE, for all μ_t, is E-stable. Therefore, the MSV REE is Evolutionary E-stable and the replicator dynamics are always stable for all

$$\begin{aligned} 1 - \frac{1}{\beta_0} &< \mu_0 \leq 1 \text{ if } \beta_0 > 1 \\ 0 &< \mu_0 \leq 1 \text{ if } \beta_0 < 1 \end{aligned}$$

if $k > 0$. ■

Proof of Proposition 6

When $2 < \beta_0 < 4$, solutions to the replicator dynamics in this case are the following:

$$\begin{aligned} \mu &= 0 \\ \mu &= 1 + \frac{\beta_0 + \sqrt{\beta_0^2 + 4\beta_1\sqrt{k\sigma^2}}}{2\beta_1\sqrt{k\sigma^2}} = \mu_1 \\ \mu &= 1 + \frac{\beta_0 - \sqrt{\beta_0^2 + 4\beta_1\sqrt{k\sigma^2}}}{2\beta_1\sqrt{k\sigma^2}} = \mu_2 \\ \mu &= 1 - \frac{\beta_0 - \sqrt{\beta_0^2 - 4\beta_1\sqrt{k\sigma^2}}}{2\beta_1\sqrt{k\sigma^2}} = \mu_3 \\ \mu &= 1 \end{aligned}$$

For $\mu_1 \in [0,1]$, it must be that

$$\frac{(1-\beta_0)^2}{\beta_1^2\sigma^2} \leq k \leq \frac{\beta_0^4}{16\beta_1^2\sigma^2}.$$

The derivative of the replicator dynamics at $\mu = 0$, $\mu = \mu_1$, and $\mu = \mu_2$ are as follows:

$$\begin{aligned} \frac{\partial \mu_t}{\delta \mu_{t-1}}|_{\mu=0} &> 1 \\ \frac{\partial \mu_t}{\delta \mu_{t-1}}|_{\mu=\mu_1} &< 1 \\ \frac{\partial \mu_t}{\delta \mu_{t-1}}|_{\mu=\mu_2} &> 1. \end{aligned}$$

Therefore, the AR(1) MEE is stable under the replicator dynamics for the stated above values.. ■

Proof of Corollary 2

Proposition 6 shows the stability properties for the replicator dynamics. For the AR(1) solution to be E-stable, it must be that:

$$\begin{aligned} \frac{1}{1-\mu} &< \beta_0 < \frac{1}{1-\mu} - (1-\mu)\beta_1 \\ \beta_1 &> 0. \end{aligned}$$

We have assumed that $(\beta_0, \beta_1) \in S \cup A$*, so that*

$$\beta_0 < \frac{1}{1-\mu} - (1-\mu)\beta_1$$

and

$$\beta_1 < 0.$$

Also note that $\mu_2 \leq 1 - \beta_0^{-1}$*, so for any* μ_0 *sufficiently close to* $\mu = \mu_1$*, it must be that* $\beta_0 > \frac{1}{1-\mu_0}$*. Therefore, the AR(1) MEE is evolutionary E-stable for*

$$k \in \left(\frac{(1-\beta_0)^2}{\beta_1^2 \sigma^2}, \frac{\beta_0^4}{16\beta_1^2\sigma^2} \right).$$

■

Proof of Proposition 7

There is only one Nash solution in this case, $\mu = \mu_3$. The derivative of the replicator dynamics is

$$\frac{\partial \mu_t}{\partial \mu_{t-1}}|_{\mu=\mu_3} < 1.$$

In this case the replicator dynamics move μ *toward* $\mu = \mu_3 > 1 - \frac{1}{\beta_0}$*. We also see that the MEE solutions as* $\mu \to 1 - \beta_0^{-1}$ *are:*

$$\begin{aligned} \lim_{\mu \to (1-\beta_0^{-1})^-} \frac{1-(1-\mu)\beta_0}{(1-\mu)\beta_1} &= 0 \\ \lim_{\mu \to (1-\beta_0^{-1})^-} \frac{A\alpha}{A - \mu\lambda - (1-\mu)A\lambda} &= \frac{\alpha}{1-\beta_0-\beta_1} \end{aligned}$$

This means that as $\mu \to 1-\beta_0^{-1}$, the MEE goes from the AR(1) solution to the MSV solution. Since we assumed fast-slow dynamics, the replicator dynamics move μ slow enough, and b_2 and a_2 will be such that $b_2 \in nbhd(b_2 = 0)$ and $a_2 \in nbhd(a_2 = \frac{\alpha}{1-\beta_0-\beta_1})$. When the dynamics move us to $\mu > 1-\beta_0^{-1}$, we are in the area of MSV E-stability. The fast-slow dynamic assumption leads us to know that $b_2 \in nbhd(b_2 = 0)$ and $a_2 \in nbhd(a_2 = \frac{\alpha}{1-\beta_0-\beta_1})$, so the MSV solution is E-stable. The inequality above implies that $k > 0$, so the MSV REE is stable under the replicator dynamics. Therefore, MSV dominance has occurred. ■

6.2 Calculation of the MSE for both of the PLM's

6.2.1 MSE for the first PLM

PLM1:

$$
\begin{aligned}
MSE_1 &= E(y-a_1)^2 \\
&= E(T_{a2} + T_{b2}y_{t-1} + v_t - a_1)^2 \\
&= E(T_{a2}^2) + E(T_{b2}^2 y_{t-1}^2) + E(v_t^2) + E(a_1^2) + 2E(T_{a2}T_{b2}y_{t-1}) - 2E(T_{a2}a_1) - 2E(a_1 T_{b2} y_{t-1}) \\
&= a_2^2 + b_2^2 E(y_{t-1}^2) + \sigma_v^2 + a_1^2 + 2a_2 b_2 a_1 - 2a_1 a_2 - 2a_1^2 b_2 \\
&= a_2^2 + \frac{b_2^2}{1-b_2^2}(a_2^2 + 2a_1 a_2 b_2 + \sigma_v^2) + \sigma_v^2 + a_1^2 + 2a_1 a_2 b_2 - 2a_1 a_2 - 2a_1^2 b_2 \\
&= (a_2^2 + 2a_1 a_2 b_2^3 + \sigma_v^2)\frac{1}{1-b_2^2} + a_1^2 + 2a_1 a_2 b_2 - 2a_1 a_2 - 2a_1^2 b_2
\end{aligned}
$$

If b=0 then the MSE from the first predictor becomes:

$$MSE_1 = \sigma_v^2$$

When we enter the MEE values in for the MSE_1 we get the following solution:

$$MSE_1 = \frac{(1-\mu)^4 \sigma^2 \beta_1^2}{(1-\mu)^4 \beta_1^2 - (1-(1-\mu)\beta_0)^2}$$

6.2.2 MSE for the second PLM

PLM2:

$$
\begin{aligned}
MSE_2 &= E(y - a_2 - b_2 y_{t-1})^2 \\
&= E(T_{a2} + T_{b2} y_{t-1} + v_t - a_2 - b_2 y_{t-1})^2 \\
&= a_2^2 + b_2^2 E(y_{t-1}^2) + \sigma_v^2 + a_2^2 + b_2^2 E(y_{t-1}^2) + 2a_1 a_2 b_2 - 2a_2^2 - 2a_1 a_2 b_2 \\
&\quad -2a_1 a_2 b_2 - 2b_2^2 E(y_{t-1}^2) + 2a_1 a_2 b_2 \\
&= \sigma_v^2
\end{aligned}
$$

So the mean square error for the second predictor will always be σ_v^2 as long as y follows a stationary process. This means that the $MSE_1 \geq MSE_2$ for all E-stable stationary values of α, β_0, and β_1. This intuitively makes sense because the AR(1) predictor is always unbiased while the MSV predictor is unbiased only when $b_2 = 0$.

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