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On the Aggregation of Quadratic Micro Equations

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Abstract

Aggregation of a flexible functional form - namely quadratic functions are considered. The representation proceeds from simple to more complicated and ends with aggregating an arbitrary number of agent-specific quadratic functions.

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1   Introduction

We study the aggregation of a flexible functional form, namely quadratic functions. It provides trivially a quadratic approximation for any twice differentiable function at a given point, say the mean point. Part of the results is presented and applied e.g. in Edgren, Turkkila, Vartia (1978). Presentation is kept intentionally elementary in its relation to mathematics, so that we see concretely the problems of aggregation and where the separate terms in the macro equation emerge from. The paper is essentially an English translation[1] of a former Finnish paper Vartia (1979).

2   Micro data and micro equations

Consider n separate agents $a_1, \ldots, a_n$ (for example individual persons, households, firms, industries or countries). For each agent there's data measured for variables ψ, x and y, which form the following <u>data matrix</u>

$$
\begin{array}{cccc}
a & \psi & x & y \\
\left[\begin{array}{cccc}
a_1 & \psi_1 & x_1 & y_1 \\
a_2 & \psi_2 & x_2 & y_2 \\
\vdots & \vdots & \vdots & \vdots \\
a_n & \psi_n & x_n & y_n
\end{array}\right]
\end{array}
$$

Variable ψ represents here the explained variable (for example household's consumption), x and y are the explanatory variables (for example income and the household's size). All of the variables are presumed to be quantitative, therefore interval-, rational- or absolute scale variables. Ordering or classifying variables are not considered explicitly, though similar results can be derived for those as well, see Edgren, Turkkila, Vartia (1978, pages 52-62). Furthermore variables are considered as continuous. Let us first consider a

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simple case. Suppose, that between the micro level variables ψ_i, x_i, y_i there's a linear constant coefficient's relation:

$$\psi_i = \alpha + \beta_1 x_i + \beta_2 y_i \quad , i = 1, \ldots, n. \tag{1}$$

So, for every a_i there is an affine ("linear") expression ψ_i of the variables (x_i, y_i), where the coefficients $(\alpha, \beta_1, \beta_2)$ <u>are the same for the each household</u>. (At first we consider only deterministic equations; later it is easy to add necessary error terms). Specification (1) means, that there exists a function $g : R^2 \rightarrow R$ such that

$$g(x, y) = \alpha + \beta_1 x + \beta_2 y \tag{2}$$

and for each $i = 1, \ldots, n$ we have

$$\psi_i = g(x_i, y_i). \tag{3}$$

This is a highly restrictive hypothesis, which is soon to be relaxed.

3   Macro data

We consider two cases here. In the first case, macro level variables are simply sums of the micro level variables, which are denoted as capital letters:

$$\begin{aligned}
\Psi &= \sum_{i=1}^{n} \psi_i \\
X &= \sum x_i \\
Y &= \sum y_i.
\end{aligned} \tag{4}$$

If ψ_i and x_i are consumption and income in euros in an household a_i, then Ψ and X are total consumption and total income in euros (for example in an economy). For instance this simple aggregation, direct addition, is used in the System of National Account SNA (as in accounting systems generally). Therefore totals are important and natural aggregates. Later

we can examine more complicated definitions of macro level variables, like geometric means, linear combinations, etc., which may be relevant in other situations. Direct sums (4) describe totals in macro level; usually for example X's size is significantly larger than any x_i's. Often it is natural and simple to examine arithmetic means instead of sums (4):

$$\overline{\psi} = \frac{1}{n} \sum \psi_i = \frac{\Psi}{n} \tag{5}$$
$$\overline{x} = \frac{1}{n} \sum x_i = \frac{X}{n}$$
$$\overline{y} = \frac{1}{n} \sum y_i = \frac{X}{n} \quad.$$

In that case one commonly thinks of studying "mean family", "mean consumer" or similar representative agents. In per capita studies means (5) are used instead of sums (4). It is to be noted, that transformation from means to sums is not necessary a trivial operation. Only, if n is a constant (for example in time) and it is natural to choose n in a single way, this transformation is almost a trivial operation. Commonly it is not clear, what are the statistical units $a_1, \ldots, a_n$ and what is the number of them: for example households and industries can be defined in several ways.

A traditional research problem in aggregation is to examine the relations between the macro level variables (Ψ, X, Y) or $(\overline{\psi}, \overline{x}, \overline{y})$.

Usual convention (almost an obsession) is that, if there exists a relation

$$\psi_i = g(x_i, y_i) \tag{6}$$

between the micro level variables, so the same (or same kind) relation is assumed to exist in the macro level:

$$\Psi = G(X, Y) \tag{7}$$

or

$$\overline{\psi} = \overline{g}(\overline{x}, \overline{y}) \tag{8}$$

3

This assumption does not necessarily hold, unless in very simple cases, which we consider in the following section. Usually Ψ depends also from other factors than totals the X and Y, and $\bar{\psi}$ depends from other quantities than the means $\bar{x}$ and $\bar{y}$. Especially variances and covariances of x and y have an effect to the size and variation of macro variables Ψ and $\bar{\psi}$, as we shall show in the following chapters. But first we shall consider the case where these complications do not arise.

4   The special case, where simple aggregation is appropriate

Equations (1)-(3) describe a simple situation, where in macro and micro levels we achieve similar simple equations. Let us examine factors which Ψ depends on:

$$
\begin{aligned}
\Psi &= \sum \psi_i && (9)\\
&= \sum (\alpha + \beta_1 x_i + \beta_2 y_i)\\
&= \sum \alpha + \sum \beta_1 x_i + \sum \beta_2 y_i\\
&= \sum \alpha + \beta_1 \sum x_i + \beta_2 \sum y_i\\
&= n\alpha + \beta_1 X + \beta_2 Y.
\end{aligned}
$$

We observe, that there exists a function $G : R^2 \to R$ such that

$$
G(x, y) = n\alpha + \beta_1 x + \beta_2 y \tag{10}
$$

and for all possible (Ψ, X, Y) values

$$
\Psi = G(X, Y). \tag{11}
$$

Here we have assumed, that $(\alpha, \beta_1, \beta_2)$-parameters and the frequency n do not change from a situation to another. If for example the number n changes

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from year to year, it must be taken into account as an argument of the G-function. But within these very restrictive assumptions, between the macro level total variables (Ψ, X, Y) there really exists a relation (10)-(11) similar to micro level relation $\psi_i = g(x_i, y_i)$. This so-called <u>principle of analogy</u> is not always logically inconsistent, but there exists "Simpletans" (possible worlds), where it realizes. This result is not a new one. It is proposed e.g. by J.S.Cramer (1971, s. 175-181) and Chipman (1976, s. 618-626), who formulates so-called conditions of the "perfect aggregation" with abstract mathematical concepts. See also Klein (1946 a, b) , Malinvaud (1954), Allen (1959, chapter 20) and Theil (1965). The problem is, how to progress when the situation is more complicated. In a similar way we get for the arithmetic mean:

$$
\begin{aligned}
\bar{\psi} &= \frac{1}{n} \sum \psi_i = \frac{\Psi}{n} \\
&= \frac{1}{n} \sum (\alpha + \beta_1 x_i + \beta_2 y_i) \\
&= \frac{1}{n} (n\alpha + \sum \beta_1 x_i + \sum \beta_2 y_i) \\
&= \alpha + \beta_1 \frac{X}{n} + \beta_2 \frac{Y}{n} \\
&= \alpha + \beta_1 \bar{x} + \beta_2 \bar{y}
\end{aligned}
\tag{12}
$$

We have shown all the intermediate phases, in order to see later complications in a correct light. So there exists a function $\bar{g} : R^2 \to R$ such that

$$
\bar{g}(x, y) = \alpha + \beta_1 x + \beta_2 y
\tag{13}
$$

and for all possible $(\bar{\psi}, \bar{x}, \bar{y})$ values

$$
\bar{\psi} = \bar{g}(\bar{x}, \bar{y}).
\tag{14}
$$

Furthermore, $\bar{g}$-function in (13) and g-function in (2) are the same function:

$$g(x,y) = \bar{g}(x,y) = \alpha + \beta_1 x + \beta_2 y \ . \tag{15}$$

Between the means $(\bar{\psi}, \bar{x}, \bar{y})$ and all micro variables (ψ_i, x_i, y_i) holds exactly same relation in this Simpletan. This result has been the main guideline in the analogy, when the principle of aggregation has been applied. It has been imagined, that also in more general situations the macro equation should necessarily be similar to the micro equation. We will see, that usually it is not so, but in the macro equation there emerges explanatory variables, which do not exist in the micro level.

5 Aggregation of linear micro equations

Consider the following case, where micro equations are, for simplicity, still linear, but the parameters vary from one micro equation to another:

$$\psi_i = \alpha_i + \beta_{1i} x_i + \beta_{2i} y_i. \tag{16}$$

Earlier we considered a case, where all the parameters a_i of the micro units were same: $\alpha_i = \alpha$, $\beta_{1i} = \beta_1$, $\beta_{2i} = \beta_2$, $i = 1, \ldots, n$. Now consider shortly those difficulties, which arise from this kind of complications. Every micro unit a_i has thus its own affine behavior function: $g_i : R^2 \to R$ such that

$$g_i(x,y) = \alpha_i + \beta_{1i} x + \beta_{2i} y \tag{17}$$

and for all possible (ψ_i, x_i, y_i) values:

$$\psi_i = g_i(x_i, y_i). \tag{18}$$

I will not discuss here common misunderstandings of the concept of a function. I just note, that equations (17) and (18) specify the meaning of (16). We can state, that the Simpletan described in these equations is still very

special, we are still considering a special case. Let us consider again the macro total Ψ:

$$
\begin{aligned}
\Psi &= \sum \psi_i \\
&= \sum (\alpha_i + \beta_{1i} x_i + \beta_{2i} y_i) \\
&= \sum \alpha_i + \sum \beta_{1i} x_i + \sum \beta_{2i} y_i.
\end{aligned} \tag{19}
$$

Most writers have stopped here, see for example J.S.Cramer (1971, p. 177)and Allen (1959, chapter 20). How to move on? At this stage we take so-called The Basic Lemma of Aggregation BLA into use, by which Edgren, Turkkila and Vartia(1978, p. 15) refer to the following identity:

$$
\sum_{i=1}^{n} w_i x_i y_i = \bar{x}\bar{y} + cov(x, y), \tag{20}
$$

where $\bar{x} = \sum w_i x_i$, $\bar{y} = \sum w_i y_i$, $cov(x, y) = \sum w_i (x_i - \bar{x})(y_i - \bar{y})$ and the sum of weights w_i is one. This transforms an average of a product to a product of averages (and a correction). Weights do not need to be positive or even non-negative, because $\sum w_i = 1$ suffices to infer (20. (If we have negative weights, averages are in fact affine combinations, not actually arithmetic averages i.e. convex combinations.) Derivation is purely algebraic and it goes as follows:

$$
\begin{aligned}
cov(x, y) &= \sum w_i (x_i - \bar{x})(y_i - \bar{y}) \\
&= \sum w_i (x_i y_i - x_i \bar{y} - \bar{x} y_i + \bar{x}\bar{y}) \\
&= \sum w_i x_i y_i - \bar{y} \sum w_i x_i - \bar{x} \sum w_i y_i + \bar{x}\bar{y} \sum w_i \\
&= \sum w_i x_i y_i - \bar{y}\bar{x} - \bar{x}\bar{y} + \bar{x}\bar{y} \\
&= \sum w_i x_i y_i - \bar{x}\bar{y}.
\end{aligned} \tag{21}
$$

Truly only the definitions and the condition $\sum w_i = 1$ are used. Non-negativity of the weights ia not needed in the derivation. Numbers x_i and

y_i can be arbitrary real numbers (or any variables) and the weights w_i can as well be arbitrary once $\sum_{i=1}^{n} w_i = 1$. Especially choosing $w_i = \frac{1}{n}$ we get following identities:

$$
\begin{aligned}
\frac{1}{n} \sum x_i y_i &= (\frac{1}{n} \sum x_i)(\frac{1}{n} \sum y_i) \\
&\quad + \frac{1}{n} \sum (x_i - \frac{1}{n} \sum x_i)(y_i - \frac{1}{n} \sum y_i) \\
&= \bar{x}\,\bar{y} + \frac{1}{n} \sum (x_i - \bar{x})(y_i - \bar{y}) \\
&= \bar{x}\,\bar{y} + cov(x,y)
\end{aligned}
\tag{22}
$$

$$
\begin{aligned}
\sum x_i y_i &= (\sum x_i)(\sum y_i)/n + \sum (x_i - \bar{x})(y_i - \bar{y}) \\
&= XY/n + ncov(x,y).
\end{aligned}
\tag{23}
$$

Understanding these results requires knowledge of some important properties of the covariance $cov(x,y)$. Generally we can write (though symbols have here somewhat more general meaning than usually):

$$
\begin{aligned}
cov(x,y) &= \sum w_i(x_i - \bar{x})(y_i - \bar{y}) \\
&= \frac{cov(x,y)}{\sqrt{cov(x,x)cov(y,y)}} \sqrt{cov(x,x)cov(y,y)} \\
&= \frac{cov(x,y)}{s(x)s(y)} s(x)s(y) \\
&= r(x,y)s(x)s(y),
\end{aligned}
\tag{24}
$$

where $s(x) = \sqrt{cov(x,x)} = \sqrt{\sum w_i(x_i - \bar{x})^2}$ and $r(x,y) = cov(x,y)/s(x)s(y)$. If weights w_i are non-negative, then $\bar{x}$ and $\bar{y}$ are weighted-means, $s(x)$ and $s(y)$ are their standard deviations, $cov(x,y)$ their covariance and $r(x,y)$ their correlation coefficient. In that case $s(x)$ and $s(y)$ are real and non-negative

and $-1 \leq r(x,y) \leq 1$. More about properties of covariance and correlation-coefficient, see Vasama-Vartia (1973, p. 422-427). Lets get back to equation (19) and apply identity (23) to the term $\sum_i \beta_{1i} x_i$:

$$
\begin{aligned}
\sum \beta_{1i} x_i &= (\sum \beta_{1i})(\sum x_i)/n + ncov(\beta_{1i}, x_i) \\
&= \bar{b}_1 X + ncov(\beta_{1i}, x_i).
\end{aligned}
\tag{25}
$$

For clarity, we sometimes use indexes in the arguments of covariances, as above. These emphasize the quantities which are varying. In same way we develop term $\sum \beta_{2i} y_i$ and get:

$$
\begin{aligned}
\Psi &= \sum \alpha_i + \sum \beta_{1i} x_i + \sum \beta_{2i} y_i \\
&= n\bar{\alpha} + \bar{\beta}_1 X + \bar{\beta}_2 Y + ncov(\beta_{1i}, x_i) + n\, cov(\beta_{2i}, y_i).
\end{aligned}
\tag{26}
$$

This is the desired presentation. Coefficients of totals X and Y are the means $\bar{\beta}_1 = \frac{1}{n}\sum \beta_{1i}$, $\bar{\beta}_2 = \frac{1}{n}\sum \beta_{2i}$ of parameters β_{1i} and β_{2i}. In addition there are two covariance terms multiplied by n in macro equation. These covariance terms disappear, if β-coefficients do not vary from micro unit to another. This was assumed in the previous chapter. Similarly, for example the covariance $cov(\beta_{1i}, x_i)$ is zero, if x_i-values (for example incomes) are the same in every micro unit (in which case $s(x) = 0$). Same happens if β_{1i} - and x_i -values do not correlate with each other, $r(\beta_{1i}, x_i) = 0$. All this appear from the equation

$$
cov(\beta_{1i}, x_i) = s(\beta_{1i})s(x_i)r(\beta_{1i}, x_i)
\tag{27}
$$

So far we have got the following result in this Simpletan under consideration. Total variable Ψ depends on totals X , Y and covariances $cov(\beta_1, x)$ and $cov(\beta_2, y)$ so, that there exists a function $H : R^4 \to R$ such that

$$
H(x, y, c_1, c_2) = n\bar{\alpha} + \bar{\beta}_1 x + \bar{\beta}_2 y + nc_1 + nc_2 \ ,
\tag{28}
$$

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where $\bar{\alpha} = \frac{1}{n} \sum \alpha_i$, $\bar{\beta}_1 = \frac{1}{n} \sum \beta_{1i}$, $\bar{\beta}_2 = \frac{1}{n} \sum \beta_{2i}$ and for all five tuples $(\Psi, X, Y, cov(\beta_1, x), cov(\beta_2, y))$ we have

$$\Psi = H\big(X, Y, cov(\beta_1, x), cov(\beta_2, y)\big). \tag{29}$$

In this exact sense <u>mentioned two covariances are exactly comparable</u> <u>explanatory variables to totals X and Y</u>, when the variation of the total Ψ is to be explained. These results are even simpler for the means. They can be derived similarly as before, but we can get the result directly dividing by n. For the means in this Simpletan holds:

$$\bar{\psi} = \bar{\alpha} + \bar{\beta}_1 \bar{x} + \bar{\beta}_2 \bar{y} + cov(\beta_1, x) + cov(\beta_2, y) . \tag{30}$$

This collapses to equation (12), if for every i-value $\beta_{1i} = \beta_1 = \bar{\beta}_1$ or in other words $s(\beta_1) = 0$ and $\beta_{2i} = \beta_2 = \bar{\beta}_2$ or $s(\beta_2) = 0$. More exactly, now there exists a function $h : R^4 \to R$ such that

$$h(x, y, c_1, c_2) = \bar{\alpha} + \bar{\beta}_1 x + \bar{\beta}_2 y + c_1 + c_2 , \tag{31}$$

where $\bar{\alpha} = \frac{1}{n} \sum \alpha_i$, $\frac{1}{n} \sum \beta_{1i}$, $\bar{\beta}_2 = \frac{1}{n} \sum \beta_{2i}$ and for all five tuples $(\bar{\psi}, \bar{x}, \bar{y}, cov(\beta_1, x), cov(\beta_2, y))$ we have

$$\bar{\psi} = h\big(\bar{x}, \bar{y}, cov(\beta_1, x), cov(\beta_2, y)\big). \tag{32}$$

In this sense mentioned covariances are explanatory variables in $\bar{\psi}$'s behavior equation. Similar results were derived for tax functions in Edgren, Turkkila, Vartia (1978). These results do not seem to be well-known in aggregation literature. [2] For example J.S. Cramer (1971, p. 177) and Allen (1959 chapter 20) advance from equation (19) towards a wrong direction. We

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[2]It was later found that van Dahl and Merkies (1984, p. 153, 159, 162-166) present similar macro equations with covariances. Also in risk analysis - say in defining the Arrow-Pratt risk measure - and in Itô Calculus in order to integrate stochastic processes the very same principles are followed.

do not stop here to comment these results in more detail, but continue to more general situations.

6 Aggregation of quadratic micro equations

We drop the assumption of linearity in micro equations and consider significantly more general quadratic situation:

$$\psi_i = \alpha_i + \beta_{1i}x_i + \beta_{2i}y_i + \beta_{11i}x_i^2 + \beta_{22i}y_i^2 + 2\beta_{12i}x_iy_i \qquad (33)$$

Parameters of the quadratic terms are twice the derivatives, e.g. $\psi_{i.11} = 2\beta_{11i}$
We examine the determination of $\bar{\psi}$:

$$
\begin{aligned}
\bar{\psi} &= \frac{1}{n}\sum \psi_i & (34)\\
&= \frac{1}{n}\sum(\alpha_i + \beta_{1i}x_i + \beta_{2i}y_i + \beta_{11i}x_i^2 + \beta_{22i}y_i^2 + 2\beta_{12i}x_iy_i)\\
&= \frac{1}{n}(\sum \alpha_i + \sum \beta_{1i}x_i + \sum \beta_{2i}y_i +\\
&\quad \sum \beta_{11i}x_i^2 + \sum \beta_{22i}y_i^2 + 2\sum \beta_{12i}x_iy_i)
\end{aligned}
$$

This starts to be quite messy. Linear terms are handled like before, so there are no difficulties with them. Let us consider the quadratic term $\frac{1}{n}(\sum \beta_{12i}x_iy_i)$ separately and apply the Basic Lemma of Aggregation BLA (22) to it in the following way:

$$
\begin{aligned}
\frac{1}{n}\sum \beta_{12i}x_iy_i &= (\frac{1}{n}\sum \beta_{12i})(\frac{1}{n}\sum x_iy_i) + cov(\beta_{12}, xy) & (35)\\
&= \bar{\beta}_{12}(\frac{1}{n}\sum x_iy_i) + cov(\beta_{12}, xy).
\end{aligned}
$$

But the sum here $(\frac{1}{n}\sum x_iy_i)$ is different from $(\frac{1}{n}\sum x_i)(\frac{1}{n}\sum y_i) = \bar{x}\bar{y}$, which is wanted as explanatory variable to macro equation. So, apply BLA (22) to the equation above again as follows:

$$\frac{1}{n}\sum \beta_{12i}x_iy_i = \bar{\beta}_{12}(\bar{x}\,\bar{y}+cov(x,y))+cov(\beta_{12},xy) \qquad (36)$$
$$= \bar{\beta}_{12}\bar{x}\,\bar{y}+\bar{\beta}_{12}cov(x,y))+cov(\beta_{12},xy)$$

This is the desired presentation! The term $\bar{\beta}_{12}\bar{x}\,\bar{y}$ in macro level corresponds exactly in form to the cross terms $\beta_{12i}x_i\,y_i$ in the micro level. In addition two covariance terms $\bar{\beta}_{12}cov(x,y)$ and $cov(\beta_{12},xy)$ have appeared to the macro equation. The former depends on the average level of the parameters β_{12i} and on the mutual variation of the explanatory variables x and y. The latter depends on the inter-dependancy of the parameters β_{12i} and the cross terms x_iy_i. Both terms are nearly zero, if the cross term x_iy_i has only a slight effect on the quantities ψ_i. In other words, if parameters β_{12i} are small. Similarly we get for the square terms

$$\frac{1}{n}\sum \beta_{11i}x_i^2 = \bar{\beta}_{11}\bar{x}^2+\bar{\beta}_{11}cov(x,x))+cov(\beta_{11},x2) \qquad (37)$$
$$= \bar{\beta}_{11}\bar{x}^2+\bar{\beta}_{11}s^2(x)+cov(\beta_{11},x^2)\ .$$

When all these the results are taken together, we get for the $\bar{\psi}$ its macro behavior equation

$$\bar{\psi} = \bar{\alpha}+\bar{\beta}_1\bar{x}+\bar{\beta}_2\bar{y} \qquad (38)$$
$$+\bar{\beta}_{11}\bar{x}^2+\bar{\beta}_{22}\bar{y}^2+2\bar{\beta}_{12}\bar{x}\,\bar{y}$$
$$+cov(\beta_1,x)+cov(\beta_2,y)$$
$$+\bar{\beta}_{11}s^2(x)+\bar{\beta}_{22}s^2(y)+2\bar{\beta}_{12}cov(x,y)$$
$$+cov(\beta_{11},x^2)+cov(\beta_{22},y^2)+2cov(\beta_{12},xy)\ .$$

Normal terms of the analog method are in the first two lines, where coefficients are intuitive average parameters. In the third line, the covariances appeared already in the linear case. These express to which extent high values

of explanatory variables correlate with households, whose reactions to these are exceptional. Actual novelties in regard to the earlier linear case are in the fourth and fifth line. These are the effects on $\bar{\psi}$ due to the variation and covariation in explanatory variables.

If x and y had been concentrated on their means, these effects would disappear, because then $s^2(x) = s^2(y) = cov(x, y) = 0$ (the last term can disappear otherwise as well). Usually these effects are ignored, which is, of course, logically incorrect. It is appropriate to compare the significance of the term $\bar{\beta}_{11}s^2(x)$ to ordinary term $\bar{\beta}_{11}\bar{x}^2$. In many cases economical variables (for example incomes, consumptions) have large variances $s^2(x)$. Because of the skewness of these distributions probably variances are larger than $\bar{x}^2$. (Namely $s^2(x) > \bar{x}^2 \Leftrightarrow s(x) > \bar{x} \Leftrightarrow s(x)/\bar{x} = $ *coefficient of variation* > 1. This is not an impossible situation!). Because the terms $\bar{\beta}_{11}\bar{x}^2$ and $\bar{\beta}_{11}s^2(x)$ have same coefficient, the effects of $\bar{x}^2$ and $s^2(x)$ on the variable $\bar{\psi}$ are directly proportional to their size. If $s^2(x)$ is left out from the list of the explanatory variables when $\bar{\psi}$ is being explained, other coefficients are probably miss-estimated as well. Especially it is probable, that $\bar{x}^2$ (which usually correlates positively with x's variance $s^2(x)$) would get overly high coefficient, for example in regression analysis.

Correspondingly we can interpret the terms $\bar{\beta}_{22}s^2(y)$ and $2\bar{\beta}_{12}cov(x, y)$. The latter term indicates, that change of the covariance of the explanatory variables would have a direct effect on ψ-variable, when the means and variances of the variables are kept the same (ceteris paribus). The covariance in the fifth line corresponds in its interpretation to the third line's covariances. These disappear, if the coefficients do not vary from micro unit to another. We formulate this mathematical result as a theorem.

Theorem 1 *Consider micro units $a_1, \ldots, a_n$ and variables ψ, x and y describing their properties. Assume, that for every micro unit there exists a*

function $g_i : R^2 \to R$ so that

$$\psi_i = g_i(x_i, y_i) \tag{39}$$

If $g_i(x, y)$ is a general quadratic function, whose coefficients can vary from micro unit to another

$$g_i(x_i, y_i) \;=\; \alpha_i + \beta_{1i}x + \beta_{2i}y \tag{40}$$
$$\beta_{11i}x^2 + \beta_{22i}y^2 + 2\beta_{12i}xy \;,$$

then between the arithmetic means of the variables (ψ_i, x_i, y_i) holds the identical equation

$$\begin{aligned}
\bar{\psi} \;=\;& \bar{\alpha} + \bar{\beta}_1\bar{x} + \bar{\beta}_2\bar{y} \\
&+\bar{\beta}_{11}\bar{x}^2 + \bar{\beta}_{22}\bar{y}^2 + 2\bar{\beta}_{12}\bar{x}\,\bar{y} \\
&+cov(\beta_1, x) + cov(\beta_2, y) \\
&+\bar{\beta}_{11}s^2(x) + \bar{\beta}_{22}s^2(y) + 2\bar{\beta}_{12}cov(x, y) \\
&+cov(\beta_{11}, x^2) + cov(\beta_{22}, y^2) + 2cov(\beta_{12}, xy).
\end{aligned} \tag{41}$$

So there exists a function $H : R^4 * R_+^2 * R^4 \to R$ such that

$$H\begin{bmatrix} x & y & \\ c_1 & c_2 & \\ s_1 & s_2 & c_3 \\ c_4 & c_5 & c_6 \end{bmatrix} = \begin{array}{ll} \bar{\alpha} \;+\; & \bar{\beta}_1 x + \bar{\beta}_2 y \\ + & \bar{\beta}_{11}x^2 + \bar{\beta}_{22}y^2 + 2\bar{\beta}_{12}xy \\ + & c_1 + c_2 \\ + & \bar{\beta}_{11}s_1^2 + \bar{\beta}_{22}s_2^2 + 2\bar{\beta}_{12}c_3 \\ + & c_4 + c_5 + 2c_6 \end{array} \tag{42}$$

and for this holds identically

$$\bar{\psi} = H\begin{bmatrix} \bar{x} & \bar{y} & \\ cov(\beta_1, x) & cov(\beta_2, x) & \\ s(x) & s(y) & cov(x, y) \\ cov(\beta_{11}, x^2) & cov(\beta_{22}, y^2) & cov(\beta_{12}, xy) \end{bmatrix}. \tag{43}$$

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Theorem (1) gives perfect solution to the aggregation oh the quadratic micro equations. Theorem states, that besides the explanatory variables in accordance with the principle of analogy, the deviances and covariances of the equation (42) must be used. If micro level's parameters are independent of i:s, covariances in the second and fourth lines disappear and we get following theorem.

Theorem 2 *Consider micro units $a_1, \ldots, a_n$ and variables describing their properties. Assume, that for every micro unit there exists a function independent of the micro unit $g : R^2 \to R$ so that*

$$\psi_i = g(x_i, y_i) \tag{44}$$

If $g(x, y)$ is a general quadratic function

$$\begin{aligned} g(x_i, y_i) \quad = \quad & \alpha + \beta_1 x + \beta_2 y \\ & + \beta_{11} x^2 + \beta_{22} y^2 + 2\beta_{12} xy \ , \end{aligned} \tag{45}$$

then between the arithmetic means of the variables (ψ_i, x_i, y_i) holds the identical equation

$$\begin{aligned} \bar{\psi} \quad = \quad & \alpha + \beta_1 \bar{x} + \beta_2 \bar{y} \\ & + \beta_{11} \bar{x}^2 + \beta_{22} \bar{y}^2 + 2\beta_{12} \bar{x} \, \bar{y} \\ & + \beta_{11} s^2(x) + \beta_{22} s^2(y) + 2\beta_{12} cov(x, y). \end{aligned} \tag{46}$$

So there exists a function $H : R^5 \to R$ such that

$$H \begin{bmatrix} x & y \\ s_1 & s_2 & c_3 \end{bmatrix} = \begin{aligned} \alpha & + & \beta_1 x + \beta_2 y \\ & + & \beta_{11} x^2 + \beta_{22} y^2 + 2\beta_{12} xy \\ & + & \beta_{11} s_1^2 + \beta_{22} s_2^2 + 2\beta_{12} c_3 \end{aligned} \tag{47}$$

and for this holds identically

$$\bar{\psi} = H \begin{bmatrix} \bar{x} & \bar{y} \\ s(x) & s(y) & cov(x, y) \end{bmatrix}. \tag{48}$$

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Theorem (2) indicates, that the micro parameters $(\alpha, \beta_1, \beta_2, \beta_{11}, \beta_{22}, \beta_{12})$ of the quadratic behavior equation (45) appear in macro equation as such as the coefficients of per capita-variables $\bar{x}, \bar{y}, \bar{x}^2, \bar{y}^2$ and $\bar{x}\bar{y}$. If you add to micro equations (45) additive error terms ε_i, whose expectations are zeros and which are independent of each other (heavy presumptions!), only the corresponding mean $\varepsilon = \frac{1}{n}\sum \varepsilon_i$ appears in the macro equation (46).

If we consider this situation in T different time periods (for which each and together) these assumptions hold and if corresponding error terms are independent of each other, the micro parameters (which are also macro parameters) can be estimated using the following data matrix:

$$
\begin{array}{cccccc}
t & \bar{x} & \bar{y} & s(x) & s(y) & cov(x,y) \\
\left[\begin{array}{c} 1 \\ 2 \\ \\ T \end{array}\right. & \begin{array}{c} \bar{x}_1 \\ \vdots \\ \vdots \\ \bar{x}_T \end{array} & \begin{array}{c} \bar{y}_1 \\ \\ \\ \bar{y}_T \end{array} & \begin{array}{c} s_1(x) \\ \\ \\ s_T(x) \end{array} & \begin{array}{c} s_1(y) \\ \\ \\ s_T(y) \end{array} & \left.\begin{array}{c} cov_1(x,y) \\ \vdots \\ \vdots \\ cov_T(x,y) \end{array}\right]
\end{array}
$$

Because the parameters are restricted, OLS-estimation is conveniently carried out from equation

$$
\begin{aligned}
\bar{\psi}_t &= \alpha + \beta_1 \bar{x}_t + \beta_2 \bar{y}_t \\
&\quad + \beta_{11}(\bar{x}_t^2 + s_t^2(x)) + \beta_{22}(\bar{y}_t^2 + s_t^2(y)) \\
&\quad + 2\beta_{12}(\bar{x}_t \bar{y}_t + cov_t(x,y)) + \bar{\varepsilon}_t.
\end{aligned} \tag{49}
$$

There are probably difficulties in estimation of the equation because of multicollinearity of the variables, see Goldberger (1964), but in principle estimation is clear. If $s_t^2(x)$, $s_t^2(y)$ and $cov_t(x, y)$ are omitted in the equation, we will get at least for the α parameter a biased and inconsistent estimator. Namely, if these variances and the covariance are constants in time, only the intercept of the equation is affected. Because variances and covariances usually don't remain exactly constant in time, but correlate with $\bar{x}_t$ and $\bar{y}_t$,

we will get at least for the β_{11}, β_{22} and β_{12} and probably for the β_1 and β_2 biased and inconsistent estimators because of omitted variable bias. Let these be examples of the dangers, which result when variances and covariances of explanatory variables are left out from the macro equation, in other words when macro equations are misspecified.

Equation (49) can be also used in the estimation of a model with varying parameters described in theorem 1, which includes error terms as described before. Reliability of the estimates of the mean parameters $(\bar{\alpha}, \bar{\beta}_1, \bar{\beta}_2, \bar{\beta}_{11}, \bar{\beta}_{22}, \bar{\beta}_{12})$ depends on how much the $cov_t(\beta_1, x)$, $cov_t(\beta_2, x), cov_t(\beta_1, x^2), cov_t(\beta_{22}, y^2)$ and $cov_t(\beta_{12}, xy)$ deviate from zero and correlate with the explanatory variables. These terms are usually almost constants and their size depends on the standard deviations of the corresponding parameters. For example

$$
\begin{aligned}
cov_t(\beta_{11}, x^2) &= cov_t(\beta_{11i}, x_i^2) \\
&= s_t(\beta_{11i})s_t(x_i^2)r(\beta_{11i}, x_i^2)
\end{aligned}
\tag{50}
$$

goes to zero with together $s_t(\beta_{11i})$. These terms depend on slowly changing structures of the society. So it is expectable, that only α parameters will be biased as described earlier, and other parameters can be estimated rather reliably. If there is knowledge about the size of the out left covariances of parameters, for example from the separate sampling study, we can naturally use this information by adding covariance terms to the model.

7   Conclusions

We have investigated how quadratic functions are aggregated. These produce trivially a flexible (namely quadratic) approximation for any twice differentiable function. Theorem 2 show how variances and covariances of all input variables emerge as new explanatory variables on the macro level even when all the micro units have the same quadratic behavior. If all the micro units

have their own quadratic behaviors, then according to theorem 1 also covariances of varying parameters and corresponding variables emerge as explanatory factors on the macro level. Estimating the macro equation is shortly sketched when we have the same micro equations from one unit to another and the data is only from the macro level. Our proposal serves also as an approximate method if the parameter-variable covariances of the individually heterogeneous functions are essentially constant. As a final conclusion, we want to remark that it is assumed in these estimations that individual data is not available and all the calculations in the estimation are based on macro information only. This may amount to large omission of possible information. If we have also the micro level observations, much more efficient estimations of the macro equation can be calculated.

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