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Control Risk Taking by Controlling Growth

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Abstract

This theoretical model explores an incentive problem between lenders and firms (=borrowers); given debt finance and limited liability, firms may take excessive risks with their lenders' funds. The model examines how the incentive problem can be eliminated by utilizing equity capital and by controlling growth. First, the firms' optimal growth path is derived. Then, the firms' optimal, incentive compatible equity ratio is shown to be dynamic: start-up firms need to maintain more equity capital than established firms. Finally, rapid growth is shown to worsen the incentive problem; hence, rapidly growing firms need to maintain a relatively high equity ratio.

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1. Introduction

There exists abundant empirical evidence that start-up firms are risky (e.g. Altman, 1993; Dunne & Hughes, 1994; Farinas & Moreno, 2000). The severe failure risk of rapidly growing firms has also been documented (Argenti, 1976). To explore the risks of start-up firms and rapidly growing firms, this paper develops a model of dynamic moral hazard.[1]

Why are rapidly growing firms risky? The paper suggests the following explanation. A firm expands its production in "novel markets" in terms of new products, lines of business or geographic areas, which are relatively non-known to the firm. As a result, the firm may have wrong expectations regarding the demands of the markets, it may underestimate the costs and times of the production and overestimate revenues. Growth may incur losses to the firm. If the losses are heavy in relation to the firm's resources, the firm collapses.

Why are start-up firms risky? Since they have no experience in production, they encounter only "novel markets". Each customer, sector, production process or geographic area is non-known for them. Besides, start-up firms lack the old reliable production with assured sales revenue.

In both cases, a firm could restrict its risk by proceeding with caution in novel markets or by exerting effort to research these markets, careful planning, product development, marketing, etc. Yet, this type of effort is costly. The costs, together with the limited liability option of firms, may tempt the firm to shirk effort and adopt a strategy of risky growth at the expense of its lenders. If risky growth into novel markets succeeds without effort, the firm earns handsome profits. If risk taking fails, the firm collapses and its lenders suffer the risk-taking costs via loan losses (see Stiglitz & Weiss, 1981; Holmström & Tirole, 1997).

In this paper, rational lenders - who cannot observe whether a firm exerts effort or not - recognize the firms' incentives to take excessive risks. To attract loans, the firm must signal its soundness to lenders by maintaining sufficient equity capital. Equity reduces the incentive problem by making it more costly for the firm to collapse (Bester, 1985, 1987; Holmström & Tirole, 1997).

[1] De Meza & Southey (1996) also model the severe risk of start-up firms. They highlight that start-up firms are mostly formed by over-optimistic individuals, who will expand their firms to a higher-than-optimal scale. As a result, start-up firms have a severe risk of collapsing.

Under this framework, the following main results are found. First, the optimal growth path for a firm is derived. With realistic parameter values, the growth is found to slow down with firm age. This finding is in the line with empirical evidence (Evans, 1987; Sutton, 1997; Farinas & Moreno, 2000). Second, the optimal, incentive compatible equity ratio of a firm is shown to be dynamic: a start-up firm needs to maintain more equity capital than an established firm. Intuitively, an established firm has old reliable products, which yield profits. The firm will not lose its future profits by taking risks with growth. Future profits thus reduce the incentive problem. In contrast, a start-up firm has no profitable old products to lose, and thus is more willing to take risks. Third, if an established firm grows rapidly, its incentive compatible equity ratio rises. Intuitively, rapid growth decreases the share of old reliable products in the firm's product mix. The relative magnitude of the future profits as a reducing factor against the incentive problem is eroded. In order to retain its incentives to exert effort, a rapidly growing firm needs to raise its equity ratio. Fourth, the incentive problem is likely to be more severe if the firm's growth is directed towards a novel sector of the economy than if the firm expands its prior production.

Consequently, equity capital and future profits can be viewed as substitutes in eliminating the incentive problem (effort aversion). Since start-up firms have no future profits and rapidly growing firms have relatively small future profits, they need to maintain high equity ratio. In contrast, established firms have relatively greater future profits and low equity ratio is sufficient to ensure their incentives to exert effort.

This paper builds on the literature on loan markets under asymmetric information (e.g. Stiglitz & Weiss, 1981). Several useful methods to reduce the problem of asymmetric information has been advanced: monitoring (Diamond, 1984), long-term lending relationships (Von Thadden, 1995), the optimal design of financial contracts (Diamond, 1984; Repullo & Suarez, 1998), loan covenants and collateral / equity capital (Bester, 1985, 1987). This paper presents how the problem of asymmetric information can be reduced by controlling growth. The paper is associated with the study by Bester (1987), in which collateral is utilized to eliminate moral hazard. While Bester sets up a model of one project, this paper investigates a firm as a going concern, which keeps on operating for ever and undertakes overlapping projects. Furthermore, Bester focuses on collateral, whereas this paper examines equity. In this context, the analysis is related to the abundant theory of the optimal capital structure (e.g. Jensen & Meckling, 1976; Myers & Majluf, 1984; Jensen, 1986). The paper extends this theory by suggesting that the optimal equity ratio may depend on the age of the firm as well as on the firm's growth speed. Finally, the paper touches the study of product quality (e.g. Klein & Leffler, 1981; Shapiro, 1983).

2. Model

Consider a model of fully competitive product markets with infinite number of periods, $t \in \{0,1,2,...\}$, lenders and a firm (= a borrower). The firm operates from period-0 to eternity and is protected by limited liability. The firm is owned by a risk neutral entrepreneur, who maximizes his expected utility. The entrepreneur receives an endowment $\frac{1}{2}E$ in periods 0 and 1, but no endowment is received thereafter. Subsection 2.1 displays a life cycle of a firm's product, whereas Subsection 2.2 points out how the product mix of the firm composes of different products. The size of the product mix in is solved in Subsection 2.3.

2.1 A product

A life cycle of a single product lasts for two production periods. The first production period is any period$-t$, $t \in \{0,1,2,...\}$, and the second production period is period$-t+1$. The economic environment is assumed to be identical in every period; that is, the economy has the same probabilities of success, interest rates, the cost of effort, etc. in every period.

In both production periods, production requires a unit of investment input. Furthermore, the quality of the product can be boosted in the first production period by exerting effort.[2] With effort, the quality is good and the products are sellable in both production periods. Without effort, the quality is bad (risky). With probability b , risk taking succeeds and bad products are sellable at the price of the good products in both production periods. With probability $1-b$, risk taking fails and no products can be sold in either production period. Effort incurs a cost c to the firm. The firm knows whether it exerts effort or not, but the choice is unobservable to outsiders. Outsiders observe the quality only if the products are non-sellable. A zero profit price of a good product, P_0, can be resolved from

$$P_0 - r_f - c \ + \ \delta\alpha(P_0 - r_f) = 0 \ , \quad \delta = \frac{1}{r_f} . \tag{1}$$

[2] Effort may include careful market analysis; learning of local laws, regulations and habits; R&D investment and product planning according to the needs of the customers; high-grade design; planning and learning of the production process (learning by doing); analysing production costs; search costs of effectual employees, suppliers, distributors and customers; personnel training; special service campaigns to attract customers to test the product; the creation of image, brand and reputation; etc. For an effort aversion problem, see also Holmström &Tirole (1997) and Chiesa (2001).

That is, during its life cycle in a fully competitive economy, a good product yields zero profits to the firm. Here r_f denotes the fixed risk-free interest rate of the economy (the cost of investment), and δ denotes the discount factor. In addition, α, which is fixed and commonly known, denotes the scale of production in the second production period, $\alpha \geq 0$. If $\alpha > 1$ ($\alpha < 1$), the scale of production is larger (smaller) in the second production period than in the first production period. In the first production period, the firm's returns amount to $P_0 - r_f - c$, whereas $\delta\alpha(P_0 - r_f)$ expresses the present value of the returns from the second production period. The zero profit price can be solved from (1) as $P_0 = r_f + c/(1+\alpha\delta)$. Setting this into $P_0 - r_f - c$ gives the rewritten returns of the first production period

$$\frac{-\alpha\delta c}{1+\delta\alpha} < 0. \tag{2}$$

Inserting $P_0 = r_f + c/(1+\alpha\delta)$ into $\delta(P_0 - r_f)$, gives the returns of the second production period

$$\frac{c}{1+\delta\alpha} > 0. \tag{3}$$

The following conclusion can be drawn.

Lemma 1. *New products are unprofitable to the firm, but old products are profitable.*

Intuitively, since a product incurs initial costs, the returns of the firm are lower in the first production period than in the second production period. Because the product yields zero returns during its life cycle, the returns of the first period need to be negative and the returns of the second period need to be positive.[3] Two assumptions are made.

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[3] The term "new product" has alternative constructions. First, it may mean a new production unit or machine, which keeps on producing the conventional products of the firm. For instance, a motorcar supplier builds up a new production unit in an unfamiliar nation, a hotel chain sets up a new hotel in an unfamiliar city, an airline company launches a new flight-path or a publisher founds a new magazine. Second, it may mean a product, which actually represents a novel sector for the firm. Whatever the case, the markets are fully competitive and the market price is fixed.

Assumption 1. *Without effort, production has negative social value,*

$$bP_0 - r_f + b\delta\alpha\lfloor P_0 - r_f \rfloor < 0. \tag{4}$$

The term, $bP_0 - r_f$, represents the returns expected in the first production period without effort. With probability b, all of the products are worth of P_0.[4] The cost of investment, r_f, materializes whether the products are sellable or not. The second term, $b\delta\alpha\lfloor P_0 - r_f \rfloor$, is the returns expected in the second production period. If the product is sellable in the first production period, it is sellable also in the second period.[5] After some manipulation, (4) simplifies to

$$bc - (1-b)r_f < 0. \tag{5}$$

Although bad products have negative social value, they may be produced. Under asymmetric information, the firm, which has debt finance and which is protected by limited liability, may gain by producing risky bad products. This is later seen in detail.

Assumption 2. *Production technology has constant returns to scale.*

Hence, the firm can select any size. This makes it possible to explore the growth process.

2.2 Product mix

Subsection 2.1 shed light on the life cycle of a single product. This subsection points out how the product mix of the firm consists of different products from period-0 to eternity. The analysis has

[4] Either all of the products are sellable or all of them are non-sellable.

[5] Alternatively, it is possible to assume that a risky product may fail in the second production period even when it succeeds in the first production period. Then, the risky product yields short-term profits, but it is not good enough to succeed in the long run. Importantly, this alternative assumption does not change the results of the paper if the products which are launched in different periods have the same completely correlated risk to fail. That is, if the risky products which are launched in period $t-1$ fail during period t, then the risky products which are launched in period t also fail. For instance, a firm transforms its whole production into a novel sector during periods $t-1$ and t. Yet, this novel sector proves to be unprofitable in the long run, the firm cannot sell its products in period t and it collapses.

only three variables (the size of the firm, the volume of new products and the returns of the firm), all of which are time dependent.

In period-0, as mentioned above, the entrepreneur receives a capital endowment $\frac{1}{2}E$ and injects it in his firm, which also attracts the amount $S_0 - \frac{1}{2}E$ of loans.[6] Here S_t denotes the size of the firm in period-t. Thus, the firm has the amount S_0 of invested funds in the production of product-0. At the end of the period, production materializes and the firm obtains selling revenue $S_0 P_0$, which is spent to cover the cost of production. The rest of the returns are paid out as dividends to the entrepreneur.

In period-1, the entrepreneur again receives an endowment $\frac{1}{2}E$, and injects it in the firm. The amount of injected equity now totals E. The amount of attracted loans is $S_1 - E$, where S_1 denotes both the period-1 size of the firm and the amount of invest funds. How are the funds invested? Since the old products – products-0 – yield profits, the firm will certainly keep on producing them. Thus, the amount αS_0 is invested in the production of products-0. The rest of the funds, $S_1 - \alpha S_0$, are invested in the production of new products, product-1. At the end of the period, production again materializes yielding sales revenue $S_1 P_0$.

In period-2, the firm's size is S_2, it maintains the amount E of equity and it attracts the amount $S_2 - E$ of loans. The life cycle of the very first products – product-0 – is over. This releases capital for new production (see Figure 1). Again, the firm will first and foremost keep on producing its old profitable products (products-1). The firm invests the amount $\alpha(S_1 - \alpha S_0)$ in the production of products-1. The rest of the funds, $S_2 - \alpha(S_1 - \alpha S_0)$, are invested in the production of new products (product-2). At the end of the period, production again materializes and the firm obtains sales revenue $S_2 P_0$.

In the next periods, the process continues. In period-t, the firm maintains the fixed amount of equity, E, and the size of the firm is S_t, which is also equal to the amount of invested funds. The life cycle of the products, which were launched for the first time in period $t-2$ (product- $t-2$) is over. The optimizing firm invests the funds first and foremost to keep on producing its old profitable products (product- $t-1$). The rest of the funds, that is,

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[6] The paper attempts to explore the problem of excessive risk taking in the loan markets. Hence, it is simply assumed that the firm cannot attract outside equity capital. For the problems of outside equity, see Jensen & Meckling (1976), Myers & Majluf (1984) and Jensen (1986).

$$L_t = S_t - \alpha L_{t-1}, \tag{6}$$

is invested in the production of new products (product- t). Here L_t denotes the volume of new production (product-t), whereas αL_{t-1} denotes the volume of old production (product- $t-1$). Hence, the volume of the new production is equal to the difference between the size of the firm and the volume of old production, (6).

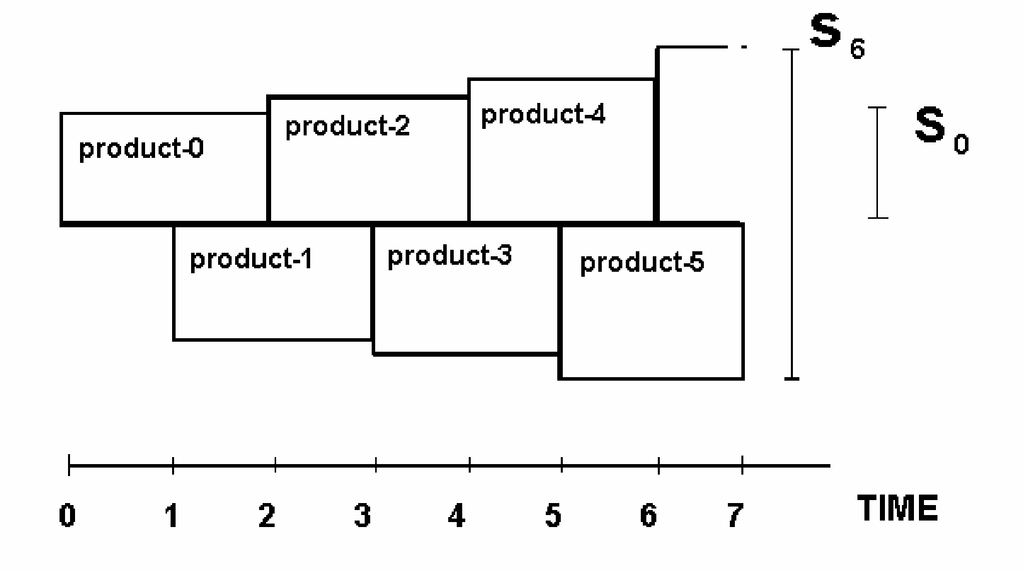

Figure 1. Product mix. The horizontal plane illustrates the change of the product mix in time. In period-0, the first endowment is received and it is invested - together with the attracted loan funds - in production (product-0). In period-1, the second endowment is received and the firm can expand its production (product-1). In period-2, the life cycle of product-0 is over. This releases capital for new production (product-2). In the next periods, the process continues and the firm has an overlapping product structure. The vertical plane illustrates the total scale of the firm's production. On the right-hand side of Figure 1, two segments of a line can be observed, (S_0, S_6). The segments measure the size of the firm. It is easy to see that in period-6 the firm is much larger than in period-0, $S_6 > S_0$. Thus, Figure 1 foresees one of the main results; the incentive compatible size of the firm grows in time. Figure 1 includes an assumption $\alpha = 1$: each product has the same volume of production in its both production periods.

Importantly, the firm is free to choose its products in every period. It could abandon its old products and reallocate its production. Yet, the firm is unwilling to abandon its old products, since it has expertise regarding these products, which thus yield profits for it.[7]

The model includes a complex assumption that the entrepreneur receives endowments in two periods (period-0 and period-1). The assumption is needed to create an overlapping structure of launched products. The overlapping product structure generates a realistic model, in which the firm operates as a going concern and has in every period both "old" production and an option to invest in new production. Maybe surprisingly, the overlapping product structure will prove to have a strong impact on the incentive problem between the firm and its lenders.

2.3 Incentive constraint

The previous subsection portrayed the production mix. Yet, the size of the production mix – that is, the exact size of the firm – was left out. This subsection puts forward the rules, which determine the incentive compatible size of the firm. To begin, the following assumption is made.

Assumption 3*. To maximize his empire, the entrepreneur maximizes the size of the firm in every period.*

When the firm produces good products - and thus each firm size yields zero profits to the entrepreneur – the entrepreneur's maximization problem is simply to maximize the size of his firm. Given the fixed amount of equity capital, E, the entrepreneur aims to minimize the equity ratio of the firm.[8]

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[7] Alternatively, it is possible to assume that the firm is tied to each product for two periods. This alternative does not, however, change the results of the paper, since the firm will keep on producing its old products even when it has an option to abandon these products.

[8] Assumption 3 offers a simple way to motivate growth and the minimization of the equity ratio. Since good products yield zero returns during their life-time, a firm with good products makes zero profits during its life-time. Hence, the size of the firm has no effect on life-time profits of the entrepreneur. Thus, profits create no incentives to grow and the entrepreneur is indifferent between alternative firm sizes. Since profits create no incentives to grow, assumption 3 is needed to motivate growth (for private benefits and growth see Jensen, 1986, and Jensen & Meckling, 1976). Alternative ways to motivate growth exist. First, if a good product yielded tiny profits, the owner would maximize the scale of production in order to maximize his profits. Second, if equity capital were relatively more expensive than loans, this would encourage the firm to maximize the scale of production – the size of the firm – in order to minimize the

Recall that the entrepreneur may exert effort in production. Alternatively, since the effort exertion is unobservable to lenders and since the firm is protected by limited liability, the entrepreneur may shirk effort and produce risky products with negative social value. If risk taking succeeds, the firm makes handsome profits. If risk taking fails, the firm passes the resulting damages via loan losses on to its lenders. Hence, there is an *incentive problem* between the firm and its lenders: the profit-maximizing firm may optimally shirk effort. Yet, in equilibrium lenders cannot be worse off than with their reservation payoffs. Thus, in equilibrium the firm's *incentive constraint* needs to be satisfied: the profit-maximizing firm prefers effort exertion to effort aversion. Then, the lenders' participation constraint is also satisfied. Sections 2c, 3 and 4 derive the firm's incentive constraint. The main result is that the firm – which has a fixed amount of equity – finds it optimal to exert effort only if its size (and thereby its lending volume) does not exceed a critical value. The incentive compatible firm size is positively related to equity capital and the volume of existing, "old" production.[9]

How can the lenders know that the incentive constraint is satisfied? The lenders can infer the effort choice by keeping track of the equity ratio of the firm and its growth path. More precisely, lenders know parameters α, b, c, E, r_f and observe the volumes of new and old products as well as the sales revenue. Lenders also remember the sizes of the previous periods, $S_0, S_1, ..., S_{t-1}$. The sizes have been such that the incentive constraint has been satisfied in every period and the firm has exerted effort. To be able to attract loans in the current period, the firm again selects its size, S_t, so that the incentive constraint is satisfied. Thereafter, the firm attracts short-term (one period) loans, $S_t - E$, and invests its funds, S_t. At the end of the period, production materializes.

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equity ratio, and thereby the costs of expensive equity per a unit of production. These alternative ways to motivate growth are rejected in order to keep the model simple.

[9] Note that the lenders are ready to grant a loan at the risk-free interest rate, r_f, if the firm will later exert effort, since production is then risk-free. On the contrary, since production has socially negative value without effort, there exist no loan rate such that i.) the lenders' participation constraint is satisfied and ii) production is profitable without effort. Hence, either the lenders grant a loan at the risk-free interest rate, r_f (when they know that the incentive constraint is satisfied and the firm will exert effort), or they grant no loan (when the incentive constraint is not satisfied and a profit-maximizing firm would shirk effort). Furthermore, any risk premium would only increase the incentive problem, which is worsening in the loan interest rate. Consequently, in order to find a equilibrium, it is sufficient to examine a loan package with risk-free loan interest rate and point out that the package satisfies the incentive constraint. See Chiesa (2001) or Holmström & Tirole (1997) for rather similar incentive problems.

Since the production is risk-free under effort exertion, the firm can repay its loans with certainty. The rest of the returns are paid out to the entrepreneur.

In every period the incentive constraint binds (it holds with equality). To see this, recall that Assumption 3 motivates the entrepreneur to maximize the size of the firm and thereby minimize the equity ratio. Without the incentive constraint, the optimal size would be infinite. Yet, then the equity ratio would approach zero and the firm would shirk effort (this is later shown in detail). Hence, the incentive problem sets a lower limit for the equity ratio – and thus an upper limit for the size of the firm – in every period. In every period- t the expected life-time profits from the production of good quality (exerting effort) equal the expected profits from the production of bad quality (effort aversion)

$$\sum_{i=0}^{\infty} \delta^i \pi_{t+i}^G = \sum_{i=0}^{\infty} (\delta b)^i \pi_{t+i}^B \quad . \tag{7}$$

Here π_t^i denotes the returns of the firm in period- t, when the quality of production is i, $i \in \{G, B\}$, $G = Good$, $B = Bad$. Besides, δ is the discount factor. The firm can keep on producing bad products until the risk realizes and the firm is unable to sell its products. Then, the firm earns no selling revenue, cannot repay its loans and it collapses. In period- $t+1$ the incentive constraint is

$$\sum_{i=1}^{\infty} \delta^{i-1} \pi_{t+i}^G = \sum_{i=1}^{\infty} (\delta b)^{i-1} \pi_{t+i}^B \quad . \tag{8}$$

Inserting (8) into (7) displays the final form of the incentive constraint of period- t,

$$\pi_t^G + \delta(1-b)\sum_{i=1}^{\infty} \delta^{i-1} \pi_{t+i}^G = \pi_t^B \ . \tag{9}$$

Again, the term π_t^G represents the returns of period- t with effort. Without effort, the returns of period- t are π_t^B. The second term on the left-hand side is the difference in the present value of future profits between the effort and non-effort strategies. If the firm shirks effort, it risks failure and loss of future profits. Hence, future profits are greater if the firm exerts effort. The exact magnitude of future profits is solved in Appendix A.

Lemma 2. *In period- t, the future profits of the firm are* $\sum_{i=1}^{\infty} \delta^{i-1} \pi_{t+i}^{G} = \alpha L_t (P_0 - r_f)$.

Recall that L_t denotes the volume of new products in period- t, and thus αL_t represents the volume of old products in period- $t+1$. Then, the profit marginal of the old products is $P_0 - r_f$. Intuitively, in period- t, the firm exerts effort in its new products, L_t. In the next period, $t+1$, these products yield profits $\alpha L_t (P_0 - r_f)$. Hence, future profits from period $t+1$ onward, $\alpha L_t (P_0 - r_f)$, arise from the old products of period- $t+1$, and these profits are based on the effort exertion in period- t. Future profits thus originate from the effort exertion in the current period. Hence, even if the firm operates under perfect competition and earns zero profits during its life-time, the firm has positive future profits when it has been operating for some time, since it has exerted effort in production.

Finally, suppose that the firm exerted effort in period $t-1$. In period- t, the firm has old profitable products. What happens if the firm does not exert effort in its new products in period- t and the risk realizes? The firm then has both sellable old products and non-sellable new products. Do the firm's profits from its old products cover loan repayments?

Lemma 3. *The firm's optimal size (the volume of new products) is so big that if a risk realizes when the firm has both good and bad products in its product mix, the firm cannot repay its loans. The firm collapses and its owner receives no dividends.*

The technical proof is omitted. Intuitively, the incentive constraint can be binding if $\pi_t^G < \pi_t^B$. This is possible only if the firm, which shirks effort, benefits from limited liability when the risk realizes.

Consequently, the incentive constraint is binding in every period. Next, the exact content of the incentive constraint is resolved in periods, $t = 0,1,2,...$. This uncovers the incentive compatible firm size in every period, and thereby the dynamic, incentive compatible equity ratio.

3. Firm's incentive compatible size in period-0

Given (9), the incentive constraint of the firm in period- 0 is

$$\pi_0^G + \delta(1-b)\sum_{i=1}^{\infty}\delta^{i-1}\pi_i^G = \pi_0^B , \qquad (10)$$

Some manipulation (see Appendix B) displays a concise form of the incentive constraint

$$0 = bcS_0 - \tfrac{1}{2}(1-b)Er_f . \qquad (11)$$

The first term expresses the benefits of effort aversion. With probability b, products are sellable even without effort, and the firm earns more returns without effort than with effort, since it avoids the costs of effort, cS_0. The second term reveals the drawback of effort aversion; with probability $1-b$ the products cannot be sold and the firm collapses, thereby losing its equity, $\frac{1}{2}E$.

Four findings follow. First, without equity the firm shirks effort since $bcS_0 > 0$. Intuitively, the firm takes advantage of its limited liability and gambles at the expense of its lenders by shirking effort. If risk taking succeeds, the firm earns handsome profits. If risk taking fails, the firm does not lose anything. Yet, rational lenders recognize the incentives of the firm and deny loans; no firms can be formed without equity. Second, a fully equity financed firm, $S_0 = \frac{1}{2}E$, exerts effort. To see this, note that the right-hand side of (11) can be rewritten as $\lfloor bc-(1-b)r_f \rfloor \frac{1}{2}E$. Given (5), the term in brackets is negative; effort aversion is unprofitable. Intuitively, a fully equity financed firm has no limited liability option and thus the incentive problem disappears. Third, since the firm is assumed to maintain a fixed amount of equity, E, effort aversion is profitable with certainty, $bcS_0 > \frac{1}{2}(1-b)Er_f$, if the firm is big enough. Fourth, the firm's optimal (=maximal) size in period-0 can be resolved from (11) as

$$S_0{}^* = \frac{1}{2cb}(1-b)Er_f . \qquad (12)$$

If the firm were bigger, the incentive constraint would not be binding, and the firm would shirk effort. Rational lenders would not grant a loan to it. If the firm were smaller, the utility of the owner could be improved by expanding the firm (Assumption 3). Hence, the optimal size is S_0*. The following conclusion can be drawn.

Proposition 1 ***(A start-up firm).*** *In period-0, the firm has no old profitable products and its effort incentives are generated entirely by equity capital. Without equity, the firm would not exert effort. Given the fixed amount of equity, the firm's optimal size is* $S_0* = (1-b)Er_f/2bc$.

4. Incentive compatible size in periods 1, 2, 3, …

In period-1 the entrepreneur receives an endowment, $\frac{1}{2}E$, which is injected in the firm. The amount of equity is E in every period from period -1 on.

The firm has exerted effort in the preceding periods; thus, old products are good. The aim is to find the incentive compatible size of period-t, S_t^*. The incentive constraint of period-t is,

$$\pi_t^G + \delta(1-b)\sum_{i=1}^{\infty}\delta^{i-1}\pi_{t+i}^G = \pi_t^B . \tag{13}$$

After some manipulation (Appendix C), a concise form of the incentive constraint can be resolved

$$0 = (S_t - \alpha L_{t-1})bc - (1-b)\alpha L_{t-1}(P_0 - r_f) - (1-b)Er_f . \tag{14}$$

The first term expresses the extra returns the firm can earn by avoiding the costs of effort. With probability b, new products prove to be sellable even without costly effort. These returns depend on the volume of new products, $S_t - \alpha L_{t-1}$. The second term indicates profits from old products. Without effort, the firm risks failure and loss of these profits. The third term reveals the risk of losing equity without effort exertion. Thus, the first term weakens the incentives to exert effort, while the last two terms strengthen the incentives. Note that old products, αL_{t-1}, reduce the incentive problem in two ways. The first term confirms that the existence of old products diminishes the need to exert effort in new production. Given the size of the firm, S_t, the larger the volume of old products, αL_{t-1}, the less is left over, $S_t - \alpha L_{t-1}$, for new production. The second term discloses that old products yield profits. Inserting $P_0 - r_f = c/1+\delta\alpha$ into (14) gives the optimal (maximal) size in period $-t$

$$S_t^* \;=\; 2S_0^* \;+\; \alpha L_{t-1}\left[1+G\right],\; where \;\; G = \frac{1-b}{b+\alpha\delta b}. \tag{15}$$

The optimal size grows along with the volume of old products, αL_{t-1}, and with the amount of equity capital, E, since $S_0{}^* = (1-b)Er_f/2bc$. The conclusion follows.

Proposition 2 ***(Established firm).*** *In periods* $1,...., \infty$*, both equity and old profitable products motivate the firm to exert effort. Old products yield profits and diminish the need to exert effort.*

Hence, the lenders can evaluate the firm by monitoring the amount of equity, the volume of the old products and the profitability of old products. Although the profitability of old products provides a valuable signal to the lenders, the signal is imperfect. If old products yield no sales revenue, the firm has shirked the effort exertion with certainty and it collapses. Yet, positive sales revenue may stem from the effort exertion or from successful risk taking without effort.

The key benefit from the overlapping structure of launched products is noteworthy of emphasis. In period-0, size $S_0{}^*$ ensures that the firm exerts effort. By exerting effort, the firm creates profitable products for period-1. In period-1, both these old products from period-0 and equity capital motivate the firm to exert effort. Given the positive incentive effects of old products, relatively less equity capital is required and the firm can lower its equity ratio by growing. Size $S_1{}^*$ ensures that the firm exerts effort in period-1, $S_1{}^* > S_0{}^*$. By exerting effort in period-1, the firm creates profitable old products for period-2. In period-2, both old products from period-1 and equity motivate to exert effort. Since the volume of old products is larger in period-2 than in period-1, the firm can grow again, $S_2{}^* > S_1{}^*$. The process continues from one period to the next – forever. In every period, the firm's choice to exert effort creates profitable products for the next period, and thus making effort aversion in the next period unprofitable. In every period, the volume of old products is larger than in the previous period, and thus the optimal size of the firm is bigger than in the previous period. The firm grows forever. This is later seen in more detail.

Importantly, the positive incentive effect of old products is based on the overlapping structure of launched products. It is not based on standard diversification, since the positive incentive effect exists even when the launched products are identical or belong to the same product group.

Unfortunately, equation (15) becomes impracticable in later periods. For example, in period-100 the size is $2S_0^* + \alpha L_{99}\left[1 + G\right]$. This is rather uninformative when the volume of old products, αL_{99}, is unknown. Fortunately, the optimal size can be resolved as a function of the firm's initial size, S_0^*. The following proposition is derived in Appendix D by induction.

Proposition 3 **(Firm' optimal size).** *A firm's optimal size in period-t is*

$$S_t^* = \left\{ 2 + 2\alpha(1+G)\sum_{i=0}^{t-2}(\alpha G)^i + (1+G)\alpha(\alpha G)^{t-1} \right\} S_0 * \ , G = \frac{1-b}{b+\alpha\delta b} \ , t \geq 1 . \tag{16}$$

5. Optimal growth speed

It is easy to resolve the following result from Proposition 3

***Proposition 4* (Growth speed).** *A firm's optimal growth speed is*

$$\Delta_t \;\equiv S_t^* - S_{t-1}^* \;=\; \alpha\, S_0^*\, (\alpha G)^{t-2}(1+G)(1+\alpha G)\,. \tag{17}$$

Proof.

$$\begin{aligned} S_t^* \; - S_{t-1}^* &= S_0^* \left\{ 2 \;+\; 2\alpha(1+G)\left[1+\alpha G+(\alpha G)^2+...+(\alpha G)^{t-2}\right] + \;(1+G)\alpha(\alpha G)^{t-2} \right\} \\ &\quad - S_0^* \left\{ 2 \;+\; 2\alpha(1+G)\left[1+\alpha G+(\alpha G)^2+...+(\alpha G)^{t-3}\right] + \;(1+G)\alpha(\alpha G)^{t-3} \right\} \\ &= \; \alpha\, S_0^*\, (\alpha G)^{t-2}(1+G)(1+\alpha G)\,. \; Q.E.D \end{aligned} \tag{18}$$

Using Proposition 4, the following results are derived in Appendix E.

***Corollary 1* (Growth factors).** *i.) The firm grows monotonously if $\alpha > 0$. ii.) Growth speeds up with α. iii.) Growth slows down with b. iv.) If $\alpha G > 1$, growth speeds up with time and approaches infinity in eternity. If $\alpha G < 1$, growth slows down with time and approaches zero in eternity.*

i.) When $\alpha = 0$, each product is produced only in one period and no old profitable products exist. Only equity motivates the firm to exert effort and it cannot grow. When $\alpha > 0$, old profitable products exist. These products strengthen the incentives to exert effort and the firm can grow. By growing, the firm creates more profitable old products, which again strengthen its incentives and help the firm to grow further. This process goes on forever. ii.) The stronger α , the larger the volume of old production. This motivates the firm to exert effort and the firm can grow rapidly. iii.) The larger b is, the higher the probability of success – and the profitability – of effort aversion. This weakens the incentives to exert effort – the firm can grow only slowly. iv.) *When* $\alpha G > 1$, growth speeds up with time; the firm creates so much old profitable products that it can grow even more rapidly in the next period. Note that $\alpha G > 1$ means $(\frac{1}{b} - 1)/(\frac{1}{\alpha} + \delta) > 1$. Hence, the growth speeds up with time if α is strong (large volume of old products exists) and b is small (the probability of success is low without effort). *When* $\alpha G < 1$, growth slows down with time. Growth creates profitable, old products, but lowers the equity ratio. In contrast to the case $\alpha G > 1$, the value of old products is insufficient to fully compensate for the lowering equity ratio. Hence, the incentive problem worsens and growth needs to slow down.

In the following, the detailed analysis of the case $\alpha G > 1$ is dropped, but the main results of the case are briefly listed.

1. When $\alpha G > 1$, growth speeds up with time.
2. The growth speed approaches infinity in eternity.
3. Since the growth speed approaches infinity, the size of the firm approaches infinity as well.
4. Since the size approaches infinity and since the firm maintains the fixed amount of equity, the equity ratio lowers to zero.

The results are a bit unrealistic. The assumption that an infinite growth speed is possible ignores any demand side considerations. Moreover, the firm could hardly retain good quality when its growth speed is approaching infinity. In addition, firms used to maintain positive equity ratio. Given the shortcomings, the case $\alpha G > 1$ is dropped and the study focuses on the case $\alpha G < 1$.

6. Optimal size in eternity

The firm's size in eternity can be solved from Proposition 3.

***Proposition 5* (Size in eternity).** *When $\alpha G < 1$, the optimal size in eternity approaches the steady state level*

$$S_{ss} = \frac{2(1+\alpha)}{1-\alpha G} S_0^*, \ \alpha G < 1. \tag{19}$$

Proof. Recall the size of the firm from (16)

$$S_t^* = \left\{ 2 + 2\alpha(1+G)\left[1+\alpha G+(\alpha G)^2+...+(\alpha G)^{t-2}\right]+ (1+G)\alpha(\alpha G)^{t-1} \right\}.$$

As t increases without bounds, the first term is fixed, 2, the second term can be expressed as a sum of an infinite geometric series, $2\alpha(1+G)/(1-\alpha G)$, and the third term, $(1+G)\alpha(\alpha G)^{t-1}$, approaches zero. The sum of 2 and $2\alpha(1+G)/(1-\alpha G)$ is equal to (19). *Q.E.D*

Intuitively, since $\alpha G < 1$, growth slows down to zero in eternity and the size settles down to the steady-state level. Furthermore, (19) can be expressed as

$$S_{ss} = \frac{2(1+\alpha)}{1-\frac{\frac{1}{b}-1}{\frac{1}{\alpha}+\delta}} S_0^*, \tag{20}$$

in which the denominator is positive. It is easy to see that the size of the steady state grows with the initial size of the firm, S_0^*, and with the scale of the old products, α (note that $dS_0^*/d\alpha = 0$) but shrinks with the success probability of bad products, b ($dS_0^*/db < 0$) and with the costs of effort, c ($dS_0^*/dc < 0$). Hence, S_{ss}^* grows with the factors which reduce the incentive problem, α , but shrinks with the factors, which worsen the incentive problem, b and c.

7. Steady state yields profits

This section points out that the firm makes positive profits in every period of the steady state (forever). To see this, recall the firm's returns

$$(S_t - \alpha L_{t-1})\lfloor P_0 - r_f - c \rfloor \ + \ \alpha L_{t-1}(P_0 - r_f)\,. \tag{21}$$

Note that the costs of effort and the costs of equity capital are included in the firm's returns. As before, the volume of new products is $L_t = S_t - \alpha L_{t-1}$. In the steady state, the volume of new products and size of the firm are fixed: L_{ss}, S_{ss}. Inserting these into $L_t = S_t - \alpha L_{t-1}$ uncovers the volume of new products as a function of the firm's size

$$L_{ss} = \frac{S_{ss}}{1+\alpha}\,. \tag{22}$$

Setting this, $S_t = S_{ss}$, and $P_0 - r_f - c = -\alpha\delta c/(1+\alpha\delta)$ from (2) into (21) displays firm's profits in every period of the steady state

$$S_{ss}\ \frac{(1-\delta)\alpha c}{(1+\alpha)(1+\alpha\delta)} \ > \ 0\,. \tag{23}$$

The present value of future profits from the current period to eternity[10] is

$$S_{ss}\ \frac{c}{(\frac{1}{\alpha}+1)(\frac{1}{\alpha}+\delta)}\,. \tag{24}$$

Given the size S_{ss}, the present value of future profits is increasing in relation to the scale of the old products, α, and in the costs of effort, c. Intuitively, the future profits originate from the old, profitable products. The stronger the scale of old products, α, the greater the future profits. Besides, the wider the profit margin of the old products, $P_0 - r_f$, the greater the future profits.

[10] Alternatively, the future profits can be solved from Lemma 2: $\alpha L_t(P_0 - r_f)$. Inserting both $P_0 - r_f = c/(1+\alpha\delta)$ from (3) and $L_{ss} = S_{ss}/1+\alpha$ from (22) into $\alpha L_t(P_0 - r_f)$ displays (24).

Since the profit margin widens with c, recall $P_0 - r_f = c/(1+\alpha\delta)$, the future profits are increasing in c.

Proposition 6 ***(Steady state profits).*** *Although product markets are fully competitive ex ante, the firm earns profits,* $S_{ss}\,\alpha c(1-\delta)/(1+\alpha)(1+\alpha\delta)$*, in every period of the steady state (forever), since the firm then has old profitable products.*

Two insights are worth of notice.

1. The future profits do not originate from a huge cost of establishment in period-0. On the contrary, they arise from old profitable products. The firm builds its future profits gradually over a long-time span by growing slowly and at the same time exerting effort in new production. Since the volume of new products mounts with time, the volume of old products also mounts with time, thereby increasing future profits.
2. The firm has everlasting positive future profits, although a life cycle of a product is two periods. The future profits last for ever since the firm optimally invests effort in every period in new products, thereby creating future profits for the next period.

Put differently, the firm has the future profits, (24), already when it achieves the steady state. Yet, the entrepreneur does not receive the whole profit (24) at once, since the firm invests in new production, which is unprofitable incurring losses $(S_{ss} - \alpha L_{ss})\lfloor P_0 - r_f - c \rfloor$ or $S_{ss}\,\alpha c\delta/(1+\alpha)(1+\alpha\delta)$. Subtracting the losses from (24) gives entrepreneur's profits in every period of the steady state, (23). Yet, the present value of everlasting profit (23) is again (24).

Since the volume of old profitable products mounts with time, the market value of the firm, $E + \alpha L_t(P_0 - r_f)$, also increases with time approaching the steady state level, $E + \alpha c S_{ss}/(1+\alpha)(1+\alpha\delta)$.[11]

[11] Since the book value of equity is fixed (E), the price-to-book (PB) ratio increases with time.

8. Dynamic equity ratio

Propositions 4 - 6 together imply the following result.

***Corollary 2* (Optimal equity ratio).** *The optimal equity ratio lowers as the firm matures; an established firm maintains a lower equity ratio than a start-up firm. In eternity, the equity ratio approaches the steady-state level,* E/S_{ss}.

The steady-state level, E/S_{ss}, is lowered by the factors which reduce the incentive problem, α , but risen by the factors, which worsen the incentive problem, b and c. If αG is almost 1, S_{ss} is very big, and the equity ratio approaches zero in the steady state.

Intuitively, given the incentive problem between the firm and its lenders, the firm needs to maintain some equity capital in order to precommit to exert effort. Equity and future profits represent substitutes in eliminating the incentive problem. The greater the future profits, the lower the needed equity ratio and vice versa.

A start-up firm has no future profits and it needs to maintain a high equity ratio. As the firm matures and grows, its future profits gradually increase. Thus, the firm can gradually lower its equity ratio. Finally, the firm achieves the steady state. The firm's size, future profits and equity ratio achieve the fixed levels of the steady state. Since the future profits are at the maximal level, the needed equity ratio is at its minimal level.

9. The problems of rapid growth

The analysis of this section differs from the rest of the paper. So far the amount of equity capital has been fixed (after period -1), and the maximal size of the firm has been found out in every period. In this section, the opposite question is posed. Given the wanted size of the firm, what is the incentive compatible equity ratio?

Suppose now that the owner can inject fresh equity in period -t.[12] Given (15), the firm's optimal size can be expressed as

$$S_t^* = \alpha L_{t-1}\left[1 + G\right] + \frac{(1-b)Er_f}{bc} + \frac{(1-b)E_{new}r_f}{bc}. \tag{25}$$

The size grows with the volume of old production, L_{t-1}, with the amount of the initial equity, E, and with the amount of the new equity, E_{new}.

It is now possible to study the incentive constraint from another point of view. Suppose that the owner will expand the firm more rapidly than the optimal growth path, (16), allows. The desired size is $\hat{S}_t$. Given $\hat{S}_t$, the incentive compatible amount of equity can be solved from (25) as a function of the desired size

$$\hat{E} = \frac{bc\hat{S}_t - \alpha bcL_{t-1}(1+G)}{(1-b)r_f}. \tag{26}$$

Dividing by $\hat{S}_t$ displays the incentive compatible equity ratio

$$\hat{e} = \frac{bc - \dfrac{\alpha bcL_{t-1}(1+G)}{\hat{S}_t}}{(1-b)r_f}. \tag{27}$$

[12] Period-t may belong to the steady state. The injection of fresh equity has no effect on the firm's optimization problem prior to period-t, since the equity ratio is at the incentive compatible level in every period and the firm can not precommit to the future ratios of equity.

Here $\hat{e}$ raises with the desired size, $\hat{S}_t$. When the firm grows without bounds, i.e. $\hat{S}_t$ approaches infinity, the incentive compatible equity ratio approaches

$$\hat{e} = \frac{bc}{(1-b)r_f}, \tag{28}$$

which is equal to the equity ratio of a de novo firm in period- 0. Therefore, the firm that has obeyed the optimal growth path up to period-t, and has thus monotonously lowered its equity ratio, must raise its equity ratio back to the high initial level. Intuitively, when the firm grows rapidly, the relative share of the old profitable production, $\alpha L_{t-1}(1+G)/S_{ss}$, declines in its production mix (see (27)). In the extreme case, the share of old products approaches zero when the size, $\hat{S}_t$, grows without bound, (28). The positive incentive effect of old production then vanishes and effort incentives must be generated exclusively with equity capital just as in the case of a start-up firm.

***Proposition** 7 **(Rapid growth).** If an established firm plans to grow rapidly, it must raise its equity ratio.*

Rapid growth makes it possible to invest a major share of the product mix in risky products at the same time. This makes risk taking profitable. In contrast, when the firm grows slowly, the share of new products is minor in every period. If the firm then shirked the effort exertion, its cost savings would be rather insignificant compared with risk of losing the equity capital and revenue from the old products.

10. Example

This section offers a brief numeric example. Imagine an agent, who forms a firm by injecting 4 million euros of equity capital in it in periods 0 and 1. The economic environment is the following.

$$b = \frac{11}{19},\ c = \ 0.64\ \textit{units / a produced unit},\ r_f = 1.1,\ \alpha = \frac{1}{2},\ E = 8\ \textit{milj. euros}\,. \qquad (29)$$

In period-0, the firm size can be solved using (12) and (29). It is 5 million euros. Given the amount of equity in period-0, 4 million, the incentive compatible equity ratio of the start-up firm is 80%.

In period-1, the optimal size is determined using (15). It is known that $L_{t-1} = S_0{}^* = \ 5$. Given (29), it is also known that $G = \frac{1}{2}$. The optimal size of period-1 is thus 13.75. Given the amount of equity, 8, the incentive compatible equity ratio is 58%.

In the steady state, the size can be calculated using (19). It is 20 million euros. Since the firm now maintains equity worth 8 million euros the incentive compatible equity ratio of an established firm is 40%.

Consider the firm in the steady state with an equity ratio of 40%. The owner will expand the firm. The desired rate of growth is 50% and thus the desired size is 30 million euros. What is the incentive compatible equity ratio? Now (27) reveals that it is 53.3%. Hence, when an established firm starts to grow rapidly, it needs to raise its equity ratio from 40% to 53.3%.

11. Overlapping structure of products

In order to highlight the effect of the overlapping product structure, an alternative framework without overlapping production is explored. Suppose that the entrepreneur sets up Firm A with equity capital $\frac{1}{2}E$ in period-0. The firm launches a new product in periods 0, 2, 4, … (Figure 2). With effort, Firm A yields zero profits due to perfect competition. Without effort, each product yields expected profits

$$p\{\ S_0P_0-(S_0-E)r_f+\alpha\delta\lfloor S_0P_0-(S_0-E)r_f\rfloor\}\ -\ Er_f\quad , \tag{30}$$

from which an incentive compatible firm size can be resolved. The incentive compatible firm size proves to be $S_0{*}$ in the first production periods (0,2,4,…). Hence, in the second production periods (1,3,5,…), the size of the firm is $\alpha S_0{*}$.

In period-1, the entrepreneur receives more capital and sets up Firm B with equity capital $\frac{1}{2}E$. The firm launches a product in periods 1, 3, 5, … . With effort, Firm B yields zero profits due to perfect competition. Without effort, the incentive compatible size is $S_0{*}$ in the first production periods (periods 1,3,5,…). In the second production periods (periods 2,4,…), the firm size is $\alpha S_0{*}$.

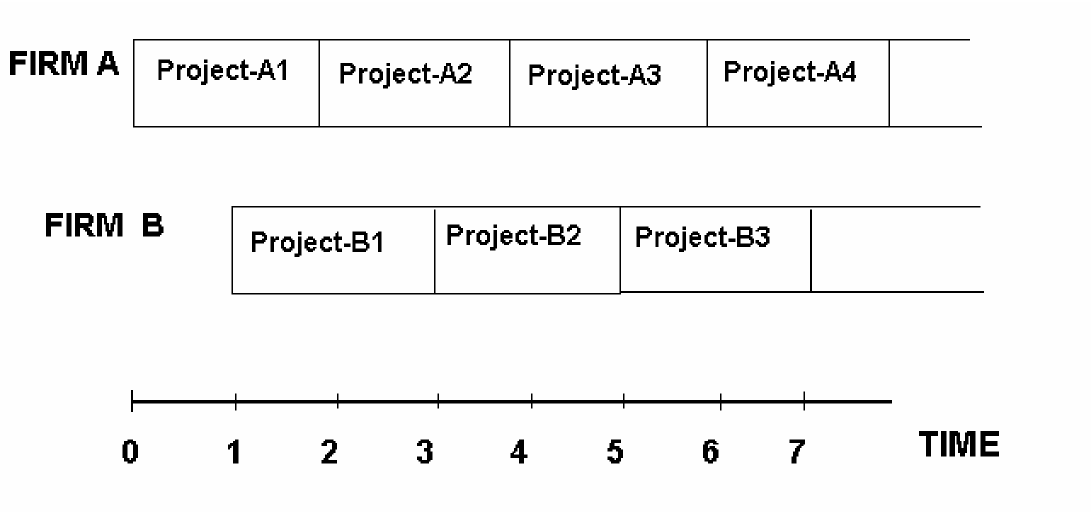

Figure 2. Split production with $\alpha = 1$.

After period-1, the total size of the two firms adds up to $(1+\alpha)S_0*$ in every period. In contrast, under the overlapping structure of products the incentive compatible size of the firm exceeded $(1+\alpha)S_0*$ (recall (16)). Hence, relatively less equity capital is needed to eliminate the problem of effort aversion under the overlapping structure of products, since the profits from old products reduce the incentives to shirk effort. This positive incentive effect does not exist when the production is split in two separated firms. Suppose that Firm A and Firm B have exerted effort until period $\hat{t}$ ($\hat{t}$ is odd). Then, Firm A has old profitable products (product-$A\hat{t}-1$) and Firm B launches a new product-$B\hat{t}$. Without effort, the product of Firm B may fail. The entrepreneur then losses his equity investment in Firm B, but earns profits from firm A, which has old profitable products (product-$A\hat{t}-1$). That is, the profits of Firm A are not used to cover the loans of Firm B (Figure 2). In contrast, under the overlapping structure of products, a single firm launches each product. If the firm does not exert effort in product-$B\hat{t}$ and the risk realizes, the returns of the firm's other products (product-$A\hat{t}-1$) are used to cover the payments to lenders (Figure 1).

Consequently, the overlapping structure of launched products has a strong effect on the incentive compatible equity ratio. Under the overlapping structure of products, higher leverage is possible.

12. Discussion

This paper has investigated an infinite-horizon model of a fully competitive product market, in which firms can choose unobservable actions (effort) that affect their yield distributions. The purpose has been to find out how the incentive problem (effort aversion problem) can be controlled by controlling growth. The following results have been derived.

Investors can gather useful information by monitoring a firm's products, revenues, investments, growth and equity ratio. Some products flourish yielding rich sales revenue, whereas the others become obsolete finishing their life cycle. The obsolete ones are replaced with new production, which requires initial effort. Both the revenue from the old products and equity capital strengthen a firm's incentives to exert effort in production, whereas the costs of effort weaken the incentives. Hence, the volume of new products should not be too large in comparison to the equity ratio and the revenue from the old products: slow growth is optimal.

The incentive problem mostly pertains to start-up firms with no old, profitable production, whereas established firms - which have profitable products - are less willing to take risks. Hence, start-up firms need to maintain more equity than established firms. The incentive compatible equity ratio of an establish firm may even approach to zero, if its expected future profits are great enough.

Rapid growth worsens the incentive problem, since a firm transfers a major share of its production in novel markets at the same time. Hence, rapidly growing firms should maintain relatively high equity ratio.

The results were derived under a few restrictive assumptions. First, production technology had constant returns of scale. Second, under effort exertion, production was completely risk-free. Third, the expected social value of bad products was not only lower than the social value of good products, but also negative. Fourth, the product markets were fully competitive. If production yielded sufficient profits under effort exertion, the firm would prefer effort exertion to effort aversion although it grew rapidly.

The model could be extended by allowing the production to be risky even under effort exertion. The following result might be derived. Let $e_{inc}(t)$ denote the incentive compatible equity ratio of period-t. It was shown above that for a start-up firm $e_{inc}(0) > 0$, but the incentive compatible equity ratio of an established firm may approach zero, $e_{inc}(\infty) = 0$. Interestingly, in the risky environment the firm might voluntarily maintain some equity, $e_{protect}(t)$, in order to protect its expected future profits against temporary shocks. A start-up firm has no future profits to protect and thus $e_{protect}(0) = 0$. In contrast, an established firm with positive future profits might have $e_{protect}(\infty) > 0$. Hence, the incentive compatible equity ratio, $e_{inc}(t)$, lowers with the future profits, but the voluntary, protective equity ratio, $e_{protect}(t)$, raises with the future profits. Consequently, the following result might emerge; for a start-up firm $e_{inc}(0) > e_{protect}(0) = 0$, but for an established firm $e_{protect}(\infty) > e_{inc}(\infty) = 0$. The equity ratio of a start-up firm is determined by the lenders' equity requirement, but the equity ratio of an established firm is determined by the firm's voluntary policy to protect its future profits.

Appendix A

This Appendix proves Lemma 2; $\sum_{i=1}^{\infty} \delta^{i-1} \pi_{t+i}^{G} = \alpha L_t (P_0 - r_f)$.

$$\begin{aligned}\sum_{i=1}^{\infty} \delta^{i-1} \pi_{t+i}^{G} &= \alpha L_t (P_0 - r_f) + (S_{t+1} - \alpha L_t)(P_0 - r_f - c) \\ &+ \delta\left[\, \alpha L_{t+1}(P_0 - r_f) + (S_{t+2} - \alpha L_{t+1})(P_0 - r_f - c) \,\right] \\ &+ \delta^2\left[\, \alpha L_{t+2}(P_0 - r_f) + (S_{t+3} - \alpha L_{t+2})(P_0 - r_f - c) \,\right] \\ &+ \delta^3\left[\, \alpha L_{t+3}(P_0 - r_f) + \ldots \right] + \;\ldots. \end{aligned} \tag{A.1}$$

Some manipulation gives

$$\begin{aligned}\sum_{i=1}^{\infty} \delta^{i-1} \pi_{t+i}^{G} &= \alpha L_t (P_0 - r_f) \\ &+ (S_{t+1} - \alpha L_t)(P_0 - r_f - c) + \delta \alpha L_{t+1}(P_0 - r_f) \\ &+ \delta\left[\, (S_{t+2} - \alpha L_{t+1})(P_0 - r_f - c) + \delta \alpha L_{t+2}(P_0 - r_f) \,\right] \\ &+ \delta^2\left[(S_{t+3} - \alpha L_{t+2})(P_0 - r_f - c) + \delta \alpha L_{t+3}(P_0 - r_f) + \ldots \right] + \;\ldots. \end{aligned} \tag{A.2}$$

Since $L_{j+1} = S_{j+1} - \alpha L_j$, $\forall j$, (recall (6)) , (A.2) can be rewritten as

$$\begin{aligned}\sum_{i=1}^{\infty} \delta^{i-1} \pi_{t+i}^{G} &= \alpha L_t (P_0 - r_f) + \\ &+ (S_{t+1} - \alpha L_t)\left[P_0 - r_f - c + \delta\alpha(P_0 - r_f)\right] \\ &+ \delta(S_{t+2} - \alpha L_{t+1})\left[\, P_0 - r_f - c + \delta\alpha(P_0 - r_f) \,\right] \\ &+ \delta^2(S_{t+3} - \alpha L_{t+2})\left[P_0 - r_f - c + \delta\alpha(P_0 - r_f) \,\right] + \;\ldots \end{aligned} \tag{A.3}$$

Given (1), $P_0 - r_f - c + \delta\alpha(P_0 - r_f) = 0$, (A.3) simplifies to

$\sum_{i=1}^{\infty} \delta^{i-1} \pi_{t+i}^{G} = \alpha L_t (P_0 - r_f)$. *Q.E.D*

Appendix B

In period- 0 , the incentive constraint of the firm is

$$\pi_0^G + \delta(1-b)\sum_{i=1}^{\infty}\delta^{i-1}\pi_i^G = \pi_0^B , \tag{B.1}$$

which consists of the following three elements.

i.) The first element represents the period-0 returns of the firm with effort

$$\pi_0^G = S_0(P_0-c)-(S_0-\tfrac{1}{2}E)r_f - \tfrac{1}{2}Er_f . \tag{B.2}$$

Here S_0 denotes the size of the firm, whereas P_0 is the price of the products, c is the cost of effort, $(S_0-\frac{1}{2}E)r_f$ is the loan repayment and $\frac{1}{2}Er_f$ is the alternative cost of the equity.

ii.) The second element, $\delta(1-b)\sum_{i=1}^{\infty}\delta^{i-1}\pi_i^G$, represents the difference in the present value of future profits between the effort and non-effort strategies.

iii.) The third element represents the period-0 profits without effort

$$\pi_0^B = b\left[S_0P_0-(S_0-\tfrac{1}{2}E)r_f\right]-\tfrac{1}{2}Er_f . \tag{B.3}$$

With probability b the products are sellable, the firm obtains sales revenue, S_0P_0, and repays its loans, $(S_0-\frac{1}{2}E)r_f$. The term $\frac{1}{2}Er_f$ is the cost of equity.[13] Inserting (B.2) and (B.3) into (B.1) and recalling Lemma (2) displays a concise form of the incentive constraint

$$0 = bcS_0-\tfrac{1}{2}(1-b)Er_f . \tag{B.4}$$

[13] With probability $1-b$ the products are non-sellable, the firm collapses and the owner receives no dividends.

Appendix C

The incentive constraint of period- t is,

$$\pi_t^G + \delta(1-b)\sum_{i=1}^{\infty}\delta^{i-1}\pi_{t+i}^G = \pi_t^B , \qquad \text{(C.1)}$$

which consists of the following three elements.

i.) The first element represents the period $-t$ returns of the firm with effort

$$\pi_t^G = \alpha L_{t-1}(P_0 - r_f) + (S_t - \alpha L_{t-1})(P_0 - r_f - c) . \qquad \text{(C.2)}$$

The first term, $\alpha L_{t-1}(P_0 - r_f)$, indicates profits from old products and the second term displays negative returns from new products.

ii.) The element $\delta(1-b)\sum_{i=1}^{\infty}\delta^{i-1}\pi_{t+i}^G$ represents future profits.

iii.) The third element represents the period $-t$ returns without effort

$$\pi_t^B = b\left[\alpha L_{t-1}(P_0 - r_f) + (S_t - \alpha L_{t-1})(P_0 - r_f)\right] - (1-b)Er_f . \qquad \text{(C.3)}$$

The square brackets contain the returns of the firm, when risk taking succeeds and the products prove to be sellable. Then, old products yield $\alpha L_{t-1}(P_0 - r_f)$ and new products yield $(S_t - \alpha L_{t-1})(P_0 - r_f)$. The term following the square brackets reveals the risk of losing equity. It is useful to rewrite (C.3) as

$$\pi_t^B = b\pi_t^G + (S_t - \alpha L_{t-1})bc - (1-b)Er_f . \qquad \text{(C.4)}$$

Inserting (C.2) and (C.4) into (C.1) and recalling Lemma 2 uncovers a concise form of the incentive constraint

$$0 = (S_t - \alpha L_{t-1})bc - (1-b)\alpha L_{t-1}(P_0 - r_f) - (1-b)Er_f . \qquad \text{(C.5)}$$

Appendix D

Using induction, Appendix D proves Proposition 3,

$$S_t^* = S_0^* \left\{ 2 + 2\alpha(1+G)\sum_{i=0}^{t-2}(\alpha G)^i + (1+G)\alpha(\alpha G)^{t-1} \right\}, \; G = \frac{1-b}{b+\alpha\delta b}, \; t \geq 1. \tag{D.1}$$

Steps 1 and 2 together imply that (D.1) determines the size correctly in periods 1 and 2.

Step 1. Recall from (15)

$$S_t^* = 2S_0^* + \alpha L_{t-1}\left[1+G\right]. \tag{D.2}$$

Given (D.2), the optimal size in period-1 is

$$\begin{aligned} S_1^* &= 2S_0^* + \alpha L_0\left[1+G\right] \\ &= 2S_0^* + \alpha S_0^*\left[1+G\right] \\ &= S_0^*\left[2+\alpha(1+G)\right]. \end{aligned} \tag{D.3}$$

Using (D.2) again gives the optimal size in period-2

$$S_2^* = 2S_0^* + \alpha L_1\left[1+G\right]. \tag{D.4}$$

Inserting first $L_1 = S_1 - \alpha L_0$ from (6) into (D.4), and using then (D.3) displays

$$\begin{aligned} S_2^* &= 2S_0^* + \alpha(S_1^* - \alpha S_0^*)\left[1+G\right] \\ &= 2S_0^* + \alpha(2+\alpha G)\left[1+G\right]S_0^* \\ &= S_0^*\left[2+2\alpha(1+G)+\alpha^2 G(1+G)\right]. \end{aligned} \tag{D.5}$$

Step 2. It is sufficient to point out that the equation (D.1) reveals the same results as (D.3) and (D.5). It is easy to see that inserting t=1 (t=2) into (D.1) displays (D.3) ((D.5)). Hence, (D.1) expresses the size of the firm correctly in periods 1 and 2.

It is next shown that if (D.1) determines the size correctly in period t-1, it will also determine the size correctly in period-t.

Step 3. It is assumed that the equation (D.1) reveals the size correctly in period t- 1

$$S_{t-1}^{*} = S_0^{*}\left\{ 2 + 2\alpha(1+G)\sum_{i=0}^{t-3}(\alpha G)^i + (1+G)\alpha(\alpha G)^{t-2} \right\}. \tag{D.6}$$

It is necessary to solve S_t^*. First, recall (D.2)

$$S_t^{*} = 2S_0^{*} + \alpha L_{t-1}\left[1+G\right],$$

and insert $L_{t-1} = S_{t-1} - \alpha L_{t-2}$ into this so that

$$S_t^{*} = 2S_0^{*} + \alpha G S_{t-1} + \alpha S_{t-1} - \alpha^2 L_{t-2}\left[1+G\right]. \tag{D.7}$$

Inserting now $S_{t-1}^{*} = 2S_0^{*} + \alpha L_{t-2}\left[1+G\right]$ from (D.2) into (D.7) uncovers

$$S_t^{*} = 2(1+\alpha)S_0^{*} + \alpha G S_{t-1}. \tag{D.8}$$

Setting (D.6) into this gives

$$\begin{aligned} S_t^{*} &= 2(1+\alpha)S_0^{*} + \alpha G\, S_0^{*}\left\{ 2 + 2\alpha(1+G)\sum_{i=0}^{t-3}(\alpha G)^i + (1+G)\alpha(\alpha G)^{t-2} \right\} \\ &= S_0^{*}\left\{ 2(1+\alpha) + 2\alpha G + \alpha G\, 2\alpha(1+G)\left[1+\alpha G+\ldots+(\alpha G)^{t-3}\right] + (1+G)\alpha(\alpha G)^{t-1} \right\} \\ &= S_0^{*}\left\{ 2 + 2\alpha(1+G) + 2\alpha(1+G)\left[\alpha G+\ldots+(\alpha G)^{t-2}\right] + (1+G)\alpha(\alpha G)^{t-1} \right\} \\ &= S_0^{*}\left\{ 2 + 2\alpha(1+G)\sum_{i=0}^{t-2}(\alpha G)^i + (1+G)\alpha(\alpha G)^{t-1} \right\}. \end{aligned} \tag{D.9}$$

This is fully in accordance with the equation (D.1). Hence, it is shown that if (D.1) is true in period t-1, it will also be true in period t. Q.E.D

Appendix E

Appendix E derives the results of Corollary 1 using Proposition 4,

$$\Delta_t \;\equiv S_t^* - S_{t-1}^* \;=\; S_0^* \,\alpha(\alpha G)^{t-2}(1+G)(1+\alpha G)\,. \tag{E.1}$$

i.) It is shown that the firm grows when $\alpha > 0$. It is easy to see from (E.1) that growth is zero if $\alpha = 0$, and that it is positive when $\alpha > 0$ since $S_0^*, G > 0$.

ii.) It is shown that growth speeds up with α.

$$\Delta_t \;=\; S_0^* \,(\alpha G)^{t-2}\left[\alpha + (\alpha G)\right](1+\alpha G)\,. \tag{E.2}$$

$$\begin{aligned} \frac{d}{d\,\alpha}\,\Delta_t \;&= S_0^*\,(t-2)\,(\alpha G)^{t-3}\,\frac{d(\alpha\, G)}{d\,\alpha}(\alpha + \alpha G)(1+\alpha G) \\ &+ S_0^*\;(\alpha G)^{t-2}\left[1+\frac{d(\alpha\, G)}{d\,\alpha}\right](1+\alpha G) \\ &+ S_0^*\;(\alpha G)^{t-2}\,(\alpha + \alpha G)\frac{d\,(\alpha G)}{d\alpha}\;>0, \end{aligned} \tag{E.3}$$

because $d\,(\alpha G)/d\alpha > 0$.

iii.) Growth slows down with b

$$\begin{aligned} \frac{d}{d\,b}\,\Delta_t \;&\equiv S_0^*\,\alpha(t-2)\,(\alpha G)^{t-3}\,\alpha\,\frac{d\,G}{d\,b}(1+G)(1+\alpha G) \\ &+ S_0^*\,\alpha\,(\alpha G)^{t-2}\,\frac{d\,G}{d\,b}(1+\alpha G) \\ &+ S_0^*\;(\alpha G)^{t-2}\,(1+G)\,\alpha^2\,\frac{d\,G}{db}\;<0, \end{aligned} \tag{E.4}$$

since $dG/db < 0$.

iv.) Growth speeds up with time if $\alpha G > 1$ and slows down with time if $\alpha G < 1$.

$$\frac{d}{dt}\Delta_t = S_0^* \alpha(1+G)(1+\alpha G)\ \frac{d}{dt}(\alpha G)^{t-2}$$
$$= S_0^* \alpha(1+G)(1+\alpha G)\ \ln(\alpha G)\ (\alpha G)^{t-2} \qquad \text{(E.5)}$$
$$= \Delta_t \ln(\alpha G).$$

Hence, growth speeds up with time if $\ln(\alpha G) > 0$, and slows down with time if $\ln(\alpha G) < 0$.

Thus, growth speeds up with time if $\alpha G > 1$ and slows down with time if $\alpha G < 1$.

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