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Abstract

I study an economy with sellers and buyers with unit supplies and unit demands. Both parties have valuations uniformly distributed on a unit interval. I quantify the inefficiency, compared to the Walrasian market, caused by a market where the agents meet randomly. There are several causes of inefficiency that I deal with separately. First, even if there is perfect information about valuations it makes a difference whether all agents participate in the markets or whether only those who would trade in the Walrasian market participate. The same applies when there is private information about valuations.

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1 Introduction

The purpose of this article is to quantify the efficiency loss associated with imperfect or non-Walrasian markets. Economies where the agents meet randomly constitute one of the best behaved classes of non-Walrasian models. A particularly applicable model of random matching is the urn-ball model where, say, buyers contact randomly sellers; when there is a continuum of buyers and sellers a seller may meet any number of buyers, and the number is given by a Poisson-distribution. This environment allows for an interesting price formation mechanism: When there are several buyers who want what the seller has a natural way to resolve the allocation problem is by auction. In a meeting where there is only one buyer and asymmetric information about the value of the object between the buyer and the seller there is no trading mechanism that results in an efficient outcome (Myerson and Satterthwaite, 1983). For simplicity, I assume that the buyer makes a take-it-or-leave-it offer.

Auction is the archetypical competitive price formation mechanism and I am interested only in market imperfections that are not too different from the Walrasian markets; it is very easy to come up with markets and price mechanisms that are so different from the Walrasian ones that they have nothing interesting in common. For instance, the standard search theoretic markets with pairwise meetings with Nash-bargaining is of no great interest (for another example see Gale and Sabourian, 2005). Further, it is not applicable in situations with uncertainty about the valuations of the parties. Non-cooperative bargaining protocols under uncertainty feature a multitude of equilibria and are technically unnecessarily difficult for my purposes.

This set-up allows me to quantify, i.e., to calculate, the exact percentages as to the inefficiency that is just due to giving up the Walrasian framework but retaining perfect information. This is accomplished by assuming that all the buyers are homogeneous and that all the sellers are homogeneous. Then I introduce heterogeneity in the valuations and study two cases. In one case only the buyers and sellers who would trade in the Walrasian markets participate in the markets where the agents are randomly matched. In the other case all agents participate in the market. To do this I cannot work with general distributions of valuations. Not too surprisingly I use the uniform distribution; it is amenable to calculations. But it is also the distribution with maximum entropy, and my conjecture is that it results in maximum inefficiency, too. This I do not, however, attempt to prove.

2 The model

Assume that there is a unit interval of both buyers and sellers. Their valuations are uniformly distributed between zero and unity. The economy lasts only for one period, meaning that the agents have only one chance to trade. This simplifies things greatly since in a dynamic economy I should calculate the agents' expected future utilities. Then the agents' reservation values should be determined in equilibrium and this is very hard. I think that limiting the study to the static case is not a serious drawback since it can still be shown that the higher a seller's valuation the higher his expected utility in both the dynamic and static model. And the same applies to the buyers, too.

The buyers contact sellers randomly in a symmetric fashion, and the number of buyers a seller meets is given by the Poisson distribution with rate $1 = \frac{\#buyers}{\#sellers}$. Thus, the probability that a seller meets exactly k buyers is $e^{-1}\frac{1}{k!}$.

Before going to calculations let me note that in the Walrasian equilibrium the price would be one half, and the maximum number of trades would be completed. In the decentralised market there are sellers who do not meet any buyers. Further, not all meetings result in a trade as either the buyers' valuations are not as high as the sellers' valuations, or the buyers strategically offer less than their own valuation trading off the high probability of trade for high gains from trade. Also trades that would not take place in the Walrasian market happen: Consider, for instance, a buyer with a low valuation (less than one half) who meets a seller alone. With positive probability the seller is a low valuation type who accepts the buyer's offer.

Auction is straightforward but I have to determine a buyer's optimal take-it-or-leave-it offer when he meets a seller alone and the valuations are private information. Consider a buyer with valuation v. Denote his offer by b. It maximises his expected utility

$$(v - b)\, F(b) \tag{1}$$

where F is the distribution function of the sellers' valuations. With a uniform distribution the first order condition yields the optimal offer

$$b = \frac{v}{2} \tag{2}$$

Note that the meeting probabilities, i.e., the ratio of buyers to sellers remains the same in both cases.

2.1 All buyers and sellers in the market or the large market

2.1.1 Private information about valuations

Assume then that a seller with valuation w meets k buyers. Now the buyers engage in an auction where the winner is the buyer with the highest valuation and he pays the second highest valuation. But there is a twist since the second highest valuation can be less than half of the highest valuation. Then the winning buyer raises his bid until it is in accordance with (2). This means that if a buyer's valuation is not at least twice the seller's valuation or if the second highest valuation of the buyers is not above the seller's valuation trade does not take place.

I need the joint density function for the highest and second highest order statistics. Remember that the valuations are assumed uniform with density

$f(x) = 1$ on $[0, 1]$. Assume that a seller meets $k \geq 2$ buyers. Denote the highest valuation by v_1 and the second highest by v_2.[1] It is given by

$$g(v_1, v_2) = k(k-1)f(v_1)f(v_2)F^{k-2}(v_2) = k(k-1)v_2^{k-2} \tag{3}$$

where the second equality follows from the assumption about uniform distribution.

When $k \geq 2$ there are three different cases to consider. First, the seller may have any valuation w between zero and one half and the highest valuation of the buyers is larger than twice the seller's valuation. Then the expected gains are given by

$$\int_0^{1/2} \int_{2w}^1 k v_1^{k-1}(v_1 - w) dv_1 dw = -\frac{1}{8} + \frac{2k+1}{4(k+2)} \tag{4}$$

Second, the seller may have any valuation w between zero and one half and the highest valuation of the buyers is less than twice the seller's valuation and the second highest valuation is larger than the seller's valuation. Then the expected gains are given by

$$\begin{aligned} &\int_0^{1/2} \int_w^{2w} \int_{v_2}^{2w} k(k-1)v_2^{k-2}(v_1 - w) dv_1 dv_2 dw \\ = &\frac{k-1}{4(k+1)(k+2)} - \frac{k-1}{k+1}\frac{1}{2^{k+3}} \end{aligned} \tag{5}$$

Third, the seller may have any valuation w between one half and unity. Then trade takes place only if the second highest valuation is higher than the seller's valuation. The expected gains are given by

$$\begin{aligned} &\int_{1/2}^1 \int_w^1 \int_{v_2}^1 k(k-1)v_2^{k-2}(v_1 - w) dv_1 dv_2 dw \\ = &\frac{2k+1}{2(k+1)} + \frac{1}{k+1}\frac{1}{2^{k+1}} - \frac{7}{8} + \frac{k-1}{k+1}\frac{1}{2^{k+3}} \end{aligned} \tag{6}$$

Summing (4), (5) and (6) I get the expected gains in a k-buyer meeting

$$\frac{k-1}{2(k+1)} + \frac{1}{k+1}\frac{1}{2^{k+1}} \tag{7}$$

If a seller meet exactly one buyer trade only takes place if the buyer's valuation is at least twice the seller's valuation. The expected gains are given by

$$\int_0^{1/2} \int_{2w}^1 k v_1^{k-1}(v_1 - w) dv_1 dw = \frac{1}{8} \tag{8}$$

[1] These should be indexed by k, too, but I omit this as there should not arise any confusion.

2.1.2 Perfect information about valuations

When a seller with valuation w meets k buyers there is trade if the highest valuation of the buyers is greater than w. The density of the highest valuation v_1 is given by $h(v_1) = kf(v_1)F^{k-1}(v_1) = kv_1^{k-1}$. Thus, the expected gains from trade in a k-buyer meeting are

$$\int_0^1 \int_w^1 kv_1^{k-1}(v_1 - w)\, dv_1 dw = \frac{1}{2} - \frac{1}{k+1} + \frac{1}{(k+1)(k+2)} \tag{9}$$

2.2 Agents with certain gains from trade in the market or the small market

2.2.1 Private information about valuations

Let me now assume that only the agents who 'should' be in the market are there. Namely, all the buyers whose valuation are above one half and all the sellers whose valuation is less than one half. Given that there are k buyers the density function, i.e., the probability that any buyer's valuation is x is given by $d(x) = 2$ and the distribution function by $D(x) = 2x - 1$. In the same vein the density of the highest valuation is given by

$$h(v_1) = kf(v_1)F^{k-1}(v_1) = 2k(2v_1 - 1)^{k-1}$$

Given that the number of buyers is $k \geq 2$ there are two cases to consider. First, when the seller's valuation w is less than one fourth. Then the buyers' valuations are necessarily at least twice as high and trade always takes place. If the seller's valuation w is between one fourth and one half trade again takes place if the buyers' highest valuation is at least double the seller's valuation. The expected gains are

$$\begin{aligned} &\int_0^{1/4} \int_{1/2}^1 2k(2v_1 - 1)^{k-1}(v_1 - w)\, dv_1 dw + \\ &\int_{1/4}^{1/2} \int_{2w}^1 2k(2v_1 - 1)^{k-1}(v_1 - w)\, dv_1 dw \\ = \; &\frac{3(2k^2 + 4k + 1)}{16(k+1)(k+2)} \end{aligned} \tag{10}$$

Second, if the seller's valuation w is between one fourth and one half trade takes place even if the highest valuation is below double the seller's valuation since the second highest valuation is necessarily above the seller's valuation. The expected gains are

$$\int_{1/4}^{1/2} \int_{1/2}^{2w} 2k(2v_1 - 1)^{k-1}(v_1 - w)\, dv_1 dw = \frac{2k+1}{16(k+1)(k+2)} \tag{11}$$

Expressions (11) and (12) sum to

$$\frac{3k+1}{8(k+1)} = \frac{3}{8} - \frac{1}{4(k+1)} \tag{12}$$

When there is only one buyer and the seller's valuation is less than one fourth trade takes place for certain. If the seller's valuation is between one fourth and one half trade takes place only if the buyer's valuation is at least double the seller's valuation. The expected gains are

$$\int_0^{1/4} \int_{1/2}^{1} 2\left(v_1 - w\right) dv_1 dw + \int_{1/4}^{1/2} \int_{2w}^{1} 2\left(v_1 - w\right) dv_1 dw = \frac{7}{32} \tag{13}$$

2.2.2 Perfect information about valuations

All buyers' valuations are higher than all sellers' valuations and consequently the agents always trade. The expected gains from trade in a k-buyer meeting are given by

$$\int_0^{1/2} \int_{1/2}^{1} 2k\left(2v_1 - 1\right)^{k-1} \left(v_1 - w\right) dv_1 dw = \frac{3}{8} - \frac{1}{4\left(k+1\right)} \tag{14}$$

3 Analysis

In the previous section the expected gains were calculated given that k buyers come to a seller with valuation w, and then I calculated the expectation of these gains over all possible values of w, i.e., I integrated over w. To get the final results it is necessary to sum over k, too, and this is what is done next. But before that let me remind of the case with homogeneous buyers and sellers. In this case it does not matter whether valuations are private information or not since all the buyers have the same valuation and all the sellers have the same valuation. Trade takes place whenever a seller meets at least one buyer and this happens with probability $1 - e^{-1} \approx 0.632$.

The expected gains in the large market can now be calculated to be with private information

$$e^{-1}\frac{1}{8} + e^{-1}\sum_{k=2}^{\infty} \frac{1}{k!}\frac{1}{k+1}\frac{1}{2^{k+1}} + e^{-1}\sum_{k=2}^{\infty} \frac{1}{k!}\frac{k-1}{2\left(k+1\right)} = e^{-1/2} - \frac{1}{2} \approx 0.107 \tag{15}$$

and with perfect information

$$e^{-1}\sum_{k=1}^{\infty} \frac{1}{k!}\left(\frac{1}{2} - \frac{1}{k+1} + \frac{1}{\left(k+1\right)\left(k+2\right)}\right) = \frac{1}{2} - e^{-1} \approx 0.132 \tag{16}$$

The expected gains in the small market with private information are given by

$$e^{-1}\frac{7}{32} + e^{-1}\sum_{k=2}^{\infty} \frac{1}{k!}\frac{3}{8} - e^{-1}\sum_{k=2}^{\infty} \frac{1}{k!}\frac{1}{4(k+1)} = \frac{3}{32}e^{-1} + \frac{1}{8} \approx 0.159 \tag{17}$$

and with perfect information

$$e^{-1}\sum_{k=1}^{\infty}\frac{1}{k!}\left(\frac{3}{8}-\frac{1}{4\,(k+1)}\right)\approx 0.171 \tag{18}$$

Now I am in a position to compare the efficiency of the two market structures and the two information structures. In the Walrasian markets all the possible trades are executed at price one half. The price is just a transfer that does not affect the total gains which add up to 0.375. To quantify the inefficiency caused by heterogeneity and not by private information the appropriate comparison is to the small market with perfect information. There $0.171/0.375 = 45.6\%$ of the possible gains are realised. When the valuations are private information the realised gains are $0.159/0.375 = 42.4\%$. In this case the difference is quite small.

In the large market under perfect information $0.132/0.375 = 35.2\%$ of the possible gains are realised. When valuations are private information $0.107/0.375 = 28.5\%$ of the gains are realised.

Proposition 1 *Both under perfect information and private information the small market is more efficient than the large market Under perfect information the small market yields* 45.6% *of the gains of the Walrasian market and the large market yields* 35.2% *of the gains of the Walrasian market . Under private information the corresponding gains are* 42.4% *in the small market, and* 28.5% *in the large market.*

Notice that the drop in efficiency in far more dramatic in the large market than in the small market when comparing perfect information and private information cases. This is because private information leads to strategic behaviour, i.e., solitary buyers offering just half of their valuation, but in the large market this leads to no trade more often than in the small market where it is known that each buyer values the goods more than any seller.

4 Conclusion

I study a parameterised situation of decentralised markets where the difficulty of co-ordinating the agents contacts keeps the economy from realising all potential gains. On top of that I introduce heterogeneity into the agents' valuations. Depending on which agents participate in the market there may be significant differences in efficiency.To my knowledge these efficiency losses have not been spelled out before, and they may be instructive even though the results are based on uniform distribution of valuations and there being equal numbers of buyers and sellers.

References

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