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Abstract

I consider an endogenous "pool of knowledge" model of growth through creative destruction with imperfect intellectual property rights. The parameters of the model include the hazard rate of imitation and a patentability requirement. The model allows for *growth traps* in which there is no growth because of slow growth in the past, although positive growth would be possible for identical parameter values. The growth-maximizing value of the imitation rate is zero, but the growth-maximizing value of the patentability requirement can have any positive value below a theoretical maximum. A low patentability requirement is growth-maximizing in very slowly growing economies.

JEL Classification: O34, O41

Keywords: Intellectual property strength; endogenous growth theory; creative destruction.

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1. Introduction

The policy instruments that are regularly considered in the literature on the economics of patents include the *required inventive step* and the *duration* of a patent, and its *lagging breadth* and *leading breadth*. Here lagging breadth refers to the minimum quality difference that the patented product must have with a lower-quality product if the latter may be produced without infringing on the patent, and similarly, leading breadth is the quality improvement which a superior product must at least have if it does not infringe on the patent (Cf. O'Donoghue et. al., 1998, p. 3). Until recently, economists have considered the problem of the optimal choice of these instruments mostly within a partial equilibrium framework. On the other hand, *endogenous growth theory* studies the incentives to conduct research and development within a general equilibrium framework. This makes it possible not just to explicitly discuss the effects of intellectual property rights on growth rate, but also to consider some aspects of IPR protection that do not have any obvious representation in a partial equilibrium model.[1]

However, the large majority of the endogenous growth models that have been put forward until now postulate that a research firm which invents a new, improved design for a product receives a permanent monopoly for producing it. This corresponds to patents which have an infinite duration, and in which both the required inventive step and the leading breadth are smaller than the quality improvement in any of the innovations that are actually made.

As Figure 1 illustrates, the model of endogenous growth theory can broadly speaking be divided into the models of growth through *specialization*, in which each newly invented design of a product increases the number of the different products on the market, and Schumpeterian models of growth through *creative destruction* in which each newly invented design of a product replaces an existing design. A model of growth through specialization provides a natural framework for analyzing the effects of the *expected length of the monopoly* on economic growth, and an interesting

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[1] E.g., the discussion of the allocation of R&D resources between the different sectors of the economy in O'Donoghue and Zweimüller (2004, pp. 103-108) does not have any obvious counterpart in a partial-equilibrium framework. Similarly, in the model of Horii and Iwaisako (2007) stronger intellectual property rights reduce the real wage in the production sector by increasing the number of the monopolist sectors of the economy, which makes working in research more attractive in comparison with working in production (see p. 79), an effect which would be hard to analyze in a partial equilibrium framework.

analysis of this kind has recently been provided in Furukawa (2007).[2] However, since in a model of growth through specialization all new designs correspond to products of a completely new kind (rather than improvements in some already existing product), the growth effects of the required inventive step and the breadth of patents cannot be addressed in their context.

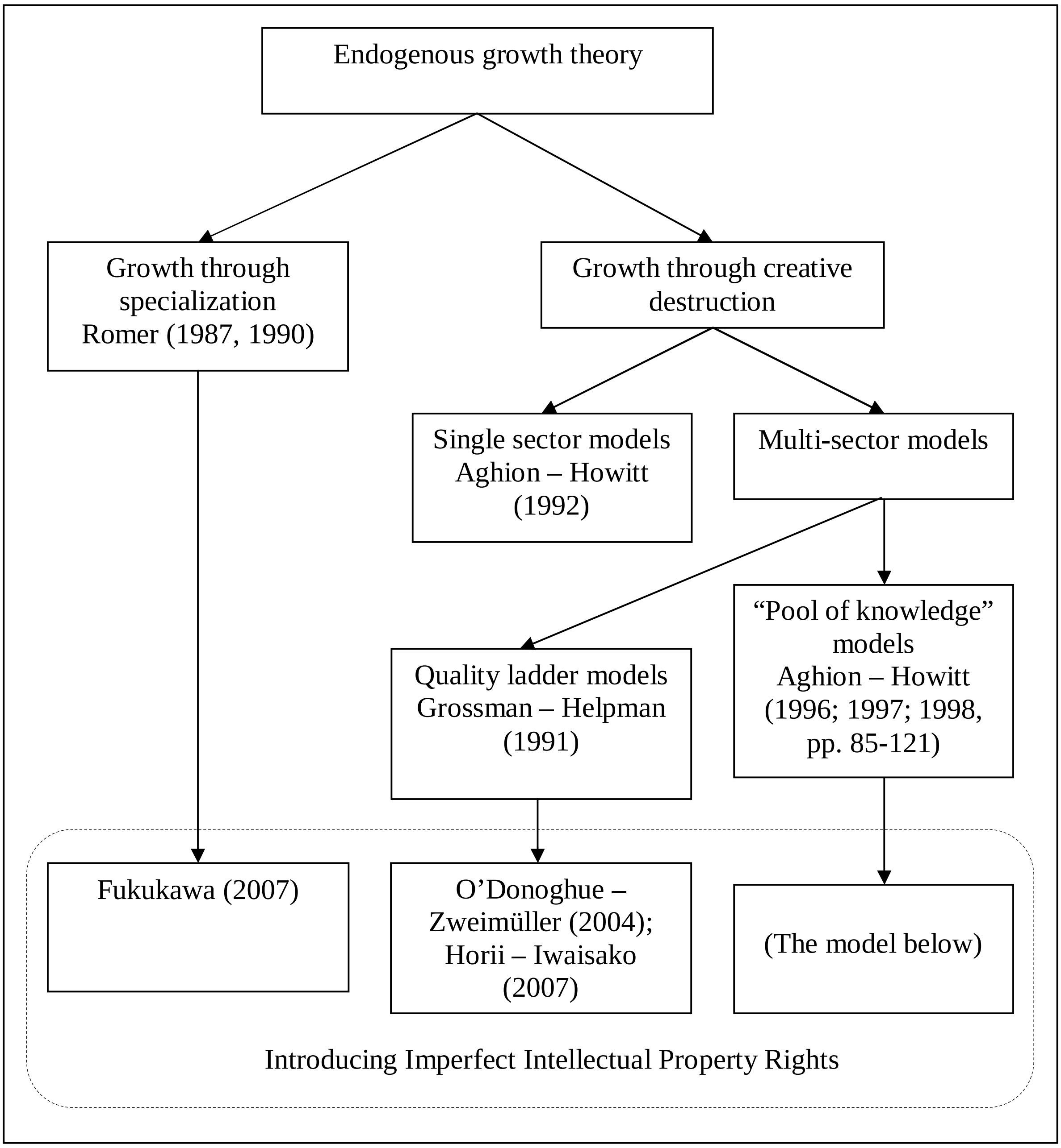

Figure 1. Endogenous Growth Models with Perfect and Imperfect Intellectual Property Rights.

[2] Furukawa (2007) shows that in the context of his model the long-run rate of innovation has an inverse-U shape as a function of the rate of imitation, and argues on the basis of this finding that too strong intellectual property protection is not growth-promoting.

The models of growth through creative destruction can be classified into the *single-sector* and *multi-sector* models on the basis of the number of the different designs that the products on the market have at each moment of time. The multi-sector models of growth through creative destruction, which constitute a natural framework for studying the effects of breadth and the required inventive step, can further be divided into two groups on the basis of the nature of the *quality improvement* that distinguishes the newly designed product from the one that it replaces. More specifically, in a *quality ladder* model, such as the one that is put forward in Grossman and Helpman (1991), each innovation constitutes a quality improvement of a fixed size relative to the previous product of the same sector (ibid., p. 45), whereas in a "leapfrogging" model each improved product receives the quality which corresponds to the current leading-edge technology (see Aghion and Howitt, 1996, 1997, and 1998, pp. 85-121). The latter assumption is motivated by the idea that the innovators make use of a shared *pool of technological knowledge*, which is represented by the quality of the newest designs of products on the market.

Time is continuous and the emergence of innovations is a Poisson process in the model of Grossman and Helpman (1991), and this easily implies that in a Nash equilibrium the expected returns from R&D are identical in all the sectors in which there is R&D.[3] It is easy to see that a model of this type can have a sensible equilibrium only if the profit that a patent to the currently used design of a product yields to its owner is independent of the quality of the product.[4] Accordingly, Grossman and Helpman (1991, p. 45) choose the utility function of the consumers in such a way that this rather implausible assumption becomes valid. The recent, interesting papers O'Donoghue and Zweimüller (2004) and Horii and Iwaisako (2007) discuss multi-sector quality ladder models with imperfect intellectual property rights,[5] and just like in the model of Grossman and Helpman (1991), in these models a quality

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[3] This is because the probability that two firms would make an innovation in the same sector at the same moment of time is infinitesimal, so that each research firm can at each moment of time choose where to put its research resources independently of the choices of the other firms. For this reason each firm can have an incentive to do research only in a sector in which the returns to it are maximal.

[4] If the profit from the monopoly to a product was an increasing function of its quality, in a continuous-time quality ladder model all research firms would choose to try improve the highest-quality product only, and the lower-quality products would permanently maintain their low quality.

[5] O'Donoghue and Zweimüller (2004) consider the optimization of the patentability requirement and the leading breadth when these are allowed to differ, and Horii and Iwaisako (2007) put forward a model in which the holder of a patent loses its monopoly with an exogenously given probability in each period.

improvement in a product does not increase its demand, which implies that an owner of a patent cannot increase her profits by making improvements to the product.[6]

Below I shall put forward a *shared pool of knowledge* model of growth through creative destruction, in which the demand of each product is an increasing function of its quality, and which provides tools for analyzing the effects of patent policy on economic growth. The adjustable parameters of the model include the *patentability requirement* (i.e. the required inventive step) and the *imitation rate*. Similarly with Horii and Iwaisako (2007) and Furukawa (2007), the imitation rate is in the current model the parameter of a Poisson process, which represents the loss of intellectual property rights through imitation.

This way of representing the fact that intellectual property rights have a finite duration can be contrasted with a representation in which the monopoly for a new product last always for a time *T*, after which the design of the product becomes non-proprietary and several competing firms start producing it. The practice of not choosing the latter option is motivated by the empirical evidence which suggests that such appropriability mechanisms as *secrecy*, *lead time*, and *complementary sales and services* would in most industries be more important than patents.[7] These findings are partially explained by the fact that a patent might fail to provide the intended temporary monopoly to its owner for several reasons.[8]

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[6] In the model of O'Donoghue and Zweimüller (2004), the innovations improve the quality of final goods, and in it the utility function of the consumers has been chosen in such a way that, keeping prices fixed, the demand of each good is independent of its quality (ibid., p. 85). Similarly, in the model of Horii and Iwaisako (2007), in which the innovations are improvements in the quality of intermediate inputs and affect the amount of a final good that results from them, the production function has been chosen in such a way that a quality improvement does not affect the amount of each intermediate good that the final-good producer buys in equilibrium (ibid., p. 51).

[7] See Levin et al. (1987, p. 794), Mansfield (1986), and Cohen et al. (2000). Mansfield presents a survey according to which in most industries, a large majority (more than 80%) of the commercially introduced inventions would have been introduced even without the patent system. However, patents were according to this survey nevertheless essentially more important within pharmaceutical and chemical industries (ibid., p. 175). Cohen et al. (2000) contains an analysis of a survey in which R&D unit or lab managers were asked to evaluate the effectiveness of various appropriability mechanisms in protecting the "firm's competitive advantage" from both product and process innovations. The considered mechanisms were secrecy, patents, other legal methods, lead time, complementary sales and services, and complementary manufacturing. It turned out that, on the average, patents were the least central of these mechanisms, whereas secrecy and lead time were the two most important ones (ibid., pp. 9-10; cf. also Figures 1-4). Levin et al. (1987) discusses a survey which has led to similar findings (see, in particular, ibid., pp. 793-398).

[8] E.g., it might be possible to "invent around" the patent, it might be uncertain whether the patent is judged to be valid if it is challenged (cf. Levin et al., 1987, pp. 802-805), and litigation costs might be so high that a patentee might choose not to defend the patent in court if it is violated. According to the survey of Cohen et al. (2000), this consideration is more relevant for the innovations which have been made by smaller firms (ibid., p. 15). Cf. also Lerner (1995).

I shall present the main features of the model in Section 2 below. Section 3 analyzes the properties of the momentary equilibria of the production sector of the model, and Section 4 discusses its balanced growth paths and more specifically, the circumstances under which growth traps and multiple equilibria are possible. Section 5 discusses the optimization of the policy variables (imitation rate and patentability requirement) of the model, and Section 6 concludes.

2. The Framework

The earlier "pool of knowledge" growth models by Philippe Aghion and Peter Howitt which are referred to in Figure 1 are models in which a single final good is produced from a continuum $[0,1]$ of intermediate goods. In these models technological progress consists in the discovery of new, improved designs for the intermediate goods. Aghion and Howitt assume that each intermediate good is produced by a monopolist, the owner of the patent to its current design, and that the monopolist can drive the previous incumbent out of the market and choose monopoly pricing without being faced with competition with the incumbent (see e.g. Aghion and Howitt 1996, p. 16). However, it seems that it would be quite difficult to give a detailed account of the economic mechanism which makes the old products disappear in the context of these models.[9]

Unlike Aghion and Howitt's earlier models, the model which is put forward below contains also an account of the competition between the producers of a superior and an inferior product of the same sector. This has been achieved by reinterpreting Aghion and Howitt's framework as describing a continuum $[0,1]$ of *final goods* which are used by consumers with a *utility function* which has a similar form with the production function of Aghion and Howitt's model. In the current model the

[9] In the model which is considered in e.g. Aghion and Howitt (1998, pp. 86-91), the intermediate goods are used for producing a homogenous final good, and the quality differences in the new and the old designs of intermediate goods are shown only in the amount of the final good that can be produced from them. In this model the owner of the patent to a superior design can use monopoly pricing without being threatened with competition with the inferior product, although the quality difference between the old and the new design may be arbitrarily small. These assumptions seem difficult to justify, although – as I shall point out below – the situation of Aghion and Howitt's model could be viewed as a representation of an inappropriately functioning patent system, in which it is possible to make trivial improvements to existing products and to patent their already existing features, and in this way exclude their producers from the market.

government has two policy instruments with which it can affect the long-run growth rate, the rate of imitation and the required inventive step, and it is easy to see that in the limit in which there is no imitation and no required inventive step, the mathematical structure of the current model becomes identical with the structure of one of the earlier models by Aghion and Howitt, despite of its different economic interpretation.[10] The structure of the current model has been depicted in Figure 2 (cf. Figure 3.1 in Aghion and Howitt, 1998, p. 86).

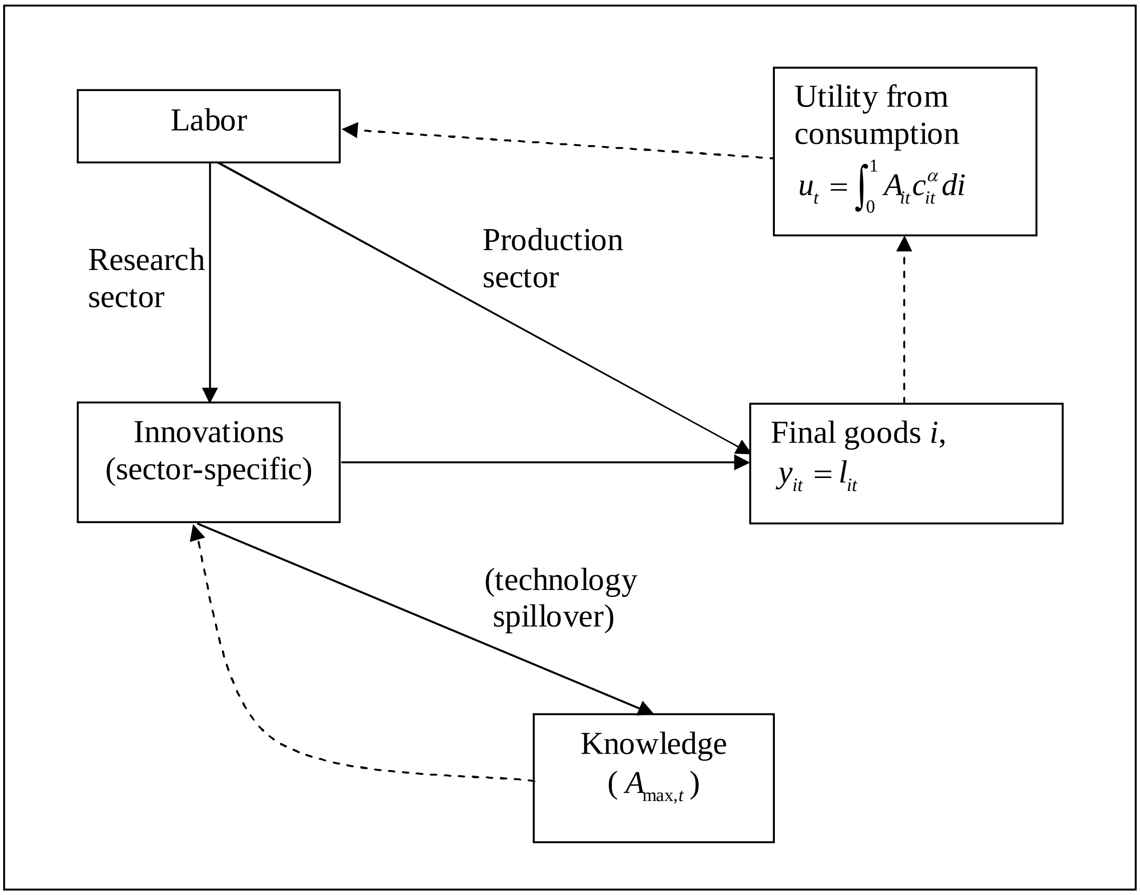

Figure 2. A multi-sector Schumpeterian growth model without capital.

[10] More specifically, the multisector "pool of knowledge" growth models that Aghion Howitt have put forward in Aghion and Howitt (1996), Aghion and Howitt (1997), and Aghion and Howitt (1998, pp. 85-121), can all be viewed as variants of a basic form of their model, which they present in ibid., pp. 86-91. It is easy to verify that if one assumes that there is no imitation, all the results in Section 3 except for the analysis between the competition of an entrant and an incumbent are valid also in this model, when the final goods are viewed as intermediate goods and the utility function (3) is interpreted as a production function of a single, homogenous final good, for which the consumers' utility function is given by (4). However, under this interpretation the formula (21) is not just an arbitrary definition, but it can be deduced by stipulating that wealth is measured in units of the final good, i.e. that the price of one unit of the final good is 1. Hence, the current model becomes essentially identical with one of Aghion and Howitt's models, when one puts $\phi = 0$ (i.e. there is no imitation) and $P = 1$ (i.e. there is no required inventive step).

There is a continuum of agents whose measure has been normalized to one and who can work either in the production sector or in the research sector. Time is continuous, and the amounts of labor force in production and in research at time *t* are denoted by L_t and D_t, respectively. The total size of the labor force has been normalized to 1 so that the *labor market clearing condition* is

(1) $$L_t + D_t = 1$$

There is a continuum $[0,1]$ of final goods which the agents consume. Each final good *i* is produced using labor only in such a way that

(2) $$y_{it} = l_{it}$$

The instantaneous utility function of the agents is

(3) $$u_t = \int_0^1 A_{it} c_{it}^{\alpha} di$$

where A_{it} is the *quality parameter* which characterizes the level of technology in sector *i* at time *t* and c_{it} the consumed amount of the good *i* at time *t*. Here $0 < \alpha < 1$, so that the utility function of the final good is concave. It will be assumed that the consumption of two different variants of the good of the same sector brings no extra utility (i.e., if the consumer consumed the amounts c'_{it} and c''_{it} of two products of the same sector *i* with the respective quality parameters A'_{it} and A''_{it}, this would make only the contribution $\max\left\{A'_{it} c'^{\alpha}_{it}, A''_{it} c''^{\alpha}_{it}\right\}$ to the integral in (3), so that consuming different products of the same sector would be pointless). The total utility of each agent is given by the utility function

(4) $$U = \int_0^\infty e^{-\rho t} u_t dt$$

where the rate of time preference $\rho > 0$ is a constant.

Similarly with the most of the endogenous growth literature, this paper is concerned with a model in which the utility of the agents is independent of the sector in which they work. However, there is empirical evidence which suggests that persons with scientific education have a preference for research, which is shown in accepting employment in research even when it has a lower salary than employment of other kinds.[11] Within the current framework, the preference for research could be modeled by postulating that the utility of an individual depends on the sector that she works in (because of e.g. a social status associated with science), leading to a wage difference between the two sectors.[12] I shall shortly consider a generalized model with this feature at the end of Section 4 below.

The firms of the research sector produce sector-specific *innovations*. An innovation in a sector $i \in [0,1]$ is a new design for the good of sector *i* whose quality parameter A_i equals the current maximum value $A_{\max,t}$ of quality parameters, which represents the current stock of knowledge, or a "technology frontier". This assumption distinguishes "pool of knowledge" growth models from the quality ladder models, and it is motivated by the idea of a *technology spillover* from the other, more advanced sectors.

When an innovation happens in a given sector $i \in [0,1]$, the product which corresponds to the innovation becomes temporarily *proprietary*. This means that its inventor receives a temporary monopoly for producing it. This monopoly can end in one of two ways: either another innovation is made in the same sector, or the considered product turns into a nonproprietary one. The latter event is governed by a Poisson process with the hazard rate ϕ. As it was explained in the introduction, the parameter ϕ is a representation of the strength of intellectual property rights and other ways of appropriating revenue for innovations, and its value can be affected by the government.

Innovations have also a *required inventive factor P*.[13] The parameter *P* specifies the minimum that the ratio of the quality parameter values before and after the innovation must at least have if it leads to the desired monopoly. In other words, since the quality parameter receives by assumption the value $A_{\max,t}$ after the innovation, an innovation leads to a monopoly in the sectors *i* for which

(5) $$\frac{A_{\max,t}}{A_{it}} \geq P$$

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[11] Cf. Stern (2004).

[12] A growth model with this feature has been put forward in e.g. Fershtman et al. (1996; cf. p. 114).

[13] The required inventive factor receives the value 1 in an economy in which arbitrarily small improvements of products lead to a monopoly. The *required inventive step*, which is 0 when trivial innovations lead to monopoly, could be defined in several different ways, e.g. as $P-1$ or as $\ln P$. However, below I shall discuss the optimal choice of *P* without explicitly introducing either of these definitions.

There are two ways of interpreting this assumption. In the earlier models which are due to Aghion and Howitt arbitrarily small improvements in products lead to a monopoly of the improved product, and in a straightforward generalization of these models (5) can be viewed simply as a representation of a *patentability requirement.* Under this interpretation the social planner chooses arbitrarily a value $P = P_{patent} \geq 1$, and then grants monopoly to those and only those innovations which satisfy the condition (5).

However, the current model contains also an analysis of the competition between the monopolist and the producer of the inferior products of the same sector, and this analysis implies that the inventor of the superior product will be able to earn monopoly profits only if the quality ratio between the new and the old product exceeds a limit P_0. If it is assumed that the producer has to take into account the competition with inferior products of the same sector, there will be no incentive for research in the sectors for $A_{\max,t}/A_{it} < P_0$, and in equilibrium there will be research only in the sector which satisfy (5) with $P = \max\left\{P_{patent}, P_0\right\}$.

Nevertheless, below I shall mostly let $P \geq 1$ be arbitrary without introducing the restriction $P \geq P_0$, and I shall consider also the situation of Aghion and Howitt's model, in which arbitrarily small improvements to a product make the producer of its inferior version disappear from the market. One way to interpret the situation in which $P < P_0$ is to think of it as a case in which the patent system does not function properly, and it is possible to obtain a monopoly for a product by patenting a combination of its already existing features and improvements to it.

The emergence of innovations in a sector *i* is governed by a Poisson process with the arrival rate

(6) $\mu_{it} = \lambda D_{it}$

where D_{it} is the amount of research labor in sector *i* at time *t* and the constant λ is the *efficiency parameter*, which represents the efficiency of the research sector in producing new innovations.

Given that the expected profit from a proprietary innovation is identical in all sectors which there is research, it will be assumed that there is the same amount of research labor in all these sectors. This implies that the Poisson parameter given by (6) is identical in all the sectors in which there is research. It is also assumed that the

Poisson processes that correspond to the innovations of the different researchers and the ones which turn proprietary innovations nonproprietary are all independent of each other. It should be observed that when $P > 1$, the value (6) is different from the quantity

(7) $$\mu_t = \lambda D_t$$

which expresses the average arrival rate of innovations in *all* sectors of economy.

Finally, the innovations also increase the "pool of knowledge" which they utilize. The size of this pool of knowledge is represented by the "technology frontier", i.e. the maximum value $A_{\max,t}$ of the quality parameter, and the time development of $A_{\max,t}$ is determined by the equation

(8) $$\dot{A}_{\max,t} = A_{\max,t}\mu_t \ln\gamma$$

Here the constant $\ln\gamma$ expresses the efficiency of the research sector in improving the level of technology.[14]

3. The Production Sector

In this section I shall deduce the properties of the momentary equilibrium of the production sector, taking the available labor force L_t and the quality parameters of both proprietary and non-proprietary goods as given. It will turn out to be convenient to specify the quality parameters in terms of the quantity

(9) $$a_i = \frac{A_{it}}{A_{\max,t}}$$

which will below be called the *relative quality parameter* of the sector *i*. Each *non-proprietary product i* is produced under perfect condition, so that its price $p_{N,t}$ equals its production costs. Given the production function (2), these are equal with the current wage w_t, so that

(10) $$p_{N,t} = w_t$$

The *proprietary products* are produced by a monopolist, i.e. their inventor. Consider now the optimization problem of a monopolist in a sector *i*, who chooses the price

[14] My motive for denoting this constant by $\ln\gamma$ is to keep my notation similar with the one used in Aghion – Howitt (1998; see p. 88).

$p_{P,t}$ for a proprietary product, assuming all other prices as given. A consumer who maximizes the instantaneous utility function (3) will distribute her consumption in such a way that the ratio of the marginal utility and the price is identical in all sectors, i.e. in such a way that the value $\tilde{u} = \left(\alpha A_{jt} c_{jt}^{\alpha-1}\right) / p_{jt}$, where p_{jt} is the price of the good *j* at time *t*, is constant for all sectors. Hence, the consumed amounts c_{it} and c_{jt} of any two goods satisfies the condition

$$c_{it} = \left(\frac{p_{jt}}{p_{it}} \frac{A_{it}}{A_{jt}} \right)^{1/(1-\alpha)} c_{jt}$$

This relation is valid for all consumers, so that in equilibrium also the produced amounts of the two goods must satisfy the relation

$$y_{it} = \left(\frac{p_{jt}}{p_{it}} \frac{A_{it}}{A_{jt}} \right)^{1/(1-\alpha)} y_{jt} \tag{11}$$

Now the profit of the monopolist of sector *i* can be expressed in the form

$$\left(p_{it} - w_t \right) y_{it} = \left(p_{it}^{-\alpha/(1-\alpha)} - p_{it}^{-1/(1-\alpha)} w_t \right) \left(p_{jt} \frac{A_{it}}{A_{jt}} \right)^{1/(1-\alpha)} y_{jt}$$

Since the number of sectors forms a continuum, the monopolist of sector *i* will in equilibrium maximize this quantity, taking not just p_{jt} but also y_{jt} as given. Hence, in equilibrium the monopolist chooses the price

$$p_{Pr,t} = \frac{w_t}{\alpha} = \frac{p_{N,t}}{\alpha} \tag{12}$$

which is at each moment of time identical in all proprietary sectors of the economy.

It is now possible to deduce the produced amounts of the goods from the result (11) and their prices, which are given by (10) and (12). If $y_{Pr,t}(a)$ and $y_{N,t}(a)$ denote respectively the produced amounts of proprietary and a non-proprietary good with relative quality parameter *a*, (10), (11), and (12) imply that

$$\begin{cases} y_{N,t}(a) = \alpha^{-1/(1-\alpha)} a^{1/(1-\alpha)} y_{Pr,t}(1) \\ y_{Pr,t}(a) = a^{1/(1-\alpha)} y_{Pr,t}(1) \end{cases} \tag{13}$$

The value of $y_{Pr,t}(1)$ – i.e. the produced amount of a proprietary good with the relative quality parameter 1 – can now be deduced from the labor market clearing condition. Since according to (2) the produced amount of each good and the amount of labor used for producing it are numerically identical, the labor market clearing condition for productive labor can be formulated as

(14) $$\int_0^1 h_{Pr,t}(a)\, y_{Pr,t}(a)\, da + \int_0^1 h_{N,t}(a)\, y_{N,t}(a)\, da = L_t$$

Here the functions $h_{Pr,t}$ and $h_{N,t}$ denote the density functions that the relative quality parameter *a* has at time *t* among the proprietary and the nonproprietary products.

Now one can conclude from (13) and (14) that

(15) $$y_{Pr,t}(1) = \frac{L_t}{\Psi_{Pr,t} + \alpha^{-1/(1-\alpha)}\Psi_{N,t}}$$

where the aggregators $\Psi_{Pr,t}$ and $\Psi_{N,t}$ are given by

(16) $$\begin{cases} \Psi_{Pr,t} = \int_0^1 h_{Pr,t}(a) a^{1/(1-\alpha)} da \\ \Psi_{N,t} = \int_0^1 h_{N,t}(a) a^{1/(1-\alpha)} da \end{cases}$$

Clearly, (13), (15), and (16) suffice to determine the produced amounts of both proprietary and non-proprietary goods. It can now be concluded that the profit of the monopolist of a sector with the relative quality parameter value *a* is

(17) $$\begin{aligned} \pi_t(a) &= y_{Pr,t}(a)\left(p_{Pr,t} - w_t\right) = a^{1/(1-\alpha)} y_{Pr,t}(1) \frac{1-\alpha}{\alpha} w_t \\ &= \frac{1-\alpha}{\alpha} a^{1/(1-\alpha)} \left(\frac{L_t}{\left[\Psi_{Pr,t} + \alpha^{-1/(1-\alpha)}\Psi_{N,t}\right]} \right) w_t \end{aligned}$$

This result shows that profits are an increasing function of the profitability (measured by $1/\alpha$), of the labor force L_t which is available in the production sector, and of the wage level w_t, and a decreasing function of the aggregators $\Psi_{Pr,t}$ and $\Psi_{N,t}$, which can be thought of as measures of the quality of the competing proprietary and non-proprietary products.

Since the measure of the agents has been normalized to 1, the produced amounts of the goods are numerically identical with the consumption of a representative consumer. Hence, the instantaneous utility of the representative consumer equals

$$\begin{aligned} u_t &= \int_0^1 A_{it} c_{it}^{\alpha} di = \int_0^1 A_{it} y_{it}^{\alpha} di = \\ &= \int_0^1 h_{Pr,t}(a)\left(aA_{\max,t}\right)\left(y_{Pr,t}(a)\right)^{\alpha} da + \int_0^1 h_N(a)\left(aA_{\max,t}\right)\left(y_{N,t}(a)\right)^{\alpha} da \\ &= A_{\max,t}\left(\Psi_{Pr,t} + \alpha^{-\alpha/(1-\alpha)}\Psi_{N,t}\right) y_{Pr,t}^{\alpha}(1) \end{aligned}$$

Together with (15), this implies that the utility of the representative consumer is given by

$$(18)\qquad u_t = \frac{\Psi_{Pr,t} + \alpha^{-\alpha/(1-\alpha)}\Psi_{N,t}}{\left[\Psi_{Pr,t} + \alpha^{-1/(1-\alpha)}\Psi_{N,t}\right]^{\alpha}} A_{\max,t} L_t^{\alpha}$$

This analysis has been based on the assumption that each monopolist has to compete with just products of the other sectors, but not with earlier, lower-quality products of the same sector. However, as it was explained in Section 2, we wish to be able to discuss both a situation in which the earlier product of the same sector always disappears when a superior product comes to the market, as it happens in Aghion and Howitt's original model, and a situation in which the new monopolist has to take into account the possibility of a competition with lower-quality products.

Turning to the analysis of this competition, it is now supposed that there are two products of the same sector with the relative quality parameters a_{OLD} and a_{NEW} and with the respective prices p_{OLD} and p_{NEW}, where $a_{OLD} < a_{NEW}$ and $p_N \le p_{OLD} < p_{NEW}$. By assumption, there are no consumers who would have an incentive for buying both products, so that each consumer makes a choice between buying some quantity c_{OLD} of the good of quality a_{OLD} and buying some quantity c_{NEW} of the good of quality a_{NEW}. Consider now the options of spending fixed amount of wealth *W* on either of the products. In this case the consumed amounts are $c_{OLD} = W/p_{OLD}$ and $c_{NEW} = W/p_{NEW}$, and according to (3) the increase in utility is in the two cases given by $(\Delta u)_{OLD} = A_{\max,t} W^{\alpha}\left(a_{OLD}/p_{OLD}^{\alpha}\right)$ and $(\Delta u)_{NEW} = A_{\max,t} W^{\alpha}\left(a_{NEW}/p_{NEW}^{\alpha}\right)$, respectively. It is now observed that *for all values of W* the utility from consuming the higher-quality product is larger than the utility from consuming the lower-quality product if and only if

$$(19)\qquad \frac{a_{OLD}}{p_{OLD}^{\alpha}} < \frac{a_{NEW}}{p_{NEW}^{\alpha}}$$

and similarly, *for all values of W* the lower-quality product yields larger utility if the converse inequality is valid. Given that this is valid for all values of *W*, all consumers will choose the higher-quality product if (19) is valid. When the older (i.e. lower-quality) product is nonproprietary so that $p_{OLD} = p_N$, and also when it is a proprietary product whose producer gets involved in a price competition which makes its price sink to p_N, the producer of the newer product will in equilibrium choose the price

$$p_{NEW} = \min\left\{ p_{Pr}, \left(\frac{a_{NEW}}{a_{OLD}}\right)^{1/\alpha} p_{OLD} \right\} = \min\left\{ \frac{w_t}{\alpha}, \left(\frac{a_{NEW}}{a_{OLD}}\right)^{1/\alpha} w_t \right\}$$

Hence, a monopolist will be able to choose the price w_t/α which would be optimal in the absence of the inferior-quality product if and only if the quality improvement in the product $P = a_{NEW}/a_{OLD}$ is at least P_0, where

(20) $\quad P_0 = \alpha^{-\alpha}$

Hence, if the inventors of new products have to compete with inferior products of the same sector, and in equilibrium there can be research only in those sectors for which $P \geq P_0$, the value P_0 forms an minimum for the required inventive step P which appears in (5).

Above I have not explicitly introduced any values for the wage level w_t or the prices $p_{Pr,t}$ and $p_{N,t}$ which are determined by it. In a growth model with capital, the choice of the units of wealth at one instant of time would suffice to determine the units at all other moments of time, but since the current model does not contain capital, and since also the stock of the other goods varies in it constantly, in it the units of wealth must be chosen for each moment of time separately. In what follows, I shall fix the units by choosing the wage level to be

(21) $$w_t = \alpha^2 \frac{A_{\max,t}}{\left(y_{Pr,t}(1)\right)^{1-\alpha}} = \alpha^2 \frac{A_{\max,t}}{L_t^{1-\alpha}} \left[\Psi_{Pr,t} + \alpha^{-1/(1-\alpha)} \Psi_{N,t} \right]^{1-\alpha}$$

at each instant of time *t*. The intuitive motivation for this definition can be seen by observing that if *i* is a proprietary good with relative quality parameter value $a = 1$, so that $A_{it} = A_{\max,t}$ and $p_{it} = w_{it}/\alpha$, (21) implies that for the representative consumer

(22) $$A_{it} \frac{d}{dc_{it}} c_{it}^{\alpha} = A_{it} \alpha c_{it}^{\alpha-1} = \frac{A_{it}\alpha}{y_{it}^{1-\alpha}} = p_{i,t}$$

and the result (13) easily implies that this formula is valid for all goods *i* if and only if it is valid for any one of them. A comparison of (22) and the definition (3) of the utility function u_t shows that with the current choice of units *the prices of the products express their marginal utility for the representative consumer*.

4. The Balanced Growth Paths and Growth Traps

In the current model it is natural to define the growth rate g of the economy to be identical with the growth rate of the "research frontier" $A_{\max,t}$, which is according to (8) given by[15]

$$g = \frac{\dot{A}_{\max,t}}{A_{\max,t}} = \mu_t \ln \gamma \tag{23}$$

A balanced growth path is, by definition, an equilibrium in which g has a positive constant value. With a *growth trap* I shall in what follows mean an equilibrium in which the growth rate is zero, although also an equilibrium with a positive growth rate would be possible for identical parameter values.

It is fairly easy to see that the current model has growth traps. This is because of the fact that unlike in the quality ladder model, in the "pool of knowledge" model researchers cannot freely choose the size of the quality improvement that their research yields in a given product. Rather, the size of the improvement is determined by the closeness of its current design to the "research frontier". Given the idealizing assumption that the shift of the "research frontier" that each single innovation causes is infinitesimal, this implies that the researchers do not have an incentive to improve on the products whose quality is so high that the improvement does not lead to the desired monopoly.

More specifically, when in the current model the required inventive factor P is larger than 1, there are no incentives for improving on proprietary products whose quality parameter a is larger than $1/P$. This immediately implies that when $P > 1$, the model has zero-growth equilibria, i.e. equilibria with no growth and no innovations, since there is no incentive for making innovations if all products have a relative quality parameter which is larger than $1/P$.

The following proposition states a somewhat stronger result.

[15] This definition is natural, since (18) implies that on a balanced growth path the value g expresses also the growth rate of the utility of the representative consumer, because – as the results (26) and (27) below show – on a balanced growth path Ψ_{Pr} and Ψ_N are constants.

PROPOSITION 1. *(a) The model has zero-growth equilibria among its Nash equilibriums whenever the required inventive factor P is larger than 1.*
(b) The considered model has zero-growth equilibria even when $P=1$ *whenever*

$$\frac{\lambda}{\rho+\phi}\frac{(1-\alpha)}{\alpha^{\alpha/(1-\alpha)}(1-\theta)}\leq 1$$

The proofs of all propositions are presented in Appendix 1.

However, the growth traps which result when all products are close to the research frontier can be viewed as an artifact of the modeling technique that we have chosen, i.e. of the fact that in the model it is impossible to make large improvements to the products which are close to the “research frontier”. More interestingly, one can ask whether incentives for research would be missing if the state of the economy corresponded to a very slow growth rate in the past. This question can be given a rigorous formulation by studying the state of the economy when the growth rate *g* stays constant, by deducing the limit of this state when *g* approaches zero, and by then asking whether there are growth traps in which the economy is in this limiting state. Below it will be seen that the model has also growth traps of this kind.

Assume now that the growth rate *g* defined by (23) is a constant. Since the quality parameter of an innovation whose age is *t* is $a=e^{-gt}$, the innovations for which

$$a=e^{-gt}>1/P$$

cannot be replaced by a better innovation. The maximum age t_0 for which this condition is valid is

(24) $\quad t_0=(\ln P)/g=(\ln P)/(\mu\ln\gamma)$

If new innovations emerge at the constant rate μ, the number of the innovations which are younger than this age is $(\ln P)/(\ln\gamma)$. However, this condition cannot be valid if $(\ln P)/(\ln\gamma)>1$, i.e. if $P>\gamma$, since the measure of the sectors of the economy has been normalized to 1. The interpretation of this result is that *the current model has no balanced growth paths with a positive growth rate when* $P>\gamma$ since in this case the products of all sectors would end up being so close to the research frontier that improvements to them would not exceed the required inventive factor. In what follows, I shall assume that

(25) $P < \gamma$

When (25) is valid and the growth rate has the constant value *g*, within a finite time the number of the sectors in which there is research will obtain the constant value $1-(\ln P)/(\ln \gamma)$, so that the rate at which innovations happen in each of these sectors is

$$\frac{\mu}{1-(\ln P)/(\ln \gamma)} = \frac{g}{\ln \gamma - \ln P}$$

The other sectors are protected from innovation, but in both kinds of sectors, a monopolist might loose the monopoly because her product might become non-proprietary through imitation. The following lemma specifies the resulting distribution of the quality parameter values.

LEMMA 1. *Suppose that* $P < \gamma$ *and that the growth rate* $g = \dot{A}_{\max,t}/A_{\max,t}$ *stays constant after the point of time* $t = 0$, *and denote the distributions of the relative quality parameters of the proprietary and the non-proprietary products at time t by* $h_{Pr,t}(a)$ *and* $h_{N,t}(a)$, *respectively. For each fixed value of a there is a value of* t_0 *which is such that, whenever* $t \geq t_0$, $h_{Pr,t}(a) = h_{Pr}(a)$ *and* $h_{N,t}(a) = h_N(a)$, *where*

$$h_{Pr}(a) = \begin{cases} (1/(\ln \gamma))\, a^{\phi/g-1} (aP)^{1/(\ln \gamma - \ln P)}, & a < 1/P \\ (1/(\ln \gamma))\, a^{\phi/g-1}, & a \geq 1/P \end{cases}$$

and

$$h_N(a) = \left(a^{-\phi/g} - 1\right) h_{Pr}(a)$$

The results of Lemma 1 can now be plugged into the definitions of the aggregators $\Psi_{Pr,t}$ and $\Psi_{N,t}$ The definition (16) implies that the steady state values Ψ_{Pr} and Ψ_N are

$$\begin{aligned} \Psi_{Pr} &= \int_0^1 h_{Pr}(a) a^{1/(1-\alpha)} da \\ &= \frac{1}{\ln \gamma} \int_0^{1/P} a^{1/(1-\alpha)+\phi/g-1} (aP)^{1/(\ln \gamma - \ln P)} da + \frac{1}{\ln \gamma} \int_{1/P}^1 a^{1/(1-\alpha)+\phi/g-1} da \end{aligned} \tag{26}$$

and

$$\Psi_N = \int_0^1 h_N(a) a^{1/(1-\alpha)} da = \frac{1}{\ln \gamma} \int_0^1 \frac{1-a^{\phi/g}}{a^{\phi/g}} h_{Pr}(a) a^{1/(1-\alpha)} da = \Psi_{\phi=0} - \Psi_{Pr} \tag{27}$$

where

(28) $$\Psi_{\phi=0}=\left(\frac{1}{\ln\gamma}\int_0^{1/P}a^{1/(1-\alpha)-1}\left(aP\right)^{1/(\ln\gamma-\ln P)}da+\frac{1}{\ln\gamma}\int_{1/P}^{1}a^{1/(1-\alpha)-1}da\right)$$

is the value that the aggregator Ψ_{Pr} would have if $\phi=0$, i.e. if there was no imitation so that all products were proprietary.

In the current model the research firms hire a part of the labor force, and in equilibrium their labor costs must be equal with the discounted value of their expected profits. Turning to the task of calculating this value, it is first observed that since in the current model the wage w_t must be fixed by a conventional definition, the equilibrium interest rate will depend on this convention. The result (22), which follows from the definition of w_t, shows that for a representative consumer a small increase in the amount of money that she consumes at time *t* produces an identical increase of utility at time *t*, and one can conclude from (4) that with our current choice of units the interest rate on a balanced growth path has to be simply the rate of time preference ρ.

As we saw above, the hazard rate with which a product is replaced by a better one is 0 as long as its age is smaller than the value t_0 defined by (24), and $g/\ln(\gamma/P)$ after that. In addition, the monopolist is threatened with losing intellectual property rights through imitation, and this event has the constant hazard rate ϕ. Given that the relative quality of a product which is invented at *t* will be $a^{-g(t'-t)}$ at a subsequent point of time t', the expected profit from an innovation which is made at time t is seen to equal

(29) $$\begin{aligned}(EV)_t &= \int_t^{t+t_0} e^{-(\rho+\phi)(t'-t)}\pi_{t'}\left(e^{-g(t'-t)}\right)dt' \\ &+\int_{t+t_0}^{\infty} e^{-(\rho+\phi)t-(g/\ln(\gamma/P))(t'-(t+t_0))}\pi_{t'}\left(e^{-g(t'-t)}\right)dt'\end{aligned}$$

Here $\pi_{t'}$ is the profit function defined by (17). Now it can be concluded from (17) that since on a balanced growth path the wage grows at the constant rate *g*,

(30) $$\pi_{t'}\left(e^{-g(t'-t)}\right)=e^{-(g/(1-\alpha))(t'-t)+g(t'-t)}\pi_t(1)$$

Here the term $(g/(1-\alpha))(t'-t)$ corresponds to the fact that the economic growth lowers the quality of the product relative to the other products on the market, and the term $g(t'-t)$ corresponds to the fact that the economic growth makes the prices of all products grow. The result (30) implies that the former, negative effect is always larger than the latter, positive effect.

Putting the results (29) and (30) together, it follows that

(31) $\left(EV\right)_t = M\pi_t\left(1\right)$

where

(32) $M = \int_0^{t_0} e^{-\left(\rho+\phi+g\left(1/(1-\alpha)-1\right)\right)t}dt + \int_{t_0}^{\infty} e^{-\left(\rho+\phi+g\left(1/(1-\alpha)-1\right)\right)t-\left(g/\ln(\gamma/P)\right)(t-t_0)}dt$

Clearly, the multiplier M is time independent, and if it were the case that $g = \phi = 0$ i.e. if there was no growth and no imitation, the value of multiplier *M* would be simply $1/\rho$. Accordingly, a natural way to think of the function *M* is to view it as a generalized discount factor for future profits, which takes into account not just the time preference of consumption, but also the other effects which were listed above in the context of (29) and (30).

Since the number of the sectors of the economy has been normalized to 1, one can conclude from (6) and (23) that $L = 1 - D = 1 - \mu/\lambda = 1 - g/\left(\lambda \ln \gamma\right)$. Together with this result, (17) and (31) imply that the expected profit from a single innovation is given by

(33) $\left(EV\right)_t = \frac{1-\alpha}{\alpha}\left(1-\frac{g}{\lambda \ln \gamma}\right)\frac{M}{\Psi_{Pr} + \alpha^{-1/(1-\alpha)}\Psi_N} w_t$

Since the research sector produces innovations at the rate λ per worker, and since in on a balanced growth path the expected profit of the research firm per worker must be equal with its labor costs, the equilibrium condition which characterizes balanced growth paths with a positive growth rate can be formulated as

$$w_t = \lambda\left(EV\right)_t$$

This condition will now be expressed in the form

(B1) $F\left(g,\phi,P\right) = \zeta$

where $\zeta = 1$ and *F* is given by

(34) $F\left(g,\phi,P\right) = \lambda\frac{1-\alpha}{\alpha}\frac{\left(1-g/\left(\lambda \ln \gamma\right)\right)M}{\Psi_{Pr} + \alpha^{-1/(1-\alpha)}\Psi_N}$

In this definition it is explicitly mentioned that *F* depends also on the parameters ϕ and *P*, which describe the strength of IPR protection and which also affect *M*, Ψ_P, and Ψ_N, since the aim of the next section will be to investigate the effects of ϕ and P on the incentives to produce innovations.

Now it is also possible to give a more precise characterization for a notion of a growth trap which is more interesting than the one we considered in the beginning of this section. The assumption that in a state of no growth the quality parameter distributions correspond to very slow growth can be formulated as

(N1) If $g = 0$, then $\Psi_{Pr} = \lim_{g' \to 0+} \left(\Psi_{Pr}\right)_{g=g'}$ and $\Psi_N = \lim_{g' \to 0+} \left(\Psi_N\right)_{g=g'}$

Clearly, when this condition is valid, a state of zero growth will be an equilibrium if

(N2) $\lim_{g \to 0+} F\left(g, \phi, P\right) < \zeta$

The conditions (B1) and (N2) contain the seemingly unnecessary parameter ζ (which has, by assumption, the value $\zeta = 1$). The motive for introducing this parameter is that it allows one to give a simple economic interpretation to the function $F\left(g, \phi, P\right)$. As it was explained in Section 2, scientists may have a preference for employment in the research sector, which is shown in a wage differential between the research sector and other sectors of the economy. Such a preference can be modeled by giving ζ a value which is smaller than 1 and similarly, a preference for employment in production can be modeled by setting $\zeta > 1$.

When ζ is interpreted as a measure of preference, each point $\left(g, \zeta\right)$ on the graph of the function $\zeta = F\left(g, \phi, P\right)$, which has been depicted in Figures 3 and 4 for two different parameter values, is such that if the preferences of the workers were represented by ζ, the growth rate g would correspond to a balanced growth rate. The points at which the curve is above the line $\zeta = 1$ correspond to cases in which the workers have a preference for working in production, and the points below it correspond to a preference for research. For this reason I shall in what follows refer to F as the *research incentive function*.

As it is customary in endogenous growth theory, all the results in this paper are concerned with the case of no preference, in which $\zeta = 1$, unless explicitly stated otherwise. However, the more general model in which ζ is arbitrary will nevertheless be used for giving the following characterization for the situations in which there are multiple equilibria.

PROPOSITION 2. *Consider the generalized model in which it is not necessarily the case that $\zeta = 1$, and let all other parameters be fixed except for ζ. The model has for each value of ζ an equilibrium which satisfies (N1) for just one value of the growth rate, if and only if the research incentive function is decreasing in g for all values of g.*

It is fairly easy to see that *F* is not decreasing for all parameters and for all growth rates, so that multiple equilibria are possible. More specifically, one can conclude from (34) that a change in the steady-state growth rate *g* has three kinds of effects on the incentives to work in research. Firstly, larger values of *g* correspond to smaller values of multiplier $\left(1 - g/\left(\lambda \ln \gamma\right)\right)$. Intuitively, this means that the supply of productive labor is smaller when the growth rate is larger, because a larger part of the population works in a research sector, and this diminishes the profits from new products. Secondly, (32) easily implies that

$$(35) \qquad \frac{\partial M}{\partial g} < 0$$

i.e. that an increase of *g* diminishes *M*. Intuitively, this means that when the growth rate is higher, the profits are lower because the danger that an inventor loses the monopoly is larger, and because the quality of other products increases faster. Finally, (26), (27), and (28) imply that

$$(36) \qquad \frac{\partial}{\partial g}\left(\Psi_{Pr} + \alpha^{-1/(1-\alpha)}\Psi_N\right) = -\left(\alpha^{-1/(1-\alpha)} - 1\right)\frac{\partial \Psi_{Pr}}{\partial g} < 0$$

so that an increase of *g* decreases the denominator in (34). Intuitively, this means that a higher growth rate *has also a positive effect* for a monopolist, because when growth is faster, a larger part of the products of competitors are proprietary, and have higher prices.

Figure 3 depicts a situation in which the last, positive effect is for small values of *g* larger than the two negative effects, causing the model to have multiple equilibria. In Figure 3 there are two positive values *g*, $g = g_1$ and $g = g_2$, which satisfy (B1) and which correspond to balanced growth paths of the model, and since also (N2) is valid in the situation of the figure, the equilibria of the model include also a *growth trap* in which the distribution of the quality parameters corresponds to slow growth in the past.

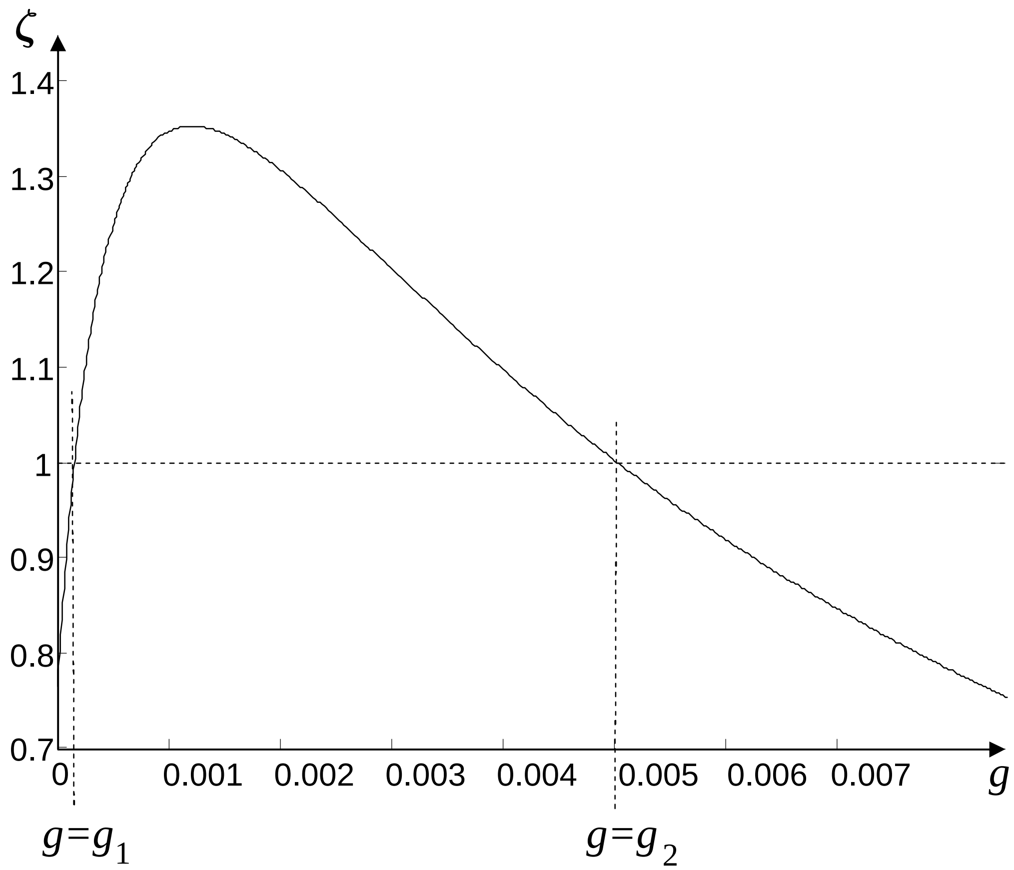

Figure 3. The research incentive function $\zeta = F(g,\phi,P)$ when $\gamma = 1.5$, $\lambda = 0.5$, $\phi = 0.005$, $\alpha = 0.95$, $\rho = 0.1$ and $P = P_0 = \alpha^{-\alpha} \approx 1.0499$.

Obviously, the positive effect of growth on the research incentive function does not exist if there is no imitation (because in this case all products are proprietary, independently of the growth rate), and it can be expected to be small also when the imitation rate is too high (because in this case most products will be non-proprietary even for high values of the growth rate). The following proposition confirms this intuition by showing that multiple equilibria exist when ϕ is small but positive, but not necessarily otherwise. Here it should be kept in mind that according to Proposition 2 multiple equilibria are impossible even in the generalized model if the research incentive function *F* is a decreasing function of *g*.

PROPOSITION 3. *(a) If* $\phi = 0$, *the research incentive function is decreasing in g.*

(b) Suppose $\phi > 0$. *For all values of* ϕ *which are sufficiently close to 0 (a sufficient condition being* $\phi < \left(\alpha^{-\alpha/(1-\alpha)} - \alpha\right)\rho$*), the model has for some values of the efficiency*

parameter λ multiple equilibria, one of which is a growth trap which satisfies (N1) (i.e. a growth trap in which the quality parameter distribution corresponds to very slow growth).

(c) If either the profit margin of the monopolists is sufficiently high (more specifically, if $\alpha^{1/(1-\alpha)} + \alpha < 1$) or the knowledge increase parameter γ is sufficiently low (more specifically, if $\ln\gamma < (1-\alpha)\left(1-\alpha^{1/(1-\alpha)}\right) \big/ \left(\alpha^{1/(1-\alpha)} + \alpha - 1\right)$) the research incentive function will be decreasing in g for sufficiently large values of the imitation rate ϕ.

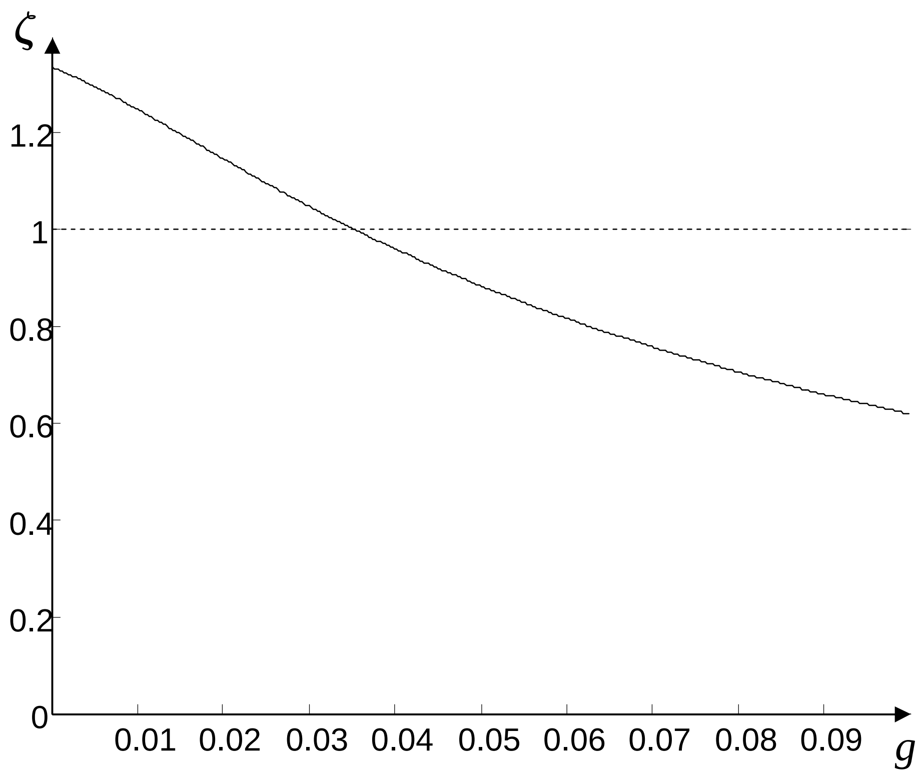

Figure 4. The research incentive function $\zeta = F(g,\phi,P)$ when $\gamma = 1.5$, $\lambda = 5$, $\phi = 0.5$, $\alpha = 0.95$, $\rho = 0.1$, and $P = P_0 = \alpha^{-\alpha} \approx 1.0499$.

Proposition 3 is illustrated with Figure 4, in which both the imitation rate ϕ and the efficiency parameter λ are larger than in Figure 3, but which corresponds to the same parameter values otherwise. A larger imitation rate decreases the incentives for research, implying that a larger value of λ is needed for obtaining a positive growth rate, and it also decreases the positive effect of growth on research incentives via the larger market share of proprietary products. This is shown in the fact that now *F* is everywhere a decreasing, and multiple equilibria are impossible.

It should be observed that in e.g. Figure 3 the balanced growth path which corresponds to the smaller of the two possible positive long-run growth rates – i.e. the one for which $g = g_1$ – has an implausible feature: in this equilibrium an arbitrarily small increase in the amount of research and the corresponding increase in *g* would cause a situation in which the profits of research firms per worker would be larger than the wage in production. The situation in which $g = g_1$ is nevertheless an equilibrium, because just like in most other endogenous growth models, in the current model the contribution of each firm to the growth rate of the economy is infinitesimal. In all actual economies the decisions of each firm have an effect on the growth rate of the economy (however small this effect might be), and this suggests making the following restriction on the equilibrium value of the growth rate *g* that one considers:

(B2) $\partial F\left(g', \phi, P\right) / \partial g' < 0$ when $g' = g$.

Excluding the implausible case in which $\partial F\left(g, \phi, P\right) / \partial g$ is precisely 0 when $F\left(g, \phi, P\right) = 1$, it is clear that the equilibrium with the largest growth rate will always satisfy the additional condition (B2), when the model has equilibria with positive growth rates. In the comparative static analysis of the next section I shall restrict attention to the case in which this additional condition is valid.

5. The Growth and Welfare Effects of Intellectual Property Policy

In the current model intellectual property policy affects growth and welfare via the required inventive factor *P* and the imitation rate ϕ. Given that the model allows for multiple equilibria, the function $g\left(\phi, P\right)$ is now defined to be the largest value of *g* which for the given ϕ and *P* satisfies (B1) and (B2) if such values exists, and to be 0 otherwise. In this section I study the growth-maximizing and welfare-maximizing choice of ϕ and *P*, assuming that $g = g\left(\phi, P\right)$.

In the current model there is no disutility of labor and the agents have identical income, and it is natural to define the welfare function to be given simply by the utility (4) of the agents. Restricting attention to balanced growth paths, I shall below consider the normalized utility function

(37) $\tilde{U} = U / A_{\max,0}$

The definition (4) and the result (18) easily imply that this is given by

$$\tilde{U} = \frac{1}{A_{\max,0}} \int_0^{\infty} e^{-\rho t} u_t dt = \frac{\Psi_{Pr} + \alpha^{-\alpha/(1-\alpha)} \Psi_N}{\left[\Psi_{Pr} + \alpha^{-1/(1-\alpha)} \Psi_N\right]^{\alpha}} \frac{L^{\alpha}}{\rho - g}$$

Remembering that according to (27) $\Psi_N = \Psi_{\phi=0} - \Psi_{Pr}$, where the quantity $\Psi_{\phi=0}$ is independent of both g and ϕ, the welfare function $\tilde{U}$ can be expressed in the form

(38) $\tilde{U} = G\left(\Psi_{Pr}\right) \dfrac{L^{\alpha}}{\rho - g}$

where

(39) $G\left(\Psi_{Pr}\right) = \dfrac{\left(\Psi_{\phi=0} - \left(1 - \alpha^{\alpha/(1-\alpha)}\right) \Psi_{Pr}\right)}{\left[\Psi_{\phi=0} - \left(1 - \alpha^{1/(1-\alpha)}\right) \Psi_{Pr}\right]^{\alpha}}$

In what follows, I shall take $\tilde{U}$ to be the welfare function that the social planner wishes to maximize. The formula (38) shows that if the social planner adjusts P and ϕ in such a way that the growth rate on the balanced growth path is increased, this has three kinds of effects on the function $\tilde{U}$. Most obviously, this has the positive effect of increasing welfare in the future, which is shown by the increase of the term $1/(\rho - g)$. Secondly, a larger growth rate demands a larger work force in research, so that the amount of labor which is available for production is smaller. This has the negative effect of decreasing the term $L^{\alpha} = \left(1 - g/(\lambda \ln \gamma)\right)^{\alpha}$ in (38). Finally, a larger growth rate corresponds to different quality parameter distributions Ψ_{Pr} and Ψ_N, which is shown in the change in $G\left(\Psi_{Pr}\right)$. This effect is characterized by the following lemma.

LEMMA 2. *For each fixed value of* $\Psi_{\phi=0}$, *the function* $G\left(\Psi_{Pr}\right)$ *receives its minimum for some value* $\Psi_{Pr,0}$ *which belongs to the interval* $\left[0, \Psi_{\phi=0}\right]$. *The function* $G\left(\Psi_{Pr}\right)$ *receives its largest value in this interval both when* $\Psi_{Pr} = 0$ *(i.e. when all products are non-proprietary) and when* $\Psi_{Pr} = \Psi_{\phi=0}$ *(i.e. when all products are proprietary), and it is decreasing in* $\left[0, \Psi_{Pr,0}\right]$ *and increasing in* $\left[\Psi_{Pr,0}, \Psi_{\phi=0}\right]$.

In other words, the resources for producing the different products are allocated most efficiently when the products are either all proprietary or all non-proprietary, but in other cases the products have price differences which cause welfare loss.

The growth and welfare effects of the rate of imitation ϕ are characterized by the following proposition. In the current model the effects of imitation on the research incentive function are purely negative, since larger values of ϕ correspond to a larger danger of the loss of monopoly, and also to lower prices of competing products, and accordingly, the growth maximizing value of ϕ is 0.

PROPOSITION 4. *Keeping the other parameters fixed, the growth rate $g(\phi, P)$ is a decreasing function of the rate of imitation ϕ whenever $g(\phi, P)$ is positive, so that the growth maximizing value of the rate of imitation is $\phi = 0$. If the growth rate g is below the discount rate ρ and the knowledge increase parameter γ satisfies the condition $\ln\gamma > \rho/\lambda$, this is also the welfare maximizing value of ϕ.*

The assumption $g < \rho$ which is made in this proposition must, of course, be valid whenever an interesting welfare analysis is possible, since the utility defined by (4) is infinite if the growth rate is larger than the discount rate. The other assumption which appears in this proposition – i.e., $\ln\gamma > \rho/\lambda$ – specifies a minimum for the extent to which each innovation increases the pool of knowledge. Intuitively, this condition means that the negative welfare effect of growth which is due to the decrease of the labor force in production is smaller than the positive effect which is due to increased future welfare.

There is no similar general and simple answer to the question which value of the required inventive factor *P* is optimal. The following proposition is concerned with $g(\phi, P)$ as a function of *P*. It should be remembered that the assumption that $P < P_0$, where $P_0 = \alpha^{-\alpha}$ in accordance with (20), implies that the producers of inferior, old products are prevented from competing with the producer of the superior, new ones, but the producers of old products exit the market voluntarily if $P \geq P_0$.

PROPOSITION 5. *Assume that the imitation rate ϕ and the other parameters of the model except for the required inventive factor P are fixed.*

(a) If $g(\phi,P)$ is positive when $P=1$, the value $g(\phi,P)$ will be increased by a sufficiently small increase of P.

(b) If $P \geq \gamma$, the model does not have balanced growth paths. If balanced growth paths with a positive growth rate exist for some value of P, the value of P which maximizes the growth rate is smaller than $e^{2(1-\alpha)}$.

(c) In the limit $P \to \gamma-$ the derivative dg/dP approaches zero. (In other words, the positive or negative growth effects of a further increase in P approach zero in the limit in which P approaches its maximum.)

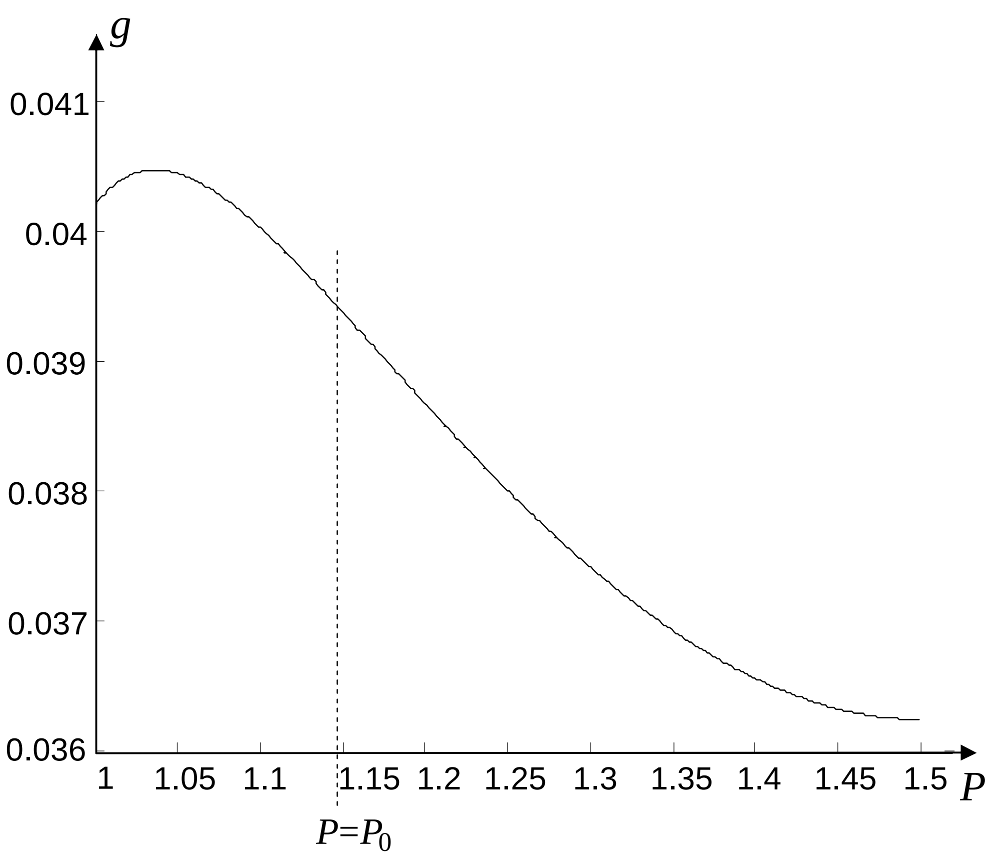

Figure 5. The growth rate as a function of the required inventive factor, when $\gamma=1.5$, $\phi=0.05$, $\lambda=1$, $\alpha=0.85$, and $\rho=0.1$.

This proposition shows that the growth-maximizing choice of *P* is never 1, but it is compatible with both a situation in which the growth-maximizing value of *P* is smaller than γ, and with a situation in which growth is an increasing function of *P* in the whole interval $[1,\gamma)$. Figure 5 illustrates the former possibility. In this case $e^{2(1-\alpha)}<\gamma$, and one can conclude from Proposition 4(b) that the problem choosing *P* so that growth is maximized must have a well-defined solution. When this is the case, (38) and Lemma 2 imply that the welfare-maximizing value of *P* will be larger or smaller than the growth-maximizing *P*, depending on the sign of $G(\Psi_{Pr})$ which represents the welfare effects of the price distribution of the products. In the situation of Figure 5 the growth-maximizing choice of *P* is smaller than P_0, which means that the growth-maximizing policy would be to protect the holder of the newest patent not just from imitation but also from competition with inferior products.

Figure 6 represents a case in which $e^{2(1-\alpha)}>\gamma$. In this case Proposition 4 does not imply that there was a growth-maximizing value of *P*. No such value exists in the situation of Figure 6, since in it growth is increased by an increase in *P* in the whole interval $[1,\gamma)$, but the model fails to have an equilibrium if $P=\gamma$.

Intuitively, an increase of *P* has a positive effect on the research incentive function because it increases *future* profits by lengthening the time during which the innovation is protected from being replaced by a superior product, and a negative effect because it increases the average *current* quality of the products of competitors, by shifting research efforts to the worst products on the market. Since the former effect is small in an economy with few innovations, it is to be expected that small values of *P*, which correspond to excluding the inferior products of the same sector from the market when a new product emerges, would be optimal in an economy with a small growth rate.

I shall conclude this section with a proposition which shows that this is, indeed, the case. The proposition is concerned with the effects of the lowering of the efficiency parameter λ. A decrease in λ shifts the research incentive function *F* downwards and decreases $g(\phi,P)$, and as Figures 3 and 4 illustrate, such a shift will make $g(\phi,P)$ decrease to zero *continuously* when *F* is decreasing in *g* (as in Figure 4) but not necessarily otherwise (e.g. not in the situation of Figure 3). We wish to consider the limit in which the growth rate is small but positive, and for this reason the following proposition contains the restrictive assumption that *F* is decreasing in *g*.

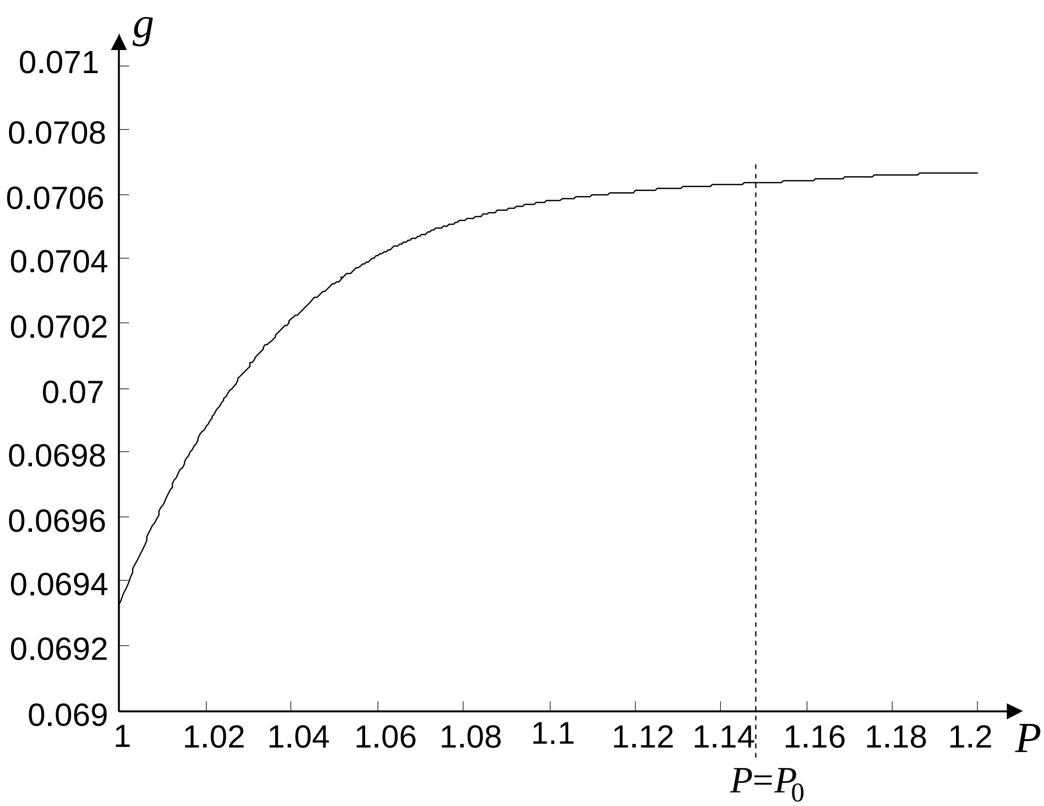

Figure 6. The growth rate as a function of the required inventive factor, when $\gamma = 1.2$, $\phi = 0.05$, $\lambda = 3$, $\alpha = 0.85$, and $\rho = 0.1$.

PROPOSITION 6. *Assume that the imitation rate* ϕ *and the parameters* α *and* γ *are fixed. Define* λ_0 *by* $\lambda_0 = \inf\left\{\lambda' \middle| \left(g(\phi, P)\right)_{\lambda=\lambda'} > 0 \text{ for some } P\right\}$ *(i.e., let* λ_0 *be the threshold value of the efficiency parameter* λ *below which the growth rate will be zero, independently of how the required inventive factor P is chosen). If the research incentive function F is decreasing in g for all P, the growth-maximizing value of the required inventive factor P approaches 1 when* $\lambda \to \lambda_0 +$. *If the welfare maximizing value of P corresponds to a positive growth rate when* $\lambda \to \lambda_0 +$, *also this value approaches 1 when* $\lambda \to \lambda_0 +$.

6. Concluding Remarks

Above I have studied the generalization of a "pool of knowledge" growth model to a situation of imperfect intellectual property rights. Unlike the *models of growth through specialization*, the framework which was developed above allows for a discussion of the required inventive step of patents, and it does not contain the implausible assumption made in *quality ladder growth models*, according to which the demand of each good would be independent of its quality.

It turned out that the model provided conceptual tools for understanding growth traps, i.e. situations in which there is no economic growth although a state of positive growth would be possible for identical parameter values. In the current model growth traps were caused by the fact that in an economy which has grown slowly in the past most products are non-proprietary and cheaper than in a quickly-growing economy with many proprietary products, and this lowers the profits of a research firm which has a monopoly to a new product.

The analysis of comparative statics revealed that the optimal value of the required inventive step is always positive, although it might be small in a slowly-growing economy. It also turned out that growth is always increased by a decrease of imitation, so that the growth-maximizing imitation rate was zero.

However, it should be observed that this result was deduced assuming that the extent to which the available "pool of knowledge" can be utilized for making new innovations is independent of whether the available products are proprietary. However, in the intended application of the model the "proprietary" products might be protected not just by patents but also by e.g. trade secrets, and in this case it is plausible to assume that the available pool of knowledge can be utilized more efficiently when there are more non-proprietary products on the market. It is clear that the growth-maximizing imitation rate might be positive in a generalized model in which this effect is taken into account.

There are several other ways in which one might wish to generalize the current model. As it was pointed out in the introduction, patent literature distinguishes between patent length, the patentability requirement, and the lagging and the leading breadth of a patent. As a natural next step, one might wish to generalize the current framework in such a way that it allowed for a discussion of patent breadth and patent length, which were not considered above explicitly.

However, it seems that a "pool of knowledge" growth model does not – at least not without dramatic modifications of the whole framework – allow for an interesting discussion of the distinction between the breadth of a patent and the patentability requirement. If one assumed that patents have a leading breadth *K* which is larger than the patentability requirement *P*, this would in the current model have precisely the same consequences as the assumption that the patentability requirement was *K*: since in the model all new products are of the quality which corresponds to the research frontier, there would in both cases be research in only those sectors whose distance from the research frontier was larger than *K*.

On the other hand, it would be fairly easy to include patents of a finite duration into the current framework, since in it the size of an innovation exceeds the patentability requirement *P* if and only if it replaces a product whose age is larger than a constant t_0. Hence, many features of the current model would remain unchanged if one replaced the patentability requirement *P* with a minimum age t_0 which a product must exceed before it can be replaced by a new one. However, in a model with this interpretation it would be natural to assume that all products become non-proprietary when they reach the age t_0, which would change the results of the above analysis.

APPENDIX 1. PROOFS OF PROPOSITIONS.

PROOF OF PROPOSITION 1. Part (a) follows trivially from the fact that if $A_i/A_{\max} > 1/P$ in all sectors *i* of the economy, it is not possible to make any innovations whose size would exceed the required inventive factor. Assume now that $P=1$, that $g=0$, and that also $\Psi_{P,t} = \int_0^1 h_{Pr,t}(a)a^{(1-\theta\alpha)/(1-\alpha)}da = 0$. Clearly, the aggregator $\Psi_{N,t} = \int_0^1 h_{N,t}(a)a^{(1-\theta\alpha)/(1-\alpha)}da$ can now have an arbitrary constant value between 0 and 1. Further, the labor supply must be $L=1$, so that the profit from a monopoly to a new product is according to (17) in this case given by

$$\pi_t(1) = \frac{1-\alpha}{\alpha}\left(\frac{1}{\left[\Psi_{Pr,t} + \alpha^{-1/(1-\alpha)}\Psi_{N,t}\right]}\right)w = \frac{1-\alpha}{\alpha^{-\alpha/(1-\alpha)}}\left(\frac{1}{\Psi_{N,t}}\right)w$$

Still assuming that $g=0$, the expected profit from an innovation is

$$EV = \int_0^\infty \pi_t(1)e^{-(\rho+\phi)t}dt = \frac{\alpha^{\alpha/(1-\alpha)}(1-\alpha)}{\rho+\phi}\frac{w}{\Psi_{N,t}}$$

Working in production is preferable to research as long as $\lambda(EV)_t < w_t$. This is equivalent with

$$\frac{\lambda}{\rho+\phi}\frac{\alpha^{\alpha/(1-\alpha)}(1-\alpha)}{\Psi_{N,t}} \le 1$$

Since the aggregator $\Psi_{N,t}$ can have any value for which $0 < \Psi_{N,t} \le 1$, *a state of zero growth is possible whenever*

$$\frac{\lambda}{\rho+\phi}\alpha^{\alpha/(1-\alpha)}(1-\alpha) < 1. \square$$

PROOF OF LEMMA 1. Define $A_0 = A_{\max,0}$, and let $H_{Pr,\tau}$ and $H_{N,\tau}$ be the density functions that the *A* parameter values have at time τ within the sets of proprietary and of nonproprietary innovations, respectively. Let *a* be arbitrary, and define t_0 by $t_0 = \ln(1/a)/g$. Further, let $t > t_0$ be arbitrary, and put $u = t - t_0$ so that

(A1) $\quad a = e^{-gt_0} = e^{-g(t-u)}$

is, intuitively, the relative quality that a product which has been invented at time *u* has at time *t*.

It is clear that during a short time interval $[u, u+\Delta t]$ an innovation happens in the part $\mu(\Delta t)$ of the sectors. Since at the same time the value of $A_{\max}$ grows with $\mu(\ln\gamma)A_{\max,u}(\Delta t)$, the value $H_{Pr,u}(A_{\max,u})$ is seen to equal

$$\text{(A2)} \quad H_{Pr,u}(A_{\max,u}) = \frac{\mu\Delta t}{\mu(\ln\gamma)A_{\max,u}\Delta t} = \frac{1}{(\ln\gamma)A_{\max,u}}$$

When $t > u$, the value of $H_{Pr,t}(A_{\max,u})$ diminishes because the products turn nonproprietary at the rate ϕ, and because of the emergence of new innovations. If an innovation has been made at the moment *u*, the moment of time at which it becomes legitimate to make a new innovation in its sector is

$$\text{(A3)} \quad t' = u + \ln P/g = u + \ln P/(\mu\ln\gamma)$$

The hazard rate of a new innovation in each of the sectors in which innovating is legitimate is $g/(\ln\gamma - \ln P)$. Hence,

$$\text{(A4)} \quad H_{Pr,t}(A_{\max,u}) = \begin{cases} e^{-\phi(t-u)}H_{Pr,u}(A_{\max,u}), & t-u \le (\ln P)/g \\ e^{-(\phi(t-u)+(g/(\ln\gamma-\ln P))(t-u-(\ln P)/g))}H_{Pr,u}(A_{\max,u}), & t-u > (\ln P)/g \end{cases}$$

and

$$\text{(A5)} \quad H_{N,t}(A_{\max,u}) = \frac{1-e^{-\phi(t-u)}}{e^{-\phi(t-u)}}H_{Pr,t}(A_{\max,u})$$

Combining (A3) and (A4), it now follows that

$$\text{(A6)} \quad H_{Pr,t}(A_{\max,u}) = \begin{cases} e^{-\phi(t-u)}\dfrac{1}{(\ln\gamma)A_{\max,u}}, & g(t-u) \le \ln P \\ e^{-(\phi(t-u)+(g/(\ln\gamma-\ln P))(t-u)-(\ln P)/(\ln\gamma-\ln P))}\dfrac{1}{(\ln\gamma)A_{\max,u}}, & g(t-u) > \ln P \end{cases}$$

Clearly, (A1) implies that the condition $g(t-u) \le \ln P$ is equivalent with $a \ge 1/P$, and further that

$$\text{(A7)} \quad h_{Pr,t}(a) = A_{\max,t}H_{Pr,t}(aA_{\max,t}) = \begin{cases} (1/(\ln\gamma))a^{\phi/g-1}(aP)^{1/(\ln\gamma-\ln P)}, & a < 1/P \\ (1/(\ln\gamma))a^{\phi/g-1}, & a \ge 1/P \end{cases}$$

Similarly,

(A8)
$$h_{N,t}(a) = A_{\max,t} H_{N,t}(aA_{\max,t}) = A_{\max,t} \frac{1-e^{-\phi(t-u)}}{e^{-\phi(t-u)}} H_{Pr,t}(aA_{\max,t}) = \frac{1-a^{\phi/g}}{a^{\phi/g}} h_{Pr,t}(a)$$
$$= \begin{cases} \left(1/(\ln\gamma)\right)\left(\left(1-a^{\phi/g}\right)/a\right)(aP)^{1/(\ln\gamma - \ln P)}, & a < 1/P \\ \left(1/(\ln\gamma)\right)\left(\left(1-a^{\phi/g}\right)/a\right), & a \geq 1/P \end{cases}$$

This completes the proof. □

PROOF OF PROPOSITION 2. Defining $F(0,\phi,P)$ to be given by

$$F(0,\phi,P) = \lim_{g\to 0+} F(g,\phi,P)$$

it is seen that an equilibrium satisfies (N1) if and only if the research incentive function expresses the wage in the research sector in it. (This is, of course, trivial if $g > 0$). However, $\lim_{g\to 0+} F(g,\phi,P) > 0$ and $\lim_{g\to(\lambda\ln\gamma)-} F(g,\phi,P) = 0$, so that there will be no values of ζ for which two possible values of the growth rate g (i.e. values for which $0 \leq g < \lambda \ln\gamma$) satisfy the equilibrium condition (B1) if and only if F is decreasing for all values of g. □

PROOF OF PROPOSITION 3. Combining (27) and (34), the function F can be expressed in the form

(A9)
$$F(g,\phi,P) = \lambda(1-\alpha)\alpha^{\alpha/(1-\alpha)} \frac{\left(1-g/(\lambda\ln\gamma)\right)M}{\Psi_{\phi=0} - \left(1-\alpha^{1/(1-\alpha)}\right)\Psi_{Pr}}$$

It immediately follows from (26) and (28) that

(A10) If $\phi = 0$, then $\Psi_{Pr} = \Psi_{\phi=0}$

According to (28), $\Psi_{\phi=0}$ is independent of g, and according to (35), $\partial M/\partial g < 0$. Together with (A10), these results imply that when $\phi = 0$

$$\frac{1}{F}\frac{\partial F}{\partial g} = \frac{-1}{\lambda\ln\gamma - g} + \frac{\partial M}{\partial g} < 0$$

This proves the validity of (a).

Turning to (b), it is now assumed that $\phi > 0$. First, it is observed that if it for some value of λ it is the case that $F(0,\phi,P) = 1$ and $\left(\partial F(g,\phi,P)/\partial g\right)_{g=0} > 0$, then the model must have an equilibrium for some positive value g_2 of g, since $\lim_{g\to(\lambda\ln\gamma)-} F(g,\phi,P) = 0$. Further, keeping ϕ and P fixed, the function $F(g,\phi,P)$

is continuous function of g and λ and a decreasing function of λ. This implies that if the value of λ is given a sufficiently small decrease $\Delta\lambda$, there will still be a positive value $g \approx g_2$ which corresponds to an equilibrium, but now $g = 0$ is a growth trap satisfying both (N1) and (N2). Accordingly, we now consider the condition which must be valid if λ can be chosen so that $F(0,\phi,P) = 1$ and $\left(\partial F(g,\phi,P)/\partial g\right)_{g=0} > 0$.

First, the function I is defined by

(A11) $I(Q,R,t_0) = \int_0^{t_0} e^{-Qt} dt + \int_{t_0}^{\infty} e^{-Qt-R(t-t_0)} dt$

Clearly, when one puts

(A12) $R = g/(\ln\gamma - \ln P)$

and

(A13) $t_0 = (\ln P)/g$

it turns out that

(A14) $M(g) = I(Q_3, R, t_0)$

where $Q_3 = \phi + \rho + (\alpha g)/(1-\alpha)$. Further, when one makes the change of variables $a = e^{-gt}$, $da = -\left(ge^{-gt}\right)dt$, it turns out that

(A15) $\Psi_P = \dfrac{g}{\ln\gamma}\left(\int_0^{t_0} e^{-(g/(1-\alpha)+\phi)t} dt + \int_{t_0}^{\infty} e^{-(g/(1-\alpha)+\phi)t+(g/(\ln\gamma-\ln P))(t-t_0)} dt\right) = \dfrac{g}{\ln\gamma} I(Q_2, R, t_0)$

where $Q_2 = \phi + g/(1-\alpha)$. In particular, since $\Psi_{\phi=0} = \left(\Psi_P\right)_{\phi=0}$, it follows that

(A16) $\Psi_{\phi=0} = \dfrac{g}{\ln\gamma} I(Q_1, R, t_0)$

where $Q_1 = g/(1-\alpha)$.

The definition (A11) easily implies that

(A17) $I(Q,R,t_0) = \dfrac{1}{Q} - \dfrac{Re^{-Qt_0}}{Q(Q+R)}$

In particular, when this result is applied to (A16), a straightforward computation shows that

(A18) $\Psi_{\phi=0} = \dfrac{(1-\alpha)}{\ln\gamma}\left(1 - \dfrac{(1-\alpha)P^{-(1/(1-\alpha))}}{1-\alpha+\ln(\gamma/P)}\right)$

Clearly, (A9) implies that

(A19)
$$\frac{\partial F(g,\phi,P)/\partial g}{F(g,\phi,P)} = \frac{-1}{\lambda \ln\gamma - g} + \frac{1}{I(Q_3,R,t_0)}\frac{\partial I(Q_3,R,t_0)}{\partial g} + \frac{\left(1-\alpha^{1/(1-\alpha)}\right)}{(\ln\gamma)\left(\Psi_{\phi=0} - \left(1-\alpha^{1/(1-\alpha)}\right)\Psi_{Pr}\right)}\left[I(Q_2,R,t_0) + g\frac{\partial I(Q_2,R,t_0)}{\partial g}\right]$$

Consider now this expression in the limit in which $g \to 0$. According to (A12), $R = 0$ when $g = 0$, and hence it must be the case both when $Q = Q_2$ and when $Q = Q_3$ that

(A20) $\lim_{g\to 0} I(Q,R,t_0) = 1/Q$

The analysis of the limits $\lim_{g\to 0} \partial I(Q,R,t_0)/\partial g$ is complicated by the fact that when $P > 1$, $\lim_{g\to 0}\left(e^{-Qt_0}\right) = \lim_{g\to 0}\left(P^{-Q/g}\right) = 0$, but if $P = 1$, it turns out that $\lim_{g\to 0}\left(e^{-Qt_0}\right) = \lim_{g\to 0}(1) = 1$. Considering each case separately, a straightforward calculation shows that

(A21) $$\lim_{g\to 0} \partial I(Q,R,t_0)/\partial g = \begin{cases} \left(-1/Q^2\right)\partial(Q+R)/\partial g, & P = 1 \\ \left(-1/Q^2\right)\partial Q/\partial g, & P > 1 \end{cases}$$

Now ξ is defined by

(A22) $$\xi = \begin{cases} \partial(Q_3+R)/\partial g, & P = 1 \\ \partial Q_3/\partial g, & P > 1 \end{cases} = \begin{cases} \alpha/(1-\alpha) + 1/\ln\gamma, & P = 1 \\ \alpha/(1-\alpha), & P > 1 \end{cases}$$

Clearly, (A15) implies that $\Psi_P = 0$ when ϕ is positive but $g = 0$. Plugging this result, (A20), and (A21) into (A19) yields

$$\left(\frac{\partial F(g,\phi,P)/\partial g}{F(g,\phi,P)}\right)_{g=0} = \frac{-1}{\lambda \ln\gamma} - \frac{1}{Q_3}\xi + \frac{\left(1-\alpha^{1/(1-\alpha)}\right)}{(\ln\gamma)\Psi_{\phi=0}}\frac{1}{Q_2}$$

Together with the definitions $Q_2 = \phi + g/(1-\alpha)$ and $Q_3 = \phi + \rho + (\alpha g)/(1-\alpha)$, this implies that

(A23) $$\left(\frac{\partial F(g,\phi,P)/\partial g}{F(g,\phi,P)}\right)_{g=0} = \frac{-1}{\lambda \ln\gamma} - \frac{\xi}{\phi+\rho} + \frac{\left(1-\alpha^{1/(1-\alpha)}\right)}{\Psi_{\phi=0}(\ln\gamma)\phi}$$

As it was explained above, we wish to find out under which circumstances this derivative is positive when $F(0,\phi,P) = 1$. Clearly, (A9) easily implies that the latter condition is equivalent with

(A24) $$\lambda \frac{(1-\alpha)\alpha^{\alpha/(1-\alpha)}}{(\rho+\phi)\Psi_{\phi=0}} = 1$$

When λ is chosen so that this condition is valid, the expression (A23) will be positive if and only if

(A25) $$\frac{-(1-\alpha)\alpha^{\alpha/(1-\alpha)}}{(\ln\gamma)(\rho+\phi)} - \frac{\xi\Psi_{\phi=0}}{\phi+\rho} + \frac{1-\alpha^{1/(1-\alpha)}}{(\ln\gamma)\phi} > 0$$

However, since (A18) implies that $\Psi_{\phi=0} < (1-\alpha)/\ln\gamma$, one can conclude that when $P > 1$,

(A26) $$\xi\Psi_{\phi=0} < \frac{1}{\ln\gamma}$$

On the other hand, when $P = 1$, (A18) and (A22) imply that

$$\left(\frac{\alpha}{1-\alpha} + \frac{1}{\ln\gamma}\right)\frac{1-\alpha}{1-\alpha+\ln\gamma} = \frac{1}{\ln\gamma}\left(\frac{1-\alpha+\alpha\ln\gamma}{1-\alpha+\ln\gamma}\right) < \frac{1}{\ln\gamma}$$

so that (A26) is valid also in this case. Hence, the result (A25) must be valid whenever

$$\frac{-(1-\alpha)\alpha^{\alpha/(1-\alpha)}}{\rho+\phi} - \frac{1}{\phi+\rho} + \frac{1-\alpha^{1/(1-\alpha)}}{\phi} > 0$$

This is equivalent with $\phi < \left(\alpha^{-\alpha/(1-\alpha)} - \alpha\right)\rho$, and it can be concluded that the model has two equilibria for some values of λ when this condition is valid, as it is stated in part (b) of this proposition.

Finally, turning to the part (c) in this proposition, it is concluded from (A19) that if the quantity Θ which is given by

(A27) $$\Theta = \frac{1}{I(Q_3,R,t_0)}\frac{\partial I(Q_3,R,t_0)}{\partial g} + \frac{\left(1-\alpha^{1/(1-\alpha)}\right)}{(\ln\gamma)\Psi}\left[I(Q_2,R,t_0) + g\frac{\partial}{\partial g}\left(I(Q_2,R,t_0)\right)\right]$$

is negative for some values of *g*, *P*, and ϕ then for these values $\partial F / \partial g < 0$, independently of how λ is chosen. In what follows Θ is viewed as a function of *g*, and the behavior of this function is studied for large values of ϕ. For this reason it is first concluded from (A15) that

(A28) $$\lim_{\phi\to\infty}\Psi_P = 0$$

Clearly, (A17) implies both when $Q = Q_2 = \phi + g/(1-\alpha)$ and when $Q = Q_3 = \phi + \rho + (\alpha g)/(1-\alpha)$ that

(A29) $\lim_{\phi\to\infty}\left(\phi I\left(Q,R,t_0\right)\right) = 1$

Further, (A17) and a straightforward calculation shows that

(A30) $$\lim_{\phi\to\infty}\left(\phi^2 \frac{\partial I\left(Q_2,R,t_0\right)}{\partial g}\right) = \begin{cases} -\left(1/(1-\alpha)+1/\ln\gamma\right), & P=1 \\ -1/(1-\alpha), & P>1 \end{cases}$$

and that

(A31) $$\lim_{\phi\to\infty}\left(\phi^2 \frac{\partial I\left(Q_3,R,t_0\right)}{\partial g}\right) = \begin{cases} -\left(\alpha/(1-\alpha)+1/\ln\gamma\right), & P=1 \\ -\alpha/(1-\alpha), & P>1 \end{cases}$$

When the results (A28)-(A31) are inserted into (A27), it turns out that for each value of g

(A32) $$\lim_{\phi\to\infty}\left(\phi\,\Theta\right) \le \frac{-\alpha}{1-\alpha} + \frac{\left(1-\alpha^{1/(1-\alpha)}\right)}{\left(\ln\gamma\right)\Psi_{\phi=0}}$$

Next it is concluded from (A18) that

(A33) $$\begin{aligned}\frac{\partial\Psi_{\phi=0}}{\partial P} &= -\frac{\left(1-\alpha\right)^2}{\ln\gamma}\frac{\partial}{\partial P}\left(\frac{e^{-\left((\ln P)/(1-\alpha)\right)}}{1-\alpha+\ln\gamma-\ln P}\right) \\ &= -\frac{\left(1-\alpha\right)^2}{\left(\ln\gamma\right)P}\left(\frac{-1}{1-\alpha}+\frac{1}{1-\alpha+\ln\left(\gamma/P\right)}\right)\frac{e^{-\left((\ln P)/(1-\alpha)\right)}}{1-\alpha+\ln\left(\gamma/P\right)} > 0\end{aligned}$$

The possible values of P range from 1 to γ and it can now be concluded from (A18) that within this range

$$\left(\Psi_{\phi=0}\right)_{P=1} \le \Psi_{\phi=0} \le \left(\Psi_{\phi=0}\right)_{P=\gamma}$$

i.e. that

(A34) $$\frac{1-\alpha}{1-\alpha+\ln\gamma} \le \Psi_{\phi=0} \le \frac{1-\alpha}{\ln\gamma}\left(1-\gamma^{-\left(1/(1-\alpha)\right)}\right)$$

Now (A32) implies that the limit of $\phi\,\Theta$ satisfies the condition

(A35) $$\lim_{\phi\to\infty}\left(\phi\,\Theta\right) \le \frac{-\alpha}{1-\alpha} + \frac{\left(1-\alpha^{1/(1-\alpha)}\right)}{\left(\ln\gamma\right)\Psi_{\phi=0}} < \frac{-1}{1-\alpha}\left(\alpha - \frac{\left(1-\alpha^{1/(1-\alpha)}\right)\left(1-\alpha+\ln\gamma\right)}{\ln\gamma}\right)$$

Clearly, this is negative if and only if

$$\frac{\left(1-\alpha^{1/(1-\alpha)}\right)\left(1-\alpha+\ln\gamma\right)}{\ln\gamma} > \alpha+\alpha^{1/(1-\alpha)}-1$$

Obviously, this is valid if either the right-hand side is negative (i.e. $\alpha+\alpha^{1/(1-\alpha)}<1$) or $\ln\gamma$ is smaller than the limit specified in this proposition, $\left(\left(1-\alpha^{1/(1-\alpha)}\right)(1-\alpha)\right)\Big/\left(\alpha+\alpha^{1/(1-\alpha)}-1\right)$. When either of these conditions is valid, $\phi\Theta$ converges to a negative limit, which implies that when ϕ is sufficiently large, Θ and accordingly also $\partial F/\partial g$ are negative for all legitimate values of λ, P and g. This completes the proof of the proposition. □

PROOF OF LEMMA 2. The definition (39) implies that

$$\text{(A36)}\; G'\left(\Psi_{Pr}\right)=\frac{(1-\alpha)\left(1-\alpha^{1/(1-\alpha)}\right)\left(1-\alpha^{\alpha/(1-\alpha)}\right)\Psi_{Pr}-\left(1-\alpha-\alpha^{\alpha/(1-\alpha)}+\alpha^{1+1/(1-\alpha)}\right)\Psi_{\phi=0}}{\left(\Psi_{\phi=0}-\left(1-\alpha^{1/(1-\alpha)}\right)\Psi_{Pr}\right)^{1+\alpha}}$$

Clearly, there is just one value of Ψ_{Pr} for which $G'\left(\Psi_{Pr}\right)=0$, and this is the value $\Psi_{Pr}=\Psi_{Pr,0}$ which is given by

$$\text{(A37)}\quad \Psi_{Pr,0}=\frac{1-\alpha-\alpha^{\alpha/(1-\alpha)}+\alpha^{1+1/(1-\alpha)}}{(1-\alpha)\left(1-\alpha^{1/(1-\alpha)}\right)\left(1-\alpha^{\alpha/(1-\alpha)}\right)}\Psi_{\phi=0}$$

The result (A36) also implies that $G'\left(\Psi_{Pr}\right)<0$ when $\Psi_{P,0}<\Psi_{Pr}$ and $G'\left(\Psi_{Pr}\right)>0$ when $\Psi_{P,0}>\Psi_{Pr}$, so that the extreme value at $\Psi_{Pr}=\Psi_{Pr,0}$ is a minimum. We are considering a case in which the values of Ψ_{Pr} have been restricted to the interval $\left[0,\Psi_{\phi=0}\right]$, and it is now observed that

$$\text{(A38)}\quad G(0)=G\left(\Psi_{\phi=0}\right)=\Psi_{\phi=0}^{1-\alpha}$$

Together with the result concerning $G'\left(\Psi_{Pr}\right)$, this implies that $0<\Psi_{Pr,0}<\Psi_{Pr}$ so that G decreases in the interval $\left[0,\Psi_{Pr,0}\right]$ and increases in the interval $\left[\Psi_{Pr,0},\Psi_{\phi=0}\right]$. □

PROOF OF PROPOSITION 4. The definitions (26) and (32) immediately imply that $\partial\Psi_{Pr}/\partial\phi<0$ and that $\partial M(g)/\partial\phi<0$. On the other hand, $\Psi_{\phi=0}$ is, obviously, independent of ϕ, as also the result (A18) shows. Hence, one can conclude from (A9) that

(A39) $$\frac{\partial F(g,\phi,P)}{\partial\phi}=\left(\frac{1}{M(g)}\frac{\partial M(g)}{\partial\phi}+\frac{1-\alpha^{1/(1-\alpha)}}{\Psi_{\phi=0}-\left(1-\alpha^{1/(1-\alpha)}\right)\Psi_{Pr}}\frac{\partial\Psi_{Pr}}{\partial\phi}\right)F(g,\phi,P)<0$$

By definition, for any given values of ϕ and P the value $g(\phi,P)$ satisfies the condition $F(g(\phi,P),\phi,P)=1$ so that

(A40) $$\frac{\partial F}{\partial g}\frac{dg}{d\phi}+\frac{\partial F}{\partial\phi}=0$$

and

(A41) $$\frac{dg}{d\phi}=-\frac{(\partial F)/(\partial\phi)}{(\partial F)/(\partial g)}$$

Since by definition $g(\phi,P)$ corresponds to an equilibrium for which $\partial F(\phi,P)/\partial g<0$ if $g(\phi,P)$ is positive, (A39) and (A41) together imply that

$$\frac{dg}{d\phi}<0$$

so that the growth maximizing value of the rate of imitation is $\phi=0$. Turning to the claim concerning welfare, consider now the term $L^{\alpha}/(\rho-g)$ which appears in the welfare function $\tilde{U}$ defined by (38).

Clearly,

$$\frac{d}{dg}\left(\frac{L^{\alpha}}{\rho-g}\right)=\frac{d}{dg}\left(\frac{\left(1-g/(\lambda\ln\gamma)\right)^{\alpha}}{\rho-g}\right)=\frac{\left(1-g/(\lambda\ln\gamma)\right)-\left(\alpha/(\lambda\ln\gamma)\right)(\rho-g)}{\left(1-g/(\lambda\ln\gamma)\right)^{1-\alpha}(\rho-g)^{2}}$$
$$=\frac{1-\left(1/(\lambda\ln\gamma)\right)\left((1-\alpha)g+\alpha\rho\right)}{\left(1-g/(\lambda\ln\gamma)\right)^{1-\alpha}(\rho-g)^{2}}$$

Together with the assumptions $g<\rho$ and $\ln\gamma>\rho/\lambda$, this implies that

(A42) $$\frac{d}{dg}\left(\frac{L^{\alpha}}{\rho-g}\right)>\frac{1-\left(1/(\lambda\ln\gamma)\right)\rho}{\left(1-g/(\lambda\ln\gamma)\right)^{1-\alpha}(\rho-g)^{2}}>0.$$

This implies that the term $L^{\alpha}/(\rho-g)$ receives its maximal value when $\phi=0$, since the growth rate is largest in this case. On the other hand, Lemma 2 states that when $\phi=0$, also the function $G(\Psi_{Pr})$ receives its maximal value $\Psi_{\phi=0}^{1-\alpha}$, which is independent of g. Putting these results together, it follows that the choice of ϕ which maximizes the welfare function $\tilde{U}=G(\Psi_{Pr})\left[L^{\alpha}/(\rho-g)\right]$ is $\phi=0$. □

PROOF OF PROPOSITION 5. The equilibrium condition $F\left(g\left(\phi,P\right),\phi,P\right)=1$ implies that

$$\frac{\partial F}{\partial g}\frac{dg}{dP}+\frac{\partial F}{\partial P}=0$$

This is equivalent with

(A43) $$\frac{dg}{dP}=-\frac{\left(\partial F\right)/\left(\partial P\right)}{\left(\partial F\right)/\left(\partial g\right)}$$

so that dg/dP has the same sign with dF/dP when (B2) is valid.

Using the notation described in (A11)-(A16), it can be concluded from (A9) that

(A44) $$\frac{1}{F\left(g,\phi,P\right)}\frac{\partial F\left(g,\phi,P\right)}{\partial P}= \\ =\frac{1}{I\left(Q_3,R,t_0\right)}\frac{\partial I\left(Q_3,R,t_0\right)}{\partial P}-\frac{\partial I\left(Q_1,R,t_0\right)/\partial P-\left(1-\alpha^{1/(1-\alpha)}\right)\partial I\left(Q_2,R,t_0\right)/\partial P}{I\left(Q_1,R,t_0\right)-\left(1-\alpha^{1/(1-\alpha)}\right)I\left(Q_2,R,t_0\right)}$$

Remembering that the values $Q_1=g/\left(1-\alpha\right)$, $Q_2=\phi+g/\left(1-\alpha\right)$, and $Q_3=\phi+\rho+\left(\alpha g\right)/\left(1-\alpha\right)$ are independent of *P*, the function *J* will now be defined as

(A45) $$J\left(Q,P\right)=\frac{\partial I\left(Q,R,t_0\right)/\partial P}{I\left(Q,R,t_0\right)}=\frac{\partial I\left(Q,g/\ln\left(g/P\right),\left(\ln P\right)/g\right)/\partial P}{I\left(Q,R,t_0\right)}$$

The following result will be made use of below:

(A46) If $J\left(Q_1,P\right)<\min\left\{J\left(Q_2,P\right),J\left(Q_3,P\right)\right\}$, then $\dfrac{\partial F\left(g,\phi,P\right)}{\partial P}>0$.

In order to prove this result, it is first observed that since the denominator $I\left(Q_1,R,t_0\right)-\left(1-\alpha^{1/(1-\alpha)}\right)I\left(Q_2,R,t_0\right)$ is always positive, the condition $J\left(Q_1,P\right)<J\left(Q_2,P\right)$ implies that

$$\frac{\partial I\left(Q_1,R,t_0\right)/\partial P-\left(1-\alpha^{1/(1-\alpha)}\right)\partial I\left(Q_2,R,t_0\right)/\partial P}{I\left(Q_1,R,t_0\right)-\left(1-\alpha^{1/(1-\alpha)}\right)I\left(Q_2,R,t_0\right)}<\frac{\partial I\left(Q_1,R,t_0\right)/\partial P}{I\left(Q_1,R,t_0\right)}$$

In a second step, it is then noted that this result, (A45), and the assumption that $J\left(Q_1,P\right)<J\left(Q_3,P\right)$ together imply that

$$\frac{1}{F\left(g,\phi,P\right)}\frac{\partial F\left(g,\phi,P\right)}{\partial P}>J_3\left(Q_3,P\right)-J_1\left(Q_3,P\right)>0$$

By modifying this proof in an obvious way, one can also prove the following result:

(A47) If $J\left(Q_1,P\right) > \max\left\{J\left(Q_2,P\right), J\left(Q_3,P\right)\right\}$, then $\dfrac{\partial F\left(g,\phi,P\right)}{\partial P} < 0$.

Since each value of Q is independent of P, (A17) implies that

(A48) $$\frac{\partial I\left(Q,R,t_0\right)}{\partial P} = -\frac{\partial}{\partial P}\left(\frac{Re^{-Qt_0}}{Q\left(Q+R\right)}\right) = -\frac{\partial}{\partial P}\left(\frac{g}{Q\left(Q\left(\ln\gamma-\ln P\right)+g\right)}e^{-Q(\ln P)/g}\right)$$

$$= -\frac{g}{Q\left(Q\left(\ln\gamma-\ln P\right)+g\right)}\left(\frac{Q}{P\left(Q\left(\ln\gamma-\ln P\right)+g\right)} - \frac{Q}{gP}\right)e^{-Q(\ln P)/g} = \frac{RQe^{-Qt_0}}{gP\left(Q+R\right)^2}$$

Hence, (A17) implies also that

(A49) $$J\left(Q,P\right) = \frac{1}{I\left(Q,R,t_0\right)}\frac{\partial I\left(Q,R,t_0\right)}{\partial P} = \frac{RQ^2e^{-Qt_0}}{gP\left(Q+R\right)\left(Q+R-Re^{-Qt_0}\right)}$$

In order to prove (a), assume that $P=1$. In this case $R = g/\ln\gamma$ and $t_0 = \left(\ln P\right)/g = 0$, and (A49) implies that

$$J\left(Q,P\right) = \frac{RQ}{g\left(Q+R\right)} = \frac{Q}{Q\ln\gamma+g}$$

Since this is an increasing function of Q and $Q_1 < \min\left\{Q_2,Q_3\right\}$, it now follows that $J\left(Q_1,P\right) < \min\left\{J\left(Q_2,P\right), J\left(Q_3,P\right)\right\}$ and one can conclude from (A46) and (A43) that if $P=1$,

$$\frac{dg}{dP} = -\frac{\left(\partial F\right)/\left(\partial P\right)}{\left(\partial F\right)/\left(\partial g\right)} > 0$$

This completes the proof of (a).

It was demonstrated in Section 4 above that the result (25) – i.e., $P<\gamma$ – is valid on any balanced growth path of the model. Turning to the other claim which was made in part (b), it is concluded from (A49) that

$$\frac{\partial J\left(Q,P\right)}{\partial Q} = \left(\frac{2}{Q} - t_0 - \frac{1}{Q+R} - \frac{1+Rt_0e^{-Qt_0}}{Q+R-Re^{-Qt_0}}\right)J\left(Q,P\right)$$

We are interested in comparing the values $J\left(Q_1,P\right)$, $J\left(Q_2,P\right)$, and $J\left(Q_3,P\right)$. It is clear that $Q_1 < \min\left\{Q_2,Q_3\right\}$ and that when $Q \ge Q_1 = g/\left(1-\alpha\right)$,

$$\frac{\partial J\left(Q,P\right)}{\partial Q} < \left(\frac{2}{Q_1} - t_0\right)J\left(Q,P\right) = \left(2\left(1-\alpha\right) - \ln P\right)\frac{J\left(Q,P\right)}{g}$$

Hence, $\partial J\left(Q,P\right)/\partial Q<0$ for all $Q\geq Q_1$ at least when $2\left(1-\alpha\right)-\ln P<0$, i.e. when $P>e^{2(1-\alpha)}$. In this case it must be the case that $J\left(Q_1,P\right)>\max\left\{J\left(Q_2,P\right),J\left(Q_3,P\right)\right\}$ and one can conclude from (A47) and (A43) that

$$\frac{dg}{dP}=-\frac{\left(\partial F\right)/\left(\partial P\right)}{\left(\partial F\right)/\left(\partial g\right)}<0$$

This proves (b).

Finally, consider the limit in which $P\rightarrow\gamma$, so that $R=g/\left(\ln\gamma-\ln P\right)\rightarrow\infty$. The results (A48) implies that in this limit

$$\lim_{P\rightarrow\gamma-}\frac{\partial I\left(Q,R,t_0\right)}{\partial P}=-\frac{g}{Qg}\left(\frac{Q}{Pg}-\frac{Q}{gP}\right)e^{-Q(\ln P)/g}=0$$

so that one can conclude from (A44) that

(A50) $$\lim_{P\rightarrow\gamma-}\left(\frac{1}{F\left(g,\phi,P\right)}\frac{\partial F\left(g,\phi,P\right)}{\partial P}\right)=0$$

This proves (c). □

PROOF OF PROPOSITION 6. Assume that $P>1$. The definitions (32), (26), and (28) imply that $M\left(0\right)=1/\left(\phi+\rho\right)$ and that

(A51) $$\left(\Psi_{Pr}\right)_{g=0}=\lim_{g\rightarrow 0}\Psi_{Pr}=\begin{cases}\Psi_{\phi=0}, & \phi=0\\ 0, & \phi>0\end{cases}$$

When these results are inserted into (A9), it is seen that

(A52) $$F\left(0,\phi,P\right)=\begin{cases}\lambda\dfrac{1-\alpha}{\alpha\left(\phi+\rho\right)\Psi_{\phi=0}}, & \phi=0\\ \dfrac{\lambda\left(1-\alpha\right)\alpha^{\alpha/(1-\alpha)}}{\left(\phi+\rho\right)\Psi_{\phi=0}}, & \phi>0\end{cases}$$

However, according to the result (A33), which was demonstrated within the proof of Proposition 2, $\partial\Psi_{\phi=0}/\partial P>0$, and one can conclude from (A51) that

(A53) $$\frac{\partial F\left(0,\phi,P\right)}{\partial P}<0$$

Hence, since F is by assumption of decreasing function of g,

$$\lambda_0 = \inf\left\{\lambda' \middle| \left(g(\phi, P)\right)_{\lambda=\lambda'} > 0 \text{ for some } P\right\}$$
$$= \inf\left\{\lambda' \middle| F(0, \phi, P) > 1 \text{ for some } P, \text{ when } \lambda = \lambda'\right\}$$
$$= \inf\left\{\lambda' \middle| F(0, \phi, 1) > 1, \text{ given } \lambda = \lambda'\right\}$$

Let $P' > 1$ be arbitrary. Since $F(0,\phi,P') < F(0,\phi,1)$ for each λ and $F(0,\phi,P)$ is a continuous and increasing function of λ, it must be the case for all values $\lambda' > \lambda_0$ which are sufficiently close to λ_0 that

$$\left(F(0,\phi,P')\right)_{\lambda=\lambda'} < \left(F(0,\phi,1)\right)_{\lambda=\lambda_0} = 1$$

In other words, when λ' is sufficiently close to λ_0 there will be no growth if $P \geq P'$, so that the growth-maximizing value of *P* must in this case be smaller than P'. Since $P' > 1$ was arbitrary, it follows that the growth-maximizing value of *P* approaches 1 when $\lambda \to \lambda_0 +$. Similarly, also the welfare-maximizing value of *P* must approach 1 if it is a value for which the growth rate is positive. □

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