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A Model of Residential Sorting with an Endogenous Wealth Distribution*

Abstract

This paper develops a dynamic multi-region model, where (i) an owner-occupying household's current location choice depends on its current wealth, and its current "match" (which may reflect the household's demographic characteristics), and (ii) the current wealth depend on the household's past fortunes in the housing market, and its past location choices. In the long-run equilibrium of the model, the size of capital gains and losses, the wealth distribution and residential sorting are all determined endogenously. We find that the larger (smaller) the capital gains and losses made in the housing market, the more residential sorting there is according to wealth (the match). Also, social welfare decreases, if the households face large housing price fluctuations. Finally, we show that under rental housing, there is more residential sorting according to the match and less sorting according to wealth, than under owner-occupation.

JEL Classification: D31, D52, R13, R21, R23.

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1 Introduction

A central theme in regional and urban economics has been to examine how households sort themselves into neighborhoods and communities according to income, wealth and various socioeconomic characteristics, such family size, the age of the household head, or education. Typically, the theoretical residential sorting models have been static, and the income/wealth distribution is taken as exogenous in these models, often with an underlying premise that a household's resources reflect its human capital. Examples include the seminal early paper by Ellickson (1971), as well as more recent important contributions by Epple and Romer (1989, 1991), Epple and Platt (1998), Epple et al. (1983,1984,1993), Henderson (1991), Pogodzinski and Sjoquist (1991,1993), Wheaton (1993), and Yinger (1995); for a recent survey, see e.g. Ross and Yinger (1999).

However, arguably the housing market often plays a major role in shaping the wealth distribution, as well as in determining an individual household's position in the distribution, and in influencing how the household's wealth evolves over time. For example in the UK households held over 60% of their total wealth in home equity in the mid 1990's (Banks et al. (2002)). As financial assets tend to be predominately owned by the wealthiest part of the population, for a "typical" British household housing was a still more important component of the portfolio.

Also, housing prices are often highly volatile, opening up the possibility of capital gains and losses. In the UK, the average inflation adjusted house price was £ 65 000 in 1980, £ 100 000 in 1989, £ 64 000 in 1995 and £ 150 000 in 2004, where these prices (as well as all monetary quantities quoted below) are denoted in 2004 pounds.¹ Moreover, in different regions property values tend to rise and fall asynchronously, so that relative regional prices may vary significantly over time. In 1981 housing was 1.5 times more expensive in London than in North-West England (Merseyside and surroundings); in 1987 the ratio of prices was 2.7, in 1992 1.25 and in 2000 2.3. Relative prices have varied quite a lot even at a more local level, e.g. between London boroughs. For example in 1995 average housing prices were 3% lower in Hackley than in Greenwich, but in 2001 Hackley was 20% more

¹Source: Nationwide Building Society, <http://www.nationwide.co.uk>.

expensive than Greenwich.^{2,3}

The capital gains and losses realized in the housing market may be quite large, compared to annual household incomes and savings. Between 1983 and 1988, when London property values more than doubled in real terms, the price of a typical London home rose on an average by £ 14 000 each year, a figure that corresponded to 72% of the mean annual disposable household income £ 19 000 in the UK over that period, and exceeded by the factor of 7.8 average yearly household savings, £ 1800. Between 1989 and 1992, when the London market lost almost half of its value, the annual capital loss of a typical homeowner was £ 17 000, or 77% of average disposable household income, and 8.4 times average household savings.⁴

As a general rule housing market risks are uninsurable and undiversifiable. For example Shiller (1993, 2003) lists home equity insurance as one of the key financial markets currently missing. In the UK, real estate futures were traded in the London Futures and Options Exchange (London Fox) in 1991, from May through October. Trading volume was low, and the market was closed when it was reported that the exchange had allegedly attempted to create a the false impression of high trading value by false trades (Shiller (1993, Ch 1)). Shiller (1993), and Shiller and Weiss (1998) discuss the potential problems, both economic and psychological, involved in providing hedging against housing price swings, as well as ways to overcome these problems.⁵

²Source: Land Registry, <http://www.landreg.gov.uk>.

³Similar patterns can be found in a number of other countries as well. For example, Shiller (1993, Ch 5) reports sustained dramatic changes in real housing prices at different times in different U.S. cities. The average annual rate of change was 9.3% in San Francisco 1976-80, 17.7% in Boston 1983-87, 16.9% in New York 1983-1987, 10.2% in Washington DC 1986-88, 19.1 in San Francisco 1987-89, 21.2% in Honolulu 1987-90, 22.3% in Seattle 1988-90, -14.6% in Houston 1985-87, -12.2% in Oklahoma City 1986-89, -10.1% in New York 1988-91, and -13.5% in Boston 1989-91.

⁴In the North-West, swings in housing prices have been more modest, both in relative and in absolute terms, but during the two-year period 1988-1989 (when the London market was already entering a recession) prices rose by 46%, so that the value of an average home increased by £ 14 000 in 1988 and by £ 11 000 in 1989.

⁵In the US, there are a few local experiments with home equity insurance. The Oak Park Experiment has been running since 1977, and the South-West Home Equity Assurance Program was initiated in 1988. Both of these programs are in Chicago, and insure homeowners against price declines caused by neighborhood change. More recently, the Yale/Neighborhood Reinvestment Corporation Home Equity Guarantee Project has developed home equity insurance products, to be initially used in Syracuse, New York. See Shiller (2003, Ch 8).

In this paper we develop a dynamic model, with a two-way relation between wealth and households' location choices: (i) A household's current location choice depends on its current wealth (and its current "match"). (ii) On the other hand, the current wealth depends on the household's past fortunes in the housing market, and its past location choices. In the long run equilibrium of the model, the size of capital gains and losses, the distribution of wealth and the pattern of residential sorting are all determined endogenously.

The assumptions of the model concern the structure of preferences (or matches) and shocks, housing market institutions and financial market incompleteness. The infinitely-lived households, or dynasties, are heterogenous with respect to their match. The match, which may reflect various demographic and socioeconomic characteristics of the household, indicates how much utility the household derives from residing in different locations. The match may also change over time, if the characteristics of the household (or the neighborhoods) change.

Swings in relative housing prices derive from regional shocks. These shocks may reflect the changing quality of local public goods, but on the other hand also the prevailing tastes and needs of the population, with respect to housing and public goods, tend to evolve, making certain locations relatively more popular and others less popular. Evidently, there may also be labor market related reasons.

In the bulk of the paper, we assume that the households are owner-occupiers, so that the house is both a consumption good and an asset for them. Also, each household must live in the house that it owns, and as a consequence the decision where to live cannot be separated from the decision where to invest. There may then be a trade-off between the consumption motive and the investment motive of housing. While a currently desirable, and expensive, location typically offers a household a high utility stream today, buying a home there may not be such a good investment, since the popularity of the area may wane in the future, and the (relative) price of the house may fall.

Importantly, financial and insurance markets are assumed to be incomplete, in the sense that regional housing price movements, as well as well idiosyncratic shocks affecting the match, are uninsurable. The households can save in terms of a safe asset, and also

borrow up to a certain limit. If a household faces the borrowing constraint, it cannot move from a less expensive location to a location with more expensive homes.

Essentially, our results relate the size of capital gains and losses, made in the housing market, to the form of residential sorting, and to social welfare. We show that small capital gains and losses tend to be associated with residential sorting according to the match, while large gains and losses are associated with residential sorting according to wealth. Social welfare decreases, when the households face larger housing price fluctuations.

Although the bulk of our analysis focuses on owner-occupation, we also briefly look at rental housing, and compare the alternative modes of housing tenure. In our framework rental housing essentially corresponds to owner-occupation with deterministic capital gains and losses, and thus the "rental market" equilibrium is just a special case of owner-occupation. Our analysis highlights the following difference between the modes of housing tenure: Under rental housing the households face in every period small, deterministic housing costs (rental payments) which depend on their housing location, while under owner-occupation there are potentially large payments and compensations (capital losses and gains) which occur only with a certain probability, and affect only a certain portion of the households in a given period. Under rental housing, households which make similar location choices, also pay similar housing costs, while under owner-occupation two households making similar location choices may face starkly different realized costs, if, say only one of the households suffers a capital loss. We then show that compared to owner-occupation, rental housing tends to result in more residential sorting according to the match, and less sorting according to wealth.

Essentially, the present paper combines two themes, which are typically addressed separately in the literature: (i) households' location choice and residential sorting, and (ii) the double role of housing as both a consumption good and an investment; housing as an important component of a household's asset portfolio. As discussed above, our paper differs from (most of the) existing residential sorting models in the sense that in this literature the income/wealth distribution is taken as given, and thus the existing studies tend to focus on one side of the two-way relation between housing and wealth. It is worth noting that while the residential sorting literature pays a lot of attention to

housing prices, including capitalization effects, typically these prices are determined by the existing wealth/income distribution, rather than affect it. There is also second key difference between the existing literature and our paper. Following the Tiebout (1956) tradition, many studies in the literature assume that the attractiveness of different housing locations essentially reflect local public policies (taxes, public goods and services), and typically public policies are also determined endogenously in equilibrium. As the title of a recent survey (Ross and Yinger (1999)) succinctly puts it, much of this literature is about "sorting and voting". In the present paper, there is no "voting": the households' matches with different locations, as well as regional shocks, are treated as exogenous.⁶

Moving to the second branch of related literature, the double role of housing as a consumption good and an asset has been recognized at least since Henderson and Ioannides (1983) who studied its implications for tenure choice. The basic problem arises from the fact that a household's consumption demand for housing may constrain its investment choice for housing, or vice versa. A subsequent paper by Henderson and Ioannides (1987) considered a model with more general institutional characteristics and then analyzed it empirically. Bruenecker (1997) further studied the interaction between the consumption demand and the investment demand for housing in a mean-variance portfolio context, while Flavin and Yamashita (2002) undertook a corresponding econometric analysis. Recent literature has also recognized that owner-occupied housing may serve as a hedge against the risk that house prices, as well as rental payments, rise in the future (see Cocco (2000), Davidoff (2003), Ortalo-Magné and Rady (2002b), and Sinai and Souleles (2003))⁷. However, none of these papers has considered the implications of housing market capital losses and gains for regional allocation of economic households.

To the best of our knowledge, the only study combining both of the themes analyzed in the present paper is the recent contribution by Ortalo-Magné and Rady (2002a), who study household mobility, volatility of house prices and income distribution.⁸ Interest-

⁶Recent empirical studies inspired by the residential sorting models include Epple and Sieg (1999) and Rhode and Strumpf (2003).

⁷These papers also analyze the role of the home as a hedging device against adverse shocks in good prices or labor income. In fact, Henderson and Ioannides (1983) already considered homeownership as a means to ensure desired housing consumption when future house prices are uncertain.

⁸Also Rady and Ortalo-Magné (1999, 2001) are somewhat related and provide explanations for why

ingly, Ortalo-Magné and Rady show that capital gains made in the housing market result in higher within-neighborhood income variability. This is because some low-income households, which have bought their home when housing prices were low, find it affordable, and indeed optimal, to stay in the newly expensive neighborhood. This result echoes our finding that large capital gains and losses in the housing market tend to weaken residential sorting according to the match. However, the study by Ortalo-Magné and Rady differs from our paper, since their analysis is partial equilibrium in nature, and focuses entirely on a single neighborhood. Also, the model has only two periods, and the first period income/wealth distribution is taken as given. Finally, in terms of tenure choice, Ortalo-Magné and Rady go further than we do in the present paper, since they let the households choose between owner-occupation and renting, while we take the mode of housing tenure (owner-occupation only or renting only) as given.⁹

The plan of the paper is as follows. Section 2 lays down the model basics. Section 3 analyzes, how the households choose their housing location, based on their current wealth and their current match. Section 4 then addresses the other side of the linkage between wealth and housing markets, and analyzes how the households' location choices induce a stationary wealth distribution. Section 5, which contains our main results, characterizes equilibrium residential sorting, and social welfare. Section 6 then briefly studies rental housing, and contrasts residential sorting under the different modes of housing tenure. Finally Section 7 concludes.

2 The basics of the economy

The economy has two of locations, or neighborhoods. Each locations has an equal, fixed, stock of identical houses. Each house is occupied by a single household and no one household is ever homeless. For convenience, assume that the stock of houses and the mass of households each comprises a continuum of size unity.

house prices tend to be very volatile - an important factor in the present theory.

⁹Empirically, the dominant mode of housing tenure varies quite significantly between countries, reflecting e.g. tax treatment, the regulation of rental housing, or the development of the credit market. See e.g. Chiiuri and Jappelli (2003).

There are infinite discrete time periods indexed by $t = 0, 1, \dots$. In each period, one of the locations is deemed to be "desirable" while the other one is "less desirable". When a period changes, the relative ranking of the locations is reversed with probability $\pi \in (0, 1]$.

We also consider a small region interpretation of the model, with a continuum of locations. Then in each period, one half of the locations are "desirable" while the remaining locations are "less desirable", and when a period changes, a measure π of the locations is hit by a regional shock. The long-run equilibrium of the model is essentially identical under both interpretations.

The households are heterogenous in the quality of their match. In every period the match θ of each household is randomly drawn from a cumulative distribution function $G(\theta)$, on some support $[\theta_L, \theta_H]$. The aggregate heterogeneity of households is unchanged over time so that G is stationary. The actual utility of a household is conditional on the neighborhood where it resides. We assume that a household receives utility θ , when living in a desirable location. The utility of anyone household living in a less desirable location is ω . In particular, we assume that $G(\omega) < \frac{1}{2}$, so that more than one half of the households prefer a desirable location to a less desirable location; if $\omega < \theta_L$, all households would rather live in a one of the popular locations. Since $\theta - \omega$ is the utility premium of living in a desirable location, the parameter ω serves to measure interregional utility differences. Also notice that if a region is hit by a shock, the benefit stream it offers changes from ω to θ , or vice versa; thus ω also measures the size of regional shocks. Finally, the households live forever and discount future utilities by a common factor $\beta \in (0, 1)$.

The sequence of events in any time period is the following. First, a random process determines which locations are desirable and which are not. Second, each household observes its current type θ , which is drawn from the distribution $G(\theta)$. Finally the households decide where they want to live. Depending on their choice, they either stay where they are or they migrate to another location. However, if the household is borrowing constrained, it cannot move from an undesirable location to a desirable location. Borrowing constraints will be discussed in the next section.

Denote the median type by θ_m . In every period, the aggregate welfare is maximized, if all households with $\theta > \theta_m$ are allocated to the (currently) desirable locations, those with

$\theta < \theta_m$ live in the less desirable locations, and the group (always of measure zero, if G is continuous) with $\theta = \theta_m$ is divided between the locations so that capacity constraints in housing are not violated. In other words, there is perfect residential sorting according to the match. If this allocation rule is followed, the expected utility of a representative household in any period is

$$W^* = \frac{1}{2}\omega + \frac{1}{2}E[\theta \mid \theta \geq \theta_m] \quad (1)$$

Notice also, that the allocation rule minimizes the within-locations variance of the match, and maximizes the between-locations variance.

3 The household's problem

In the market outcome, the location choice depends on not only the match, but also on housing prices and wealth. We fix the housing price in (currently) desirable locations to 1, and normalize the price of housing in (currently) less desirable regions to 0. These normalizations can be made without loss of generality, since in this model the only choice available to the households is whether to own a house in a popular neighborhood or in an unpopular neighborhood; a household cannot sell a house without buying another one.¹⁰ Notice also that the normalizations adopted here mean that capital gains and losses are of size unity.

Markets are incomplete in the sense that capital gains and losses (as well as idiosyncratic shocks affecting the match θ) are uninsurable. The simple incomplete markets setting we consider here resembles the Huggett (1993) pure credit model and the Bewley (1983) model with outside money; for a recent text-book treatment, see also Ljungqvist and Sargent (2004, Ch. 17). In addition to owning a home, the households can transfer wealth from the current period to the next by holding financial assets. The households are also allowed to borrow up to a certain limit $-b$, so that asset holdings can be either

¹⁰Only owner-occupation is allowed at this point, and being homeless would entail a large negative utility stream.

positive or negative. There are two sorts of financial assets. First there is pure credit, or inside money, which is available at zero net supply: a household with negative assets has (at least indirectly) borrowed from other households, with positive holdings. Second, there is outside money, with fixed nominal supply M . If the price of money, in terms of housing (in good locations), is $1/p$, the real supply of outside money is M/p . In the stationary equilibrium, that we analyze in this paper, inside and outside money are perfect substitutes, and the real interest rate is zero.¹¹

Denote the household's financial asset holdings by a and let h be housing: h is equal to 1, if the household owns a house in a desirable location, and equal to 0, if the house is in an undesirable location. Then in any given period t , the household's budget constraint reads

$$a_t + h_t = a_{t-1} + (1 - s_t) h_{t-1} + s_t (1 - h_{t-1}) \quad (2)$$

$$a_t \geq -b \quad (3)$$

where s_t is an indicator function which is equal to 1 if the location where the household lives is hit by a regional shock and 0 otherwise. The wealth of the household in period t , figuring on the right-hand side of (2), consists of period $t - 1$ (positive or negative) financial asset holdings and housing. If the region where the household lived and owned a house in period $t - 1$ has not been hit by a shock, the value of the household's house has remained unaltered (0 or 1); if there has been a shock, the household has made a capital gain or a loss. The left-hand side of (2) then tells that the household divides its total resources between period t housing and financial assets. Finally, equation (3) reminds

¹¹In this model, there are no stationary equilibria with a positive rate of interest. This is because the households have no sources of income outside the housing sector. If a household wants to increase its asset holdings, its best strategy is to live in an unpopular region in every period, and hope for capital gains. However, with a positive probability, the household does not make (enough) gains in the housing market. With a positive interest rate, the logic of compound interest implies that a household with negative initial asset holdings exceeds any finite borrowing limit $-b$ with a positive probability at some point in the future. At the aggregate level, a strictly positive fraction of households with a negative initial position ends up with asset holdings below $-b$. Then to avoid breaching the borrowing limit, all households should have non-negative asset positions. This is however impossible, since there are no outside sources of credit, and the aggregate supply of inside money is zero (obviously the degerate allocation, where all households have zero assets, is feasible).

that asset holdings are not allowed to fall below the debt limit $-b$.

In any given period, a household's total wealth (z) consists of financial assets and housing,

$$z_t = a_t + h_t \tag{4}$$

Then from (2) we get a law of motion for z_t ,

$$z_{t+1} = z_t + s_{t+1}(1 - 2h_t), \tag{5}$$

which immediately reveals that a household's wealth changes if and only if the neighborhood where it resides is hit by a shock, and housing prices rise or fall. Also, plugging (4) into (3) shows, how the borrowing constraint limits location choices at low wealth levels:

$$h_t = 0 \text{ if } z_t < 1 - b \tag{6}$$

Looking at equations (5) and (6), two observations can be made: (i) A household's wealth always changes in steps of size unity. (ii) All households with wealth $z = \varepsilon - b$, where $\varepsilon \in [0, 1)$ are borrowing constrained. Likewise all households with wealth $z = n + \varepsilon - b$, where $n \in \{0, 1, 2, \dots\}$, are n capital losses away from the borrowing constraint.

Definition 1 *A household has n units of disposable wealth, if $z = n + \varepsilon - b$, where $n \in \{0, 1, 2, \dots\}$ and $\varepsilon \in [0, 1)$.*

In the analysis, disposable wealth n will serve as the state variable. It evolves according to the law of motion

$$n_{t+1} = n_t + s_{t+1}(1 - 2h_t) \tag{7}$$

Without loss of generality, we assume that $\varepsilon = 0$ for all households, so that the households do not carry any "superfluous" assets.¹² If a household has no disposable wealth, it is

¹²Allowing for $\varepsilon > 0$ would affect the nominal, or monetary, price of housing p , but otherwise the

borrowing constrained and can only live in an unpopular area:

$$h_t = 0 \text{ if } n_t = 0 \tag{8}$$

Consider the optimization problem of any one household. In every period it chooses its location so as to maximize the expected discounted utility stream

$$E_\theta \sum_{t=0}^{\infty} \beta^t [h_t \theta + (1 - h_t) \omega],$$

subject to (7) and (8). The problem can be conveniently presented in a recursive form. Let $V(n)$ denote the optimal value of the problem for a household which has n units of disposable wealth. $V(n)$ satisfies the Bellman equations

$$\begin{aligned} V(n) = E_\theta [\max \{ & \theta + \beta [(1 - \pi)V(n) + \pi V(n - 1)], \\ & \beta [\omega + (1 - \pi)V(n) + \pi V(n + 1)] \}] \text{ for } n \geq 1 \end{aligned} \tag{9}$$

and

$$V(0) = \omega + \beta [(1 - \pi)V(0) + \pi V(1)] \tag{10}$$

Note that the value function $V(n)$ is evaluated after the regional shock has been realized but before the type θ has been revealed to the household. The maximization problem then defines the optimal choice of location that takes place when the value of θ becomes known. Inside the maximum operator, the first expression is the value of living in a desirable neighborhood, while the second expression is the value of choosing a less desirable location. If the household's optimal decision is to live in a good neighborhood, it can immediately "eat" whatever value θ realized. its prospects for the next period are discounted by β and are given in the square brackets in the first argument of the maximization problem. There is a probability $1 - \pi$ that the location will be popular also tomorrow so that it

long-run equilibrium of the model would remain intact.

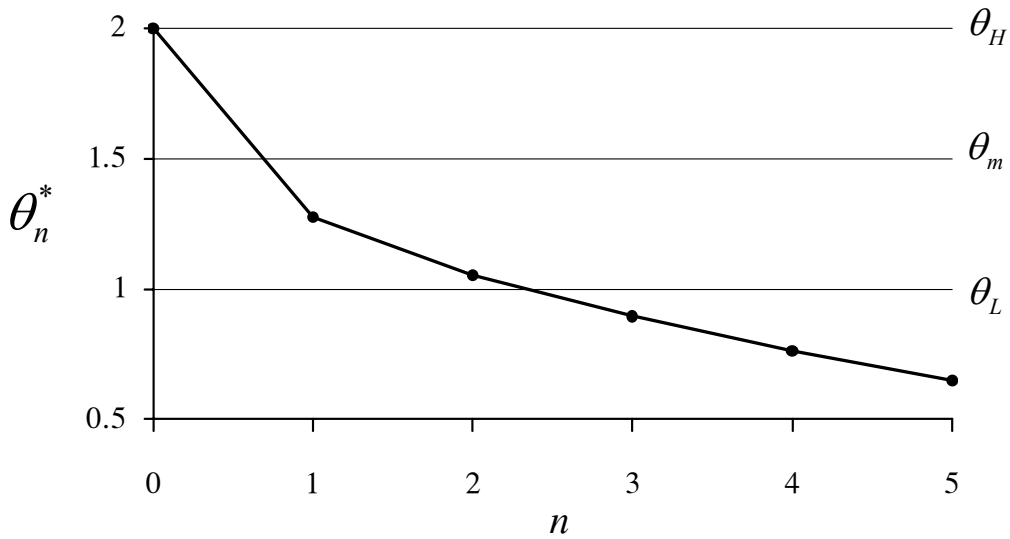


Figure 1: The θ_n^* -curve when θ is uniformly distributed on $[1, 2]$, $\omega = 0$, $\beta = .95$, and $\pi = .3$.

will be facing the same value function as today, while with probability π the location loses its appeal and the household suffers a capital loss. If the household chooses an undesirable neighborhood, its utility in the current period is ω . With probability $(1 - \pi)$ its neighborhood is out of vogue also in the next period, while with probability π its status improves, and the household makes a capital gain.

The location choice reflects a tension between the consumption and the investment aspect of housing. While a popular region typically offers a household a higher utility stream today, currently less popular and less expensive areas are more attractive from the investment point of view. The investment motive is forward-looking and arises from the household's need to have enough wealth also in future periods. Indeed, the maximization problem involves a trade-off between present benefits and future options. The household wants to avoid the situation where its location choices are limited by the borrowing constraint. Choosing a desirable location when the match is not so good entails the possibility that this option may not be available in the future when the match is better.

In each wealth class $n \geq 1$ there is then a critical quality of match

$$\theta_n^* = \omega + \pi\beta[V(n+1) - V(n-1)] \quad (11)$$

that equates the two arguments in the maximization operator in (9). households with $\theta > \theta_n^*$ choose a desirable neighborhood, while those with $\theta < \theta_n^*$ go to a currently less desirable neighborhood in the hope of making capital gains in the housing market. Finally, households with $\theta = \theta_n^*$ are indifferent between the two alternatives; if the distribution $G(\theta)$ is continuous, this group is always of measure zero. Figure 1 shows θ_n^* with different values of n when the earnings are uniformly distributed on $[1, 2]$, $\omega = 0$, $\beta = .95$, and $\pi = .3$. Clearly, θ_n^* decreases with n . This is a general property of θ_n^* , and it stems from the fact that the value function is concave. (Concavity is proved in the appendix.) Also, this finding has a natural interpretation. If a household is wealthy, additional assets are of less value. To put it differently, the more assets the household has, the more distant is the prospect of being borrowing constrained at some point in the future.

Essentially, Figure 1 illustrates the following finding:

Proposition 1 *Residential sorting takes place both according to the match and according to wealth. Wealthy households and households with a good match tend to choose a popular neighborhood, while less wealthy households and households with a poorer match live in the less desirable locations.*

This result parallels the findings in Epple and Platt's (1998) model with two-dimensional heterogeneity, where both income and tastes affect equilibrium stratification of the population. Compared to Epple and Platt (1998), a key difference here is that heterogeneity in terms of wealth will be derived endogenously, in the next section.

Next we analyze how the size and the frequency of regional shocks affect location choices.

The parameter ω provides a measure of how different the desirable and the undesirable locations are from each other. It also gauges the size of regional shocks: if a location is hit by a shock, the utility stream it offers to households changes from ω to θ , or vice

versa. An increase in regional differences, so that ω decreases, strengthens the households' incentives to choose a desirable location in the current period (consumption motive). On the other hand, it also reinforces the incentives to accumulate assets (investment motive), since a household stands to lose more if it faces the borrowing constraint at some point in the future. However, since the future losses are discounted, and only occur with a certain probability, the effect on current consumption demand dominates. The larger the regional differences, and the regional shocks, the more likely it is, *ex ante* (i.e. before the realization of the match), that a household with a given level of disposable wealth n chooses a currently desirable location. The following lemma states these findings more formally:

Lemma 1 *The θ_n^* -schedule shifts up (down) when ω increases (decreases). That is, for all $n \geq 1$, $\frac{d\theta_n^*}{d\omega} > 0$.*

Proof See the appendix. ■

The parameter π measures the frequency of regional shocks. As the parameter π does not affect the utility streams, θ and ω , available in different location, an increase in π leaves the consumption demand of housing unaltered; however the investment motive becomes stronger, encouraging the households to choose a currently less popular and less expensive location. This is because households living in a popular neighborhood are increasingly likely to suffer capital losses, while capital gains are an increasingly likely prospect if the household buys a property in a currently less desirable area. Viewing the household's dilemma from a somewhat different angle, when transitions up and down the wealth ladder take place with a higher probability, the prospect of facing the borrowing constraint at some point in the future becomes an increasingly powerful deterrent. Then in any unconstrained wealth class, a household needs a better match before it chooses to live in a currently popular neighborhood.

Lemma 2 *The θ_n^* -schedule shifts up (down) as π increases (decreases). That is, for all $n \geq 1$, $\frac{d\theta_n^*}{d\pi} > 0$.*

Proof See the appendix. ■

4 Stationary wealth distribution

The previous section showed, how a household chooses its location based on its current wealth and its current match. On the other hand, a household's current wealth depends on its past fortunes in the housing market, and its past location choices. Then the long-run wealth distribution arises as a result of households' moving policies.

Denote by f_n the size of wealth class n , and let

$$f_n^b = G(\theta_n^*) f_n, \quad f_n^g = (1 - G(\theta_n^*)) f_n$$

be the frequency of households with disposable wealth n who live in bad (or undesirable) locations and good (or desirable) locations, respectively. Notice also that

$$\frac{f_n^b}{f_n^g} = \frac{G(\theta_n^*)}{1 - G(\theta_n^*)} \equiv \gamma_n \quad (12)$$

The "odds ratio" γ_n decreases with n as wealthy households are more likely to choose a good neighborhood.

Consider first the two-region interpretation of the model. If there is no regional shock in a given period ($s = 0$), the wealth distribution remains unaltered. If there is a shock ($s = 1$), all f_n households who were previously in wealth class n either climb one step up or fall one step down, depending on their house location. They are replaced by f_{n-1}^b class $n - 1$ households who have made a capital gain, and f_{n+1}^g class $n + 1$ households who have suffered a capital loss. The distribution is stationary if and only if

$$f_n = (1 - s) f_n + s (f_{n-1}^b + f_{n+1}^g) \quad (13)$$

for all n . Next, if there is a continuum of atomistic regions, in every period, the fraction π of the locations is hit by a regional shock, and the wealth distribution is stationary if and only if

$$f_n = (1 - \pi) f_n + \pi (f_{n-1}^b + f_{n+1}^g) \quad (14)$$

for all n . It is easy to see that the equations (13) and (14) both boil down to the same stationarity condition

$$f_n \equiv f_n^b + f_n^g = f_{n-1}^b + f_{n+1}^g \quad (15)$$

As a consequence, both model variants have the same long-run wealth distribution, and the same long-run equilibrium.

There are no wealth classes below the borrowing constrained class 0, and at wealth level 0 the borrowing constrained households can only choose an unpopular location. Observing that $f_{-1} = f_0^g = 0$, equation (15) implies $f_0 \equiv f_0^b = f_1^g$; that is, the group of households with minimum financial asset holdings, but a house in a good location (f_1^g) must be of the same size as the borrowing constrained group (f_0^b). But then $f_1 \equiv f_1^b + f_1^g = f_0^b + f_2^g$ implies $f_1^b = f_2^g$, and iterating forward leaves us with the sequence of equations

$$f_n^b = f_{n+1}^g \quad (16)$$

for all $n \geq 0$. In words, each two groups with the same financial asset holdings, but different housing location, must be of equal size. Summing over all wealth classes yields $\sum_n f_n^b = \sum_n f_n^g$ and given that the aggregate mass of households is unity $\sum_n f_n = \sum_n (f_n^b + f_n^g) = 1$, it follows that

$$\sum_n f_n^i = \frac{1}{2} \quad i \in \{b, g\} \quad (17)$$

But these equations indicate that the demand for housing, on the left-hand side, is equal to the supply of housing ($\frac{1}{2}$), in both location types.

To obtain an explicit characterization of the wealth distribution, we next combine (12) and (16). What we get is a simple first order difference equation for both f_n^g and f_n^b .

$$f_{n+1}^g = \gamma_n f_n^g \quad \text{and} \quad f_{n+1}^b = \gamma_{n+1} f_n^b \quad (18)$$

and the frequency of any node can be linked to the size of the liquidity constrained group

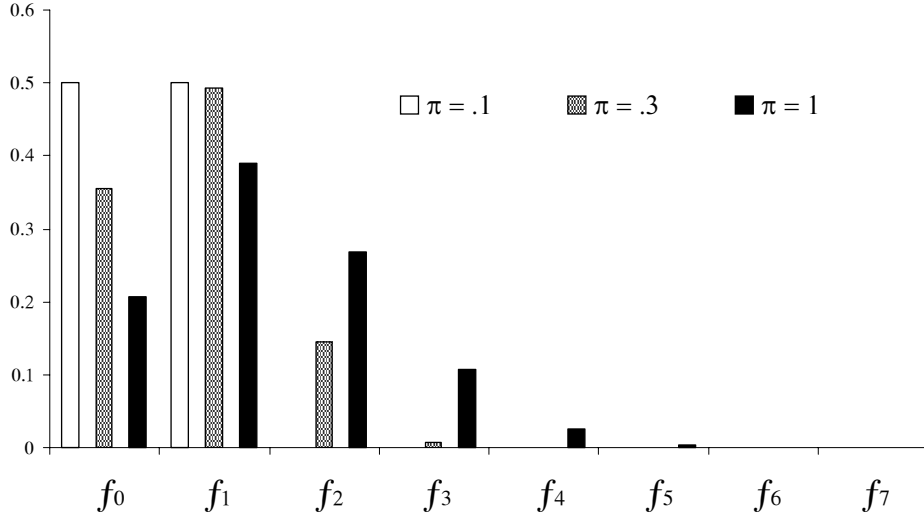


Figure 2: Wealth distributions with different values of π when θ is uniformly distributed on $[1, 2]$ and $\beta = .95$

f_0

$$f_n^b = f_{n+1}^g = f_0 \prod_{i=0}^n \gamma_i \quad n \geq 0$$

where $\gamma_0 \equiv 1$. These equations, combined with the housing market equilibrium (17) lead to the formulae

$$f_{n+1}^g = f_n^b = \frac{\prod_{i=0}^n \gamma_i}{\sum_{k=0}^{\bar{n}-1} \prod_{i=1}^k \gamma_i} \quad \text{for } n = 0, \dots, \bar{n} - 1$$

which, together with the equalities $f_n = f_n^b + f_n^g$, determine the stationary distribution.

Figure 2 shows the stationary wealth distribution for three different values of π , when θ is uniformly distributed on $[1, 2]$ and $\beta = .95$. The distributions are single-peaked, with wealth classes in the middle typically having more mass than those on the tails. The single-peakedness is a general property, and follows from the fact that γ_n is decreasing in n . Intuitively, households with fewer assets are likely to choose the less popular location and make capital gains. For wealthy households, capital losses are more probable. Thus

transitions in the wealth distribution tend to happen towards the middle.

We also notice that increasing π from .1 first to .3 and then to 1 shifts the distribution to the right, towards higher wealth classes. When capital gains and losses become more probable, the households adopt increasingly cautious strategies, and are more likely to choose the less desirable location, in the hope of capital gains. As a result a larger proportion of the households reach higher wealth levels. Notice also that with bigger values of π , there is less mass on the tails of the distribution. This observation is worth mentioning since households in the extreme wealth classes are unwilling or unable to move in response to a changing match.

These findings can be restated more precisely, if we define the cumulative distribution function

$$F(n; \omega, \pi) = \sum_{i=0}^n f_i.$$

Lemma 3 *When ω or π increases, the wealth distribution shifts to the right, in the sense of first-order stochastic dominance. That is $\frac{\partial F(n; \omega, \pi)}{\partial \omega} \leq 0$ and $\frac{\partial F(n; \omega, \pi)}{\partial \pi} \leq 0$. In particular, the size of the borrowing constrained group (f_0) decreases.*

Proof By Lemmas 1 and 2, the θ_n^* -schedule shifts up when ω or π grows. This then increases the "odds ratio" γ_n , and equations (18) imply that the ratio f_{n+1}^i / f_n^i , $i \in \{b, g\}$ goes up. But then it follows immediately that for each n , $\frac{\partial F(n; \omega, \pi)}{\partial \omega} \leq 0$ and $\frac{\partial F(n; \omega, \pi)}{\partial \pi} \leq 0$.

■

The aggregate long-run equilibrium of the economy is defined by the stationary wealth distribution, combined with the households' location choices. In this section we have essentially studied how location choices induce the distribution. It is however also useful to move the other way round. The wealth distribution, and especially Lemma 3, allows us to further characterize the environment that the households face, assess how wealthy they are, and to reinterpret the location choices they make.

First, the average wealth level in the economy, $E[n]$, can be used as a yardstick, against which the size of capital gains and losses, normalized to 1, can be measured.

Remark 1 *The larger, or the less frequent (and more unexpected), the regional shocks, the larger the capital gains and losses are, compared to average household wealth $E[n]$.*

Proof Since the size of capital gains and losses is normalized to 1, their relative magnitude, proportioned to the average wealth in the economy, is $1/E[n]$. Lemma 3 implies that $E[n]$ increases together with π and ω . ■

The next remark points out how the households' wealth can be assessed, and reinterprets Lemmas 1 and 2, in revealing a relation between a household's relative position in the wealth distribution, and its location choice.

Remark 2 *Consider a household with disposable wealth n . The smaller (larger) the values of ω and π , (i) the wealthier (poorer) the household is, relative to other households and (ii) the more likely the household is to choose a currently desirable (undesirable) location.*

Proof The result follows from Lemmas 1-3. ■

Finally, we characterize the asset market equilibrium. The market clearing condition is

$$E[a] = \frac{M}{p} \quad (19)$$

where the left-hand side of (19) is the aggregate demand for assets, and the right hand side is the net supply, equal to real outside money. Using (4) and Definition 1, (19) can be rewritten as

$$E[n] = \frac{1}{2} + b + \frac{M}{p} \quad (20)$$

This equation states that the average, and aggregate, disposable wealth $E[n]$ is equal to the value of the housing stock $\frac{1}{2}$, inside money, captured by the borrowing limit b , and outside money $\frac{M}{p}$. In equilibrium, the value of outside money in terms of housing, $1/p$, or its reciprocal the monetary value of housing p , which also measures the monetary value of capital gains and losses, adjusts to guarantee market clearing. In light of Lemma 3, it is evident that p increases when regional shocks become larger, or less frequent. Also

notice that since real net supply of outside money $\frac{M}{p}$ cannot be negative, the equation (20) implies a maximum allowed value for the borrowing limit

$$b \leq b^{\max} \equiv E[n] - \frac{1}{2} \quad (21)$$

If this condition is not met, the model does not have a stationary equilibrium. If the borrowing limit is set in monetary terms, so that $b = B/p$, where the monetary borrowing limit B is exogenously given, the adjustment of the price level p guarantees that the real-valued borrowing limit b is never too lax, and the condition (21) always holds. In particular, with a monetary borrowing limit, the monetary value of housing is

$$p = \frac{B + M}{E[n] - \frac{1}{2}}$$

5 Residential sorting

This section studies the aggregate behavior of the economy. We begin by analyzing social welfare. Addressing this normative issue will eventually also allow us to characterize residential sorting, since in the present model high social welfare is essentially associated with location choices based on the match, rather than wealth.

Consider a household with n units of disposable wealth. In any period it chooses the desirable neighborhood if its match $\theta \geq \theta_n^*$ and otherwise goes to the less desirable neighborhood. Given this strategy, the expected utility (before the draw of θ) of the household, or alternatively the average realized utility of all households in class n , is

$$u_n = \Pr(\theta \leq \theta_n^*)\omega + \Pr(\theta \geq \theta_n^*)E[\theta \mid \theta \geq \theta_n^*] = G(\theta_n^*)\omega + v_n$$

where $v_n \equiv \int_{\theta_n^*}^{\theta_H} \theta dG(\theta)$. Aggregation then involves summing over all wealth classes. Notice that $\sum_n f_n G(\theta_n^*) = \sum_n f_n^b = \frac{1}{2}$ (by the housing market equilibrium (17)), and

overall welfare in any given period is

$$\widetilde{W} = \sum_{n=0}^{\bar{n}} f_n u_n = \frac{1}{2} \omega + W$$

where

$$W = \sum_{n=0}^{\bar{n}} f_n v_n \equiv E[\theta \mid h = 1]$$

is a measure of net welfare, reflecting the regional allocation of households. Notice that if the parameter ω , which captures the size of regional differences and regional shocks, changes, this affects net welfare W only as long as the location policies followed by the households change. This measure, $W \equiv E[\theta \mid h = 1]$, is the key when we study sorting according to the match: if the average match in the desirable locations $E[\theta \mid h = 1]$ improves, the regions or neighborhoods become increasingly stratified along the match dimension.

An alternative way to approach social welfare is to imagine that a new household enters the economy. The entrant is assigned to wealth class n with probability f_n , and its expected intertemporal prospects are then given by the value function $V(n)$. The household's prospects ex ante, i.e. before it knows its wealth, are

$$\overline{W} = \sum_{n=0}^{\bar{n}} f_n V(n) \tag{22}$$

We can also define a value function based measure of net welfare

$$\widehat{W} = \overline{W} - \frac{1}{2} \frac{\omega}{1 - \beta}$$

The appendix shows that, up to a constant multiplier, the per period utility and value

function based measures of (net) welfare are equivalent:

$$\widetilde{W} = (1 - \beta) \overline{W} \quad (23)$$

$$W = (1 - \beta) \widehat{W} \quad (24)$$

These equalities will be needed in the proof of the following proposition.

Proposition 2 *The smaller, or the more frequent, the regional shocks, the higher is the level of social (net) welfare.*

Proof It suffices to show that the results hold for the measure of net welfare. The results concerning gross welfare then follow immediately.

The required derivations are more succinct and clearer if some vector notation is introduced. Let V , v , and f be column vectors capturing the value function, the expected utility per period, and the stationary wealth distribution, respectively. As proving the result with respect to π and ω , involves the same steps, we also introduce a generic parameter ρ , where $\rho \in \{\pi, \omega\}$.

Now totally differentiating (22) yields

$$\begin{aligned} \frac{d\widehat{W}}{d\rho} &= \frac{df'}{d\rho} \left(V - \frac{\omega}{1-\beta} \right) + f' \frac{d}{d\rho} \left(V - \frac{\omega}{1-\beta} \right) \\ &= \frac{df'}{d\rho} V + f' \frac{\partial}{\partial \rho} \left(V - \frac{\omega}{1-\beta} \right), \end{aligned} \quad (25)$$

where the second equality is obtained by using (i) the fact that (as $f'\omega = \omega$) $\frac{df'}{d\rho}\omega = 0$ and (ii) the envelope theorem: as the threshold θ_n^* is chosen optimally in every wealth class $n \geq 1$, the indirect effect on the value function can be ignored. Next we use the identity (24) to show that also the weighted sum of direct effects $f' \frac{\partial}{\partial \rho} \left(V - \frac{\omega}{1-\beta} \right)$ vanishes:

$$f' \frac{\partial}{\partial \rho} \left(V - \frac{\omega}{1-\beta} \right) = \frac{\partial}{\partial \rho} \left[f' \left(V - \frac{\omega}{1-\beta} \right) \right] = (1 - \beta)^{-1} \frac{\partial(f'v)}{\partial \rho} = 0$$

The first equality exploits the fact that the stationary distribution f depends on the parameters π, ω only indirectly, through the choice of policy and the second equality uses (24). The final equality follows from the observation that expected net utility in a given

period (v) does not depend directly on ρ .

Thus only the effect through the stationary wealth distribution remains. By Lemma 3 we know that the distribution shifts to the right, towards higher wealth classes, when π or ω increases. As the value function V is increasing in n , this shift in the stationary distribution translates into higher aggregate net welfare \widehat{W} :

$$\frac{d\widehat{W}}{d\rho} = \frac{df'}{d\rho} V \geq 0$$

Notice that this also implies that the average match in the desirable location improves

$$\frac{dE[\theta | h = 1]}{d\rho} > 0$$

■

The following remark may help in understanding what the proposition means in more concrete terms.

Remark 3 *Social welfare is higher, when capital gains and losses made in the housing market are small, compared to the average wealth in the economy, than when these gains and losses are large.*

Proof This finding follows by combining Proposition 2 and Remark 1. ■

We try to interpret these results with the help of the (preliminary) findings and remarks made earlier in this paper. In what follows we will focus on the case with large and/or infrequent regional shocks, resulting in low welfare. The opposite situation, with small and/or frequent shocks, and high welfare, can be interpreted simply by inverting the arguments.

The inefficiencies arising when regional shocks are large or infrequent can be traced back to the household's problem, studied in Section 3. When ω and π are small, Lemmas 1 and 2 indicate that when households choose their location, the forward-looking investment motive tends to be weak, compared to the consumption motive. This is already a sign that the location choices made in the market equilibrium may substantially deviate from the socially optimal rule, since the investment motive provides the households the incentives

not to choose a desirable location, when the match is not so good. Next, the weakness of the investment motive, and the resulting location choices, means that typically the households do not accumulate lots of financial assets; see Lemma 3. This then has its implications on the wealth distribution; in particular, the borrowing constrained group tends to be rather large. This is a further sign of inefficiency, as borrowing constrained households can only live in undesirable, and less expensive, neighborhoods; they cannot choose their location based on the current match.

Proposition 2 also allow us to characterize how heterogeneity within and between locations, in terms of the match, changes, when π or ω increases.

Proposition 3 *The smaller, or the more frequent the regional shocks are, (i) the more distinct the locations are from each other in terms of the match and (ii) the less heterogeneity in terms of the match there is within locations.*

Proof (i) By Proposition 2, the average match in the desirable location $E[\theta | h = 1]$ improves when π or ω increases. As the average match in the whole population, $E[\theta] \equiv \bar{\theta}$, is constant, the expected value of θ in the undesirable location $E[\theta | h = 0]$ must decrease. But then it follows immediately that the between-groups variance

$$Var(E[\theta | h]) = \frac{1}{2} (E[\theta | h = 0] - \bar{\theta})^2 + \frac{1}{2} (E[\theta | h = 1] - \bar{\theta})^2$$

increases. ■

(ii) The overall variance of the match in the economy $Var(\theta) \equiv \sigma^2$ can be decomposed into within-groups variance and between-groups variance, $E[Var(\theta | h)] + Var(E[\theta | h]) = \sigma^2$. As the economy-wide variance σ^2 is constant, the within-groups component $E[Var(\theta | h)]$ necessarily decreases, if the between-groups component $Var(E[\theta | h])$ increases. ■

When the Proposition 2 was interpreted, it became apparent that whenever social welfare is low, and there is little residential sorting according to the match, the investment motive plays a minor role in location choice and the households' incentives to accumulate financial assets are weak. Then capital gains and losses, as well as housing price differences

between desirable and undesirable locations tend to be large, compared to average wealth in the economy. Given this background, it should hardly be surprising that, when we next turn to residential sorting along the wealth dimension, we will see a mirror image, compared to sorting according to the match.

The household possesses both housing wealth and financial capital. Then, in principle the neighborhoods could differ in terms of both wealth categories. However, equations (16) indicate that in the long-run equilibrium the distribution of financial assets is identical in both types of location. Then given that $E[a | h = 1] = E[a | h = 0]$, interregional wealth differences derive entirely from different housing values

$$E[n | h = 1] - E[n | h = 0] = E[h | h = 1] - E[h | h = 0] = 1$$

and proportioning regional wealth differences to the average wealth gives

$$\frac{E[n | h = 1] - E[n | h = 0]}{E[n]} = 1/E[n] \quad (26)$$

Proposition 4 *The larger, or the more infrequent / persistent, the regional shocks, the more the locations differ from each other in terms of wealth.*

Proof The result follows from equation (26) and Lemma 3. ■

Finally, combining Propositions 3 and 4, and Remark 1 allows us to characterize the relation between the size of capital gains and losses made in the housing market, and the form of residential sorting.

Proposition 5 *Small capital gains and losses in the housing market are associated with residential sorting according to the match. Large capital gains and losses are associated with residential sorting according to wealth.*

Residential sorting according to the match and according to wealth tend to produce different neighborhoods and regions. Remember that a household's match may be interpreted as reflecting certain socioeconomic characteristics of the household, or the current life situation of its members. Thus if two households have a similar current match, one

might think that the households resemble each other in terms of, say, size, or that the members of the two households are undergoing a similar episode of life. By contrast, (in this model) a household's current wealth essentially just indicates whether the household has been lucky or unlucky in the housing market. Two households with the same wealth level may be very different in other respects. Proposition 5 then indicates that when capital gains and losses made in the housing market are small, neighborhoods or regions tend to be internally homogenous along the "relevant" dimension. Large capital gains and losses muddle this picture: within the same region or neighborhood there may reside households, which have little in common, apart from the value of their home.

6 Rental housing

The model also allows us to study rental housing, and to contrast the rental arrangement to owner-occupation.

Assume that the whole housing stock is managed by real estate companies, owned by the households. We normalize the price system, so that the per period rent is 2 in desirable locations and 0 in less desirable locations. The companies collect the rental payments and distribute them back to the households, so that in each period each household receives a revenue stream 1. Then in each period all households residing in a desirable location pay 1 unit more than they earn, while those residing in a less desirable location earn one unit more than they pay. Then the disposable wealth of a household evolves according to the law of motion

$$n_{t+1} = n_t + (1 - 2h_t) \tag{27}$$

It is easy to notice that (27) is a special case of the wealth dynamics under owner-occupation (7): (27) corresponds to the situation, where $\pi = 1$, and $s_t = 1$ in every period. Rental payments and revenues are equivalent to capital losses and gains which occur with certainty. More generally, the incentives that the renters face when choosing their housing location correspond to the incentives that owner-occupiers face, if regional

shocks take place with probability one in every period. As a consequence, also the pattern of residential sorting is equivalent to that in an owner-occupation economy, with deterministic shocks. Then we can state the following result.

Proposition 6 *Under rental housing, there is more residential sorting according to the match, and less sorting according to wealth, than under owner-occupation.*

Proof Rental housing corresponds to owner-occupation in the special case, with $\pi = 1$. Then the result follows from Propositions 3 and 4. ■

In the present model, location choices which are based on the match, rather than wealth, result in high social welfare. Thus Proposition 5 suggests, that rental housing might be superior to owner-occupation, in terms aggregate utility. However, it is well-known that, compared to owner-occupation, rental relationships may involve various agency problems, which can then lower the quality of housing services. To provide a simple welfare comparison between the modes of housing tenure, we model these welfare costs by assuming that under rental housing, a household's per period utility is $\theta - \alpha$ in a desirable location and $\omega - \alpha$ in a less desirable location, where $\alpha \geq 0$. Notice that the size of agency costs is the same in all locations. Then while agency costs lower the households' well-being, they do not affect location choices.

It is easy to see that per period social (net) welfare under rental housing (W^R) is given by

$$W^R = W_{\pi=1} - \alpha$$

where $W_{\pi=1}$ is the measure of net social welfare under owner-occupation, when $\pi = 1$.

Proposition 7 *Social welfare tends to be higher (lower) under rental housing than under owner-occupation, when regional shocks are infrequent (frequent), and agency costs in rental housing are small (large).*

Proof According to Proposition 2, social welfare under owner-occupation increases when regional shocks become more frequent, and π increases. Under rental housing, the

households do not face capital gains and losses and their incentives do not depend on the frequency of the regional shocks. Thus the "true" value of π does not affect social welfare under rental housing. ■

7 Conclusions

In typical models of residential sorting, wealth affects a household's choice of location and housing quality, but there is no opposite connection from housing and location choices to wealth. However, empirical evidence indicates that the housing market often plays a major role in shaping the wealth distribution, in determining an individual household's position in the distribution, and in influencing how the household's wealth evolves over time.

In this paper, we developed a dynamic multi-region model, where (i) a household's current location choice depends on its current wealth, and its current "match" (where the match may reflect various demographic or socioeconomic characteristics of the household), and (ii) the current wealth depends on the household's past fortunes in the housing market, and its past location choices. In the long-run equilibrium of the model, the size of capital gains and losses, the wealth distribution and residential sorting are all determined endogenously.

Essentially, our results relate the size of capital gains and losses, made in the housing market, to the form of residential sorting, and to social welfare. We show that small capital gains and losses tend to be associated with residential sorting according to the match, while large gains and losses are associated with residential sorting according to wealth. Social welfare decreases, when the households face larger housing price fluctuations. In addition, we compare owner-occupation and rental housing. Under rental housing there is more residential sorting according to the match, and less sorting according to wealth, than under owner-occupation.

Appendix

Proof of Lemmas 1 and 2

In this appendix we show that the value function is concave. We also demonstrate that the θ_n^* -schedule shifts up when π or ω increases. To establish these result, first notice that a household's strategy, telling how it chooses its location in each wealth class, essentially involves finding an optimal threshold value θ_n^* for each $n \geq 1$. As there is a one-to-one mapping between the threshold θ_n^* , and the corresponding value of the cumulative distribution function $G(\theta_n^*)$, also $x_n \equiv G(\theta_n^*)$ can be equivalently used as the choice variable. Given this reinterpretation of the problem, and using matrix notation, the Bellman equations (9) can be reexpressed in the following form

$$V = \max_{\{x_n\}} u + \beta [(1 - \pi) I + \pi P] V \quad (28)$$

for $n \geq 1$ (and $x_0 = 1$) where V is the value function, stacked as a column vector, u is the column vector of expected immediate utility, with elements

$$u_n(x_n) \equiv x_n \omega + \int_{x_n}^1 G^{-1}(x) dx$$

and P is a transition matrix, with elements

$$P_{i,j} = \begin{cases} 1 - x_i & \text{if } j = i - 1 \\ x_i & \text{if } j = i + 1 \\ 0 & \text{otherwise} \end{cases} \quad i, j \in \{0, \dots, \bar{n}\}$$

Notice that $\frac{du_n^2}{dx_n^2} = -\frac{1}{G'(\theta_n^*)} < 0$. Thus (28) defines a maximization problem with a concave objective function and linear constraints. As a consequence the value function $V(n)$ is concave.

Also the first order conditions can be rephrased using matrix notation

$$\theta^* = \omega \mathbf{1} + \pi \beta D V \quad (29)$$

where $\theta^* = (\theta_1^*, \dots, \theta_{\bar{n}}^*)'$ is the vector of threshold values¹³, $\mathbf{1} = (1, \dots, 1)'$ and D is the difference matrix, with elements

$$D_{i,j} = \begin{cases} 1 & \text{if } j = i + 1 \\ -1 & \text{if } j = i - 1 \\ 0 & \text{otherwise} \end{cases} \quad i \in \{1, \dots, \bar{n}\}, j \in \{0, \dots, \bar{n}\}$$

Next we want to study what happens to optimal location choices, when the parameters π and ω change. Differentiating the right hand side of (29) with respect to π yields

$$\frac{d(\pi\beta DV)}{d\pi} = \beta D \left(V + \pi \frac{dV}{d\pi} \right) \quad (30)$$

Then differentiating the Bellman equation (28) with respect to π , and using the envelope theorem, yields

$$\frac{dV}{d\pi} = \frac{\partial V}{\partial \pi} = \frac{\delta}{\pi} (I - \delta P)^{-1} (P - I) V \quad (31)$$

where $\delta \equiv \frac{\pi\beta}{1-\beta(1-\pi)}$, $\delta \in [0, 1)$ is the uncertainty adjusted discount rate. Finally we plug (31) into (30):

$$\frac{d(\pi\beta DV)}{d\pi} = \beta(1 - \delta) D (I - \delta P)^{-1} V > 0 \quad (32)$$

In signing the expression, the following facts have been used: (i) The value function V is increasing n . (ii) Then also $(I - \delta P)^{-1} V = \sum_{i=0}^{\infty} (\delta P)^i V$ is increasing in n . To see this, notice that $(1 - \delta P)^{-1} V$ is the value of a Markov process, with transition matrix P and immediate gain in state n given by $V(n)$. As this immediate gain increases with n , the expected present value of the program also increases. (iii) When we premultiply an increasing vector by the difference matrix D , the result is positive.

¹³Notice that θ_0^* cannot be freely chosen, as the agents are liquidity constrained.

Now we can get the desired result:

$$\frac{d\theta^*}{d\pi} = \beta(1 - \delta)D(I - \delta P)^{-1}V > 0$$

In words, when the probability of capital gains and losses increases, the θ_n^* -schedule shifts upwards.

Next we turn to the parameter ω . The derivative of the right hand side of (29) with respect to ω is

$$\mathbf{1} + \pi\beta D \frac{dV}{d\omega}$$

and differentiating the Bellman equation (28) with respect to ω yields

$$\frac{dV}{d\omega} = \frac{\delta}{\pi\beta} (I - \delta P)^{-1} x$$

where $x \equiv (x_0, \dots, x_{\bar{n}})$. Then

$$\frac{d\theta^*}{d\omega} = \mathbf{1} + \pi\beta D \frac{dV}{d\omega} = \mathbf{1} + \delta D (I - \delta P)^{-1} x$$

As x is a decreasing sequence, the elements of the vector $\delta D (I - \delta P)^{-1} x$ are negative. Next we want to establish that these terms are smaller than one in absolute value. To do so, we adopt the notation $k \equiv \delta D (I - \delta P)^{-1} x$. The elements of the vector k satisfy the recursive equations

$$k(n) = x_{n+1} - x_{n-1} + \delta [x_{n+1}k(n+1) + (1 - x_{n-1})k(n-1)] \quad (33)$$

Assume that

$$k(m) = \min_n k$$

Then in particular, $k(m+1), k(m-1) \geq k(m)$, and it is immediately clear that

$$k(m) \geq x_{m+1} - x_{m-1} + \delta [x_{m+1}k(m) + (1 - x_{m-1})k(m)]$$

or

$$k(m) \geq \tilde{k} \equiv \frac{\Delta x}{1 - \delta(1 + \Delta x)}$$

where $\Delta x \equiv x_{m+1} - x_{m-1}$. As x is a non-increasing sequence, Δx can take values over the interval $[-1, 0]$. Differentiating \tilde{k} with respect to Δx yields $\frac{d\tilde{k}}{d\Delta x} = \frac{1-\delta}{(1-\delta(1+\Delta x))^2} > 0$, and $\tilde{k} \geq -1$. Thus $\min_n k \geq \tilde{k} \geq -1$, and

$$\frac{d\theta^*}{d\omega} = \mathbf{1} + \delta D(I - \delta P)^{-1} x \geq (1 - \delta) \mathbf{1} > \mathbf{0}$$

The derivation of equations (23) and (24)

In this appendix we derive the equations (23) and (24).

First, with a given moving policy θ^* , the Bellman equations (28) can be reexpressed using matrix algebra, and the notation introduced in the previous section:

$$V = (1 - \delta) \frac{u}{1 - \beta} + \delta P V \tag{34}$$

Now solving (34) for u we get

$$u = \frac{1 - \beta}{1 - \delta} (I - \delta P) V \tag{35}$$

On the other hand the stationary distribution f can be solved from the equation

$$f'(I - P) = 0 \Leftrightarrow (I - P')f = 0 \tag{36}$$

which determines f as the eigenvector associated to the unit eigenvalue of P' . With these

preliminaries, we can now derive the equations (24):

$$\begin{aligned}\widetilde{W} &= f'u = f'\frac{1-\beta}{1-\delta}(I - \delta P)V = \frac{1-\beta}{1-\delta}f'(I - P + (1-\delta)P)V \\ &= (1-\beta)f'PV = (1-\beta)f'V = (1-\beta)\overline{W}\end{aligned}\tag{37}$$

The second equality follows from (35), elementary manipulations lead to the third equality, the fourth and the fifth equality then use (36).

Also

$$W = f'v = f'u - f'x\omega = f'u - \frac{1}{2}\omega = (1-\beta)\left(\overline{W} - \frac{1}{2}\frac{\omega}{1-\beta}\right) = (1-\beta)\widehat{W}$$

where the second equality follows from the fact that $v = u - x\omega$, the third equality uses the housing market equilibrium $f'x = \frac{1}{2}$, and the fourth equality follows from (37).

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