



Helsinki
Center
of
Economic
Research

Discussion Papers

Learning in a Model of Exit

Pauli Murto

Helsinki School of Economics and HECER

and

Juuso Välimäki

Helsinki School of Economics, University of Southampton,
CEPR and HECER

Discussion Paper No. 110
August 2006

ISSN 1795-0562

HECER – Helsinki Center of Economic Research, P.O. Box 17 (Arkadiankatu 7), FI-00014
University of Helsinki, FINLAND, Tel +358-9-191-28780, Fax +358-9-191-28781,
E-mail info-hecer@helsinki.fi, Internet www.hecer.fi

Learning in a Model of Exit*

Abstract

We analyze information aggregation in a stopping game with uncertain common payoffs. Players learn from their own private experiences as well as by observing the actions of other players. We show that when the number of players is large, information aggregation is efficient in the long-run sense. By this we mean that almost all players take the efficient action in the long run. At the same time, information is not aggregated well in the ex ante sense as the payoffs of all players are well below those attainable with information sharing.

JEL Classification: D82, D83

Keywords: Experimentation, Observational Learning, Exit, Timing Games.

Pauli Murto

Department of Economics
Helsinki School of Economics
P.O. Box 1210
FI-00101 Helsinki
FINLAND

e-mail: pauli.murto@hse.fi

Juuso Välimäki

Department of Economics
Helsinki School of Economics
P.O. Box 1210
FI-00101 Helsinki
FINLAND

e-mail: juuso.valimaki@hse.fi

*We would like to thank numerous seminar audiences and, in particular, Dirk Bergemann, Hikmet Gunay, Godfrey Keller, Elan Pavlov and Peter Sorensen for useful comments. This paper supersedes HECER Working Paper No. 70 'Experimentation and Observational Learning in a Market with Exit'.

1 Introduction

In this paper, we analyze the informational performance of a simple stopping game where players collect private information during the play of the game and also observe the actions of other players. For concreteness, we consider a market whose viability is initially uncertain. A number of firms have entered, and they observe new information as long as they are active in the market. At each instant, the firms decide whether to exit. In addition to their direct observations about the state of the market, they observe the behavior of the other firms. Each decision by a currently active firm creates an informational externality. By exiting, a firm delivers bad news to the remaining firms. Staying in the market, on the other hand, is good news to the others. We assume that exit is irreversible in the sense that once a firm exits the market, it is not possible to re-enter. This informational structure is in line with the recent literature on observational learning models where agents infer each others' information from the actions taken by others. In the conclusion, we outline an alternative interpretation for the model as one with irreversible investments.

Our main result is that in the sense of long-run allocation of firms to the market, the model aggregates information efficiently if there are many firms. By this we mean that almost all firms stay in a good market and all firms exit eventually from a bad market in all equilibria of the game. This is in contrast to the previous literature on observational learning including the herding models discussed below. At the same time the sum of equilibrium payoffs is well below the efficient level. We show that the unique symmetric equilibrium payoff provides a lower bound for Nash equilibrium payoffs. We also show that the unique (asymmetric) pure strategy equilibrium of the game yields the highest sum of payoffs within the class of Nash equilibria.

We model the game as a discrete time, infinite horizon stopping game. Our main results are derived for the case where the time interval between consecutive periods is arbitrarily short. When the market is good, each firm meets a customer with probability λ per unit of time, but when the market is bad, there are no customers. The arrivals of customers are assumed to be independent across the firms and across

time periods (conditional on the state of the market). Furthermore, we assume that if the market is known to be good, then it is in each firm's best interest to stay in the market. Under these assumptions, not seeing customers is bad news to each firm. Other things equal, firms become more pessimistic about the market, and eventually they exit. At the same time, the decisions of other firms convey information and this gives the uncertain firms an incentive to stay in the market. The equilibria in the model strike a balance between the bad news from own experiences and good news from observations on others.

Most equilibria of our model use mixed strategies. To see this, consider a symmetric equilibrium where all firms take the same decisions (conditional on having the same information). If an individual firm exits with probability 1 when it has seen no customers, other firms learn its private history in a single period. If the time interval between periods is short, the informational gains outweigh the losses from waiting and it is optimal for all the other firms to stay. Hence there cannot be symmetric equilibria in pure strategies. In the unique symmetric equilibrium of our model, the firms exit with probabilities that keep them indifferent between exiting and staying. As long as no firm exits, these probabilities are small. However, exit by any firm triggers an immediate stronger randomization from the others. If no other firm leaves, play resumes to the mode of small exit probabilities. If any other firms leave, there is a need for an even stronger randomization, and consequently there is a possibility that the market collapses in the sense that most or all of the remaining firms exit. Hence, the equilibrium path exhibits phases of inaction during which firms learn only little from each other, and randomly arriving waves of exit during which the firms learn a lot from each other.

In obtaining the limiting results for the case where the number of firms grows large, a key role is played by the relative probabilities of an exit wave ending up in market collapse and returning to the phase of inaction with fewer firms. We show that when the state of the market is good, the probability of a market collapse goes to zero when the number of firms in the market grows large. It is clear that a bad market must eventually collapse. Note that when the number of firms is increased towards

infinity, the noise in the aggregate information held by the firms washes out. Yet, the aggregate behavior of the firms conditional on the market state remains random in this limit; exit waves arrive randomly and each such wave results in market collapse with a non-trivial probability if the market is bad.

We have assumed somewhat unrealistically that the profitability of the market does not depend on the number of active firms. The reason for this assumption is to maintain comparability with other models of observational learning with pure informational externalities. We verify that the main qualitative features of our model remain valid in a model where the probability of receiving a customer in any period depends negatively on the number of active firms as long as a good market is profitable even in the case that no firms exit. If this is not the case, then the analysis is complicated by considerations reminiscent of war of attrition. We also verify the robustness of our results to two other extensions: relaxation of the extreme signal structure according to which the firms become fully informed upon seeing a customer, and introduction of private information on the opportunity costs of staying in the market.

This paper is related to two strands of literature. The literature on herding and observational learning has studied the informational performance of games where players have private information at the beginning of the game. Many of these models also assume an exogenously given order of moves for the players, e.g. Banerjee (1992), Bikhchandani, Hirshleifer & Welch (1992), and Smith & Sorensen (2000). This latter assumption has been relaxed by a number of papers. Among those, the most closely related to ours is Chamley & Gale (1994).¹ In that paper a number of firms are contemplating entry into an industry. Each firm has private information about the profitability of the market and the resulting game is a waiting game that mirrors our setting. Chamley and Gale show that when actions can be taken at arbitrarily short intervals, the symmetric equilibrium of the game exhibits herding with positive probability: the firms' beliefs may get trapped in an inaction region even if taking the action would be optimal. In our model the additional information that arrives

¹See also a more general model ?. An early contribution along these lines is also Mariotti (1992).

during the game prevents the beliefs from getting trapped. This leads to different properties of information aggregation as best seen by comparing the two models in the limit of short periods and large number of players. In Chamley and Gale information aggregates quickly but incompletely (leading to an incorrect herd at a positive probability), whereas in our model information aggregates slowly but completely (in the sense that almost all players eventually choose the correct action). Other papers that have studied the effects of endogenous timing on observational learning include Gul & Lundholm (1995), Zhang (1992) and Aoyagi (1998*a*). Their main emphasis is on determining whether better informed agents move first.

Caplin & Leahy (1994) is the paper closest to ours in the sense of having both endogenous timing and arrival of private information. While the motivation in that paper is quite close to ours, there is a difference in the modeling strategies that turns out to be important. In contrast to our game that has a finite number of firms, Caplin and Leahy assume a continuum of firms from the beginning. As a result, they are forced to pose specific restrictions on their model parameters to achieve existence of equilibrium. They correctly point out that this potential non-existence of equilibrium is an artifact of their assumption of a continuum of agents, but one may ask whether some of the very properties of their equilibrium might be artifacts of this assumption as well. Our model indicates that working with a finite number of firms not only solves the existence problem, but more importantly, leads to a different pattern of information aggregation. In our model information is revealed gradually over time even in the limit where the number of firms goes to infinity, whereas in Caplin and Leahy all uncertainty is resolved at the first instant of public information revelation.²

The second strand of literature that is directly relevant to our paper is the litera-

²At a late stage in writing this paper, we became aware of a paper by Rosenberg, Solan & Vieille (2005) that also analyzes endogenous timing of irreversible action in a game with private information arriving over time. Their informational assumptions on signals that are observed at each stage are different from ours and as a result both the analysis and the results in the two papers are quite different. In particular, they have signals that give rise to unbounded variation in beliefs, which means that the true state of the world is revealed quickly as the number of firms is increased. Furthermore, they do not analyze the case where the time interval between periods is small.

ture on strategic experimentation. We have borrowed the analytical framework from a recent paper Keller, Rady & Cripps (2005). Their paper explores the Markov perfect equilibria of a model where observations by all of the agents are publicly observable. As a result, the motivation as well as the analysis of the two models are very different in the end. Our model also differs from that in Keller et al. in that we assume exit to be irreversible. The reason for this assumption is that in a continuous time model with reversible entry and exit, the firms would find it easy to communicate to each other their observations through an exit followed by quick re-entry. In order to respect our assumption of imperfect observability, we assume exit decisions to be irreversible. This property also distinguishes our model from Aoyagi (1998*b*), which studies multi-armed bandits without publicly observed outcomes and asks whether agents with the possibility to repeatedly choose between different actions eventually converge to the same action. On the other hand, Décamps & Mariotti (2004) and Moscarini & Squintani (2004) are stopping games with experimentation and irreversible actions, but in contrast to our model they have publicly observed outcomes.

The paper is organized as follows. Section 2 sets up the discrete time model. Section 3 provides the analysis of the symmetric and asymmetric equilibria of the model. In section 4, we prove our main theorem that in all equilibria of the exit game, almost all firms stay in the market if and only if the market is good when the number of firms is large and the time interval between periods is small. In Section 5, we compute the symmetric equilibrium explicitly in the limiting continuous time version of the model. In Section 6 we verify the robustness of our main conclusions to a number of extensions. Section 7 concludes.

2 Model

In this section we present the model in discrete time. Time periods are denoted by $t = 0, 1, \dots, \infty$. We denote by a constant $\Delta t > 0$ the time interval between any two consecutive periods t and $t + 1$. The discount factor between two periods is

$$\delta = \frac{1}{1 + r\Delta t},$$

where r is the discount rate. Since it is not our purpose to analyze the effect of observation lags, we are ultimately interested in the limit where the firms can react to the observed actions instantaneously, which we obtain by letting $\Delta t \rightarrow 0$.

At the beginning of the game, N risk neutral firms have entered the market whose true profitability is uncertain.³ We assume for simplicity that the market is either good or bad and use notation $M = g$ and $M = b$ to refer to these two possibilities. Define $P_g(\cdot) \equiv P(\cdot | M = g)$ and $P_b(\cdot) \equiv P(\cdot | M = b)$ to refer to probabilities of various events conditional on market being good and bad, respectively.

Initially all firms are equally optimistic about the state of the market. The common prior probability that the market is good is denoted by p^0 . If the market is good, a customer arrives at a firm with a constant probability $\lambda \cdot \Delta t$ within each period. The value of each customer to the firm is v . If the market is bad, no customer will ever arrive. We say that a firm is *informed* if it has seen a customer, otherwise a firm is *uninformed*. The state of the market is the same for all firms, i.e. we have a setting with symmetric payoffs and common values. Conditional on the market state, the arrivals of customers are independent across firms.

At the beginning of each period, all active firms make a binary decision: either stay in the market or leave. Leaving is costless but irreversible. Once the firm has exited, it will never again face any costs or revenues. If the firm stays, it pays the per period (opportunity) cost $c \cdot \Delta t$, observes a signal indicating either an arrival or no arrival of a customer, and moves to the next period. We assume that $c < \lambda v$, which means that an informed firm will never exit regardless of what the other firms do. Within each period the firms act simultaneously, but they know each other's previous actions. However, they do not observe the arrivals of customers at other firms, and thus they do not know whether other firms are informed or uninformed.⁴ Note that new

³It makes no difference to the model that follows whether the firms have entered subject to a zero profit condition or not.

⁴Since exit is irreversible, we do not need to worry about the information of those firms that have

information arrives to the firms through two channels: their own market experience and observations on other firms' behavior. In the terminology of learning models, each firm engages simultaneously in experimentation and observational learning.

The history of firm i consists of the private history recording its own market experience (i.e. the arrivals of its customers), and the public history recording the actions of all the firms. However, since observing a customer reveals fully that the market is good, the only thing that matters in each firm's own market experience is whether it has seen at least one customer. As it is a strictly dominant strategy for any firm that has observed a customer to stay in the market, we simplify the analysis by *postulating* that these firms stay in the market. This has no effect on the analysis, but it allows us to restrict our attention to uninformed firms only. For those firms, the only relevant history is the public history, and from now on we call this simply the *history*. We denote the history in period t by h^t and define it recursively as follows:

$$\begin{aligned} h^0 &= \emptyset, \\ h^t &= h^{t-1} \cup a^{t-1} \quad \forall t \in \{1, 2, \dots\}, \end{aligned}$$

where $a^t = (a_1^t, \dots, a_N^t)$ is a vector where each $a_i^t \in \{0, 1\}$ denotes an indicator for i staying in the market at period t . Denote by H^t the set of all possible histories up to t and let $H = \bigcup_{t=0}^{\infty} H^t$. Since exit is irreversible, $a_i^t = 0$ implies that $a_i^{t'} = 0$ for all $t' > t$ in all elements of H^t . Denote by $H_i \equiv \{h^t \in H \mid a_i^{t-1} = 1\}$ the set of histories, in which i has not yet left the market. Denote by $A(h^t) \equiv \{i \in \{1, \dots, N\} \mid h^t \in H_i\}$ the set of firms that remain in the market at the beginning of period t after history h^t and by $n(h^t)$ the number of such firms.

A strategy for an uninformed firm i is a mapping

$$\sigma_i : H_i \rightarrow [0, 1]$$

already left the market. Hence, when we refer to *informed* and *uninformed* firms, we only mean those firms that are still active.

that maps all histories where i is still active to a probability of exiting the market. The strategy profile is $\sigma = (\sigma_1, \dots, \sigma_N)$.

Active firms learn from each other through the following mechanism. If a firm exits, the other firms learn for sure that this firm has not seen a customer. If a firm stays, the other firms become somewhat more convinced that this firm has seen a customer.

As the game proceeds, the firms update their probability assessments about the state of the market, and also about whether the other firms are informed or not. Given a history h^t and a strategy profile σ , firm i that has not observed a customer yet forms a probability assessment that the market is good by Bayes' rule. We denote this belief of an uninformed firm by $p_i(h^t; \sigma)$.⁵ Note that different uninformed firms may have different beliefs after the same public history, because their strategies may be different and thus reveal different information to each other. On the other hand, firms also update their probability assessments about whether a particular firm is informed or not. We denote by $q_i(h^t; \sigma)$ the probability assessment calculated by others that firm i has seen a customer after history h^t , conditional on that the market is good. Since this conditional probability is based only on the past behavior of this particular firm, we may equivalently think that $q_i(h^t; \sigma)$ is the probability assessment made by a Bayesian outside observer. Note an important difference between $p_i(h^t; \sigma)$ and $q_i(h^t; \sigma)$: the former is the belief held by i on the common state of the market, while the latter is the commonly held belief (or equivalently, a belief held by an outside observer) on the characteristic specific to i (i.e. whether i has seen a customer).

Note also that there are histories that are inconsistent with some strategy profiles, making Bayes' rule inapplicable. In particular, assume that at history h^t some firm j exits in period t even if this should not happen with a positive probability according to σ . Then we may simply assume that all remaining firms update their beliefs to a level that would prevail if firm j did not exist in the first place, and then continue the subgame with one less firm present leaving firm j out in all subsequent belief updates. This arbitrary assumption concerning off-equilibrium beliefs has no effect

⁵For an informed firm the probability assessment that the market is good is trivially equal to 1.

on any results, but ensures that all equilibria that we will consider are Perfect Bayesian Equilibria.

The payoff of a firm is the expected discounted sum of future cash flows as estimated by each firm on the basis of its own market experience, observations of other firms' behavior, and initial prior probability p^0 . Denote by $V_i(h^t; \sigma)$ the payoff of an uninformed firm i after history h^t and with profile σ . An informed firm will stay for ever, and its payoff is easy to calculate:

$$V^+ = \frac{(\lambda v - c) \Delta t}{1 - \frac{1}{1+r\Delta t}} = \frac{(1 + r\Delta t)(\lambda v - c)}{r}.$$

In Sections 3 and 4, we analyze the equilibria of the model formally. A reader who wants to get an intuitive characterization first may want to go directly to Section 5.

3 Equilibrium

As a useful starting point, consider a monopoly firm that can only learn from its own market experiments. This firm faces an optimal stopping problem, where it decides whether to stay for at least one more period or to exit permanently. Denote by p the current probability assessment that the market is good if the firm has not seen a customer yet. If the firm stays for a period of length Δt , but still receives no customer, the new posterior $p + \Delta p$ is obtained by Bayes' rule:

$$p + \Delta p = \frac{p(1 - \lambda\Delta t)}{p(1 - \lambda\Delta t) + 1 - p} = \frac{p(1 - \lambda\Delta t)}{1 - p\lambda\Delta t} = \frac{1 - \lambda\Delta t}{\frac{1}{p} - \lambda\Delta t}. \quad (1)$$

Consider next the monopoly value function $V_m(p)$. If the firm exits, the stopping value is 0. On the other hand, if the firm stays, it receives a customer with probability $p\lambda\Delta t$ in which case p jumps to 1 and the firm's value jumps to $V_m(1) = V^+ = \frac{(1+r\Delta t)(\lambda v - c)}{r}$. If there is no customer, p falls to $p + \Delta p$. Bellman's equation can thus be written as:

$$V_m(p) = \max \left[0 ; -c\Delta t + pv\lambda\Delta t + \frac{1}{1+r\Delta t} \left\{ p\lambda\Delta t \left(\frac{(1+r\Delta t)(\lambda v - c)}{r} \right) + (1 - p\lambda\Delta t) V_m \left(\frac{1 - \lambda\Delta t}{\frac{1}{p} - \lambda\Delta t} \right) \right\} \right]. \quad (2)$$

It is well known that the solution to this type of a stopping problem can be written as a threshold level p^* such that it is optimal to stop when $p < p^*$, while it is optimal to stay otherwise. Under the assumptions of the model, it must be that $0 < p^* < 1$. Furthermore, $V_m(p)$ must be strictly increasing and convex when $p > p^*$, while it must be pasted to stopping value 0 at $p = p^*$. We will see that the monopoly threshold p^* plays a crucial role also in the model with many firms. Denote $t^* = \min \{t | p_m^t < p^*\}$.

Let us now consider the model with N firms. We will consider symmetric and asymmetric equilibria separately, but we start with a result that is valid in all equilibria. Since the model has no payoff externalities, it is easy to see that a firm can always guarantee at least the payoff of a monopoly firm in equilibrium. Hence it follows immediately that no firm exits earlier than the monopoly firm would. Proposition 1 below states this, but shows also that there cannot be equilibria, where *all* firms earn a higher payoff than the monopoly firm.

Proposition 1 *Let σ be an equilibrium profile. After any h^t , it must be that $V_i(h^t; \sigma) \geq V_m(p_i(h^t; \sigma))$ for all $i \in A(h^t)$ and $V_i(h^t; \sigma) = V_m(p_i(h^t; \sigma))$ for some $i \in A(h^t)$. Further, whenever $p_i(h^t, \sigma) > p^*$, it must be that $\sigma_i(h^t) = 0$.*

Proof. *In the Appendix.* ■

Since $p_i(h^t, \sigma) > p^*$ for all $t < t^*$, we have:

Remark 1 *In any equilibrium, all firms stay with probability one in all periods $t < t^*$.*

This means that there can never be any information sharing before time t^* , because the firms reveal information only through exit.

3.1 Symmetric Equilibrium

In this section we consider equilibria in symmetric strategy profiles. A profile σ is symmetric if $\sigma_i(h^t) = \sigma_j(h^t)$ for all i and j and for all h^t . When σ is symmetric, all uninformed firms update their beliefs in the same way, and hence they all share a common probability $p(h^t; \sigma)$ that the state of the market is g . When analyzing symmetric equilibria, we may simply use $p \in (0, 1)$ to denote this common belief. Similarly, the probability that a given firm has seen a customer conditional on the market being good, as estimated by a Bayesian observer, is the same for all firms, and we may use $q \in (0, 1)$ to denote this.

Note that all uninformed firms have also the same (expected) payoff in the symmetric equilibrium. It follows from Proposition 1 that this common payoff must be the same as that of a monopoly firm. Hence, after an arbitrary history h^t , any firm would be just as well off if it decided to ignore all observations of the other firms from time t onwards. This means that in a symmetric equilibrium no firm is able to benefit from the information that the firms reveal to each other.⁶ This observation facilitates our analysis in the remainder of this section.

We discuss next the inference from other firms' actions when the firms use arbitrary symmetric strategies. Consider a period where n firms remain in the market and play a strategy according to which each of them exits with probability $\pi \in [0, 1]$ if uninformed. Define $X(\pi, n, q)$ to be the random variable counting the number of firms that exit in the period. Using $q^- \equiv 1 - q$ as a shorthand for the probability that an arbitrary firm is *uninformed* conditional on the market being good, this random variable has the following conditional distributions:

$$P_g(X(\pi, n, q) = k) = \binom{n}{k} (q^- \pi)^k (1 - q^- \pi)^{n-k},$$

$$P_b(X(\pi, n, q) = k) = \binom{n}{k} \pi^k (1 - \pi)^{n-k},$$

and the following unconditional distribution:

⁶This property is not robust to some natural extensions of the model (see Section 6).

$$\begin{aligned}
P(X(\pi, n, q) = k) &= pP_g(X(\pi, n, q) = k) + (1-p)P_b(X(\pi, n, q) = k) \\
&= \binom{n}{k} \pi^k \left[p(q^-)^k (1 - q^- \pi)^{n-k} + (1-p)(1-\pi)^{n-k} \right]. \quad (3)
\end{aligned}$$

Let us now describe how p evolves over time. Consider an individual firm with belief p , who stays in the market, and at the same time observes the behavior of $n-1$ other firms that exit with probability π . This firm gets two different pieces of information that affect p . First, the firm observes that $X(\pi, n-1, q) = k$ other firms exit. Second, the firm observes that no customer arrives, which we may write as $Y = 0$ (where Y is the indicator random variable for the arrival of a customer). Given this, the firm's belief jumps to a new value given by:

$$\begin{aligned}
&p + \Delta p \\
&= \frac{pP_g(X(\pi, n-1, q) = k \wedge Y = 0)}{pP_g(X(\pi, n-1, q) = k \wedge Y = 0) + (1-p)P_b(X(\pi, n-1, q) = k \wedge Y = 0)} \\
&= \frac{p(q^-)^k (1 - q^- \pi)^{n-k} (1 - \lambda \Delta t)}{p(q^-)^k (1 - q^- \pi)^{n-k} (1 - \lambda \Delta t) + (1-p)(1-\pi)^{n-k}}. \quad (4)
\end{aligned}$$

Obviously, the greater the number of other firms that exit, the lower the new belief of this particular firm.

It is also straightforward to describe how q evolves over time. Consider an individual firm, that randomizes according to π , but does not exit. The probability assessment of the other firms for this firm having seen a customer, conditional on the market being good, changes as a result of two forces. First, simply as a result of time passing, the probability that the firm has seen a customer increases. Second, observing that a randomizing firm stays gives a signal that makes others more convinced that the firm has seen a customer. Note that both of these effects increase q (q can only decrease when a firm exits, in which case q falls to zero). Since the exact formula for the change in q is not central to our results in this section, we skip that. In section 5 we will derive the law of motion for q in the continuous time limit of the model.

To derive a symmetric equilibrium, we use the fact that whenever all firms apply mixed strategies, they must be indifferent between exiting and staying. In the following lemma we establish the conditions under which a unique probability $\pi^*(n, p, q)$ exists such that if $n - 1$ firms exit according to this probability, then this provides the n^{th} firm just enough information to keep him indifferent between exiting and staying:

Lemma 1 *Consider the optimal decision of an individual firm with belief p , who may either exit the market now or stay one more period to observe the behavior of $n - 1 \in \{1, 2, \dots\}$ firms, each of whom exits with probability π if uninformed, and with probability 0 if informed. Let $q \in (0, 1)$ be the probability that each individual firm is informed given that the market is good. Then there is a lower threshold belief $\underline{p}(n, q) \in (0, p^*)$ such that:*

1. *If $p \leq \underline{p}(n, q)$, then it is optimal to exit irrespective of π*
2. *If $p \geq p^*$, then it is optimal to stay irrespective of π*
3. *If $p \in (\underline{p}(n, q), p^*)$, then there is a unique $\pi^*(n, p, q) \in (0, 1)$ such that when $\pi = \pi^*(n, p, q)$, the firm is indifferent between staying and exiting. When $\pi < \pi^*(n, p, q)$, it is optimal to exit while if $\pi > \pi^*(n, p, q)$, it is optimal to stay. Furthermore, if $X(\pi^*(n, p, q), n - 1, q) = 0$, then $p + \Delta p > p^*$.*

Function $\underline{p}(n, q)$ is continuous in q and decreasing in both n and q . Function $\pi^(n, p, q)$ is continuous in p and q and decreasing in n , p , and q .*

Proof. In the Appendix. ■

The following proposition establishes the existence and uniqueness of a symmetric equilibrium, and uses Lemma 1 to characterize it:

Proposition 2 *The exit game has a unique symmetric equilibrium. The strategy profile $\sigma^S = \{\sigma_1^S, \dots, \sigma_N^S\}$ in this symmetric equilibrium can be defined recursively as follows:*

For initial histories $h^0 \in H^0$:

$$\sigma_i^S(h_i^0) = \begin{cases} 0 & , \text{ if } p^0 \geq p^* \\ 1 & , \text{ if } p^0 < p^* \end{cases} , i = 1, \dots, N.$$

For histories $h^t \in H^t$ extending to period $t \in \{1, 2, \dots\}$:

$$\sigma_i^S(h^t) = \begin{cases} 0 & , \text{ if } p^t \geq p^* \\ \pi^*(n^t, p^t, q^t) & , \text{ if } \underline{p}(n^t, q^t) < p^t < p^* \\ 1 & , \text{ if } p^t \leq \underline{p}(n^t, q^t) \end{cases} , i \in A(h^t),$$

where $n^t = n(h^t)$, q^t is the probability assessment of a Bayesian observer that an arbitrary active firm has seen a customer conditional on the market being good, and p^t is the common belief consistent with Bayesian updating held by all uninformed firms after history h^t .

Proof. In the Appendix. ■

The symmetric equilibrium path can be verbally described as follows. In the beginning, given that p^0 is above the monopoly exit threshold p^* , all firms stay in the market with probability one. The firms continue to experiment in this manner until $t = t^*$ where the beliefs of the uninformed firms fall below p^* . At this point they start to randomize. All firms exit with probability $\pi^*(n^t, p^t, q^t)$ that keeps them indifferent between exiting and continuing. In each period, the remaining uninformed firms update their current beliefs after observing the number of exits. If no firm exits in $t \geq t^*$, then according to Lemma 1 the belief of each uninformed firm jumps strictly above p^* . Following this jump, all firms stay in the market with probability one until p falls back below p^* at which point the randomization starts over again. This is continued until all firms have either observed a customer or left the market. If at some point the belief of the uninformed firms falls below $\underline{p}(n^t, q^t)$, the market collapses as all remaining uninformed firms exit. In such a case, the uninformed firms are so pessimistic that they do not have enough information to release in order to keep each other indifferent between staying and exiting. Note that if the market is bad, all firms must eventually exit.

When Δt shrinks to zero, the equilibrium path can be described more explicitly. We will do that in Section 5.

3.2 Asymmetric Equilibria

The exit game has a number of asymmetric equilibria in addition to the symmetric one discussed above. For example, there is an asymmetric equilibrium in pure strategies that Pareto dominates the symmetric mixed strategy equilibrium. This equilibrium gives the firms a particularly high total payoff.

In the pure strategy equilibrium the firms exit sequentially in a pre-determined order. In each period, each uninformed firm exits either with probability zero or with probability one. Since no firm ever exits if informed, a firm that exits with probability one conditional on being uninformed reveals fully its payoff relevant private history to the other firms. As soon as such a firm stays, all firms at later positions in the "exit sequence" learn that this firm has observed a customer, and consequently no firm will ever exit after that. The equilibrium is characterized in the following proposition:

Proposition 3 *The exit game has a unique (up to a permutation of the players) equilibrium in pure strategies that Pareto dominates the symmetric equilibrium. In this equilibrium, no firm exits in periods $t < t^*$, but at all periods $t \geq t^*$, $k^t > 0$ firms exit with probability one (if uninformed) until either i) all firms have exited, or ii) at some period $t' \geq t^*$ some firm that was supposed to exit stays, in which case all the remaining firms stay ever after. There is a unique sequence $\{k^t\}_{t=t^*}^T$ of positive integers for which $\sum_{t=t^*}^T k^t = N$ such that this behavior constitutes an equilibrium.*

Proof. In the Appendix. ■

To define an equilibrium, the sequence $\{k^t\}_{t=t^*}^T$ must be such that on the one hand all k^t uninformed firms that exit at period t are better off by doing so than by staying and observing the behavior of $k^t - 1$ firms, and on the other hand, all uninformed firms that stay must be better off by observing the behavior of k^t firms than by exiting. This condition is formalized in the proof of Proposition 3.

When the periods are short enough, the firms reveal their information in the pure strategy equilibrium sequentially one firm at a time:

Proposition 4 *There is an $\epsilon > 0$ such that if $\Delta t < \epsilon$, then at most one firm exits in each period in the pure strategy equilibrium.*

Proof. In the Appendix. ■

We conclude this section by proving that the pure strategy equilibrium delivers the maximal Nash equilibrium payoff to the players in the exit game. Taken together with the lower bound derived in the previous subsection for the symmetric mixed strategy equilibrium, we have obtained a partial characterization for the equilibrium payoff set of the game.

Proposition 5 *There is an $\epsilon > 0$ such that if $\Delta t < \epsilon$, the pure strategy equilibrium maximizes the sum of payoffs in the set of Nash equilibrium payoffs.*

Proof. In the Appendix. ■

It is also worth pointing out that as $N \rightarrow \infty$ and $\Delta t \rightarrow 0$, the average expected continuation payoff of uninformed agents at date t^* approaches the first best optimal payoff of $\frac{\lambda v - c}{r}$.

4 Large Markets

In this section, we analyze the equilibria of the game as the number of firms gets large. We are interested in the case where firms can react to the observed actions of the competitors quickly and therefore we consider the double limit where $\Delta t \rightarrow 0$ and $N \rightarrow \infty$.

The main result in this section and perhaps the main result of the entire paper is that in large markets, the long run equilibrium outcome is efficient with a probability converging to unity. To make this statement precise, we calculate the total number of exits in the market when the time interval between periods is Δt and the total

number of firms in the market is N . Denote this random variable by $X(\Delta t, N)$. Our main theorem shows that for all $\varepsilon > 0$,

$$\lim_{N \rightarrow \infty, \Delta t \rightarrow 0} P_g \left(\frac{X(\Delta t, N)}{N} < \varepsilon \right) = 1$$

and

$$\lim_{N \rightarrow \infty, \Delta t \rightarrow 0} P_b \left(\frac{X(\Delta t, N)}{N} = 1 \right) = 1.$$

Hence almost all firms stay when the market is good, but all firms exit when the market is bad. The second statement follows immediately from the arguments in the previous section and therefore we concentrate on the first assertion in this section.

It is clear from the previous analysis that the result cannot hold for a finite N . It is not hard to see that the result also fails in the case where Δt is bounded away from zero. For a given positive Δt , the cost of staying in the market for an additional period is not negligible and hence for sufficiently pessimistic beliefs, it is a dominant strategy for the firms to exit. It is then easy to see that in e.g. the symmetric equilibrium outlined above, there is an $\hat{N} < \infty$ such that if at least \hat{N} firms exit, then the remaining firms exit as well. As a result, all firms exit the market with a positive (but quite possibly small) probability even when the market is good.

Theorem 1 *In all equilibria of the exit game, for all $\varepsilon > 0$,*

$$\lim_{N \rightarrow \infty, \Delta t \rightarrow 0} P_g \left(\frac{X(\Delta t, N)}{N} < \varepsilon \right) = 1.$$

Proof. In the Appendix. ■

The idea of the proof is that in a large market with no delays between observations and actions, it is very unlikely that a large number of firms exit, and at the same time their posterior beliefs remain so low that their decisions to exit are consistent with equilibrium behavior.

5 Computing the Symmetric Equilibrium in Continuous Time

In this section, we compute and characterize the continuous time limit of the symmetric equilibrium given in Proposition 2. We have two reasons for doing that. First, we want to illustrate the properties of the model in a notationally simpler and hopefully more transparent environment. Second, since the period length in discrete time may be interpreted as a delay between observations and reactions, it is of interest to analyze the model as $\Delta t \rightarrow 0$ to separate out any effects such observation lags might have on the results.

To build intuition, we first use simple reasoning to derive the properties of the equilibrium directly in continuous time, without using the analysis of Section 3.1 or even formally defining strategies. However, we then check rigorously that we indeed end up with the equilibrium given in Proposition 2 as $\Delta t \rightarrow 0$.

In continuous time the firms discount future at flow rate $r > 0$, pay the flow opportunity cost $c > 0$, and meet customers at a Poisson rate λ (assuming the market is good; in a bad market no customers ever arrive). At each instant, the firms choose simultaneously whether to stay in the game or to take an irreversible exit decision. The firms are able to react to other firms' exit decisions instantaneously (that is, if a firm i exits at time t , another firm j is able to react to the bad news induced by i 's exit and follow suit essentially at that same time moment, yet strictly after i). Note that this is a property of the discrete time model in the limit $\Delta t \rightarrow 0$.

Formalizing mixed strategies in continuous time is more subtle than in discrete time, because a firm may either exit at some flow probability ϕ such that the probability of exiting between t and $t + dt$ is ϕdt , or at a discrete probability π that gives a strictly positive probability measure to the event of exit exactly at t . It will be seen that in symmetric equilibrium all firms apply flow exit probabilities as long as information arrives gradually, which is the case as long as no one exits. However, as soon as a firm exits, a discrete amount of bad news is released, and this induces the remaining firms to apply a discrete exit probability to release enough information to

keep each other indifferent between staying and exiting. A sequence of such discrete randomizations takes place within an infinitesimal time interval, and stops either when enough good news has been released to move the game back to the flow randomization mode, or when all the firms have exited. Hence, the equilibrium exhibits phases of inaction and waves of exit.

Consider first a monopoly firm experimenting in the market. The evolution of p as long as no customers arrive is given by a continuous time counterpart to (1):

$$\frac{dp}{dt} = -\lambda p(1-p). \quad (5)$$

Denote by $V(p)$ the value function of a monopoly. Bellman function in the continuation region is:

$$\begin{aligned} rV(p) dt &= p\lambda v dt + E(dV(p)) \\ &= p\lambda v dt + p\lambda dt \left(\frac{\lambda v}{r} - V(p) \right) + (1-p\lambda dt) V'(p) \lambda p(1-p) dt. \end{aligned}$$

The optimal stopping threshold p^* can be solved using value matching, i.e. $V(p^*) = \frac{c}{r}$ and smooth pasting, i.e. $V'(p^*) = 0$ to yield:

$$p^* = \frac{rc}{\lambda(v(r+\lambda) - c)}. \quad (6)$$

Moving to the case of multiple firms, we start by some immediate observations. First, since it is always possible to mimic the monopolist firm, it is never optimal to exit at a belief above p^* , regardless of the number of firms in the market. Second, there cannot be symmetric equilibria in pure strategies. To see why, suppose on the contrary that all uninformed firms exit with probability one at some $0 < p \leq p^*$ in the symmetric equilibrium. Since each firm has seen a customer at a positive probability, any individual firm then observes instantaneously that the market is good with a strictly positive probability. Since the cost of waiting to get this information vanishes in the continuous time limit, the capital gain from staying outweighs this cost and it can not be optimal to exit. On the other hand, pure strategy profile commanding every firm to stay forever cannot be an equilibrium, because then observations

regarding other firms would be uninformative and any individual firm should employ the optimal strategy of the monopolist.

Third, in any symmetric equilibrium, the firms must exit with a positive probability at $p = p^*$. To see why, suppose on the contrary that all firms stay at probability one until p falls to $p' < p^*$. Then there is no observational learning for $p \in (p', 1]$ and by the solution to the monopolist's problem, we know that there is a profitable deviation to exit at all $p \in (p', p^*]$.

Finally, the probability with which the firms exit at $p = p^*$ must be interpreted in the sense of flow exit probabilities. If, on the contrary, the firms exited with a strictly positive instantaneous probability at $p = p^*$, then the posterior would jump with a positive probability to a value strictly above p^* . In that case the capital gain from staying for an additional dt would outweigh the cost of waiting cdt and this would contradict the optimality of exit. On the other hand, the randomizations must be "strong" enough to prevent p from falling below p^* in case of no firm exiting, because otherwise the capital gain from staying could not cover the cost of waiting. Therefore, the requirement for equilibrium randomizations is that conditional on no firms exiting, the posterior of uninformed firms must remain exactly at p^* . Let us denote by $\phi(n, q)$ the exit rate used by each uninformed firm that keeps the beliefs of all uninformed firms at a constant level, given the number of firms n , and conditional probability q with which an arbitrary firm has seen a customer given that the market is good. Using Bayes' rule, we find:

$$\phi(n, q) = \frac{\lambda}{(n-1)q}. \quad (7)$$

Note that q varies over time, so that even if $\phi(n, q)$ does not depend directly on calendar time, it does so through q . Let us now consider how q changes over time. There are two forces that move it. First, as time goes by, there is a positive probability that within each dt a given firm sees a customer. Denote by dq_1 the change in q due to this effect:

$$dq_1 = \lambda(1 - q) dt. \quad (8)$$

Second, as a randomizing firm stays, it becomes more likely to an observer that the reason for staying is that this firm has seen a customer. Within a short dt , the probability that the firm exits is $\phi(n, q)$ if he has seen a customer, and 0 if he has not seen a customer. The change in q due to this second effect is then:

$$\begin{aligned} q + dq_2 &= \frac{q}{q + (1 - \phi(n, q) dt) (1 - q)}, \text{ or} \\ dq_2 &= \frac{(1 - q) q \phi(n, q) dt}{1 - (1 - q) \phi(n, q) dt}. \end{aligned} \quad (9)$$

Combining (7) and (8) and letting dt be small, we get

$$\frac{dq}{dt} = \frac{dq_1 + dq_2}{dt} = \lambda(1 - q) + (1 - q) q \phi(n, q).$$

Inserting (6), we may write this as:

$$\frac{dq}{dt} = \frac{n}{n-1} \lambda(1 - q).$$

The evolution of q is thus as follows. In the beginning of the game q starts from zero, that is, $q(0) = 0$. Until $t = t^*$, firms do not randomize, and only the effect (7) is present. This means that for $t \leq t^*$, $\frac{dq}{dt} = \lambda(1 - q)$, or $q(t) = 1 - e^{-\lambda t}$. However, from t^* onwards, the firms randomize at intensity $\phi(n, q)$, and as a result, the rate of growth in q jumps to a higher level $\frac{dq}{dt} = \frac{n}{n-1} \lambda(1 - q)$. Note that this rate depends on the number of the firms. As $n \rightarrow \infty$, this rate approaches the level at which it would be in the absence of randomizations (because when n is large, each individual firm randomizes at a low rate).

In order to complete the description of the symmetric equilibrium, we must specify what happens when firms exit. When $p = p^*$ and a single firm exits, the posterior falls immediately to level

$$p^-(q) = \frac{p^*(1 - q)}{1 - p^*q} < p^*. \quad (10)$$

When $p < p^*$, the firms must exit with a discrete probability, because otherwise their beliefs would stay below p^* with probability 1 after an instant dt . By previous arguments, firms must exit with positive probability at all such p and hence the

continuation payoff would be 0. Given that there is the positive opportunity cost cdt from staying in the market, such a strategy cannot be optimal. On the other hand, using the same argument as above, symmetric equilibrium randomization require that for all possible outcomes in the randomization, posterior beliefs stay below p^* . We must therefore construct an equilibrium by requiring that the posterior rises exactly to p^* conditional on no exits in the randomization.

Denote by $\pi(n, p, q)$ the required exit probability of the uninformed firms when there are n firms left in the market. Firm i exits with probability $\pi(n, p, q)$ if the market is bad. If the market is good, firm i has become informed with probability q and exits with probability $(1 - q)\pi(n, p, t)$. Hence requiring that the posterior be p^* conditional on no exits amounts to:

$$\frac{p(1 - (1 - q)\pi(n, p, q))^{n-1}}{p(1 - (1 - q)\pi(n, p, t))^{n-1} + (1 - p)(1 - \pi(n, p, t))^{n-1}} = p^*.$$

Rewriting, we get

$$\frac{1 - p^*}{p^*} \frac{p}{1 - p} = \frac{(1 - \pi(n, p, q))^{n-1}}{(1 - (1 - q)\pi(n, p, q))^{n-1}}, \quad (11)$$

and we can solve for the unique $\pi(n, p, q)$ that satisfies this equation.

In order to analyze the equilibria as n grows, it is useful to take logarithms on the two sides of (10) and use the approximation $\ln(1 - x) \approx -x$ for x small to get:

$$\pi(n, p, q) \xrightarrow{n \rightarrow \infty} \frac{-\ln\left(\frac{1-p^*}{p^*} \frac{p}{1-p}\right)}{(n-1)q} \equiv \bar{\pi}(n, p, q). \quad (12)$$

Note that the number of firms that actually exit follows a binomial distribution. If the market is bad, the binomial parameters are $\bar{\pi}(n, p, q)$ and n , and if the market is good, the parameters are $(1 - q)\bar{\pi}(n, p, q)$ and n . According to (??), $\bar{\pi}(n, p, q) \cdot n$ converges to $-\ln\left(\frac{1-p^*}{p^*} \frac{p}{1-p}\right)/q$ as n grows. This means that as $n \rightarrow \infty$, the distribution of the number of firms that exit approaches the Poisson distribution with parameter $-\ln\left(\frac{1-p^*}{p^*} \frac{p}{1-p}\right)/q$ if the market is bad, and parameter $\left((1 - q)\ln\left(\frac{1-p^*}{p^*} \frac{p}{1-p}\right)\right)/q$ if the market is good.

Note that when the firms apply discrete exit probabilities during an exit wave, q jumps up by discrete amounts. Given that a firm applies the exit probability $\pi(n, p, q)$

and stays, q changes by:

$$q + dq = \frac{q}{q + (1 - q)(1 - \pi(n, p, q))}.$$

We have now constructed informally a symmetric equilibrium in the continuous time game. Its main features are: *i)* No firm exits at beliefs above the monopoly exit level p^* . *ii)* At posterior $p = p^*$, uninformed firms exit at a flow rate that keeps the beliefs of the uninformed unchanged as long as no other firm exits. *iii)* When a firm exits, the posterior of the uninformed firms falls below p^* . This starts a sequence of discrete exit randomizations - a wave of exit - such that at each round all uninformed firms exit with a strictly positive probability. *iv)* As $N \rightarrow \infty$ the probability that an individual firm exits when the market is good converges to 0.

This exit wave described in property *iii)* consisting of many rounds of exit takes place within an infinitely short time interval and stops either when all firms have exited (we call this a market collapse), or when no firm exits at some round, which causes p to jump back to p^* starting another phase of flow randomizations. To see why individual exit probabilities must vanish as stated in property *iv)*, note that the probability distribution of the number of exiting firms within each round of an exit wave follows a Poisson distribution independent of the total number of firms. Therefore, as $N \rightarrow \infty$, the proportion of those firms that actually need to exit before the true market state is revealed to all firms reduces to zero.

To connect the continuous and discrete time models, we consider the properties of the equilibrium characterized in Proposition 2 in the limit $\Delta t \rightarrow 0$. As long as no firm is exiting, the posterior of the uninformed firms falls according to the Bayes' rule (1), which converges to (??) as $\Delta t \rightarrow 0$. As the step size in the Bayes' rule is continuous in Δt , randomizations conditional on no exits take place at p close to p^* when Δt is small. At the same time, conditional on no exit in any randomization, $p + \Delta p \rightarrow p^*$ as $\Delta t \rightarrow 0$, because the cost of staying in the market converges to zero. Hence conditional on no exit, the posterior stays arbitrarily close to p^* and this is possible in the limit only if all firms randomize at the flow exit rates calculated in (6). On the other hand, as soon as a firm exits, p falls substantially below p^* , and

equilibrium randomizations $\pi^*(n, p, q)$ given in Lemma 1 converge to the solution of (10) as $\Delta t \rightarrow 0$. Therefore, what we have been describing in this section is indeed the equilibrium of Proposition 2 in the limit $\Delta t \rightarrow 0$.

In the symmetric equilibrium the payoff of each individual firm is the same as it would be in the absence of observational learning. Firms exit the market at a much lower rate, however. In particular, when the number of firms is large, exit is slow enough to allow for almost perfect learning of the true market state in the long run. The cost of this learning is that firms stay in the market too long when the market is bad. To see this explicitly, consider the arrival rate of market collapse in a large market conditional on the market being bad. In a large bad market the exit waves arrive at rate $\lim_{n \rightarrow \infty} \phi(n, q) = \frac{\lambda}{q}$, but not all exit waves lead to a market collapse. The exit wave can only end at p jumping back to p^* , or at a market collapse, which in the case of a large market effectively means that p falls to (almost) zero. It is then easy to show that in order to preserve the martingale property of p , it must be that the arrival rate of market collapse in a bad market is exactly the same as the arrival rate of a customer in a good market, that is, λ . This also means that the probability that a given exit wave leads to a market collapse is q (as calculated at the moment when the exit wave starts). By the same line of reasoning we may conclude that in a small market, the market collapses arrive at higher intensity than in a large market (in a small market, collapse does not push p all the way to zero, so the martingale property on p is preserved by increasing the arrival rate of market collapses).

Finally, let us contrast the symmetric equilibrium with the pure strategy equilibrium. In continuous time, the pure strategy equilibrium is easy to describe. At time t^* , the firms reveal their private history by exiting in sequence until either all firms have exited, or until one firm reveals that the market is good by staying. All of this takes place at time t^* , so the difference to the symmetric equilibrium is that the true state of the market is revealed faster. This explains why the payoffs are greater than in the symmetric equilibrium (except for the first firm in sequence to exit). Even if in a large market there is almost perfect learning in all equilibria (Theorem 1), different equilibria differ from each other in how long the firms stay in a bad market. The

symmetric equilibrium is the worst in this sense, whereas the pure strategy equilibrium is the best. However, even in this equilibrium the firms stay in a bad market too long; information is never aggregated before $t = t^*$.

6 Extensions

In this section we discuss the robustness of our results to three extensions.

6.1 Payoff Externalities

Consider a modification of the model where the rate at which customers arrive to each active firm is given by a decreasing function $\lambda(n)$, where n is the number of firm remaining in the market.⁷ We assume that $\lambda(N) > \frac{c}{v}$, which means that the good market is profitable even if no firm exits (we do not want to complicate the model by considerations reminiscent of war of attrition). We argue that the main result for large markets continues to hold in this case.

It is relatively straightforward to see that this modified model must also have a unique symmetric equilibrium in mixed strategies. There cannot be a symmetric equilibrium in pure strategies for the same reason as before: simultaneous exit by all firms would provide so much information that it would be optimal to wait. On the other hand, exiting with probability zero provides no information, and at some point it would become individually optimal to exit. The firms' exit decisions are strategic substitutes, and it is clear that whenever neither all firms exiting nor all firms staying with probability one is compatible with equilibrium, there must be a unique intermediate exit probability that keeps all firms indifferent. Taking the limit to continuous time, it is also easy to see that the qualitative properties remain the same as in the original model: the equilibrium path exhibits phases of inaction during which all firms apply flow randomization, but as soon as one firm exits, this starts an exit wave that ends when all firms have exited or when no firm exits at some round,

⁷Under this assumption, it is optimal for an individual firm to hide any good news from others and hence communication between firms is not likely to be effective.

in which case the play resumes to the flow randomization mode. The main difference to the original model is that in this modified model the threshold belief at which the firms are indifferent between staying and exiting cannot be constant over time. To see why, assume on the contrary that the firms exit at a rate that keeps their belief unchanged conditional on no exits. Over time, the uninformed firms become convinced that other firms have seen customers (in order to compensate the negative news of having seen no consumers themselves). Hence, it becomes more likely that in a good market, many firms stay. Because there is a negative payoff externality, this is bad news for the firms. To compensate for this, it must be that the threshold belief increases over time as long as no firm exits. Finding an analytical expression for the exit rate that keeps the firms indifferent is likely to be hard, though.

On the other hand, as a firm exits and an exit wave starts, the level to which the belief must rise in order to end the wave keeps getting lower as more firms have exited. Again, the reason is simple: the less there are firms left, the better the market will be for those who stay (if the market turns out to be good), hence the belief that makes firms indifferent between staying and exiting is lower.

In the limit of a large market, the equilibrium path converges to that of the original model. To keep the market viable for all firms in that limit, we must require that $\lim_{n \rightarrow \infty} \lambda(n) = \lambda_\infty > \frac{c}{v}$. Using the same argument as in the original model, it is clear that if a large number of firms exit, the true state of the market is almost fully revealed to all remaining firms. Hence, in equilibrium either a small fraction of firms exit, or all firms exit. In the former case, the rate of arrival of customers is close to λ_∞ , while in the latter case the market must be bad, and there are no customers. Hence, the equilibrium path is essentially the same as with the original model where customers arrive at rate λ_∞ irrespective of the number of firms in the market.

6.2 Imperfect Signals

One of the main properties of our original model is that when the number of firms is large, there are no incorrect herds (i.e. situations, where a non-negligible fraction of the

firms exit a good market). One might suspect that this result hinges on the extreme assumption that as soon as a firm sees one customer, this firm becomes fully informed about the state of the market. In this subsection we consider briefly an extension of our model, where no firm ever gets fully informed about the state of the market, and argue that the main properties of our model still continue to hold.

We modify the model so that customers arrive in both states of the world, but the rate of arrival is greater if the market is good. Denote by λ_g and λ_b the arrival rates of customers in good and bad market, respectively, and assume $\lambda_b < \frac{c}{v} < \lambda_g$ (so that it is optimal to stay if and only if the market is good). All firms are now uninformed, but at each date they are divided into types $k = 0, 1, 2, \dots$, where k counts the cumulative number of customers. Our Poisson assumption implies that the number of customers is a sufficient statistic for the private history of each firm.

Again, there is a unique symmetric equilibrium. The logic is the same as before: because the information revealed to others by exit decisions increases value of waiting, the firms' exit probabilities are strategic substitutes. Hence, within each type, either it is optimal for all firms to exit or stay, or there is a unique probability of exit that keeps all firms of that type indifferent between exiting and staying. Moreover, if all firms within a type are indifferent, then all firms with a lower type find it strictly optimal to exit (because they are more pessimistic about the state). The firms with a higher type find it strictly optimal to stay (because they are more optimistic). Hence, there must be a symmetric equilibrium, where within each period at most one type of firms randomize, and it is common knowledge that all the firms that have seen fewer firms than this randomizing type, have already exited (or exit with probability 1 within the same period).

Let us now briefly characterize this equilibrium in continuous time. Denote by $p^k(t)$ the belief of type k firms at time t , and let p^* be the monopoly exit threshold (which is a straight-forward modification of (5) to fit the extended model). It is easy to show that $p^0(t)$ defines unambiguously $p^k(t)$ for $k = 1, 2, \dots$ ⁸ In the beginning, no

$${}^8 p^k(t) = \frac{\frac{p^0(t)}{1-p^0(t)} \left(\frac{\lambda_g}{\lambda_b}\right)^k}{1 + \frac{p^0(t)}{1-p^0(t)} \left(\frac{\lambda_g}{\lambda_b}\right)^k}.$$

firm exits until $p^0(t)$ falls to p^* . At that point those firms who have seen no customer start to randomize at a rate that keeps the beliefs of type 0 unchanged as long as no firm exits. Note, however, that contrary to the original model, there is a t_0^* such that beyond this time even knowing for sure that all other firms have seen at least one customer cannot keep the belief of a type $k = 0$ firm from falling below p^* . This means that those firms must exit at a rate that increases to infinity towards time t_0^* so that if no firm has exited by that time, it becomes common knowledge that all remaining firms have seen at least one customer. Then beliefs start falling again as there are no type $k = 0$ firms left to randomize until $p^1(t)$ falls to p^* , at which point type $k = 1$ firms start to randomize. Whenever one firm exits, an exit wave starts. At every round of an exit wave, the lowest type of firms left in the market exit at a probability strictly greater than zero (and possibly at probability one). An exit wave ends when either all firms have exited, when no firm of the randomizing type exits at some round (in which case play resumes to flow randomization mode), or when all firms of type k exit at probability 1, and the number of those firms turns out to be so low that p^{k+1} after this round is above p^* (in which case there will be a period of no randomizations until $p^{k+1}(t)$ falls to p^*).

When the number of firms grows large, knowing the exact number of firms that have seen no customers is enough to fully reveal the true state of the world. The game is thus essentially played by only those firms. Therefore, $t_0^* \rightarrow \infty$, and only those firms who have never seen customers randomize until at some point, in case the market is bad, so many firms have exited that the beliefs of all remaining firms are so low that it is almost certain for every firm that the market is indeed bad; market collapse is then inevitable.

Although the detailed characterization of the equilibrium is more complicated in this modified model, there is not much that would be qualitatively different to the original model. The main difference is that in contrast to the original model, the ex-ante payoff is now higher in equilibrium than for a monopoly firm. The reason is that a firm who sees many customers is able to benefit from information revealed by those firms who have seen fewer customers. Yet, the ex-ante payoff of all firms is still

well below what they would get under full information sharing.

6.3 Private Information on Opportunity Costs

We showed in the previous sections that when the firms are identical, the unique symmetric equilibrium is in mixed strategies. We now argue that the main insights from that model continue to hold in the symmetric pure strategy equilibrium of a model where the opportunity cost c_i of firm i is privately known. Assume that $c_i \sim F(\cdot)$ for all i , and the costs are independent across firms. We assume throughout that the support of F is a convex set $[\underline{c}, \bar{c}]$, F is continuously differentiable on the support, and $f(c) > \varepsilon > 0$ for all $c \in [\underline{c}, \bar{c}]$. We also assume that $\bar{c} < \lambda v$ and hence it is optimal for all firms to stay in a good market.⁹

In symmetric equilibrium, all uninformed firms have the same belief, and it is straight-forward to show that the model has a unique symmetric equilibrium where at each period all firms with a cost parameter above some cutoff level exit at probability 1 while the rest stay at probability 1. Let us derive the properties of this equilibrium directly in continuous time. We argue first that the posterior belief $p(t)$ of the uninformed players is nonincreasing in t as long as no exits are observed. To see this, recall from the previous section that the highest posterior belief p that may induce player i of type c_i to exit is given by

$$p^*(c_i) = \frac{rc_i}{\lambda(v(r + \lambda) - c_i)}. \quad (13)$$

Since this level is calculated by equating the gain from own experiments to the cost of staying in the market, it is clear that whenever the posterior is increasing in t , all firms exiting at p would have exited at $p' < p$. This would, however, imply a jump in the posterior conditional on no exits and by the arguments of the previous section, this is not compatible with equilibrium.

As soon as we know that $p(t)$ is nonincreasing, it is clear that it is optimal for a firm of type c_i to exit as soon as the belief falls to the level $p^*(c_i)$ calculated in (11).

⁹A recent paper by Levin & Peck (2006) considers a similar extension to the model of Chamley & Gale (1994).

To characterize equilibrium, we must now show how $p(t)$ evolves over time when the firms apply such cutoff strategies. As $p^*(c_i)$ is an increasing function, we can invert it to get the cost type that is indifferent between exiting and staying at posterior p :

$$c(p) = \frac{p\lambda v(r + \lambda)}{r + \lambda p}. \quad (14)$$

To calculate the density of firms exiting at p we then compute

$$g(p) = f(c(p)) c'(p) = f\left(\frac{p\lambda v(r + \lambda)}{r + \lambda p}\right) \frac{\lambda v(r + \lambda)(r + \lambda p) - p\lambda^2 v(r + \lambda)}{(r + \lambda p)^2}.$$

Using this distribution, we can write the Bayes' rule for updating beliefs when there are n remaining firms conditional on not seeing customers and conditional on not observing any exits:

$$p + dp = \frac{p(1 - \lambda dt + (n - 1)e^{-\lambda t}g(p)dp)}{p(1 - \lambda dt + (n - 1)e^{-\lambda t}g(p)dp) + (1 - p)(1 + (n - 1)g(p)dp)}.$$

From this, we can deduce the law of motion for p when no customers and no exits are observed:

$$\frac{dp}{dt} = \frac{-\lambda p(1 - p)}{1 + (n - 1)g(p)(1 - e^{-\lambda t})p(1 - p)}. \quad (15)$$

Observe that as $n \rightarrow \infty$, the time change in p slows down. Notice that since $f(c)$ is bounded away from zero, $g(p)$ is also bounded away from zero on its support. As the probability that a firm exits is $ng(p)\left(-\frac{dp}{dt}\right)$, we observe from (12) that the probability of an exit per unit of time approaches $\frac{\lambda}{1 - e^{-\lambda t}}$ as $n \rightarrow \infty$. This is the same limit that we discovered for the symmetric equilibrium of the symmetric model.

Consider next the beliefs after an exit is observed. As in the previous section, an exit leads to a discrete drop in the posterior to

$$p^-(p) = \frac{p(n - 1)e^{-\lambda t}g(p)}{p(n - 1)e^{-\lambda t}g(p) + (1 - p)(n - 1)g(p)} = \frac{pe^{-\lambda t}}{pe^{-\lambda t} + (1 - p)} < p.$$

Observe that this is the same equation as in the previous section when $q = (1 - e^{-\lambda t})$. The equation for q is simpler in the current case as the equilibrium is in pure strategies and informed as well as uninformed players of type c remain in the market with

probability 1 until the posterior hits $p(c)$. When $n \rightarrow \infty$, the expressions become identical.

Once the posterior drops to $p^-(p)$, the market is not viable for all remaining firms any longer. Hence an immediate additional exit round is called for. As there must be a strictly positive probability for exits, the new posterior $p^+(p^-)$ (conditional on no new exits) cannot rise back to the original level. In other words, $p^+(p^-(p)) < p$. The exact level of the new posterior is determined by Bayes' rule and indifference at p^+ as follows. Let $\xi(p^+, p) = G(p) - G(p^+)$. In equilibrium, all uninformed players with costs above $c(p^+)$ exit. Conditional on no further exits, the posterior is calculated from Bayes' rule and must equal the cutoff belief p^+ for optimal exit:

$$\frac{p^- (1 - e^{-\lambda t} \xi(p^+, p))^{n-1}}{p^- (1 - e^{-\lambda t} \xi(p^+, p))^{n-1} + (1 - p^-) (1 - \xi(p^+, p))^{n-1}} = p^+, \quad (16)$$

where n is the number of remaining firms immediately after p dropped to p^- . Again, it can be shown that the left hand side is decreasing in p^+ and that a unique solution to the equation exists. When $n \rightarrow \infty$, $p^+(p^-(p)) \uparrow p$.

If there are further exits, then the beliefs are adjusted in the same way as with our original model, and yet another round of exits is called for. The cutoff belief for exiting types is within every round chosen such that in case no firm exits, belief rises exactly to this cutoff level. Exit wave continues in this manner until no firm exits in some round, or when all firms have exited.

The remaining task is then to show that if the market is good, the expected number of firms that can exit in any exit wave is bounded. This follows from logic similar to our main theorem. As we must have $p^+ < p$, there cannot be positive probability events along the exit wave that result in posteriors above p . But this puts a bound on the total number of firms that can exit in a good market.

It is now clear that the main properties of this model are the same as in our original model. The only qualitative difference is again that firms' payoffs are increased above the monopoly payoff level. This is because a firm of type $c_i < \bar{c}$ gets some valuable information about the exit behavior of those with higher cost before it becomes optimal for this particular firm to exit. However, when we shrink the support of F ,

i.e. let $\underline{c} \uparrow \bar{c}$, these rents disappear, and the properties of the model converge to the original model. This provides a purification for our mixed strategy equilibrium of the original model.

7 Conclusion

This paper shows that privately collected information is aggregated in large markets with exit in the long run sense. At the same time, we show that this does not imply that welfare of the firms would be close to the welfare resulting from full information sharing. When learning from others operates through irreversible exit, each individual who reveals information loses at the same time his own opportunity to learn from others later on. This induces reluctance in revealing information, and thereby, leads to slow aggregation of information. Welfare is then dissipated by the fact that the firms stay in a bad market too long. Note that in the symmetric equilibrium of the model, the firms stay in a bad market not only longer than they would in the presence of full information sharing, but also longer than they optimally would without any learning from others. Hence, while learning from others is valuable in that it decreases the likelihood of an erroneous exit from a good market, the value of this improved ability to take the correct action is dissipated by the costly delay in exiting a bad market.

We have called the stopping action in the game a decision to exit. We can reinterpret the model as one where firms are timing their investment decisions in the following manner. An irreversible investment opportunity yields a positive net present value in a good state of the world and a negative present value in the bad state. Bad states are characterized by disruptions that arrive according to a Poisson process. In the good state, there are no disruptions. With this formulation, a model of irreversible investment is equivalent to the model of exit outlined in the text, and our main conclusion would be that with a large number of investors contemplating similar projects with common uncertainty and observing each others decisions, most firms will eventually invest if and only if it is optimal to do so, yet they will do so

inefficiently late.

8 Appendix

Proof of Proposition 1. If a firm would get less than a monopoly in σ , then this firm could deviate by ignoring the information obtained by observing the behavior of the other firms, and replicate the behavior of a monopoly firm. Since the model has no payoff externalities, this would guarantee the same payoff as a monopoly firm, and thus for all active firms $V_i(h^t; \sigma) \geq V_m(p_i(h^t; \sigma))$. In particular, a firm that would exit at $p_i(h^t, \sigma) > p^*$ would have a lower payoff than a monopoly firm, thus in equilibrium $p_i(h^t, \sigma) > p^*$ implies that $\sigma_i(h^t) = 0$. To show that $V_i(h^t; \sigma) = V_m(p_i(h^t; \sigma))$ for at least one active firm, it suffices to note that at any history, there must be some firm that is the next to exit at a positive probability, and since this firm chooses to do so without any further observations on the exit behavior of the other firms, this firm can not have a better payoff than a monopoly firm. ■

Proof of Lemma 1. Define a "one-step" continuation payoff function as the value of a hypothetical firm that stays in the market one more period to observe the actions of $n - 1$ other firms, each of whom exits independently at probability π in case of being uninformed and at probability 0 in case of being informed, but after this specific period will ignore all observations about other firms, and instead will behave like a monopoly:

$$C_n(\pi, p, q) \equiv -c\Delta t + pv\lambda\Delta t + \frac{1}{1+r\Delta t} \left\{ p\lambda\Delta t \left(\frac{(1+r\Delta t)(\lambda v - c)}{r} \right) + (1-p\lambda\Delta t) \sum_{k=0}^n P(X(\pi, n-1, q) = k) \cdot V_m(p + \Delta p) \right\}, \quad (17)$$

where $V_m(\cdot)$ is defined by (2), and $P(X(\pi, n-1, q) = k)$ and $p + \Delta p$ are given by (3) and (4), respectively.

Take any parameter values in the range $\pi \in (0, 1)$, $p \in (0, 1)$, and $q \in (0, 1)$. Clearly, $C_n(\pi, p, q)$ is continuous in all parameters and strictly increasing in p . Since $V_m(\cdot)$ is convex and an increase in π induces a mean preserving spread in $p + \Delta p$, it

follows that $C_n(\pi, p, q)$ is also increasing in π . In particular, $V_m(\cdot)$ is strictly convex for $p > p^*$, and hence $C_n(\pi, p, q)$ is strictly increasing in π whenever a randomization of the firms induces p to jump above p^* at a positive probability. This means that $C_n(\pi, p, q)$ is strictly increasing in π at such parameter values that $C_n(\pi, p, q) = 0$.

When $\pi = 0$, observation gives no information, and hence $C_n(0, p, q)$ gives the payoff of a monopoly firm that is constrained to stay for at least one more period. Since at $p = p^*$ a monopoly firm is indifferent between continuing and staying, we must have $C_n(0, p^*, q) = V_m(p^*) = 0$. For any $\pi \in (0, 1]$, we have $C_n(\pi, p^*, q) > 0$. In particular, $C_n(1, p^*, q) > 0$, while on the other hand it follows by direct calculation from (??) that $C_n(1, 0, q) = -c\Delta t < 0$. From the fact that $C_n(\cdot)$ is continuous and strictly increasing in p , it immediately follows that there is a unique $\underline{p}(n, q) \in (0, p^*)$ such that $C_n(1, p, q) = 0$ for $p = \underline{p}(n, q)$. Since $C_n(\cdot)$ is strictly increasing in p and increasing in π , it follows that for $p < \underline{p}(n, q)$, $C_n(\pi, p, q) < 0$ for any $\pi \in [0, 1]$. Thus, for $p < \underline{p}(n, q)$ it is optimal to exit irrespective of π . On the other hand, from the fact that $C_n(\pi, p, q)$ is everywhere continuous and increasing in π , and strictly increasing in π when $C_n(\pi, p, q) = 0$, it follows that for any $p \in (\underline{p}(n, q), p^*)$ there is a unique $\pi^*(n, p, q) \in (0, 1)$ such that $C_n(\pi^*(n, p, q), p, q) = 0$, meaning that the firm is indifferent between staying and exiting. It also follows that $C_n(\pi, p, q) < (>) 0$ for $\pi < (>) \pi^*(n, p, q)$, and hence it is strictly optimal to exit (stay). The fact that it is optimal to stay irrespective of π for $p \geq p^*$ follows trivially from the monopoly optimization problem.

The continuity and monotonicity properties of $\underline{p}(n, q)$ and $\pi^*(n, p, q)$ can be established by implicit differentiation of the conditions $C_n(\pi^*(n, p, q), p, q) = 0$ and $C_n(1, \underline{p}(n, q), q) = 0$, respectively. The fact that p must jump above p^* when no firm exits follows from the fact that in order to make the firm indifferent between staying one more period and continuing, $\pi^*(n, p, q)$ must induce a positive probability of moving p to a level that gives a strictly positive monopoly payoff, that is, above p^* . This must in particular be the case if no firm exits, because this is the event that induces the most optimistic belief to the firm. ■

Proof of proposition 2. Since in a symmetric equilibrium all firms must

have the same payoff after any history, it follows from Proposition 1 that $V_i(h^t; \sigma) = V_m(p_i(h^t; \sigma))$ for all $i \in A(h^t)$. This means that in checking whether a particular profile is an equilibrium, it suffices to consider the optimality of the *current period* actions at all possible histories of the game by taking as given that the payoff in the next period is the monopoly payoff $V_m(p_i(h^t; \sigma))$. It is then straight-forward to see that in all histories, where the current belief of the uninformed firms is $p^t \in (\underline{p}(n^t, q^t), p^*)$, the only symmetric action that leaves no possibility for a profitable deviation for any firm is the randomization with an exit probability that gives the one-step continuation payoff equal to zero to all uninformed firms. Since each of the n^t active firms have access to the randomization of $n^t - 1$ other firms, the unique exit probability that satisfies this requirement is, according to Lemma 1, $\pi^*(n^t, p^t, q^t)$. Lemma 1 implies that for all histories where $p^t \geq p^*$, it is the dominant strategy for all firms to stay at probability one, and for all histories where $p^t \leq \underline{p}(n^t, q^t)$, it is the dominant strategy for all firms to exit at probability one. Thus, σ^S as defined in Proposition 2 is an equilibrium, and there can not be other symmetric equilibria. ■

Proof of Proposition 3. Take a profile σ that defines the behavior of the firms as it is described in Proposition 3. Let the number of firms that reveal information in σ within each period $t > t^*$ be given by a sequence $\{k^t\}_{t=t^*}^T$. Define this sequence so that for $t = t^*, t^* + 1, \dots$:

$$k^t \equiv \min \left[N - \bar{k}^{t-1}; \min \{n \in \{1, 2, \dots\} \mid C_n(1, p^t, q^t) \geq 0\} \right], \quad (18)$$

where $\bar{k}^t = 0$ for $t = t^*$ and $\bar{k}^t = \sum_{t'=t^*}^t k_{t'=0}^t$ for $t > t^*$. Function $C_n(\cdot)$ is the one-step continuation payoff function defined in the Proof of Lemma 1, $q^t = 1 - (1 - \lambda \Delta t)^t$, and p^t is the belief of an uninformed firm, who has observed the exit of $\bar{k}^{t'}$ firms at periods $t' = t^*, \dots, t - 1$ (and thus learnt that those firms have not observed a customer). Then, the sequence $\{k^t\}_{t=t^*}^T$ defining the number of exiting firms within each period is obtained by taking the strictly positive terms from the sequence $\{k^t\}_{t=t^*}^\infty$. It is clear that condition (??) defines a unique sequence. Starting from $t = t^*$, k^t is given by the smallest positive integer such that $C_n(1, p^t, q^t) \geq 0$, until this condition can not

be satisfied by an integer smaller than $N - \bar{k}^{t-1}$. When this happens, $k^t = N - \bar{k}^{t-1}$ (meaning that all the remaining firms exit), and at all periods after this $k^t = 0$.

The description of the equilibrium strategies is completed as follows. In each period t , k^t firms with the smallest indices amongst the active firms are the ones to exit. If in any period t' an exit by a firm that exits with probability zero in equilibrium is observed, the strategies of the active firms remain exactly as on the equilibrium path. In other words, the remaining firms assign no informational content to such exits.¹⁰ To see that σ is an equilibrium, note that k^t is defined in (??) by taking the *smallest* number of firms such that when those k^t firms reveal their information, the remaining firms have a positive one-step continuation payoff. Thus, none of those k^t firms has an incentive to stay, because by deviating a firm would induce all the remaining firms to stay forever, and therefore this deviating firm would never receive any information from the remaining firms in the future. Hence, the appropriate payoff is given by the one-step payoff function, which in this case is negative as only $k^t - 1$ would reveal information to this deviating firm. On the other hand, (??) requires that a firm that does not belong to the group of those k^t firms has a positive one-step continuation payoff. For these firms, the total payoff can be even higher than the one-step payoff, since they may get even more information from other firms in the future. By deviating (and exiting) such a firm would only get a payoff equal to zero, which obviously would not be optimal.

The equilibrium as described here is the only pure strategy equilibrium, because it is always a dominant strategy for all firms to stay in periods $t < t^*$, and for all $t \geq t^*$, any number $\tilde{k}^t \neq k^t$ representing the number of exiting firms would allow a profitable deviation. If \tilde{k}^t were greater than k^t , any of the exiting firms would gain by staying, and if \tilde{k}^t were smaller than k^t , any of the staying firms would gain by exiting.

Finally, note that the uniqueness is up to a permutation of the firms, because we have not fixed the order in which the firms exit. Any permutation is an equilibrium,

¹⁰The full description of the equilibrium strategies is available from the authors upon request. They are notationally cumbersome but otherwise straightforward and we have omitted displaying them in order to save space.

as long as it allocates k^t firms to exit at period t . ■

Proof of Proposition 4. When $\Delta t \rightarrow 0$, the cost of waiting one more period approaches zero. Therefore, for a firm with an arbitrary belief $p > 0$, there must be an $\epsilon(p)$ such that when $\Delta t < \epsilon(p)$, it is optimal for this firm to wait one more period if waiting fully reveals the information of another firm. Fix an arbitrary period length $\widetilde{\Delta t}$ and take the lowest belief that an uninformed firm can ever have before all firms have exited when the firms reveal their information one at the time in succeeding periods $t^*, t^* + 1, \dots$. Denote this lowest belief by p^- and take $\epsilon(p^-)$. When $\Delta t < \min(\epsilon(p^-), \widetilde{\Delta t})$, observing the behavior of one firm is enough to keep the remaining firms better off than exiting, meaning that $C_1(1, p^t, q^t) > 0$ for all remaining firms at all p^t and q^t that are reached when firms exit one at a time in succeeding periods. Then condition (??) defines $k^t = 1$ for all $t = t^*, t^* + 1, \dots, t^* + N - 1$. ■

Proof of proposition 5. Consider the planner's problem of choosing strategies $\sigma = (\sigma_1, \dots, \sigma_n)$ to

$$\begin{aligned} \max_{\sigma} \sum_{i=1}^n V_i(h; \sigma) \\ \text{s.t. } \sigma_i \max\{0, p_i(h) - p^*\} = 0. \end{aligned}$$

In other words, player i can be chosen to exit with positive probability only if her posterior on g is at or below p^* . Observe that in this problem, the only information available to the planner is that released by the exit decisions. Since all Nash equilibria of the game satisfy the constraint, the claim is proved if we show that the pure strategy equilibrium solves the problem. The proof has two main steps. In the first, it is shown that a pure strategy solution exists for the program. In the second, we show that at most one firm exits after each history.

Consider an arbitrary history h and suppose that $0 < \sigma_i(h) < 1$. Denote the continuation histories by $(h'_{-i}, 0)$ and $(h'_{-i}, 1)$ depending on whether player i exited or not. Consider next an alternative randomization scheme where first with probability $(1 - \sigma_i(h))$, firm i stays regardless of her information and with probability $\sigma_i(h)$ she exits if and only if she hasn't seen a customer. If the planner doesn't see the results of

the first stage randomization, the posteriors from this scheme are the same as those induced by the mixed strategy $\sigma_i(h)$. By a simple revealed preference argument, it is (weakly) optimal for the planner to observe the first stage randomization. Since the payoffs are then linear combinations of payoffs from pure strategies, an optimal pure strategy exists.

The second step is analogous to the proof of proposition 4 and therefore omitted.

Finally, it is an easy exercise to show that the optimal time to first exit in the planner's problem is decreasing in the number of firms in the market (due to the informational externality) and hence it is optimal to exit at t^* . ■

Proof of Theorem 1. Denote by $P_g^\sigma(A)$ the probability that some event A occurs given that the market is good and the firms adopt strategy profile σ . We want to show that for any ε and δ , $\exists \overline{\Delta t}(\varepsilon, \delta) > 0$ and $\overline{N}(\varepsilon, \delta) > 0$ such that

$$P_g^\sigma \left(\frac{X(\Delta t, N)}{N} \geq \varepsilon \right) < \delta$$

whenever σ is equilibrium, $\Delta t < \overline{\Delta t}(\varepsilon, \delta)$ and $N > \overline{N}(\varepsilon, \delta)$.

Assume throughout the proof that the true state of the market is $M = g$. Fix some $\varepsilon > 0$ and $\delta > 0$. Denote by $A_\varepsilon^{N, \Delta t}$ the set of such histories h^t where the number of firms that have exited exceeds εN at period t :

$$A_\varepsilon^{N, \Delta t} \equiv \{h^t \in H \mid n(h^t) < N(1 - \varepsilon) \wedge n(h^t \setminus a^{t-1}) \geq N(1 - \varepsilon)\}.$$

Let $B_\sigma^{N, \Delta t}$ be the set of histories h^t such that the belief of the outside observer is below p^* after history h^t :

$$B_\sigma^{N, \Delta t} \equiv \{h^t \in H \mid p_0(h^t; \sigma) < p^*\}.$$

Denote by $\Psi^{N, \Delta t}$ the set of all such strategy profiles in a game with N players and period length Δt for which it holds that $\sigma_i(h^t) = 0$ for all $i = 1, \dots, N$ and for all $h^t \in H^t$ when $t < t^*$. We have already shown that in any equilibrium all firms stay at probability 1 for all $t < t^*$, so all equilibria strategy profiles are automatically contained in this set. Consider now an arbitrary $\sigma \in \Psi^{N, \Delta t}$. Whenever σ induces a firm to exit at a positive probability, there is a strictly positive probability that this

firm is in fact informed (because $\sigma \in \Psi^{N,\Delta t}$ means that all firms have experimented the market for a strictly positive time interval before starting to exit). Hence, if after any history h^t , σ induces at least one firm to exit at a high probability, it also leads to a release of positive information at a high probability. It is then clear that if σ induces a large number of firms to exit at a high probability, it also makes the probability that an observer becomes optimistic about the state of the world high (remember that we assume that the true state of the market is good). Thus, by choosing a high enough N we may always make sure that with any $\sigma \in \Psi^{N,\Delta t}$, the probability that at least εN firms exit and at the same time the belief of the outside observer remains below p^* is arbitrarily small. This means that there is some $\bar{N}(\varepsilon, \delta)$ such that if $N > \bar{N}(\varepsilon, \delta)$, the following must hold for any $\sigma \in \Psi^{N,\Delta t}$:

$$P_g^\sigma (A_\varepsilon^{N,\Delta t} \cap B_\sigma^{N,\Delta t}) < \frac{\delta}{2}. \quad (19)$$

Up to this point, the only requirement we have put on σ is that $\sigma_i(h^t) = 0$ for all firms at the beginning of the game. This is only one specific property that an equilibrium must have. Let us now utilize further the fact that σ is an equilibrium. Consider the possibility that after some period during which at least one firm exits, the belief of the outside observer is at or above p^* , that is, $p_0(h^t; \sigma) \geq p^*$ for some h^t for which $n(h^t) > n(h^t \setminus a^{t-1})$. This means that $p_0(h^t; \sigma)$ would have been above p^* by a fixed margin if no firm had exited, and therefore ex ante there was a positive probability that the belief of the outside observer would be strictly above p^* after that period. Since at the end of any period the outside observer has exactly the same information as those firms who did exit would have if they had stayed, it must be that any of those firms who actually did exit, faced a positive ex-ante probability that their own belief would be above p^* after this period, had they not exited. This probability must reduce to zero as $\Delta t \rightarrow 0$, because otherwise their decision to exit could not be consistent with equilibrium behavior ($\Delta t \rightarrow 0$ means that the cost of waiting one more period vanishes, so even a small probability of p jumping above p^* would make it optimal to wait). This then means that the probability that the belief of an outside observer would be above p^* after any period at which at least one firm

exists must become arbitrarily small as Δt is reduced towards zero. Hence, there must be some $\overline{\Delta t}(\varepsilon, \delta)$ such that whenever σ is an equilibrium and $\Delta t < \overline{\Delta t}(\varepsilon, \delta)$, we have:

$$P_g^\sigma \left(A_\varepsilon^{N, \Delta t} \cap (B^{N, \Delta t, \sigma})^C \right) < \frac{\delta}{2}. \quad (20)$$

Note that $\sigma \in \Psi^{N, \Delta t}$ for any equilibrium σ , and hence (14) holds in equilibrium if N is large enough. Note also that (14) holds for any $N > \overline{N}(\varepsilon, \delta)$, no matter what Δt , and (15) holds for any $\Delta t < \overline{\Delta t}(\varepsilon, \delta)$, no matter what N . Therefore, we may put (14) and (15) together to see that whenever σ is an equilibrium, $\Delta t < \overline{\Delta t}(\varepsilon, \delta)$ and $N > \overline{N}(\varepsilon, \delta)$, the following holds:

$$P_g^\sigma (A_\varepsilon^{N, \Delta t}) = P_g^\sigma (A_\varepsilon^{N, \Delta t} \cap B^{N, \Delta t, \sigma}) + P_g^\sigma (A_\varepsilon^{N, \Delta t} \cap (B^{N, \Delta t, \sigma})^C) < \frac{\delta}{2} + \frac{\delta}{2} = \delta.$$

Since $P_g^\sigma (A_\varepsilon^{N, \Delta t}) = P_g \left(\frac{X(\Delta t, N)}{N} \geq \varepsilon \right)$, this completes the proof. ■

References

- Aoyagi, M. 1998a. “Equilibrium Delay in a Simple Dynamic Model of Investment.” *Economic Theory* pp. 123–146.
- Aoyagi, M. 1998b. “Mutual Observability and the Convergence of Actions in a Multi-Person Two-Armed Bandit Model.” *Journal of Economic Theory* 82:405–424.
- Banerjee, A.V. 1992. “A Simple Model of Herd Behavior.” *Quarterly Journal of Economics* 107:797–817.
- Bikhchandani, S., D. Hirshleifer & I. Welch. 1992. “A Theory of Fads, Fashion, Custom, and Cultural Change as Informational Cascades.” *Journal of Political Economy* 100:992–1026.
- Caplin, A. & J. Leahy. 1994. “Business as Usual, Market Crashes, and Wisdom After the Fact.” *American Economic Review* 84:548–565.
- Chamley, C. & D. Gale. 1994. “Information Revelation and Strategic Delay in a Model of Investment.” *Econometrica* 62:1065–1086.

- Décamps, J.-P. & T. Mariotti. 2004. "Investment Timing and Learning Externalities." *Journal of Economic Theory* 118:80–102.
- Gul, F. & R. Lundholm. 1995. "Endogenous Timing and the Clustering of Agents' Decisions." *Journal of Political Economy* 103:1039–1066.
- Keller, G., S. Rady & M. Cripps. 2005. "Strategic Experimentation with Exponential Bandits." *Econometrica* 73:39–68.
- Levin, D. & J. Peck. 2006. "Investment Dynamics with Common and Private Values." mimeo.
- Mariotti, M. 1992. "Unused Innovations." *Economics Letters* 38:367–371.
- Moscarini, G. & F. Squintani. 2004. "Competitive Experimentation with Private Information."
- Rosenberg, D., E. Solan & N. Vieille. 2005. "Social Learning in One Arm Bandit Problems."
- Smith, L. & P. Sorensen. 2000. "Pathological Outcomes of Observational Learning." *Econometrica* pp. 371–398.
- Zhang, J. 1992. "Strategic Delay and the Onset of Investment Cascades." *Rand Journal of Economics* pp. 188–205.