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Experimentation and Observational Learning in a Market with Exit

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Abstract

We analyze a model where n firms are initially uncertain about the state of the market. Information arrives from two sources: own experiences in the market and observations of other firms' decisions to exit. We show that equilibria of this model aggregate information in the sense that almost all firms stay in the market if and only if the market is good. Nevertheless, payoffs are well below the full information level as firms tend to stay in a bad market too long.

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1 Introduction

In this paper, we examine the informational performance of a simple market where a number of firms have entered a market whose viability is initially uncertain. In a good market, all firms make a positive profit whereas in a bad market, all firms make a loss. The only decision that the firms take in this model is whether to stay in the market or exit. The firms observe new information as long as they are active in the market. In addition to their direct observations about the state of the market, they observe the behavior of the other firms. Each decision by a currently active firm in the market creates an informational externality. By exiting, a firm delivers bad news to the other firms. Staying in the market, on the other hand, is good news to the other firms. We assume that exit is irreversible in the sense that once a firm exits the market, it is not possible to re-enter. This informational structure is in line with the recent literature on observational learning models, where agents infer each others' information from the actions taken by others.

The informational model that we adopt is quite simple. We model the game as an infinite horizon game of timing where each firm has to decide on the policy to exit the market. When the market is good, each firm gets a customer according to a flow rate λ. The arrivals of customers are assumed to be independent across the firms (conditional on the state of the market). If the market is bad, then no firm ever observes a customer. Not seeing customers is then bad news to each firm and other things equal, would lead to a more pessimistic belief about the state of the market and eventually to a decision to exit.

We show that the equilibria of this model involve mixed strategies where sufficiently pessimistic firms exit the market at a positive rate. Not seeing the other firms exit is then good news to an individual firm that is still uncertain about the state of the market. In equilibrium, the negative news from not seeing customers is balanced with the good news from not seeing exits. At the moment of exit by one of the firms in the market, the posterior beliefs of the remaining firms about the market state jump down. If the equilibrium payoffs of all the firms were strictly increasing in the probability of being in a good market, this would lead all the remaining firms to exit immediately. This is nevertheless not consistent with equilibrium because firms that have seen a customer in the past would never leave and hence an individual firm would get an extremely informative signal by waiting just a bit longer.

Our main result is that in the sense of long-run efficient allocation of firms to the market, the market aggregates information efficiently in the limit as the number of firms gets large. By this we mean that in markets with many firms, almost all firms stay in a good market and all firms exit eventually from a bad market in all equilibria of the game. This is in contrast to the previous literature on observational learning including the herding models discussed below. At the same time the sum of payoffs to the firms is well below the efficient level. We show that the payoffs in the unique symmetric equilibrium of the model provide a lower bound for the Nash equilibrium payoffs in the game. We also show that the unique (asymmetric) pure strategy equilibrium of the game provides the players the highest sum of payoffs within the class of Nash equilibria.

In the symmetric equilibrium of the game, the exit of a firm triggers an immediate randomization by the remaining firms. If no other firm leaves, play resumes according to the symmetric equilibrium play of the model with one fewer firm and where no exits have been observed. If other firms exit, there is need for an additional randomization by the remaining firms and there is a possibility that the market collapses in the sense that most or all of the remaining firms exit.

In obtaining the limiting results for the case where the number of firms grows large, a key role is played by the relative probabilities of market collapse and returning to the equilibrium path of a model with fewer firms. We show that when the state of the market is good, the probability of a market collapse goes to zero when the number of firms in the market grows. It is clear that a bad market must eventually collapse.

We have made the somewhat unrealistic assumption that the profitability of the market is not affected by the number of active firms. The reason for this assumption is to maintain comparability with other models of observational learning where the only strategic effect between the players is through the informational externality. We have verified that the qualitative features of our model remain valid in a model where the probability of receiving a customer in any period depends negatively on the number of active firms as long as a good market is profitable even in the case that no firms exit. If this is not the case, then the analysis is complicated by considerations reminiscent of war of attrition.

This paper is related to two strands of literature. The literature on herding and observational learning has concentrated on the situation where the information is held by the agents at the beginning of the game. Many of the models also assume an exogenously given order of moves for the players. If we adopted these assumptions in our current model, we would get a result similar to the conclusions in e.g. the model of herding by Banerjee (1992). By relaxing these assumptions in a direction that we see as being quite natural, we see that the results also change considerably. Within this strand of literature, the most closely related paper to ours is Chamley & Gale (1994).[1] In that paper a number of firms are contemplating entry into an industry. Each firm has private information about the profitability of the market and the resulting game is a waiting game that mirrors our setting. The main result of Chamley and Gale shows that as actions can be taken at arbitrarily short intervals, the symmetric equilibrium of the game exhibits herding with positive probability. The key difference to our model is that in Chamley and Gale, no additional information arrives and this leads to very different conclusions in the end. Other papers that have studied the effects of endogenous timing on observational learning models include Caplin & Leahy (1994) and Gul & Lundholm (1995). Our model is quite close to Caplin & Leahy (1994) in its motivation, but as that paper assumes a continuum of firms in the market, the analytics of the model are quite different. For example, at the first instant of public information revelation all uncertainty is resolved in their model. In our model, information is revealed gradually over time even in the limit where the number of firms goes to infinity. In Gul & Lundholm (1995), the main emphasis is on determining whether better informed agents move first.

The second strand of literature that is directly relevant to our paper is the liter-

[1] An early contribution along these lines is also Mariotti (1992).

ature on strategic experimentation. We have borrowed the analytical model from a recent paper Keller, Rady & Cripps (2005).[2] Their model explores the Markov perfect equilibria of a model where all the observations by all of the agents are publicly observable. As a result, the motivation as well as the analysis of the two models turns out to be quite different in the end. Our model also differs from that in Keller et al. in that we assume exit to be irreversible. The reason for this assumption is that in a continuous time model with reversible entry and exit, the firms would find it easy to communicate to each other their observations through an exit followed by quick re-entry. In order to respect our assumption of imperfect observability, we assume exit decisions to be irreversible.[3] Finally, Moscarini & Squintani (2004) introduce privately held prior beliefs into a model of R&D race where the success of opponents is publicly observed.

The paper is organized as follows. Section 2 sets up the discrete time model. Section 3 provides the analysis of the symmetric and asymmetric equilibria of the model. In section 4, we prove our main theorem that in all equilibria of the exit game, almost all firms stay in the market if and only if the market is good when the number of firms is large and the time interval between periods is small. In Section 5, we compute the symmetric equilibrium explicitly in the limiting continuous time version of the model. Section 6 concludes.

[2] Another example of stopping games with publicly observed randomness is Décamps & Mariotti (2004).

[3] At a late stage in writing this paper, we became aware of a paper by Rosenberg, Solan & Vieille (2005) that analyzes games similar to ours. Their informational assumptions on signals that are observed at each stage are different from ours (they have a continuum of signals and as a result, they can concentrate on pure strategy equilibria). Furthermore, they do not analyze the case where the time interval between periods is small and as a result both the analysis and the results in the two papers are quite different. An earlier paper on multi-armed bandits and observational learning is Aoyagi (1998).

2 Model

In this section we present the model in discrete time. Time periods are denoted by $t = 0, 1, ..., \infty$. We denote by a constant $\Delta t > 0$ the time interval between any two consecutive periods t and $t+1$. The discount factor between two periods is

$$\delta = \frac{1}{1 + r\Delta t},$$

where r is the discount rate. Since it is not our purpose to analyze the effect of observation lags, we are ultimately interested in a limit where the firms can react to the observed actions instantaneously, which we obtain by letting $\Delta t \to 0$.

At the beginning of the game, N risk neutral firms have entered the market whose true profitability is uncertain.[4] We assume for simplicity that the market is either good or bad and use notation $M = g$ and $M = b$ to refer to these two possibilities. Define $P_g(\cdot) \equiv P(\cdot | M = g)$ and $P_b(\cdot) \equiv P(\cdot | M = b)$ to refer to probabilities of various events conditional on market being good and bad, respectively.

Initially all firms are equally optimistic about the state of the market. Denote the common prior probability that the market is good by p^0. If the market is good, a customer arrives at a firm at a constant probability $\lambda \cdot \Delta t$ within each period. The value of each customer to the firm is v. If the market is bad, no customer will ever arrive. This means that as soon as a firm observes a customer for the first time, it becomes evident for this firm that the market is good. We say that a firm is *informed* if it has seen a customer, otherwise a firm is *uninformed*. The state of the market is the same for all firms, i.e. we have a setting with symmetric payoffs and common values. Conditional on the market state, the arrivals of customers at different firms are independent.

At the beginning of each period, a firm that is still active in the market has a binary decision to make: either stay in the market or leave. Leaving is costless but irreversible. Once the firm has exited, it will never again face any costs or

[4] It makes no difference to the model that follows whether the firms have entered subject to a zero profit condition or not.

revenues. If the firm stays, it pays the per period (opportunity) cost $c \cdot \Delta t$, observes a signal indicating either an arrival or no arrival of a customer, and moves to the next period. We assume that $c < \lambda v$, which means that an informed firm will never want to exit the market, no matter what the other firms do. Within each period the firms act simultaneously, but they know each other's actions at all previous periods. However, they do not observe the arrivals of customers at other firms, and thus they do not know whether the other firms are informed or uninformed.[5] Note that new information arrives to the firms through two channels: their own market experience and observations on other firms' behavior. In the terminology of learning models, each firm engages simultaneously in experimentation and observational learning.

The history of firm i consists of the private history recording its own market experience (i.e. the arrivals of its customers), and the public history recording the actions of all the firms. However, since observing a customer reveals fully that the market is good, the only thing that matters in firms' own market experience is whether or not they have seen at least one customer. As it is always a strictly dominant strategy for any firm who has observed at least one customer to stay in the market forever, we simplify the analysis by *postulating* that any firm that has seen a customer always stays in the market. This has no effect on any results, but it allows us to restrict our analysis on the uninformed firms only. For those firms, the only relevant history is the public history, which we from here on call simply the *history.* We denote the history at period t by h^t and define it recursively as follows:

$$
\begin{aligned}
h^0 &= \emptyset \, , \\
h^t &= h^{t-1} \cup a^{t-1} \; \forall t \in \{1, 2, ...\} \, ,
\end{aligned}
$$

where $a^t = (a_1^t, ..., a_N^t)$ is a vector where each $a_i^t \in \{0, 1\}$ denotes an indicator for i staying in the market at period t. Denote by H^t the set of all possible histories up to

[5] Since exit is irreversible, we do not need to worry about the information of those firms that have already left the market. Hence, when we refer to *informed* and *uninformed* firms, we only mean those firms that are still active.

t and let $H = \bigcup_{t=0}^{\infty} H^t$. Since exit is irreversible, $a_i^t = 0$ implies that $a_i^{t'} = 0$ for all $t' > t$ in all elements of H^t. Denote by $H_i \equiv \left\{ h^t \in H \left| a_i^{t'} = 0 \forall t' < t \right. \right\}$ the set of histories, in which i has not yet left the market. Denote by $A\left(h^t\right) \equiv \left\{ i \in \{1, ..., N\} \left| h^t \in H_i \right. \right\}$ the set of firms that remain in the market at the beginning of period t after history h^t and by $n\left(h^t\right)$ the number of such firms.

A strategy for an uninformed firm i is a mapping

$$\sigma_i : H_i \rightarrow [0, 1]$$

that maps all histories where i is still active to a probability of exiting the market. The strategy profile is $\sigma = \{\sigma_1, ..., \sigma_N\}$.

As the game proceeds, the firms update their beliefs on the state of the market on the basis of their own market experience and on the exit behavior of the other firms. Given a history h^t and a strategy profile σ, consider a firm i that has not yet observed a customer. Then i's assessment for the probability that the market is good is defined by Bayes' rule. We denote this belief of an uninformed firm by $p_i\left(h^t; \sigma\right)$.[6] Note that different uninformed firms may have different beliefs after the same public history, because their strategies may be different and thus reveal different information to each other.

Note also that there are histories that are inconsistent with some strategy profiles, making Bayes' rule inapplicable. In particular, assume that at history h^t some firm j exits in period t even if this should not happen with a positive probability according to σ. Then we simply assume that all remaining firms update their beliefs to a level that would prevail if firm j did not exist in the first place, and then continue the subgame with one less firm present leaving firm j out in all subsequent belief updates. This arbitrary assumption concerning off-equilibrium beliefs has no effect on any results, but ensures that all equilibria that we will consider are Perfect Bayesian Nash equilibria.

The payoff of a firm is the expected discounted sum of future cash flows as estimated by each firm on the basis of its own market experience, observations of other

[6]For an informed firm the probability assessment that the market is good is trivially equal to 1.

firms' behavior, and initial prior probability p^0. Denote by $V_i\left(h^t;\sigma\right)$ the payoff of an uninformed firm i after history h^t and with profile σ. An informed firm will stay for ever, and its payoff is easy to calculate:

$$V^+ = \frac{(\lambda v - c)\,\Delta t}{1 - \frac{1}{1+r\Delta t}} = \frac{(1+r\Delta t)\,(\lambda v - c)}{r}.$$

In Sections 3 and 4, we analyze the equilibria of the model formally. A reader who wants to get an intuitive characterization first may want to go directly to Section 5.

3 Equilibrium

As a useful starting point, consider a monopoly firm that can only learn from its own market experiments. This firm faces an optimal stopping problem, where at the beginning of each period it must decide whether to stay for at least one more period or to exit permanently. Denote by p the current probability assessment that the market is good held by a monopoly firm, who is still in the market at the beginning of an arbitrary period, but has not seen a customer yet. If the firm stays one more period of length Δt in the market, but still receives no customer, then the new value $p+\Delta p$ is obtained by Bayes' rule:

$$p + \Delta p = \frac{p\,(1-\lambda\Delta t)}{p\,(1-\lambda\Delta t) + 1 - p} = \frac{p\,(1-\lambda\Delta t)}{1 - p\lambda\Delta t} = \frac{1-\lambda\Delta t}{\frac{1}{p} - \lambda\Delta t}. \tag{1}$$

Consider next the monopoly value function $V_m\left(p\right)$. If the firm exits, there is nothing more to receive or pay, and the stopping value must be 0. On the other hand, if the firm stays, it receives a customer at probability $p\lambda\Delta t$ in which case p jumps to 1 and the firm's value jumps to $V_m\left(1\right) = V^+ = \frac{(1+r\Delta t)(\lambda v - c)}{r}$. If there is no customer, p falls to $p+\Delta p$. The Bellman function can thus be written as:

$$V_m(p) = \max\left[0\,; -c\Delta t + pv\lambda\Delta t + \frac{1}{1+r\Delta t}\left\{ p\lambda\Delta t\left(\frac{(1+r\Delta t)(\lambda v - c)}{r}\right) + (1 - p\lambda\Delta t)\, V_m\left(\frac{1-\lambda\Delta t}{\frac{1}{p}-\lambda\Delta t}\right) \right\} \right]. \quad (2)$$

It is well known that the solution to this type of a sopping problem can be written as a threshold level p^* such that it is optimal to stop when $p < p^*$, while it is optimal to stay otherwise. Under the assumptions of the model, it must be that $0 < p^* < 1$. Further, $V_m(p)$ must be strictly increasing and convex when $p > p^*$, while it must be pasted to stopping value 0 at $p = p^*$. We will see that the monopoly threshold p^* has a crucial role also in the model with many firms. Denote $t^* = \min\{t\,|p_m^t < p^*\}$.

Let us now consider the model with N firms. We will consider symmetric and asymmetric equilibria separately, but before that we state a result valid in all equilibria. Since the model has no payoff externalities, it is easy to see that a firm can always guarantee at least the payoff of a monopoly firm in equilibrium, from which it follows immediately that no firm wants to exit earlier than a monopoly firm would. Proposition 1 below states this, but shows also that there can not be an equilibrium, where *all* firms would get a higher payoff than a monopoly firm.

Proposition 1 *Let σ be an equilibrium profile. After any h^t, it must be that $V_i(h^t;\sigma) \geq V_m(p_i(h^t;\sigma))$ for all $i \in A(h^t)$ and $V_i(h^t;\sigma) = V_m(p_i(h^t;\sigma))$ for some $i \in A(h^t)$. Further, whenever $p_i(h^t,\sigma) > p^*$, it must be that $\sigma_i(h^t) = 0$.*

Proof. *In the Appendix.* ■

Since $p_i(h^t,\sigma) > p^*$ for all $t < t^*$, we have:

Corollary 1 *In any equilibrium, all firms stay with probability one at all periods $t < t^*$.*

This means that there can never be any information sharing before time t^*, because the firms reveal information only via their exit behavior.

3.1 Symmetric equilibrium

In this section we consider equilibria in symmetric strategy profiles. A profile σ is symmetric if $\sigma_i(h^t) = \sigma_j(h^t)$ for all i and j and for all h^t. When σ is symmetric, all uninformed firms update their beliefs in the same way, and hence they all share a common probability $p(h^t; \sigma)$ that the state of the market is g. When analyzing symmetric equilibria, we may simply use $p \in (0,1)$ to denote this common belief.

Note that all uninformed firms have also the same (expected) payoff in the symmetric equilibrium. It follows from Proposition 1 that this common payoff must be the same as that of a monopoly firm. Hence, after an arbitrary history h^t, any firm would be just as well off if it decided to ignore all observations of the other firms from time t onwards. This means that in a symmetric equilibrium no firm is able to benefit from the information that the firms reveal to each other. This observation facilitates the analysis of the symmetric equilibrium.

We next model the information that the firms extract from each other when they use arbitrary symmetric strategies. Consider some arbitrary period when n firms remain in the market and play a strategy according to which they exit at probability $\pi \in [0,1]$ if uninformed. The probability that an arbitrary firm has observed a customer conditional that the market is good is crucial in determining the amount of information that can be extracted by observing the other firms' actions. Throughout the paper we use letter q to denote this probability. In period t, this conditional probability is $q^t \equiv 1 - (1 - \lambda \Delta t)^t$ and q without a superscript denotes an arbitrary value for this probability without referring to the calender time. Use $q^- = 1 - q$ as a shorthand for the complement, that is, the probability that an arbitrary firm is *un*informed conditional on the market being good.

Using the definitions above, we may denote by $X(\pi, n, q)$ the number of firms that exit in the period under consideration. This random variable has the following conditional distributions:

$$\begin{aligned} P_g\left(X\left(\pi,n,q\right)=k\right) &= \begin{pmatrix} n \\ k \end{pmatrix} \left(q^{-}\pi\right)^{k}\left(1-q^{-}\pi\right)^{n-k}, \\ P_b\left(X\left(\pi,n,q\right)=k\right) &= \begin{pmatrix} n \\ k \end{pmatrix} \pi^{k}\left(1-\pi\right)^{n-k}. \end{aligned}$$

As a first step, consider an individual firm with belief p, who observes the behavior of the n firms. This firm attaches the following unconditional distribution to this random variable:

$$\begin{aligned} P\left(X\left(\pi,n,q\right)=k\right) &= pP_g\left(X\left(\pi,n,q\right)=k\right)+\left(1-p\right)P_b\left(X\left(\pi,n,q\right)=k\right) \\ &= \begin{pmatrix} n \\ k \end{pmatrix} \pi^{k}\left[p\left(q^{-}\right)^{k}\left(1-q^{-}\pi\right)^{n-k}+\left(1-p\right)\left(1-\pi\right)^{n-k}\right]. \end{aligned} \tag{3}$$

Given that k firms exit, the belief of the outside observer jumps to a new value given by:

$$\begin{aligned} p+\Delta p &= \frac{pP_g\left(X\left(\pi,n,q\right)=k\right)}{pP_g\left(X\left(\pi,n,q\right)=k\right)+\left(1-p\right)P_b\left(X\left(\pi,n,q\right)=k\right)} \\ &= \frac{p\left(q^{-}\right)^{k}\left(1-q^{-}\pi\right)^{n-k}}{p\left(q^{-}\right)^{k}\left(1-q^{-}\pi\right)^{n-k}+\left(1-p\right)\left(1-\pi\right)^{n-k}}. \end{aligned} \tag{4}$$

Obviously, the greater the number of firms that exit, the lower the new belief of the observer.

To derive a symmetric equilibrium, we use the fact that whenever all firms apply mixed strategies, they must be indifferent between exiting and staying. In the following lemma we establish the conditions under which a unique probability $\pi^{*}\left(n,p,q\right)$ exists such that if $n-1$ firms exit according to this probability, then this provides the "n:th" firm just enough information to keep him indifferent between exiting and staying:

Lemma 1 *Consider the optimal decision of an individual firm with belief* p*, who may either exit the market now or stay one more period to observe the behavior of* $n-1 \in \{1, 2, ...\}$ *firms, each of whom exits with probability* π *if uninformed, and with probability* 0 *if informed. Let* $q \in (0,1)$ *be the probability that each individual firm is informed given that the market is good. Then there is a lower threshold belief* $\underline{p}(n,q) \in (0, p^*)$ *such that:*

1. *If* $p \leq \underline{p}(n,q)$*, then it is optimal to exit irrespective of* π
2. *If* $p \geq p^*$*, then it is optimal to stay irrespective of* π
3. *If* $p \in \left(\underline{p}(n,q), p^*\right)$*, then there is a unique* $\pi^*(n,p,q) \in (0,1)$ *such that when* $\pi = \pi^*(n,p,q)$*, the firm is indifferent between staying and exiting. When* $\pi < \pi^*(n,p,q)$*, it is optimal to exit while if* $\pi > \pi^*(n,p,q)$*, it is optimal to stay. Furthermore, if* $X(\pi, n-1, q) = 0$*, then* $p + \Delta p > p^*$*.*

Function $\underline{p}(n,q)$ *is continuous in* q *and decreasing in both* n *and* q*. Function* $\pi^*(n,p,q)$ *is continuous in* p *and* q *and decreasing in* n*,* p*, and* q*.*

Proof. In the Appendix. ■

The following proposition establishes the existence and uniqueness of a symmetric equilibrium, and utilizes Lemma 1 to characterize it:

Proposition 2 *The exit game has a unique symmetric equilibrium. The strategy profile* $\sigma^S = \left\{\sigma_1^S, ..., \sigma_N^S\right\}$ *in this symmetric equilibrium can be defined recursively as follows:*

For initial histories $h^0 \in H^0$*:*

$$\sigma_i^S\left(h_i^0\right) = \begin{cases} 0 & , \text{ if } p^0 \geq p^* \\ 1 & , \text{ if } p^0 < p^* \end{cases} , \quad i = 1, ..., N .$$

For histories $h^t \in H^t$ *extending to period* $t \in \{1, 2, ...\}$*:*

$$\sigma_i^S\left(h^t\right) = \begin{cases} 0 & , \text{ if } p^t \geq p^* \\ \pi^*\left(n^t, p^t, q^t\right) & , \text{ if } \underline{p}\left(n^t, q^t\right) < p^t < p^* \\ 1 & , \text{ if } p^t \leq \underline{p}\left(n^t, q^t\right) \end{cases} , i \in A\left(h^t\right),$$

where $n^t = n(h^t)$, $q^t = 1 - (1 - \lambda \Delta t)^t$, *and* p^t *is the common belief of all uninformed firms induced by* h^{t-1} *and* $\sigma^S(h^{t'})$, $t' < t$, *according to Bayesian rule.*

Proof. In the Appendix. ■

The symmetric equilibrium path can be verbally described as follows. In the beginning, given that p^0 is above the monopoly exit threshold p^*, all firms stay in the market at probability one. The firms continue to experiment in this manner until $t = t^*$ where the beliefs of the uninformed firms fall below p^*. At this point they start to randomize. All firms exit with probability $\pi^*(n^t, p^t, q^t)$ that keeps them indifferent between exiting and continuing. In each period, the remaining uninformed firms update their current beliefs after observing the number of exits. If no firm exits in $t = t^*$, then according to Lemma 1 the belief of each uninformed firm jumps strictly above p^*. Following this jump, all firms stay in the market with probability one until p falls back below p^* at which point the randomization starts over again. This is continued until all firms have either observed a customer or left the market. If at some point the belief of the uniformed firms falls below $\underline{p}(n^t, q^t)$, the market collapses as all remaining uninformed firms exit. In such a case, the uninformed firms are so pessimistic that they do not have enough information to release in order to keep each other indifferent between staying and exiting. Note that if the market is bad, all firms must eventually exit.

When Δt shrinks to zero, the equilibrium path can be described more explicitly. We will do that in Section 5.

3.2 Asymmetric Equilibria

The exit game has a number of asymmetric equilibria in addition to the symmetric one discussed above. For example, there is an asymmetric equilibrium in pure strategies that Pareto dominates the symmetric mixed strategy equilibrium. This equilibrium is interesting, because it gives the firms a particularly high total payoff.

In the pure strategy equilibrium the firms exit sequentially in a pre-determined order. At every period, each uninformed firm exits either at probability zero or at probability one. Since no firm ever exits if informed, a firm that exits at probability one conditional on being uninformed reveals fully its payoff relevant private history to the other firms. As soon as such a firm stays, all firms at later positions in the "exit sequence" learn that this firm has observed a customer, and consequently no firm will ever exit after that. The equilibrium is characterized in the following proposition:

Proposition 3 *The exit game has a unique (up to a permutation of the players) equilibrium in pure strategies that Pareto dominates the symmetric equilibrium. In this equilibrium, no firm exits at periods $t < t^*$, but at all periods $t \geq t^*$, $k^t > 0$ firms exit at probability one (if uninformed) until either i) all firms have exited, or ii) at some period $t' \geq t^*$ some firm that was supposed to exit stays, in which case all the remaining firms stay ever after. There is a unique sequence $\{k^t\}_{t=t^*}^T$ of positive integers for which $\sum_{t=t^*}^{T} k^t = N$ such that this behavior constitutes an equilibrium.*

Proof. In the Appendix. ■

To define an equilibrium, the sequence $\{k^t\}_{t=t^*}^T$ must be such that on the one hand all k^t uninformed firms that exit at period t are better off by doing so than by staying and observing the behavior of $k^t - 1$ firms, and on the other hand, all uninformed firms that stay must be better off by observing the behavior of k^t firms than by exiting. This condition is formalized in the proof of Proposition 3.

When the periods are short enough, the firms reveal their information in the pure strategy equilibrium sequentially one firm at a time:

Proposition 4 *There is an $\epsilon > 0$ such that if $\Delta t < \epsilon$, then at most one firm exits in each period in the pure strategy equilibrium.*

Proof. In the Appendix. ■

We conclude this section by proving that the pure strategy equilibrium delivers the maximal Nash equilibrium payoff to the players in the exit game. Taken together with the lower bound derived in the previous subsection for the symmetric mixed strategy equilibrium, we have obtained a partial characterization for the equilibrium payoff set of the game.

Proposition 5 *The pure strategy equilibrium maximizes the sum of payoffs in the set of Nash equilibrium payoffs.*

Proof. In the Appendix. ■

It is also worth pointing out that as $N \to \infty$ and $\Delta t \to 0$, the average expected continuation payoff of uninformed agents at date t^* approaches the first best optimal payoff of $\frac{\lambda v - c}{r}$.

4 Large Markets

In this section, we analyze the equilibria of the exit game as the number of firms gets large. We are interested in the case where firms can react to the observed actions of the competitors quickly and therefore we consider the double limit of the market where $\Delta t \to 0$ first and then $N \to \infty$.

The main result in this section and perhaps the main result of the entire paper is that in large markets, the long run equilibrium outcome is efficient with a probability converging to unity. To make this statement precise, we calculate the total number of exits in the market when the time interval between periods is Δt and the total number of firms in the market is N. Denote this random variable by $X(\Delta t, N)$. Our main theorem shows that for all $\varepsilon > 0$,

$$\lim_{N \to \infty, \Delta t \to 0} P_g \left(\frac{X(\Delta t, N)}{N} < \varepsilon \right) = 1$$

and

$$\lim_{N \to \infty, \Delta t \to 0} P_b \left(\frac{X(\Delta t, N)}{N} = 1 \right) = 1.$$

Hence almost all firms stay when the market is good, but all firms exit when the market is bad. The second statement follows immediately from the arguments in the previous section and therefore we concentrate on the first assertion in this section.

It is clear from the previous analysis that the result cannot hold for a finite N. It is not hard to see that the result also fails in the case where Δt is large. For large Δt, the cost of staying in the market for an additional period is not small and hence for sufficiently pessimistic beliefs, it is a dominant strategy for the firms to exit. It is then easy to see that in e.g. the symmetric equilibrium outlined above, there is an $\widehat{N} < \infty$ such that if at least $\widehat{N}$ firms exit, then the remaining firms exit as well. As a result, all firms exit the market with a positive (but quite possibly small) probability even when the market is good.

Theorem 1 *In all equilibria of the exit game, for all* $\varepsilon > 0$,

$$\lim_{N\to\infty,\Delta t\to 0} P_g \left(\frac{X(\Delta t, N)}{N} < \varepsilon \right) = 1.$$

Proof. In the Appendix. ■

The idea of the proof is that in a large market with no delays between observations and actions, it is very unlikely that a large number of firms exit, and at the same time their posterior beliefs remain so low that their decisions to exit are consistent with equilibrium behavior.

5 Computing the Equilibrium in Continuous Time

In this section, we compute the symmetric equilibrium in continuous time. We have two reasons for doing that. First, we want to illustrate the properties of the model in a notationally simpler and hopefully more transparent environment. Second, since the period length in discrete time may be interpreted as a delay between observations and reactions, it is natural to analyze the model as $\Delta t \to 0$ to eliminate any effects such observation lags might have on the results.

To build intuition, we use simple reasoning to derive the properties of the equilibrium from the first principles. We work directly in continuous time, but it is easy to check rigorously that where we end up is nothing but the equilibrium given in Proposition 2 as $\Delta t \to 0$.

In continuous time the firms discount future at flow rate $r > 0$, pay the flow opportunity cost $c > 0$, and meet customers at a Poisson rate λ (assuming the market is good; in a bad market no customers ever arrive). At each instant, the firms choose simultaneously whether to stay in the game or to take an irreversible exit decision. The firms are able to react to other firms' exit decisions instantaneously (that is, if a firm i exits at time t, another firm j is able to react to the bad news induced by i's exit and follow suit essentially at that same time moment, yet strictly after i). Note that this is a property of the discrete time model in the limit $\Delta t \to 0$.

Formalizing mixed strategies in continuous time is more subtle than in discrete time, because a firm may either exit at some flow probability ϕ such that the probability of exiting between t and $t + dt$ is ϕdt, or at a discrete probability π that gives a strictly positive probability measure to the event of exit exactly at t. It will be seen that in symmetric equilibrium all firms apply flow exit probabilities as long as information arrives gradually, which is the case as long as no one exits. However, as soon as a firm exits, a discrete amount of bad news is released, and this induces the remaining firms to apply a discrete exit probability to release enough information to keep each other indifferent between staying and exiting. A sequence of such discrete randomizations continues either until enough good news has been released to move the game back to the flow randomization mode, or until all firms have exited the game. Hence, the equilibrium exhibits phases of inaction coupled with waves of exit.

Consider first a monopoly firm experimenting in the market. The evolution of p as long as no customers arrive is given by a continuous time counterpart to (1):

$$\frac{dp}{dt} = -\lambda p \left(1 - p\right). \tag{5}$$

Denote by $V\left(p\right)$ the value function of a monopoly. Bellman function in the continuation region is:

$$\begin{aligned} rV\left(p\right)dt &= p\lambda v dt + E\left(dV\left(p\right)\right) \\ &= p\lambda v dt + p\lambda dt\left(\frac{\lambda v}{r} - V\left(p\right)\right) + \left(1 - p\lambda dt\right)V'\left(p\right)\lambda p\left(1-p\right)dt. \end{aligned}$$

The optimal stopping threshold p^* can be solved using value matching, i.e. $V\left(p^*\right) = \frac{c}{r}$ and smooth pasting, i.e. $V'\left(p^*\right) = 0$ to yield:

$$p^* = \frac{rc}{\lambda \left(v \left(r + \lambda \right) - c \right)}. \tag{6}$$

Moving to the case of multiple firms, we start by some immediate observations. First, since it is always possible to mimic the monopolist firm, it is never optimal to exit at a belief above p^*, regardless of the number of firms in the market. Second, there cannot be symmetric equilibria in pure strategies. To see why, suppose on the contrary that all uninformed firms exit at probability one at some $0 < p \leq p^*$ in the symmetric equilibrium. Since each firm has become informed with probability $p \left(1 - e^{-\lambda t}\right) > 0$, any individual firm observes instantaneously that the market is good with probability

$$p \left(1 - \left(e^{-\lambda t}\right)^{N-1}\right) > 0$$

by staying for dt in the market. The capital gain from staying for dt is hence

$$\left(\frac{\lambda v - c}{r} \right) p \left(1 - \left(e^{-\lambda t}\right)^{N-1}\right) > cdt$$

as dt is small. On the other hand, pure strategy profile commanding every firm to stay forever cannot be an equilibrium, because then observations regarding other firms would be uninformative and any individual firm should employ the optimal strategy of the monopolist.

Third, in any symmetric equilibrium, the firms must exit at a positive probability at $p = p^*$. To see why, suppose on the contrary that all firms stay at probability one until p falls to $p' < p^*$. Then there is no observational learning for $p \in (p', \infty]$ and by the solution to the monopolist's problem, we know that there is a profitable deviation to exit at all $p \in (p', p^*]$.

Finally, the probability at which the firms exit at $p = p^*$ must be interpreted in the sense of flow exit probabilities. If, on the contrary, the firms exited with a strictly positive instantaneous probability at $p = p^*$, then the posterior would jump with a positive probability to a value strictly above p^*. In that case the capital gain from staying for an additional dt would outweigh the cost of waiting cdt and this would contradict the optimality of exit for an individual firm. On the other hand, the randomizations must be "strong" enough to prevent p from falling below p^* if no firm exits, because otherwise the capital gain from staying could not cover the cost of waiting. Therefore, the requirement for equilibrium randomizations is that conditional on no firms exiting, the posterior of uninformed firms must remain exactly at p^*. Let us denote by $\phi(n,t)$ the equilibrium exit rate of each individual firm at the threshold belief p^*, given the number of firms n, and time t that induces conditional probability $q(t) \equiv 1 - e^{-\lambda t}$ with which a firm has seen a customer given that the market is good. Using the Bayes' rule, we get:

$$\phi(n,t) = \frac{\lambda}{(n-1)(1-e^{-\lambda t})}. \tag{7}$$

Notice that this implies that the total probability with which each firm i observes another firm exiting is independent of the number of other firms in the market.

On the other hand, when a single firm exits, the posterior falls immediately to level

$$p^-(t) = \frac{p^* e^{-\lambda t}}{1 - p^*(1-e^{-\lambda t})}. \tag{8}$$

In order to complete the description of the symmetric equilibrium, we need to specify the behavior of the firms at beliefs below p^*. At $p < p^*$, the firms must exit with a discrete probability. If they didn't, then beliefs would stay below p^* with probability 1 after an instant dt. By previous arguments, firms must exit with positive probability at all such p and hence the continuation payoff would be 0. Given that there is the positive opportunity cost cdt from staying in the market, such a strategy cannot be optimal. On the other hand, using the same argument as above, symmetric equilibrium randomization require that for all possible outcomes in the randomization, posterior beliefs stay below p^*. We must therefore construct an equilibrium by requiring that the posterior rises exactly to p^* conditional on no exits in the randomization.

Denote by $\pi(n,p,t)$ the symmetric exit probability of the uninformed firms at posterior p, when there are n firms left in the market. Firm i exits with probability

$\pi(n,p,t)$ if the market is bad. If the market is good, firm i has become informed with probability $1-e^{-\lambda t}$ and exits with probability $e^{-\lambda t}\pi(n,p,t)$. Hence requiring that the posterior be p^* conditional on no exits amounts to:

$$\frac{p\left(1-e^{-\lambda t}\pi(n,p,q)\right)^{n-1}}{p\left(1-e^{-\lambda t}\pi(n,p,t)\right)^{n-1}+(1-p)\left(1-\pi(n,p,t)\right)^{n-1}}=p^*.$$

Rewriting, we get

$$\frac{1-p^*}{p^*}\frac{p}{1-p}=\frac{\left(1-\pi(n,p,t)\right)^{n-1}}{\left(1-e^{-\lambda t}\pi(n,p,t)\right)^{n-1}}, \tag{9}$$

and we can solve for the unique $\pi(n,p,t)$ that satisfies this equation.

In order to analyze the equilibria as n grows, it is useful to take logarithms on the two sides of (9) and use the approximation $\ln(1-x)\approx -x$ for x small to get:

$$\pi(n,p,t)\underset{n\to\infty}{\to}\frac{-\ln\left(\frac{1-p^*}{p^*}\frac{p}{1-p}\right)}{(n-1)\left(1-e^{-\lambda t}\right)}\equiv\overline{\pi}(n,p,t). \tag{10}$$

Note that the number of firms that actually exit follows a binomial distribution. If the market is bad, the binomial parameters are $\overline{\pi}(n,p,t)$ and n, and if the market is good, the parameters are $e^{-\lambda t}\overline{\pi}(n,p,t)$ and n. According to (10), $\overline{\pi}(n,p,t)\cdot n$ converges to $-\ln\left(\frac{1-p^*}{p^*}\frac{p}{1-p}\right)/\left(1-e^{-\lambda t}\right)$ as n grows. This means that as $n\to\infty$, the distribution of the number of firms that exit approaches the Poisson distribution with parameter $-\ln\left(\frac{1-p^*}{p^*}\frac{p}{1-p}\right)/\left(1-e^{-\lambda t}\right)$ if the market is bad, and parameter $\left(-e^{-\lambda t}\ln\left(\frac{1-p^*}{p^*}\frac{p}{1-p}\right)\right)/\left(1-e^{-\lambda t}\right)$ if the market is good.

We have now constructed informally a symmetric equilibrium in the continuous time game. Its main features are: *i)* No firm exits at beliefs above the monopoly exit level p^*. *ii)* At posterior $p=p^*$, uninformed firms exit at a flow rate that keeps the beliefs of the uninformed unchanged as long as no other firm exits. *iii)* When a firm exits, the posterior of the uninformed firms falls below p^*. This starts a sequence of discrete exit randomizations - a wave of exit - such that at each round all uninformed firms exit with a strictly positive probability such that the number of exiting firms follows a Poisson distribution with a parameter that depends on the current level of p. This exit wave consisting of many such rounds takes place within an infinitely short time interval and stops either when all firms have exited (we call this a market collapse), or when no firm exits at some round, which causes p to jump back to p^* starting another phase of flow randomizations. Furthermore, *iv)* As $N \to \infty$ the probability that an individual firm exits when the market is good converges to 0. To see why, note that the probability distribution of the number of exiting firms within each round of an exit wave is independent of the total number of firms, as long as N is large. Therefore, as $N \to \infty$, the proportion of those firms that actually need to exit before the true market state is revealed to all firms reduces to zero.

It is useful to check that the equilibrium as described here corresponds to the equilibrium in discrete time. To do this, let us now consider the properties of the equilibrium characterized in Proposition 2 in the limit $\Delta t \to 0$. As long as no firm is exiting, the posterior of the uninformed firms falls according to the Bayes' rule (1), which converges to (5) as $\Delta t \to 0$. As the step size in the Bayes' rule is continuous in Δt, randomizations conditional on no exits take place at p close to p^* when Δt is small. At the same time, conditional on no exit in any randomization, $p + \Delta p \to p^*$ as $\Delta t \to 0$, because the cost of staying in the market converges to zero. Hence conditional on no exit, the posterior stays arbitrarily close to p^* and this is possible in the limit only if all firms randomize at the continuous exit rates calculated in (7). On the other hand, as soon as a firm exits, p falls substantially below p^*, and equilibrium randomizations $\pi^* (n, p, q)$ given in Lemma 1 converge to the solution of (9) as $\Delta t \to 0$. Therefore, what we have been describing in this section is indeed the equilibrium of Proposition 2 in the limit $\Delta t \to 0$.

At this stage, we can summarize our main economic findings. In the symmetric equilibrium that we constructed, the payoff of each individual firm is the same as it would be in the absence of observational learning. Firms exit the market at a much slower rate, however. In particular, when the number of firms is large, exit is slow enough to allow for almost perfect learning of the true market state in the long run. The cost of this learning is that firms stay in the market too long when the market is bad. To see this explicitly, let us now compute the arrival rate of market collapse in a large market conditional on the market being bad.

In a large bad market the exit waves arrive at rate $\lim_{n\to\infty} \phi (n, t) = \frac{\lambda}{\left(1-e^{-\lambda t}\right)}$. However, not all exit waves lead to market collapse. Denote by $p^c(t)$ the probability that an exit wave taking place at time t leads to market collapse, given that the market is bad (in a good market, probability of market collapses vanishes as $N \to \infty$). The posterior belief at the beginning of the exit wave is $p^-(t)$ as given in (8). The exit wave can only end at p jumping back to p^*, or at p going to zero in a market collapse. The subjective probability of an uninformed firm for the former possibility as calculated at the beginning of the exit wave is $p^-(t) + (1 - p^-(t))(1 - p^c(t))$, and for the latter $(1 - p^-(t))\, p^c(t)$. Since unconditional p is a martingale, the subjective expected value of p after the exit wave must be $p^-(t)$. This martingale condition can be written as:

$$p^* \left[p^-(t) + \big(1 - p^-(t)\big)(1 - p^c(t))\right] = p^-(t),$$

which, after using (6) and (8), is easy to solve for $p^c(t)$:

$$p^c(t) = 1 - e^{-\lambda t}.$$

Therefore, the rate at which the market collapse arrives must be

$$\eta = \lim_{n \to \infty} \phi(n, t) \cdot p^c(t) = \lambda,$$

that is, in a bad large market the collapse arrives at the same rate as a customer arrives to each firm in a good market. The intuition for this result is as follows. When N is large, the probability that a given small firm exits before the true market state has been (almost) fully revealed vanishes. However, we know that in a symmetric equilibrium each firm has the same payoff as a monopoly firm, which means that an uninformed firm must be kept indifferent between exiting and staying. This is only possible when the arrival of the signal that fully reveals that the market is bad (market collapse) arrives at the same rate as the signal that fully reveals that the market is good (a customer). In essence, after p has dropped to p^*, an individual uninformed firm in a large market sees the world as if waiting for a fully revealing signal that arrives at rate λ: with probability p^* the contents of the signal is a customer that pays a lump payment v and indicates that the market is good, and with probability $(1 - p^*)$ this signal is a market collapse that indicates that the market is bad and pays nothing. As long as no signal arrives, p stays at p^*, and the firm is indifferent between waiting and exiting.

Let us now contrast the symmetric equilibrium to the pure strategy equilibrium. In continuous time, the pure strategy equilibrium is easy to describe. At time t^*, the firms reveal their private history by exiting in sequence until either all firms have exited, or until one firm reveals that the market is good by staying. Everything takes place at time t^*, so the difference to the symmetric equilibrium is that the true state of the market is revealed faster. This explains why the payoffs are greater than in the symmetric equilibrium (except for the first firm in sequence to exit). Even if in a large market there is almost perfect learning in all equilibria (Theorem 1), different equilibria differ from each other in how long the firms stay in a bad market. The symmetric equilibrium is the worst in this sense, whereas the pure strategy equilibrium is the best. However, even in this equilibrium the firms stay in a bad market too long; information can never aggregate before $t = t^*$.

6 Conclusion

This paper shows that information is aggregated in large markets with exit in the long run sense. At the same time, we show that this does not imply that welfare of the firms would be close to the welfare resulting from full information sharing. In fact, in the symmetric equilibrium of the model, no firm benefits from observing the others. In asymmetric equilibria, some firms' payoffs are above the monopoly level, but even in large markets, welfare is always strictly below the payoffs in the case where past histories are publicly observed.

We have kept the model as simple as possible in order to highlight the mechanics of information generation. There are a number of directions for extending the model. In a market context it might be natural to assume that the rate of arrival of customers at a given firm depends negatively on the number of firms in the market. The main results in the paper would not change if the rate at which customers arrive is given by $\lambda(n)$.

Another possibility is to assume that the opportunity cost of staying in the market is private information to each firm. In this setting, the game has a symmetric pure strategy equilibrium and the model can address the issue of information aggregation about a common values variable in a setting with incomplete information about private values components of uncertainty. Our initial results suggest that the limit of this game as the heterogeneity of the firms is reduced towards zero corresponds exactly to the symmetric mixed strategies equilibrium of the present paper. Therefore, incomplete information concerning the firms' cost parameters could be used as a purification argument for our mixed strategy equilibrium. It may prove fruitful to consider more general specifications for the private benefits of the market participants in this setting.

7 Appendix

Proof of Proposition 1. If a firm would get less than a monopoly in σ, then this firm could deviate by ignoring the information obtained by observing the behavior of the other firms, and replicate the behavior of a monopoly firm. Since the model has no payoff externalities, this would guarantee the same payoff as a monopoly firm, and thus for all active firms $V_i(h^t;\sigma) \geq V_m(p_i(h^t;\sigma))$. In particular, a firm that would exit at $p_i(h^t,\sigma) > p^*$ would have a lower payoff than a monopoly firm, thus in equilibrium $p_i(h^t,\sigma) > p^*$ implies that $\sigma_i(h^t) = 0$. To show that $V_i(h^t;\sigma) = V_m(p_i(h^t;\sigma))$ for at least one active firm, it suffices to note that at any history, there must be some firm that is the next to exit at a positive probability, and since this firm chooses to do so without any further observations on the exit behavior of the other firms, this firm can not have a better payoff than a monopoly firm. ■

Proof of Lemma 1. Define a "one-step" continuation payoff function as the value of a hypothetical firm that stays in the market one more period to observe the actions of $n-1$ other firms, each of whom exits independently at probability π in case of being uninformed and at probability 0 in case of being informed, but after this specific period will ignore all observations about other firms, and instead will behave like a monopoly:

$$\begin{aligned} C_n(\pi, p, q) \quad \equiv \quad & -c\Delta t + pv\lambda\Delta t + \frac{1}{1+r\Delta t}\Bigg\{ \quad p\lambda\Delta t\left(\frac{(1+r\Delta t)(\lambda v - c)}{r}\right) \\ & + (1-p\lambda\Delta t)\sum_{k=0}^{n} P(X(\pi, n-1, q) = k)\cdot V_m(p+\Delta p) \quad \Bigg\}, \quad (11) \end{aligned}$$

where $V_m(\cdot)$ is defined by (2), and $P(X(\pi, n-1, q) = k)$ and $p + \Delta p$ are given by (3) and (4), respectively.

Take any parameter values in the range $\pi \in (0,1)$, $p \in (0,1)$, and $q \in (0,1)$. Clearly, $C_n(\pi, p, q)$ is continuous in all parameters and strictly increasing in p. Since $V_m(\cdot)$ is convex and an increase in π induces a mean preserving spread in $p + \Delta p$, it follows that $C_n(\pi, p, q)$ is also increasing in π. In particular, $V_m(\cdot)$ is strictly convex for $p > p^*$, and hence $C_n(\pi, p, q)$ is strictly increasing in π whenever a randomization of the firms induces p to jump above p^* at a positive probability. This means that $C_n(\pi, p, q)$ is strictly increasing in π at such parameter values that $C_n(\pi, p, q) = 0$.

When $\pi = 0$, observation gives no information, and hence $C_n(0, p, q)$ gives the payoff of a monopoly firm that is constrained to stay for at least one more period. Since at $p = p^*$ a monopoly firm is indifferent between continuing and staying, we must have $C_n(0, p^*, q) = V_m(p^*) = 0$. For any $\pi \in (0,1]$, we have $C_n(\pi, p^*, q) > 0$. In particular, $C_n(1, p^*, q) > 0$, while on the other hand it follows by direct calculation from (11) that $C_n(1, 0, q) = -c\Delta t < 0$. From the fact that $C_n(\cdot)$ is continuous and strictly increasing in p, it immediately follows that there is a unique $\underline{p}(n,q) \in (0, p^*)$ such that $C_n(1, p, q) = 0$ for $p = \underline{p}(n,q)$. Since $C_n(\cdot)$ is strictly increasing in p and increasing in π, it follows that for $p < \underline{p}(n,q)$, $C_n(\pi, p, q) < 0$ for any $\pi \in [0,1]$. Thus, for $p < \underline{p}(n,q)$ it is optimal to exit irrespective of π. On the other hand, from the fact that $C_n(\pi, p, q)$ is everywhere continuous and increasing in π, and strictly increasing in π when $C_n(\pi, p, q) = 0$, it follows that for any $p \in \left(\underline{p}(n,q), p^*\right)$ there is a unique $\pi^*(n, p, q) \in (0,1)$ such that $C_n(\pi^*(n, p, q), p, q) = 0$, meaning that the firm is indifferent between staying and exiting. It also follows that $C_n(\pi, p, q) < (>) \, 0$ for $\pi < (>) \, \pi^*(n, p, q)$, and hence it is strictly optimal to exit (stay). The fact that it is optimal to stay irrespective of π for $p \geq p^*$ follows trivially from the monopoly optimization problem.

The continuity and monotonicity properties of $\underline{p}(n,q)$ and $\pi^*(n,p,q)$ can be established by implicit differentiation of the conditions $C_n(\pi^*(n,p,q),p,q) = 0$ and $C_n\left(1,\underline{p}(n,q),q\right) = 0$, respectively. The fact that p must jump above p^* when no firm exits follows from the fact that in order to make the firm indifferent between staying one more period and continuing, $\pi^*(n,p,q)$ must induce a positive probability of moving p to a level that gives a strictly positive monopoly payoff, that is, above p^*. This must happen in particular if no firm exits, because this is the event that induces the most optimistic belief to the firm. ■

Proof of proposition 2. Since in a symmetric equilibrium all firms must have the same payoff after any history, it follows from Proposition 1 that $V_i(h^t;\sigma) = V_m(p_i(h^t;\sigma))$ for all $i \in A(h^t)$. This means that in checking whether a particular profile is an equilibrium, it suffices to consider the optimality of the *current period* actions at all possible histories of the game by taking as given that the payoff in the next period is the monopoly payoff $V_m(p_i(h^t;\sigma))$. It is then straight-forward to see that in all histories, where the current belief of the uninformed firms is $p^t \in \left(\underline{p}(n^t,q^t),p^*\right)$, the only symmetric action that leaves no possibility for a profitable deviation for any firm is the randomization with an exit probability that gives the one-step continuation payoff equal to zero to all uninformed firms. Since each of the n^t active firms have access to the randomization of $n^t - 1$ other firms, the unique exit probability that satisfies this requirement is According to Lemma 1 $\pi^*(n^t,p^t,q^t)$. Lemma 1 implies that for all histories where $p^t \geq p^*$, it is the dominant strategy for all firms to stay at probability one, and for all histories where $p^t \leq \underline{p}(n^t,q^t)$, it is the dominant strategy for all firms to exit at probability one. Thus, σ^S as defined in Proposition 2 is an equilibrium, and there can not be other symmetric equilibria. ■

Proof of Proposition 3. Take a profile σ that defines the behavior of the firms as it is described in Proposition 3. Let the number of firms that reveal information in σ within each period $t > t^*$ be given by a sequence $\{k^t\}_{t=t^*}^{T}$. Define this sequence so that for $t = t^*, t^* + 1, ...$:

$$k^t \equiv \min \left[N - \overline{k}^{t-1}; \min \left\{ n \in \{1, 2, ...\} \left| C_n \left(1, p^t, q^t\right) \geq 0 \right. \right\} \right], \tag{12}$$

where $\overline{k}^t = 0$ for $t = t^*$ and $\overline{k}^t = \sum_{t'=t^*}^{t} k_{t=0}^t$ for $t > t^*$. Function $C_n(\cdot)$ is the one-step continuation payoff function defined in the Proof of Lemma 1, $q^t = 1-(1-\lambda\Delta t)^t$, and p^t is the belief of an uninformed firm, who has observed the exit of $\overline{k}^{t'}$ firms at periods $t' = t^*, ..., t-1$ (and thus learnt that those firms have not observed a customer). Then, the sequence $\{k^t\}_{t=t^*}^T$ defining the number of exiting firms within each period is obtained by taking the strictly positive terms from the sequence $\{k^t\}_{t=t^*}^{\infty}$. It is clear that condition (12) defines a unique sequence. Starting from $t = t^*$, k^t is given by the smallest positive integer such that $C_n(1, p^t, q^t) \geq 0$, until this condition can not be satisfied by an integer smaller than $N - \overline{k}^{t-1}$. When this happens, $k^t = N - \overline{k}^{t-1}$ (meaning that all the remaining firms exit), and at all periods after this $k^t = 0$.

The description of the equilibrium strategies is completed as follows. In each period t, k^t firms with the smallest indices amongst the active firms are the ones to exit. If in any period t' an exit by a firm that exits with probability zero in equilibrium is observed, the strategies of the active firms remain exactly as on the equilibrium path. In other words, the remaining firms assign no informational content to such exits.[7]

To see that σ is an equilibrium, note that k^t is defined in (12) by taking the *smallest* number of firms such that when those k^t firms reveal their information, the remaining firms have a positive one-step continuation payoff. Thus, none of those k^t firms has an incentive to stay, because by deviating a firm would induce all the remaining firms to stay forever, and therefore this deviating firm would never receive any information from the remaining firms in the future. Hence, the appropriate payoff is given by the one-step payoff function, which in this case is negative as only $k^t - 1$ would reveal information to this deviating firm. On the other hand, (12) requires

[7] The full description of the equilibrium strategies is available from the authors upon request. They are notationally cumbersome but otherwise straightforward and we have omitted displaying them in order to save space.

that a firm that does not belong to the group of those k^t firms has a positive one-step continuation payoff. For these firms, the total payoff can be even higher than the one-step payoff, since they may get even more information from other firms in the future. By deviating (and exiting) such a firm would only get a payoff equal to zero, which obviously would not be optimal.

The equilibrium as described here is the only pure strategy equilibrium, because it is always a dominant strategy for all firms to stay in periods $t < t^*$, and for all $t \geq t^*$, any number $\widetilde{k}^t \neq k^t$ representing the number of exiting firms would allow a profitable deviation. If $\widetilde{k}^t$ were greater than k^t, any of the exiting firms would gain by staying, and if $\widetilde{k}^t$ were smaller than k^t, any of the staying firms would gain by exiting.

Finally, note that the uniqueness is up to a permutation of the firms, because we have not fixed the order in which the firms exit. Any permutation is an equilibrium, as long as it allocates k^t firms to exit at period t. ■

Proof of Proposition 4. When $\triangle t \to 0$, the cost of waiting one more period approaches zero. Therefore, for a firm with an arbitrary belief $p > 0$, there must be an $\epsilon(p)$ such that when $\triangle t < \epsilon(p)$, it is optimal for this firm to wait one more period if waiting fully reveals the information of another firm. Fix an arbitrary period length $\widetilde{\triangle t}$ and take the lowest belief that an uninformed firm can ever have before all firms have exited when the firms reveal their information one at the time in succeeding periods t^*, $t^* + 1$, ... Denote this lowest belief by p^- and take $\epsilon(p^-)$. When $\triangle t < \min\left(\epsilon(p^-), \widetilde{\triangle t}\right)$, observing the behavior of one firm is enough to keep the remaining firms better off than exiting, meaning that $C_1(1, p^t, q^t) > 0$ for all remaining firms at all p^t and q^t that are reached when firms exit one at a time in succeeding periods. Then condition (12) defines $k^t = 1$ for all $t = t^*, t^* + 1, ..., t^* + N - 1$. ■

Proof of proposition 5. Consider the problem of choosing strategies $\sigma = (\sigma_1, ..., \sigma_n)$ to

$$\max_{\sigma} \sum_{i=1}^{n} V_i(h; \sigma)$$
$$\text{s.t. } \sigma_i \max\{0, p_i(h) - p^*\} = 0.$$

In other words, player i can be chosen to exit with positive probability only if her posterior on g is at or below p^*. Since all Nash equilibria of the game satisfy the constraint, the claim is proved if we show that the pure strategy equilibrium solves the problem.

It is easy to see that the pure strategy equilibrium maximizes the sum of payoffs. The principle of unimprovability states that a path is optimal if there is no profitable one-step deviation to it. Given the definition of the pure strategy equilibrium, it is clear that it is not optimal for a firm to stay when it should exit (following firms stay in the market when they should not). Also, it is never optimal for a firm to exit when it should stay as the other firms' continuation strategies are unaffected by such exits and it is privately optimal for the firm to stay. ■

Proof of Theorem 1. We want to show that for any ε and δ, $\exists\ \overline{\Delta t} > 0$ and $\overline{N} > 0$ such that

$$P_g\left(\frac{X(\Delta t, N)}{N} \geq \varepsilon\right) < \delta.$$

whenever $\Delta t < \overline{\Delta t}$ and $N > \overline{N}$.

We start by various definitions. First, denote by $P_g^\sigma(\cdot)$ the probability of a given event conditional that the market is good when the players adopt strategy σ. Next, consider an *outside observer* who starts with the initial belief p^0 and observes the behavior of all the firms, but receives no signals of her own. We denote by $p_0(h^t; \sigma)$ the posterior belief of such an outside observer that has observed a history h^t and knows that the firms play according to σ. Finally, given σ, define for each history h^t a *randomization history* r^t as a list containing the data of the actual randomization probabilities used by all firms so far, *given that they are uninformed.* Formally, any pair (σ, h^t) induces a randomization history r^t as follows:

$$r^t = \left\{\left[\sigma_1\left(h^0\right), ..., \sigma_N\left(h^0\right)\right], ..., \left[\sigma_1\left(h^t\right), ..., \sigma_N\left(h^t\right)\right]\right\},$$

where $h^\tau \subset h^t$ is the truncation of history h^t up to $\tau < t$. Let us denote by R the set of all possible randomization histories, and by R^σ the set of all possible randomization histories that σ may induce at a positive probability. Note that in this definition, we have actually extended the definition of strategies so that they define exit probabilities also for the firms that have already exited. More precisely, if firm i

has exited during history h^t, we define the probability of exit for this firm to be zero, i.e. $\sigma_i(h^t) = 0$ for $i \notin A(h^t)$.

Note that a randomization history r^t is nothing but a series of independent Bernoulli trials, and hence every r^t induces a probability that a given history h^t occurs. Hence, we write $P_g^{r^t}(A)$ to refer to the probability that some $h^t \subset A$ occurs as a result of randomization history r^t (note that this probability has nothing to do with σ). Since every h^t maps to a single r^t (given σ), it must hold that for any $A \subset H$:

$$P_g^{\sigma}(A) \leq \max_{r^t \in R^{\sigma}} P_g^{r^t}(A). \tag{13}$$

Let us now begin the actual proof by fixing any $\varepsilon > 0$ and $\delta > 0$. Denote by $A_{\varepsilon}^{N,\Delta t}$ the set of such histories h^t where the number of firms that have exited exceeds εN at period t:

$$A_{\varepsilon}^{N,\Delta t} \equiv \left\{ h^t \in H \left| n\left(h^t\right) < N(1-\varepsilon) \right. \;\; \wedge n\left(h^t \backslash a^{t-1}\right) \geq N(1-\varepsilon) \right\}.$$

Let $B^{N,\Delta t,\sigma}$ be the set of histories h^t such that the belief of the outside observer is below p^* after history h^t:

$$B^{N,\Delta t,\sigma} = \left\{ h^t \in H \left| p_0\left(h^t;\sigma\right) < p^* \right. \right\}.$$

Take any equilibrium σ. Consider the possibility that after some period during which at least one firm exits, the belief of the outside observer is at or above p^*, that is, $p_0(h^t;\sigma) \geq p^*$ for some h^t for which $n(h^t) > n(h^t \backslash a^{t-1})$. This means that $p_0(h^t;\sigma)$ would have been above p^* by a fixed margin if no firm had exited, and therefore ex ante there was a positive probability that the belief of the outside observer would be strictly above p^* after that period. Since at the end of any period the outside observer has exactly the same information as those firms who did exit would have if they had stayed, it must be that any of those firms who actually did exit, faced a positive ex-ante probability that their own belief would be above p^* after this period, had they not exited. Since the cost of waiting one more period to observe the behavior of the other firms vanishes as $\Delta t \to 0$, their exit decision would not be consistent with equilibrium behavior if the period length is short enough. Hence, there must be some threshold level $\overline{\Delta t}$ such that whenever $\Delta t < \overline{\Delta t}$, it must hold in any equilibrium that $p_0(h^t;\sigma) < p^*$ whenever $n(h^t) > n(h^t \backslash a^{t-1})$. This means that $A_\varepsilon^{N,\Delta t} \subset B^{N,\Delta t,\sigma}$ whenever $\Delta t < \overline{\Delta t}$. Therefore, if σ is an equilibrium and $\Delta t < \overline{\Delta t}$, we have:

$$P_g^\sigma\left(A_\varepsilon^{N,\Delta t}\right) = P_g^\sigma\left(A_\varepsilon^{N,\Delta t} \cap B^{N,\Delta t,\sigma}\right).$$

Using (13), we may now write:

$$P_g^\sigma\left(A_\varepsilon^{N,\Delta t} \cap B^{N,\Delta t,\sigma}\right) \leq \max_{r^t \in R^\sigma} P_g^{r^t}\left(A_\varepsilon^{N,\Delta t} \cap B^{N,\Delta t,\sigma}\right)$$

Whenever a firm randomizes, there is a strictly positive probability that she is in fact informed (given $M = g$). Hence, if on a r^t a given randomization leads to an exit at a high probability, it must also lead to a release of positive information at a high probability. It is then clear that for any $r^t \in R^t$, there must be some $N_A(r^t)$ such that $\left\{P_{r^t}\left(A_\varepsilon^{N,\Delta t}\right) > \delta\right\} \Longrightarrow \left\{P_{r^t}\left(B_\gamma^{N,\Delta t,\sigma}\right) < \delta\right\}$ whenever $N > N_A(r^t)$. On the other hand, there must be a $N_B(r^t)$ such that $\left\{P_{r^t}\left(B_\gamma^{N,\Delta t,\sigma}\right) > \delta\right\} \Longrightarrow \left\{P_{r^t}\left(A_\varepsilon^{N,\Delta t}\right) < \delta\right\}$ whenever $N > N_B(r^t)$. Thus, whenever $N > \max(N_A(r^t), N_B(r^t))$, we must have $\min\left[P_{r^t}\left(A_\varepsilon^{N,\Delta t}\right), P_{r^t}\left(B_\gamma^{N,\Delta t,\sigma}\right)\right] < \delta$, which means that $P_{r^t}\left(A_\varepsilon^{N,\Delta t} \cap B_\gamma^{N,\Delta t,\sigma}\right) < \delta$. Define $\overline{N} = \max_{r^t \in R^t}\left[\max(N_A(r^t), N_B(r^t))\right]$. Then, we have

$$P_g^\sigma\left(A_\varepsilon^{N,\Delta t}\right) \leq \max_{r^t \in R^t} P_{r^t}\left(A_\varepsilon^{N,\Delta t} \cap B_\gamma^{N,\Delta t,\sigma}\right) < \delta$$

whenever $\Delta t < \overline{\Delta t}$ and $N > \overline{N}$. Since $P_g^\sigma\left(A_\varepsilon^{N,\Delta t}\right) = P_g\left(\frac{X(\Delta t,N)}{N} \geq \varepsilon\right)$, this completes the proof. ■

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