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Citation: [Journal of Mathematical Physics](#) **53**, 122108 (2012); doi: 10.1063/1.4770050

View online: <http://dx.doi.org/10.1063/1.4770050>

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## Relation between primes and nontrivial zeros in the Riemann hypothesis; Legendre polynomials, modified zeta function and Schrödinger equation

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(Received 14 September 2012; accepted 20 November 2012;  
published online 18 December 2012; publisher error corrected 26 December 2012)

We study the dependence between prime numbers and the real and imaginary parts of the nontrivial zeros of the Riemann zeta function. The Legendre polynomials and the partial derivatives of the Riemann zeta function are used to investigate the above dependence along with the Riemann hypothesis with physical interpretations. A modified zeta function with finite terms is defined as a new implement for the study of the zeta function and its zeros. © 2012 American Institute of Physics. [<http://dx.doi.org/10.1063/1.4770050>]

### I. INTRODUCTION

The distribution of prime numbers has been considered as one of the darkest mysteries in mathematics, despite its usefulness in various fields.<sup>1</sup> Analogically, prime numbers can be thought as the key for the number system or arithmetic.<sup>1</sup> In 1859, Bernhard Riemann in his memoir<sup>2</sup> extended the Euler definition<sup>3</sup> to a complex variable, proved its meromorphic continuation and functional equation, and established a relation between its zeros and the distribution of prime numbers.<sup>4</sup> In the paper, an explicit formula for  $\pi(x)$ , the number of primes less than a given number  $x$ , was derived, which indicates that the zeros of the Riemann zeta function control the oscillations of primes around their “expected” positions. According to Riemann, all of its nontrivial zeros must lie in the range  $0 \leq \text{Re}(s) \leq 1$ , and furthermore the nontrivial zeros of the zeta function

$$\zeta(s) = 1 + 1/2^s + 1/3^s + \dots \quad (\text{analytic for } \text{Re}(s) > 1)$$

are symmetrically distributed around the critical line

$$s = 1/2 + it.$$

Especially, he addressed that all nontrivial zeros of the zeta function have the real part equal to  $1/2$ , which is called the Riemann hypothesis.

Currently, it is widely believed that the distribution of the nontrivial zeros of the Riemann zeta function must have the answer to the distribution of prime numbers. The Riemann hypothesis is actually against the conventional idea that prime numbers should be randomly distributed because if the Riemann hypothesis is right, this alludes that a sort of functional relation exists between prime numbers and the imaginary parts of the nontrivial zeros of the Riemann zeta function.

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With its importance and intense scholarly interest, the Riemann hypothesis is now considered as one of the greatest unsolved problems in mathematics.<sup>1,13</sup> Over the earlier centuries, there have been numerous attempts to substantiate the Riemann hypothesis<sup>5-13</sup> as well as diverse trials of physicists to relate the zeta function and the distribution of its zeros to physical properties.<sup>14-47</sup> Thus, many physicists still have a keen interest in this issue.<sup>48-55</sup> Not merely has the secret of prime numbers been believed to be crucially important in mathematics and physics, but this is significantly of a great importance in many fields including information science.

Motivated by the importance of the problem, first, the functional relations in between primes and the real and imaginary parts of the nontrivial zeros are studied in this article. Moreover, we reconfirm such dependences once more by using the Legendre polynomials and by investigating the partial derivatives of the Riemann zeta function, and we extend the ideas to some possible problems in physics. Especially, expressing a single term of the Euler product formula in terms of Legendre polynomials gives rise to an interesting mathematical structure providing functions which show regularity and define an interesting virtual potential. Finally, we introduce a simple approach to the numerical calculation of the zeros on the critical line.

## II. RELATION BETWEEN IMAGINARY PARTS OF NONTRIVIAL ZEROS AND PRIME NUMBERS

Let us write a complex number as  $s = \sigma + it$  and call the  $N$ th nontrivial zero of the Riemann zeta function  $s_N = \sigma_N + it_N$ . Also, let the  $j$ th prime number be  $p_j$ . In fact, it is readily observed that there is a relation between the imaginary part of a nontrivial zero of the Riemann zeta function and every prime number, so that a relation between nontrivial zeros also exists. Considering the absolute value of the Riemann zeta function, we find that it is actually a sum of products where one is the function of only  $\sigma$  and the other is the function of only  $t$ . [Refer to Eq. (15).] Especially, note that  $t$  is included in the cosine terms expressed as  $\cos(t \ln p_j)$ . Varying  $t$ , every cosine term with respect to every prime number also varies at the same time. This implies that all of the cosine terms are functionally bound together. Restricting  $s \in \{z \in \mathbb{C} | \zeta(z) = 0\}$  in the critical strip, if the Riemann hypothesis is true, the imaginary parts will vary as  $N$  changes, totally independent of the constant  $\sigma_N$  for every  $N$ ; the truth is that the positions of the ordinates of the nontrivial zeros are dependent only on how primes are distributed.

For any  $N$ , observe that there is a uniquely defined  $f_{N,j \rightarrow i}$  such that

$$\cos(t_N \ln p_i) = f_{N,j \rightarrow i}(\cos(t_N \ln p_j)), \quad (1)$$

where  $i$  and  $j$  are arbitrarily chosen. Rewriting the above relation as

$$\cos(t_N \ln p_i) = \cos\left(t_N \ln p_j + \theta_N^{ji}\right), \quad (2)$$

we find

$$p_i = p_j e^{\frac{2n\pi + \theta_N^{ji}}{t_N}}. \quad (3)$$

Here,  $n$  can easily be evaluated once  $t_N$  is given while at this stage, we do not know whether or not  $\theta_N^{ji}$  has a relation with  $\sigma_N$ . Fortunately, the property of the ordinates of the nontrivial zeros that are uniformly distributed over the reals in the proper interval<sup>56-58</sup> allows us to know the information of the characteristics of  $t_N$ , including the fact that  $t_N$  is determined only by the distribution of prime numbers. Even if the exact formula of  $\theta_N^{ji}$  is unknown, given that  $t_N$ 's are uniformly distributed as is mentioned above, we have the following simultaneous Eq. (4) from (2) and the theorem of

directional statistics,<sup>59,60</sup>  $\bar{z} = \frac{1}{N} \sum_{n=1}^N e^{i\theta_n}$  with  $z_n = e^{i\theta_n}$  drawn from a circular uniform distribution. That is,

$$\sum_N z_N = \sum_N \bar{z} = 0, \text{ so that } \sum_N e^{i\theta_N^{ji}} = 0 \quad (4)$$

which are defined for each  $(i, j)$ , where  $z_N = e^{it_N \ln(\frac{p_i}{p_j})}$ . From this, we know that  $t_N$  is not determined by  $\sigma_N$  and that all of the ordinates of the nontrivial zeros of the Riemann zeta function and prime numbers are entangled all together.

On the other hand, if we vary  $N$ ,  $\sigma_N$  might vary as every  $x_i$  does, and so let us define a relation  $g_N$  such that  $\sigma_N = g_N(t_N)$  for any  $N \in \mathbb{N}$ . Then, observe that all of the imaginary parts of the nontrivial zeros and prime numbers can be thought as being responsible for the value of each  $\sigma_N$ , and thus we finally recognize the fact that there is a relation between  $\sigma_N$  and  $\sigma_{N'}$  for any  $N$  and  $N'$ . Suppose that the relation is not the identity transformation, i.e.,  $\sigma_N \neq \sigma_{N'}$ . Taking into account  $t_N = g_N^{-1}(\sigma_N)$  and  $t_{N'} = g_{N'}^{-1}(\sigma_{N'})$ , we directly face with the contradiction against the fact that the value of  $t_N$  is not dependent on  $\sigma_N$ . Hence, the relation is in fact the identity transformation, and so  $\sigma_N = \sigma_{N'}$ .

### III. RIEMANN ZETA FUNCTION AND LEGENDRE POLYNOMIALS

Expanding the Euler product form of the zeta function in terms of the Legendre polynomials and making use of their orthogonal property, we reproduce the results in Sec. II through reduction to the absurd. Also, an interesting mathematical structure is found by expressing a single term of the Euler product formula in terms of Legendre polynomials, providing functions which show regularity. Then, a virtual potential can be defined, the plot of which shows very intriguing patterns.

#### A. Theorems

**Theorem 1:** *The absolute value of the Riemann zeta function can be expressed as*

$$|\zeta(s)| = \sum_{\substack{l_1, l_2, \dots \\ m_1, m_2, \dots}} \left( \prod_{i=1}^{\infty} \alpha_{l_i, m_i} P_{l_i}(x_i) P_{m_i}(x_i) \right),$$

where  $P_n(x)$  is the  $n$ th Legendre polynomial,  $x_i = \cos(t \ln p_i)$ ,  $\alpha_{l_i, m_i} = \frac{2m_i+1}{2} \beta_{l_i, m_i}$  and

$$\beta_{l_i, m_i} = \begin{cases} 2p_i^{-l_i \sigma} & \text{if } m_i = 0 \\ 0 & \text{if } m_i \in \mathbb{N}. \end{cases}$$

*Proof:* Before proceeding, keep in mind that it is not necessary to obtain the analytic continuation for  $\text{Re}(s) > 0$  in this argument. See Appendix A.

The Riemann zeta function can be written in terms of every prime number as

$$\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s} = \prod_{i=1}^{\infty} \left\{ \sum_{j=0}^{\infty} \left( \frac{1}{p_i^s} \right)^j \right\}. \quad (5)$$

In the critical strip,  $0 < \sigma < 1$  (for any  $\sigma > 0$  indeed; the analytic region is  $\sigma > 1$ ),  $\left| \frac{1}{p_i^s} \right| = \frac{1}{p_i^\sigma} < 1$ , and so the convergence theorem of the complex geometric series allows us to write

$$\sum_{j=0}^{\infty} \left( \frac{1}{p_i^s} \right)^j = \frac{1}{1 - p_i^{-s}}. \quad (6)$$

Hence, we have the Euler product formula,

$$\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s} = \prod_{i=1}^{\infty} \frac{1}{1 - p_i^{-s}}. \quad (7)$$

For each  $p_i$ ,

$$\frac{1}{1 - p_i^{-s}} = \frac{1}{\sqrt{1 - 2p_i^{-\sigma} \cos(t \ln p_i) + p_i^{-2\sigma}}} e^{i\phi_i(\sigma,t)}, \tag{8}$$

where  $\phi_i(\sigma, t) = \tan^{-1} \left( \frac{\sin(t \ln p_i)}{\cos(t \ln p_i) - p_i^\sigma} \right)$ .

If we take the absolute value of the Riemann zeta function, then the phase factor,  $e^{i \sum_{i=1}^{\infty} \phi_i(\sigma,t)}$ , will vanish, i.e.,

$$|\zeta(s)| = \prod_{i=1}^{\infty} \frac{1}{\sqrt{1 - 2p_i^{-\sigma} \cos(t \ln p_i) + p_i^{-2\sigma}}}. \tag{9}$$

Expanding (9) with the Legendre polynomials as is done in the multipole expansion for the axial multipole moment in electrostatics, we have

$$\frac{1}{\sqrt{1 - 2p_i^{-\sigma} \cos(t \ln p_i) + p_i^{-2\sigma}}} = \sum_{l_i=0}^{\infty} p_i^{-l_i\sigma} P_{l_i}(x_i). \tag{10}$$

Writing

$$p_i^{-l_i\sigma} = \sum_{m_i=0}^{\infty} \alpha_{l_i m_i} P_{m_i}(x_i), \tag{11}$$

where

$$\alpha_{l_i m_i} = \frac{2m_i + 1}{2} \beta_{l_i m_i}, \tag{12}$$

$$\beta_{l_i m_i} = \int_{-1}^1 p_i^{-l_i\sigma} P_{m_i}(x_i) dx_i = \begin{cases} 2p_i^{-l_i\sigma} & \text{if } m_i = 0 \\ 0, & \text{if } m_i \in \mathbb{N}, \end{cases} \tag{13}$$

then

$$|\zeta(s)| = \prod_{i=1}^{\infty} \left\{ \sum_{l_i=0}^{\infty} \left( \sum_{m_i=0}^{\infty} \alpha_{l_i m_i} P_{m_i}(x_i) \right) P_{l_i}(x_i) \right\} \tag{14}$$

or equivalently,

$$|\zeta(s)| = \sum_{\substack{l_1, l_2, \dots \\ m_1, m_2, \dots}} \left( \prod_{i=1}^{\infty} \alpha_{l_i m_i} P_{l_i}(x_i) P_{m_i}(x_i) \right). \tag{15}$$

Here,  $\alpha_{l_i m_i}$  is the function of only  $\sigma$ , while  $P_{l_i}(x_i)$  and  $P_{m_i}(x_i)$  is the function of only  $t$ . □

**Theorem 2:** For any fixed  $\alpha \in \mathbb{R}$ , the fractional parts of the numbers  $\alpha\gamma$ , where  $\beta + i\gamma$  runs over all of the zeros of the Riemann zeta function in the critical strip with  $0 < \gamma < T$ , become uniformly distributed in  $\mathbb{R}/k\mathbb{Z}$  for any  $k \in \mathbb{R}$  as  $T \rightarrow \infty$ .

*Proof:* See the theorems of Fujii and Hlawka.<sup>56-58</sup> □

*Corollary 2.1:* For any fixed  $i \in \mathbb{N}$  and  $t \in \mathbb{R} - \{0\}$ ,

$$X_i = \{x_i | x_i = \cos(t_N \ln p_i), N \in \mathbb{N}\}$$

$$X_t = \{x_t | x_t = \cos(t \ln p_i), i \in \mathbb{N}\}$$

are dense.

*Proof:* Theorem 2 directly guarantees that  $X_i$  is dense, while the fact that  $X_i$  is dense can be proved with the Prime Number Theorem, providing that  $\ln p_{i+1} - \ln p_i < \frac{\epsilon}{2i}$  for a large enough  $i$  where  $\epsilon > 0$  is fixed. This supports the argument that given  $\epsilon > 0$ , for any  $x \in [-1, 1]$ , there exists  $p_i$  such that  $|\ln p_i - \cos^{-1}x - 2n\pi| < \epsilon$  for a large enough  $n$ .  $\square$

*Lemma 3:* At least for one pair of  $i$  and  $j$ , there exists  $N$  such that

$$\cos(t_N \ln p_i) = f_{N,j \rightarrow i}(\cos(t_N \ln p_j)), \tag{16}$$

where  $f_{N,j \rightarrow i}$  is a relation between  $(t_N, p_i)$  and  $(t_N, p_j)$ .

*Proof:* First, let us restrict

$$s \in \{z = 1/2 + it \in \mathbb{C} | \zeta(z) = 0\} \text{ in (15). Then,}$$

we have

$$\sum_{\substack{l_1, l_2, \dots \\ m_1, m_2, \dots}} \left( \prod_{i=1}^{\infty} \alpha_{l_i m_i} P_{l_i}(x_i) P_{m_i}(x_i) \right) = 0. \tag{17}$$

Suppose that there is no  $N$  satisfying the above relation for any  $i$  and  $j$ . Namely, let us assume  $x_i$  to be an independent variable. If we could (actually impossible as seen in Sec. II), we would not face any problem after taking integrations for every  $x_i$ . (The integrations can be taken because of Corollary 2.1.) Let us rewrite (17),

$$\sum_{\substack{l_1, l_2, \dots \\ m_1, m_2, \dots}} \left( \prod_{i=1}^{\infty} \beta_{l_i m_i} \right) \left( \prod_{i=1}^{\infty} \frac{2m_i + 1}{2} P_{l_i}(x_i) P_{m_i}(x_i) \right) = 0. \tag{18}$$

Here, not merely is  $x_i$  dependent on  $t_N$ , but also the corresponding real part in  $\beta_{l_i m_i}$  might be so. Considering  $\sigma(t_N) = \frac{1}{2} + \varepsilon(t_N)$ , we may want to redefine (13) as

$$\beta_{l_i m_i}(t_N) = 2p_i^{-l_i(\frac{1}{2} + \varepsilon(t_N))} \delta_{0, m_i}. \tag{19}$$

If  $s_N$  is on the critical line,  $\varepsilon(t_N) = 0$ . Otherwise, there would always exist the unique  $s_{N'}$  such that  $t_N = t_{N'}$  with  $\varepsilon(t_N) + \varepsilon(t_{N'}) = 0$  which would probably hinder us from making use of integrations. However, we are not going to face with such a problem later once we accept the result in Sec. II as a fact, taking  $\varepsilon(t_N) = 0$  for any  $N$ .

Since  $\beta_{l_i m_i} \in [0, 2]$  with  $\lim_{l_i \rightarrow \infty} \beta_{l_i m_i} = 0$  and  $\beta_{l_i m_i} |_{m_i > 0} = 0$ , the following integration in relation to (18) has the property:

$$\prod_{i=1}^{\infty} \frac{2m_i + 1}{2} \int_{-1}^1 P_{l_i}(x_i) P_{m_i}(x_i) dx_i = \prod_{i=1}^{\infty} \delta_{l_i, m_i} \in \{0, 1\}, \tag{20}$$

where  $\delta_{l_i, m_i}$  is the Kronecker delta. Therefore, for every combination of  $l_i$ 's and  $m_i$ 's, it is guaranteed to have

$$\left( \prod_{i=1}^{\infty} \int_{-1}^1 dx_i \right) \left( \prod_{i=1}^{\infty} \frac{2m_i + 1}{2} \beta_{l_i m_i} P_{l_i}(x_i) P_{m_i}(x_i) \right) \geq 0. \tag{21}$$

Especially, considering the case that  $l_i = 0$  and  $m_i = 0$  for every  $i \in \mathbb{N}$ , i.e.,  $\beta_{00} = 2$  and  $\delta_{0,0} = 1$ , we can simply notice the fact that

$$\sum_{\substack{l_1, l_2, \dots \\ m_1, m_2, \dots}} \prod_{i=1}^{\infty} \int_{-1}^1 \frac{2m_i + 1}{2} \beta_{l_i m_i} P_{l_i}(x_i) P_{m_i}(x_i) dx_i = \infty, \tag{22}$$

which is contradictory to (18), resulting from the wrong assumption that there is no relation between  $x_i$ 's. Therefore, the proof is complete.  $\square$

*Corollary 3.1:* The function  $f_{N,j \rightarrow i}$  in Lemma 3 is uniquely defined.

*Proof:* Suppose that there exists another function  $g$  satisfying the relation in Lemma 3. Then, we immediately get  $g = f_{N,j \rightarrow i}$  from

$$\cos(t_N \ln p_i) = f_{N,j \rightarrow i}(\cos(t_N \ln p_j)) = g(\cos(t_N \ln p_j))$$

$$g^{-1} \circ f_{N,j \rightarrow i}(\cos(t_N \ln p_j)) = \cos(t_N \ln p_j). \tag{23}$$

□

*Remark:*

- (1)  $f_{N,j \rightarrow i}$  is invertible.
- (2) Using  $\theta_N^{ji}$ , the relation  $f_{N,j \rightarrow i}$  can be re-expressed as follows:  $\cos(t_N \ln p_i) = \cos(t_N \ln p_j + \theta_N^{ji})$ .
- (3)  $\theta_N^{ji}$  is independent of  $\sigma_N$ .

**Theorem 4:** For any  $i$  and  $j$ , Lemma 3 holds.

*Proof:* Repeating the similar argument in Lemma 3, it is generalized to Theorem 4. That is, suppose that only for two different  $q$  and  $r$ , there exists  $N$  satisfying Lemma 3. Here, we proceed the same steps in the proof of Lemma 3 except that we do not integrate for the (supposedly) independent variables  $x_q$  and  $x_r$ , and so let us start from the following:

$$\sum_{\substack{l_1, l_2, \dots \\ m_1, m_2, \dots}} \prod_{i \in \mathbb{N}/\{q,r\}} \int_{-1}^1 \alpha_{l_i m_i} P_{l_i}(x_i) P_{m_i}(x_i) dx_i = \left[ \sum_{\{l_i, m_i\} \in \mathbb{N}/\{q,r\}} \left( \prod_{i \in \mathbb{N}/\{q,r\}} \int_{-1}^1 \frac{2m_i + 1}{2} \beta_{l_i m_i} P_{l_i}(x_i) P_{m_i}(x_i) dx_i \right) \right] \\ \times \left[ \sum_{\substack{l_q, l_r \\ m_q, m_r}} \frac{1}{4} (2m_q + 1) (2m_r + 1) \beta_{l_q m_q} \beta_{l_r m_r} P_{l_q}(x_q) P_{m_q}(x_q) P_{l_r}(x_r) P_{m_r}(x_r) \right]. \tag{24}$$

Similar to (22), the former square bracket part diverges to positive infinity. Moreover, note that the latter one – let us call it  $L$  – can be expressed as

$$L = \left[ \sum_{l_q} p_q^{-l_q \sigma} P_{l_q}(x_q) \right] \left[ \sum_{l_r} p_r^{-l_r \sigma} P_{l_r}(x_r) \right] = \frac{1}{\sqrt{1 - 2p_q^{-\sigma} \cos(t \ln p_q) + p_q^{-2\sigma}}} \\ \times \frac{1}{\sqrt{1 - 2p_r^{-\sigma} \cos(t \ln p_r) + p_r^{-2\sigma}}} \tag{25}$$

and that  $L$  can never be zero. Thus, these facts finally lead to contradiction. Note that we can arbitrarily choose  $x_q$  and  $x_r$ , and therefore, the proof of Theorem 4 is complete. □

*Corollary 4.1:* For any  $i, j$  and  $N \in \mathbb{N}$ , there exists  $\theta_N^{ji}$  such that  $p_i = p_j e^{\frac{2n\pi + \theta_N^{ji}}{t_N}}$ , where  $n = \left\lceil \frac{t_N \ln(\frac{p_i}{p_j})}{2\pi} \right\rceil$ .

*Proof:* Refer to Sec. II. □

TABLE I. The first few  $\gamma_{l,m_t}$ .

$m_t$	$\gamma_{l,m_t}$
0	$\frac{1}{l^2 R^2 + 1}$
1	$\frac{1}{l^2 R^2 + 4}$
2	$\frac{(0+l_t^2 R^2)}{(1+l_t^2 R^2)(9+l_t^2 R^2)}$
3	$\frac{(1+l_t^2 R^2)}{(4+l_t^2 R^2)(16+l_t^2 R^2)}$
4	$\frac{(0+l_t^2 R^2)(4+l_t^2 R^2)}{(1+l_t^2 R^2)(9+l_t^2 R^2)(25+l_t^2 R^2)}$
5	$\frac{(1+l_t^2 R^2)(9+l_t^2 R^2)}{(4+l_t^2 R^2)(16+l_t^2 R^2)(36+l_t^2 R^2)}$
6	$\frac{(0+l_t^2 R^2)(4+l_t^2 R^2)(16+l_t^2 R^2)}{(1+l_t^2 R^2)(9+l_t^2 R^2)(25+l_t^2 R^2)(49+l_t^2 R^2)}$
7	$\frac{(1+l_t^2 R^2)(9+l_t^2 R^2)(25+l_t^2 R^2)}{(4+l_t^2 R^2)(16+l_t^2 R^2)(36+l_t^2 R^2)(64+l_t^2 R^2)}$
8	$\frac{(0+l_t^2 R^2)(4+l_t^2 R^2)(16+l_t^2 R^2)(36+l_t^2 R^2)}{(1+l_t^2 R^2)(9+l_t^2 R^2)(25+l_t^2 R^2)(49+l_t^2 R^2)(81+l_t^2 R^2)}$
9	$\frac{(1+l_t^2 R^2)(9+l_t^2 R^2)(25+l_t^2 R^2)(49+l_t^2 R^2)}{(4+l_t^2 R^2)(16+l_t^2 R^2)(36+l_t^2 R^2)(64+l_t^2 R^2)(100+l_t^2 R^2)}$
10	$\frac{(0+l_t^2 R^2)(4+l_t^2 R^2)(16+l_t^2 R^2)(36+l_t^2 R^2)(64+l_t^2 R^2)}{(1+l_t^2 R^2)(9+l_t^2 R^2)(25+l_t^2 R^2)(49+l_t^2 R^2)(81+l_t^2 R^2)(121+l_t^2 R^2)}$

**Theorem 5:** For any fixed  $\sigma$ ,  $t \in \mathbb{R}$  with  $R = \frac{\sigma}{l}$ ,

$$\frac{1}{|1 - p_i^{-s}|} = \sum_{l_t=0}^{\infty} \sum_{m_t=0}^{\infty} \alpha_{l,m_t} P_{m_t}(x_t) P_{l_t}(x_t),$$

where

$$\alpha_{l,m_t} = \frac{2m_t + 1}{2} \beta_{l,m_t} \gamma_{l,m_t}, \quad \beta_{l,m_t} = 1 + (-1)^{m_t} e^{-l_t R \pi},$$

$$\gamma_{l,m_t} = \prod_{j=0}^{m_t+1} (j^2 + (l_t R)^2)^{S(j,m_t)} \text{ and } S(j,m_t) = \begin{cases} 1, & j + m_t \in 2\mathbb{N} \\ 0, & \{(j = 0) \wedge (m_t \in 2\mathbb{N} - 1)\} \\ -1, & j + m_t \in 2\mathbb{N} - 1. \end{cases}$$

*Proof:* The proof is omitted; through the mathematical induction, we can get the general forms of  $\beta_{l,m_t}$  and  $\gamma_{l,m_t}$ . Actually, they can empirically be deduced, and we might go through the following evaluations: (26)–(29), to find the rule. See Table I.

$$\begin{aligned} \int_{-1}^1 e^{-l_t R \cos^{-1} x_t} P_0(x_t) dx_t &= \int_{-1}^1 e^{-l_t R \cos^{-1} x_t} dx_t = - \int_{\pi}^0 e^{-l_t R u_t} \sin u_t du_t = \left[ \frac{e^{-l_t R u_t} (\cos u_t + l_t R \sin u_t)}{l_t^2 R^2 + 1} \right]_{\pi}^0 \\ &= \frac{1}{l_t^2 R^2 + 1} + \frac{e^{-l_t R \pi}}{l_t^2 R^2 + 1} = (1 + e^{-l_t R \pi}) \frac{1}{l_t^2 R^2 + 1} \Rightarrow \beta_{l,0} = 1 + e^{-l_t R \pi}, \quad \gamma_{l,0} = \frac{1}{l_t^2 R^2 + 1}, \end{aligned} \tag{26}$$

$$\begin{aligned} \int_{-1}^1 e^{-l_t R \cos^{-1} x_t} P_1(x_t) dx_t &= \int_{-1}^1 x e^{-l_t R \cos^{-1} x_t} dx_t = - \int_{\pi}^0 e^{-l_t R u_t} \cos u_t \sin u_t du_t = - \frac{1}{2} \int_{\pi}^0 e^{-l_t R u_t} \sin 2u_t du_t \\ &= \left[ \frac{e^{-l_t R u_t} (l_t R \sin 2u_t + 2 \cos 2u_t)}{2(l_t^2 R^2 + 4)} \right]_{\pi}^0 = (1 - e^{-l_t R \pi}) \frac{1}{l_t^2 R^2 + 4} \Rightarrow \beta_{l,1} = 1 - e^{-l_t R \pi}, \quad \gamma_{l,1} = \frac{1}{l_t^2 R^2 + 4}, \end{aligned} \tag{27}$$



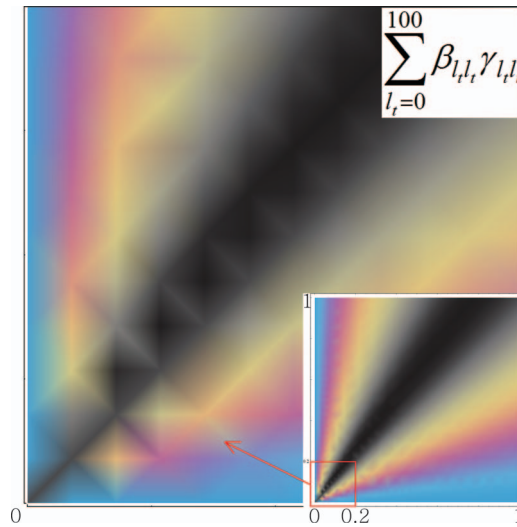


FIG. 1. The density plot of the integral in (30). The first 101 terms of  $\sum_{l_t} \beta_{l_t} \gamma_{l_t}$  are considered. The abscissa and ordinate correspond to  $\sigma$  and  $t$ , respectively.

$$\begin{aligned}
 \int_{-1}^1 e^{-l_t R \cos^{-1} x_t} P_2(x_t) dx_t &= \int_{-1}^1 \frac{1}{2} (3x^2 - 1) e^{-l_t R \cos^{-1} x_t} dx_t = -\frac{1}{4} \int_{\pi}^0 e^{-l_t R u_t} (3 \cos 2u_t + 1) \sin u_t du_t \\
 &= -\frac{3}{4} \int_{\pi}^0 e^{-l_t R u_t} \cos 2u_t \sin u_t du_t - \frac{1}{4} \int_{\pi}^0 e^{-l_t R u_t} \sin u_t du_t = -\frac{3}{8} \int_{\pi}^0 e^{-l_t R u_t} (\sin 3u_t - \sin u_t) du_t \\
 &= -\frac{1}{4} \int_{\pi}^0 e^{-l_t R u_t} \sin u_t du_t = \left[ -\frac{e^{-l_t R u_t} (l_t R \sin u_t + \cos u_t)}{8 (l_t^2 R^2 + 1)} + \frac{e^{-l_t R u_t} (3l_t R \sin 3u_t + 9 \cos 3u_t)}{8 (l_t^2 R^2 + 9)} \right]_{\pi}^0 \\
 &= (1 + e^{-l_t R \pi}) \frac{l_t^2 R^2}{l_t^4 R^4 + 10l_t^2 R^2 + 9} \Rightarrow \beta_{l_t,2} = 1 + e^{-l_t R \pi}, \gamma_{l_t,2} = \frac{l_t^2 R^2}{(1 + l_t^2 R^2)(9 + l_t^2 R^2)},
 \end{aligned} \tag{28}$$

$$\begin{aligned}
 \int_{-1}^1 e^{-l_t R \cos^{-1} x_t} P_3(x_t) dx_t &= \int_{-1}^1 \frac{1}{2} (5x^3 - 3x) e^{-l_t R \cos^{-1} x_t} dx_t = \frac{1}{4} \int_{\pi}^0 e^{-l_t R u_t} (1 - 5 \cos 2u_t) \cos u_t \sin u_t du_t \\
 &= \frac{1}{8} \int_{\pi}^0 e^{-l_t R u_t} \sin 2u_t du_t - \frac{5}{16} \int_{\pi}^0 e^{-l_t R u_t} \sin 4u_t du_t \\
 &= \left[ -\frac{e^{-l_t R u_t} (l_t R \sin 2u_t + 2 \cos 2u_t)}{8 (l_t^2 R^2 + 4)} + \frac{e^{-l_t R u_t} (5l_t R \sin 4u_t + 20 \cos 4u_t)}{16 (l_t^2 R^2 + 16)} \right]_{\pi}^0 \\
 &= (1 - e^{-l_t R \pi}) \frac{1 + l_t^2 R^2}{l_t^4 R^4 + 20l_t^2 R^2 + 64} \Rightarrow \beta_{l_t,3} = 1 - e^{-l_t R \pi}, \gamma_{l_t,3} = \frac{1 + l_t^2 R^2}{(4 + l_t^2 R^2)(16 + l_t^2 R^2)}.
 \end{aligned} \tag{29}$$

□

*Remark:* A virtual potential at a fixed  $s = \sigma + it$  can be defined as follows. The density plot with respect to  $\sigma$  and  $t$  is given in Fig. 1.

$$\begin{aligned}
 \left[ \int_{i \in \mathbb{N}} \frac{1}{|1 - p_i^{-s}|} \right]_s &= \int_{-1}^1 dx_t \sum_{l_t, m_t} \alpha_{l_t, m_t} P_{m_t}(x_t) P_{l_t}(x_t) = \sum_{l_t} \beta_{l_t} \gamma_{l_t} \\
 &= \beta_{00} + \frac{\beta_{11}}{1R^2 + 4} + \frac{\beta_{22}(0 + 4R^2)}{(1 + 4R^2)(9 + 4R^2)} + \frac{\beta_{33}(1 + 9R^2)}{(4 + 9R^2)(16 + 9R^2)} + \frac{\beta_{44}(0 + 16R^2)(4 + 16R^2)}{(1 + 16R^2)(9 + 16R^2)(25 + 16R^2)} + \dots
 \end{aligned} \tag{30}$$

**B. Expatiation upon the above mathematical results**

Theorem 1 allows us to express the zeta function in terms of the Legendre polynomials. In order to make use of integrations to reach Corollary 4.1 and Theorem 5, it is necessary to prove Corollary 2.1 since  $x_i$  and  $x_r$  are not continuous variables. Also, reduction to the absurd is applied from Lemma 3 to Theorem 4, which gives rise to Corollary 4.1, reconfirming the dependence between the nontrivial zeros and primes in Sec. II.

On the other hand, Theorem 5 shows that each prime number is related to the interesting mathematical functions,  $\beta_{l,m_t}$  and  $\gamma_{l,m_t}$ . Here,  $\beta_{l,m_t}$  oscillates as  $m_t$  increases, while it converges to 1 as  $l_t$  goes to infinity. Meanwhile,  $\gamma_{l,m_t}$  itself decreases as  $l_t$  and  $m_t$  increase, but the numbers appearing in  $\gamma_{l,m_t}$  have a very regular pattern. For instance, as shown in Table I, if  $m_t = 2$ , then  $0^2$ ,  $1^2$ , and  $3^2$  are found in  $\gamma_{l,m_t}$ , but  $2^2$  does not appear unless  $l_t = m_t$ . This pattern is shown for the cases of even  $m_t$ 's. Similarly, if  $m_t = 5$ , then  $1^2$ ,  $2^2$ ,  $3^2$ ,  $4^2$ , and  $6^2$  are found in  $\gamma_{l,m_t}$ , but  $5^2$  does not appear unless  $l_t = m_t$ ; when  $m_t$  is odd,  $\gamma_{l,m_t}$  has such a pattern. Defining the virtual potential related to the Legendre polynomials as well as both  $\beta_{l,m_t}$  and  $\gamma_{l,m_t}$ , as is in (30), every term except  $l_t = m_t$  vanishes by the orthogonality of the Legendre polynomials. Although infinitely many primes do not uniquely define a specific path of the integral, Theorem 5 makes it possible to provide the uniquely defined value of the integral as in (30). Surprisingly, the density plot of the virtual potential with respect to  $\sigma$  and  $t$  (Fig. 1) gives very intriguing patterns showing various kinds of diamonds. We conjecture that such patterns would be related to the distribution of prime numbers.

**IV. SCHRÖDINGER EQUATION FROM RIEMANN ZETA FUNCTION**

**A. Theorems**

**Theorem 6:** For  $\sigma > 0$ ,  $\frac{\partial \zeta(s)}{\partial t} = i \frac{\partial \zeta(s)}{\partial \sigma}$  with

$$\frac{\partial \zeta(s)}{\partial \sigma} = \zeta(s) \ln \left[ \prod_{i=1}^{\infty} p_i \left( 1 + \frac{p_i^\sigma - \cos(t \ln p_i)}{p_i^{-\sigma} - \cos(t \ln p_i)} \right)^{-1} \right] + i \zeta(s) \ln \left[ \prod_{i=1}^{\infty} p_i \frac{\sin(t \ln p_i)}{p_i^\sigma + p_i^{-\sigma} - 2 \cos(t \ln p_i)} \right],$$

$$\frac{\partial \zeta(s)}{\partial t} = i \zeta(s) \ln \left[ \prod_{i=1}^{\infty} p_i \left( 1 + \frac{p_i^\sigma - \cos(t \ln p_i)}{p_i^{-\sigma} - \cos(t \ln p_i)} \right)^{-1} \right] - \zeta(s) \ln \left[ \prod_{i=1}^{\infty} p_i \frac{\sin(t \ln p_i)}{p_i^\sigma + p_i^{-\sigma} - 2 \cos(t \ln p_i)} \right].$$

*Proof:* From (7) with the chain rule being applied, we readily have

$$\frac{\partial \zeta(s)}{\partial \sigma} = \zeta(s) \ln \left( \prod_{i=1}^{\infty} p_i^{\frac{1}{1-p_i^\sigma}} \right) \text{ and } \frac{\partial \zeta(s)}{\partial t} = i \zeta(s) \ln \left( \prod_{i=1}^{\infty} p_i^{\frac{1}{1-p_i^\sigma}} \right). \tag{31}$$

To decompose (31) into real part and imaginary part, let us differentiate both sides of (8). Then, we have

$$\begin{aligned} \frac{\partial \zeta(s)}{\partial \sigma} &= -\zeta(s) \sum_{i=1}^{\infty} \frac{(\cos(\frac{1}{2}t \ln p_i) - i \sin(\frac{1}{2}t \ln p_i)) \ln p_i}{(p_i^\sigma - 1) \cos(\frac{1}{2}t \ln p_i) + i (p_i^\sigma + 1) \sin(\frac{1}{2}t \ln p_i)} \\ &= \zeta(s) \sum_{i=1}^{\infty} \frac{(1 - p_i^\sigma e^{-it \ln p_i}) \ln p_i}{p_i^{2\sigma} - 2p_i^\sigma \cos(t \ln p_i) + 1} = \zeta(s) \sum_{i=1}^{\infty} \frac{(1 - p_i^\sigma \cos(t \ln p_i) + i p_i^\sigma \sin(t \ln p_i)) \ln p_i}{p_i^{2\sigma} - 2p_i^\sigma \cos(t \ln p_i) + 1} \\ &= \zeta(s) \sum_{i=1}^{\infty} \frac{\ln p_i}{1 + \frac{p_i^\sigma - \cos(t \ln p_i)}{p_i^{-\sigma} - \cos(t \ln p_i)}} + i \zeta(s) \sum_{i=1}^{\infty} \frac{\sin(t \ln p_i) \ln p_i}{p_i^\sigma + p_i^{-\sigma} - 2 \cos(t \ln p_i)} \\ &= \zeta(s) \ln \left[ \prod_{i=1}^{\infty} p_i \left( 1 + \frac{p_i^\sigma - \cos(t \ln p_i)}{p_i^{-\sigma} - \cos(t \ln p_i)} \right)^{-1} \right] + i \zeta(s) \ln \left[ \prod_{i=1}^{\infty} p_i \frac{\sin(t \ln p_i)}{p_i^\sigma + p_i^{-\sigma} - 2 \cos(t \ln p_i)} \right]. \end{aligned} \tag{32}$$

Also,

$$\frac{\partial \zeta(s)}{\partial t} = i \zeta(s) \ln \left[ \prod_{i=1}^{\infty} p_i \left( 1 + \frac{p_i^\sigma - \cos(t \ln p_i)}{p_i^{-\sigma} - \cos(t \ln p_i)} \right)^{-1} \right] - \zeta(s) \ln \left[ \prod_{i=1}^{\infty} p_i \frac{\sin(t \ln p_i)}{p_i^\sigma + p_i^{-\sigma} - 2 \cos(t \ln p_i)} \right]. \tag{33}$$

□

*Corollary 6.1:* For  $\sigma > 0$ , the Riemann zeta function satisfies the Schrödinger equation

$$i \frac{\partial \zeta(s)}{\partial t} = - \frac{1}{2M(s)} \frac{\partial^2 \zeta(s)}{\partial \sigma^2} + V(s)\zeta(s)$$

$$\text{with } M(s) = \frac{1}{2} \ln \left( \prod_{i=1}^{\infty} p_i \frac{1}{1-p_i^s} \right) \text{ and } V(s) = \left\langle \frac{2p^s}{1-p^s} \right\rangle \Big|_{\frac{\ln p_i}{1-p_i^s}, i \in \mathbb{N}}.$$

*Proof:* Only a few further evaluations in the proof of Theorem 6 are needed. In fact,

$$M(s) = \frac{1}{2} \ln \left[ \prod_{i=1}^{\infty} p_i \left( 1 + \frac{p_i^\sigma - \cos(t \ln p_i)}{p_i^{-\sigma} - \cos(t \ln p_i)} \right)^{-1} \right] + \frac{i}{2} \ln \left[ \prod_{i=1}^{\infty} p_i \frac{\sin(t \ln p_i)}{p_i^\sigma + p_i^{-\sigma} - 2 \cos(t \ln p_i)} \right], \tag{34}$$

$$V(s) = \left\langle \frac{2p^s}{1-p^s} \right\rangle \Big|_{\frac{\ln p_i}{1-p_i^s}, i \in \mathbb{N}} = \frac{\sum_{i=1}^{\infty} \left( \frac{2p_i^s}{1-p_i^s} \right) \left( \frac{\ln p_i}{1-p_i^s} \right)}{\sum_{i=1}^{\infty} \frac{\ln p_i}{1-p_i^s}}. \tag{35}$$

□

*Definition 1:* For  $\sigma > 0$  and any set of prime numbers  $P$ ,

$$\zeta_P(s) = \prod_{p_i \in P} \left\{ \sum_{j=0}^{\infty} \left( \frac{1}{p_i^s} \right)^j \right\} = \prod_{p_i \in P} \frac{1}{1 - p_i^{-s}}.$$

*Corollary 6.2:*  $\zeta_P(s)$  satisfies

$$i \frac{\partial \zeta_P(s)}{\partial t} = - \frac{1}{2M_P(s)} \frac{\partial^2 \zeta_P(s)}{\partial \sigma^2} + V_P(s)\zeta_P(s)$$

$$\text{with } M_P(s) = \frac{1}{2} \ln \left( \prod_{p_i \in P} p_i \frac{1}{1-p_i^s} \right) \text{ and } V_P(s) = \left\langle \frac{2p^s}{1-p^s} \right\rangle \Big|_{\frac{\ln p_i}{1-p_i^s}, p_i \in P}.$$

**B. Analysis**

From Secs. I and II, we are aware of the fact that the positions of the imaginary parts of the nontrivial zeros of the Riemann zeta function are independent of the real parts. Actually, it is the distribution of prime numbers that determines the positions. Theorem 6 agrees with these conclusions in that the variation of the behavior of the zeta function is not much controlled by the real part but by the imaginary part and the distribution of prime numbers. See Fig. 2.

On the other hand, the behaviors observed in Fig. 2 are actually the behaviors of the mass of the virtual particle in Corollary 6.1, which is under the mean prime potential,  $V(s)$ . From this point of view, we observe that only the imaginary parts of the zeros of the Riemann zeta function determine the positions where the mass of the particle drastically increases. On the other hand, the probability that the particle exists on the critical line at the corresponding imaginary parts is zero.

Meanwhile,  $M(s)$  and  $V(s)$  defined above have not yet obtained the analytic continuation in the critical strip at this stage since we started from the Euler product formula, which is not analytically

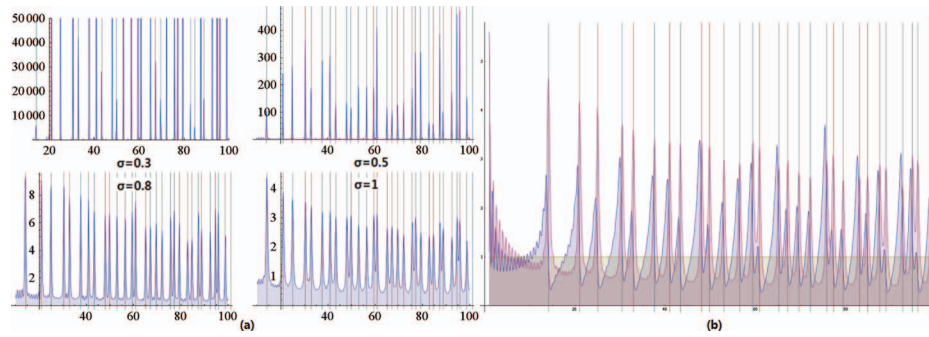


FIG. 2. (a)  $\prod_{i=1}^{100} p_i \left( 1 + \frac{p_i^\sigma - \cos(t \ln p_i)}{p_i^{-\sigma} - \cos(t \ln p_i)} \right)^{-1}$  with the vertical lines corresponding to  $t = t_n$  from  $n = 1$  to  $n = 29$   
 (b)  $\prod_{i=1}^{1000} p_i \left( 1 + \frac{p_i^\sigma - \cos(t \ln p_i)}{p_i^{-\sigma} - \cos(t \ln p_i)} \right)^{-1}$  and  $\prod_{i=1}^{1000} p_i \frac{\sin(t \ln p_i)}{p_i^\sigma + p_i^{-\sigma} - 2\cos(t \ln p_i)}$  for  $\sigma = 1$  and  $1 < t < 100$  with the same vertical lines.

defined in that region. Thus, further researches are required in order to show analytic behaviors in the critical strip by introducing new analytic functions as if the Dirichlet Eta function is employed to express the zeta function instead of using its p-series form in the critical strip. Nevertheless, the above definition of  $M(s)$  and  $V(s)$  itself is still able to numerically show their singular behaviors at the imaginary parts of the nontrivial zeros as mentioned above.

**V. MODIFIED RIEMANN ZETA FUNCTION WITH FINITE TERMS**

In this section, we introduce a modified zeta function with finite terms which converges to the Riemann zeta function in the limit case.

**A. Theorems**

*Definition 2:*  $Z(s, N) = \zeta(s, N) - L(s, N)$ ,

$$\text{where } \zeta(s, N) = \sum_{k=1}^N \frac{1}{k^s},$$

$$L(s, N - 1) = \int_1^N \frac{dk}{k^s}$$

and

$$\text{Re}[L(s, N - 1)] = \frac{\sigma - 1 + N^{\sigma-1}t \sin(t \ln N) + N^{\sigma-1}(1 - \sigma) \cos(t \ln N)}{1 - 2\sigma + \sigma^2 + t^2},$$

$$\text{Im}[L(s, N - 1)] = \frac{-t + N^{\sigma-1}t \cos(t \ln N) + N^{\sigma-1}(\sigma - 1) \sin(t \ln N)}{1 - 2\sigma + \sigma^2 + t^2}.$$

**Theorem 7:**  $\lim_{N \rightarrow \infty} Z(s, N) \simeq \zeta(s)$  for  $\sigma > 0$ .

*Proof:* Let us first consider the real part. Note that there exist functions  $f$  and  $g$  such that the oscillation of  $\text{Re}[L(s, N)]$  with increasing  $N$  is enveloped by  $f(N)$  and  $g(N)$ . Observe that

$$\{f(N + 1) - g(N + 1)\} - \{f(N) - g(N)\} \rightarrow 0 \tag{36}$$

as  $N \rightarrow \infty$ .

Since  $f(N) - g(N) > \text{Re}[L(s, N)]$ ,

$$\lim_{N \rightarrow \infty} |\text{Re}[L(s, N + 1)] - \text{Re}[L(s, N)]| = 0 \tag{37}$$

or simply

$$\lim_{N \rightarrow \infty} \{\text{Re}[Z(s, N + 1)] - \text{Re}[Z(s, N)]\} = 0 \tag{38}$$

due to the fact that

$$\text{Re}[Z(s, N)] - \text{Re}[\zeta(s)] \simeq \text{Re}[L(s, N)]. \tag{39}$$

Moreover, for any  $N$ , at least one of the following inequalities is satisfied:

$$\begin{aligned} \frac{\cos(t \ln N)}{N^\sigma} &\leq \text{Re}[\zeta(s, N)] - \text{Re}[Z(s, N)] \leq \text{Re}[Z(s, N + 1)] - \text{Re}[Z(s, N)] \\ \frac{\cos(t \ln N)}{N^\sigma} &\geq \text{Re}[\zeta(s, N)] - \text{Re}[Z(s, N)] \geq \text{Re}[Z(s, N + 1)] - \text{Re}[Z(s, N)] \end{aligned} \tag{40}$$

in which both  $\frac{\cos(t \ln N)}{N^\sigma}$  and  $\text{Re}[Z(s, N + 1)] - \text{Re}[Z(s, N)]$  converge to zero as  $N \rightarrow \infty$ . Hence,

$$\text{Re}[\zeta(s)] \simeq \lim_{N \rightarrow \infty} \text{Re}[\zeta(s, N)] = \lim_{N \rightarrow \infty} \text{Re}[Z(s, N)]. \tag{41}$$

A similar argument applies to the case of  $\text{Im}[\zeta(s)]$  which finally leads to

$$\zeta(s) = \lim_{N \rightarrow \infty} \zeta(s, N) \simeq \lim_{N \rightarrow \infty} Z(s, N). \tag{42}$$

The symbol,  $\simeq$ , is used because there is a disagreement between the behavior of  $\lim_{N \rightarrow \infty} Z(s, N)$  and the Riemann zeta function for small  $t < t_1$ . See Fig. 4. □

### B. Analysis

Theorem 7 allows us to effectively deal with the problematic summation part even if we keep using the p-series form of the zeta function in the critical strip. The more terms we consider here, the more vibrating noises we have from  $t = 0$  to some extent (Fig. 3). Summing up simply in the order does not give very satisfactory results as is expected from the fact that the p-series form can only conditionally be convergent, i.e., it is not analytically defined in the critical strip. In Fig. 3, the noises in the left figure hinder us from finding  $s = \sigma + it$  such that  $\text{Re}[\zeta(s, N)] \simeq 0$ . Once we choose  $N$  to which we summed up from  $n = 1$  subsequently, there is a determined region of  $t$  where we can successfully analyze the behavior of the zeta function. However, it could ill-behave in other regions.

Nevertheless, we might try to find out a way with which we can utilize the p-series form of the zeta function for the half plane  $\text{Re}(s) > 0$ . The method suggested in Theorem 7 focuses on the overlap between  $\zeta(s, N)$  and  $L(s, N)$ . Thus, if we would like to find zeros, we can try to find out  $t$  such that the overlap is maximized, i.e.,  $Z(s, N) = 0$ . Note that the oscillation axis of  $\zeta(s, N)$  is  $\zeta(s)$ , whereas  $L(s, N)$  oscillates always about 0. Once  $s$  reaches one of the nontrivial zeros  $s_n$ ,  $\zeta(s, N)$  also has its oscillation center at zero. Then, we expect  $Z(s_n, N) \simeq 0$ . See Fig. 4.

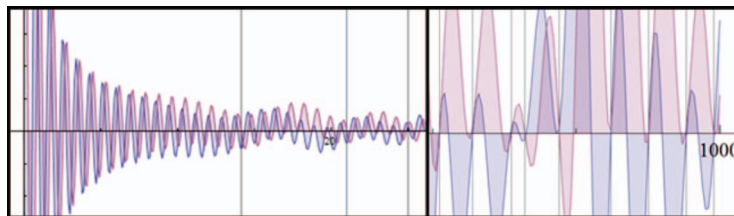


FIG. 3.  $\text{Re}[\zeta(0.5 + it, 2000)]$  (in red) and  $\text{Im}[\zeta(0.5 + it, 2000)]$  (in blue) for  $0 < t < 27$  (left) and  $990 < t < 1000$  (right) with the vertical lines corresponding to the imaginary parts of the nontrivial zeros.

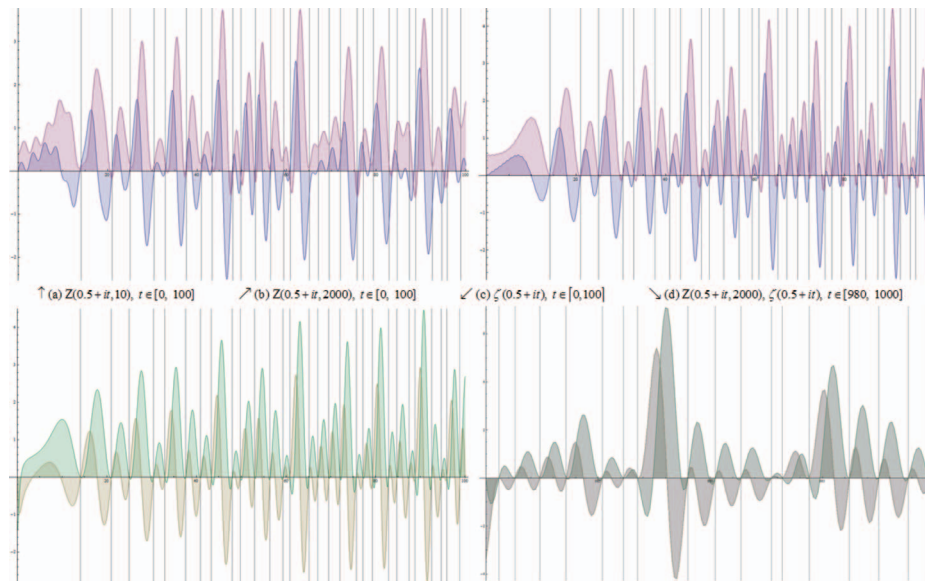


FIG. 4.  $\text{Re}[Z(0.5 + it, N)]$  (red),  $\text{Im}[Z(0.5 + it, N)]$  (blue),  $\text{Re}[\zeta(0.5 + it)]$  (green), and  $\text{Im}[\zeta(0.5 + it)]$  (gold). In (a) and (b), we do not have any noises which appear in Fig. 3. Furthermore, we have smoother curves as  $N$  increases, as Theorem 7 implies. For a large  $t$  with a large enough  $N$ , it also behaves well in accordance with the zeta function as is seen in (d) where the colors of the curves indicate that  $Z(0.5 + it, 2000)$  and  $\zeta(0.5 + it)$  almost overlap.

## VI. CONCLUSION

We carried out a logical approach to the nontrivial zeros of the Riemann zeta function with the uniformly distributed property of the imaginary parts of the nontrivial zeros in relation to the functional dependence between the zeros and primes. It is noted that there is a uniquely defined relation between prime numbers and imaginary parts of the zeros, independent of their real part. From this result, it is inferred that the real parts of the zeros are related by the identity transformation. Also, the Legendre polynomials are used for the reconfirmation of the existence of the relation between primes and the ordinates of the nontrivial zeros and for some further investigations, noting on their orthogonal property. Especially, the term,  $|1 - p_i^{-s}|^{-1}$ , which may possibly be related to the concept of Green's function, can be integrated. The integral can be thought of as a virtual potential defined by primes of which the resultant value is invariable even though no unique path of the integral is defined. Owing to the connection with the Legendre polynomials, the virtual potential is easily evaluated where the result consists of an interesting mathematical structure. Here, the density plot of the potential provides enigmatic patterns. Furthermore, the partial derivatives of the Riemann zeta function can give rise to the Schrödinger equation of which the solution is the zeta function itself, permitting us to think of a virtual particle under the mean prime potential. Only the imaginary parts of the zeros of the Riemann zeta function determine the positions where the mass of the particle drastically increases coincident with the positions of the zeros on the critical line in the Riemann hypothesis. On the other hand, the probability that the particle exists on the critical line at the corresponding imaginary parts is always zero. Lastly, the modified zeta function, which finally converges to the actual zeta function with  $N$  approaching to the positive infinity, helps us readily observe the behavior of the Riemann zeta function for the right half-plane except for small  $t < t_1$ .

From this study, we anticipate that it can make a path way to understand prime numbers. By extension, further studies to find the  $\theta_N^{ji}$  could lead to the key to the exact relation between primes. If the Riemann zeta function can also satisfy the Klein-Gordon equation or Dirac equation and the position of the nontrivial zeros retain the same kind of meaning as in the case of the Schrödinger equation, then prime numbers can be thought that they are more likely to have a real relation with nature.

## ACKNOWLEDGMENTS

This research was supported by the Global Ph.D. Fellowship Program (2011-0006466) and National Honor Scientist Program (2010-0020414) of the National Research Foundation of Korea funded by the Ministry of Education, Science and Technology of Korea.

## APPENDIX A: ANALYTIC CONTINUATION

In mathematics, an analytic function is a function that is locally given by a convergent power series, and so the Riemann zeta function  $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$  is an analytic function for  $\text{Re}(s) > 1$  because the series itself is absolutely convergent. However, this expression is not analytic for  $\text{Re}(s) \leq 1$ , so that we generally need to extend the analytic domain of  $\zeta(s)$ . In fact, the analytic continuation of the Riemann zeta function is obtained for all complex numbers  $s$  except for a simple pole at  $s = 1$ . To investigate the critical strip,  $\{s \in \mathbb{C} | 0 < \text{Re}(s) < 1\}$ , analytically, one may start from the zeta alternating series, which is called the Dirichlet eta function. (Note that the continuation is independent of whatever techniques we use because of the uniqueness of analytic continuations.) Since the eta function  $\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s}$  is an analytic function for  $\text{Re}(s) > 0$ , it is permitted to extend the domain of definition of the Riemann zeta function in the following way:

$$\begin{aligned} \eta(s) &= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} = \sum_{n=1}^{\infty} \frac{1}{n^s} - \sum_{n=1}^{\infty} \frac{2}{(2n)^s} \\ &= \zeta(s) - \frac{2}{2^s} \zeta(s) \Leftrightarrow \zeta(s) = \frac{1}{1 - 2^{1-s}} \eta(s). \end{aligned} \quad (\text{A1})$$

That is, the technique using the eta function allows us to extend the analytic domain of the Riemann zeta function for all complex number  $s$  with positive real part except  $s = 1$ . (Using a different way, we can furthermore do the analytic continuation to  $\text{Re}(s) \leq 1$ , also.) Even though it is almost impossible to get a convergent value in the critical strip from the p-series form of the Riemann zeta function,  $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ , the existence of the values of the function is still guaranteed by the analytic continuation. Notice that

$$\begin{aligned} \zeta(s) &= \frac{1}{1 - 2^{1-s}} \eta(s) = \frac{1}{1 - 2^{1-s}} \left( 1 - \frac{1}{2^s} + \frac{1}{3^s} - \dots \right) \\ &= \frac{1}{1 - 2^{1-s}} \left\{ 1 - \frac{1}{2^s} - \left( \frac{1}{2^s} \right)^2 - \dots \right\} \prod_{i=2}^{\infty} \frac{1}{1 - p_i^{-s}} \\ &= \frac{1}{1 - 2^{1-s}} \left\{ 2 - \left( 1 + \frac{1}{2^s} + \left( \frac{1}{2^s} \right)^2 + \dots \right) \right\} \prod_{i=2}^{\infty} \frac{1}{1 - p_i^{-s}} \\ &= \frac{1}{1 - 2^{1-s}} \left( 2 - \frac{1}{1 - 2^{-s}} \right) \prod_{i=2}^{\infty} \frac{1}{1 - p_i^{-s}} \\ &= \frac{1}{1 - 2^{1-s}} \frac{1 - 2^{1-s}}{1 - 2^{-s}} \prod_{i=2}^{\infty} \frac{1}{1 - p_i^{-s}} = \prod_{i=1}^{\infty} \frac{1}{1 - p_i^{-s}}. \end{aligned} \quad (\text{A2})$$

From the function analytically defined for  $\text{Re}(s) > 0$  except for  $s = 1$ , we deduce the Euler product formula, which is analytic only for  $\text{Re}(s) > 1$ . Although the Euler product formula cannot locally be given by a convergent power series, we use this formula since it is enough for the development of our logical arguments. See Appendix B.

## APPENDIX B: THE ARGUMENT SIMILAR TO ONE IN SEC. II WITH THE RIEMANN ZETA FUNCTION DEFINED BY THE ETA FUNCTION

The absolute value of the Riemann zeta function is defined as follows:

$$|\zeta(s)| = \frac{1}{\sqrt{1 - 2(2^{1-\sigma}) \cos(t \ln 2) + (2^{1-\sigma})^2}} |\eta(s)|. \quad (\text{B1})$$

Observe that (B1) is zero if and only if  $\eta(s) = 0$ , so that the zeros of  $\eta(s)$  are actually the zeros of the Riemann zeta function. From Theorem 2, it is easily seen that

$$X_k = \{x_k | x_k = \cos(t_N \ln k), N \in \mathbb{N}\} \quad (\text{B2})$$

is also dense for any  $k \in \mathbb{N}/\{1\}$ . For any  $(m, n) \in \mathbb{N}^2$ , suppose that there is no  $N \in \mathbb{N}$  such that

$$\cos(t_N \ln m) = f(\cos(t_N \ln n)), \quad (\text{B3})$$

where  $f$  is a relation between  $(t_N, m)$  and  $(t_N, n)$ .

Restricting  $s \in \{z = 1/2 + it \in \mathbb{C} | \zeta(z) = 0\}$  and integrating both sides of  $\eta(s) = 0$  with  $\prod_{k=2}^{\infty} \int_{-1}^1 dx_k$ , we face with the contradiction that the real part diverges to the positive infinity. The appearance of this kind of contradiction is actually irrelevant to whether or not the function we handle is analytically defined.

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