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Emergent Schrödinger geometries from mass-deformed CFT

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Abstract: We study an M-theory solution for the holographic flow of AdS_4 times Sasaki-Einstein 7-manifolds with skew-whiffing, perturbed by a mass operator. The infrared solution contains the 5 dimensional Schrödinger geometry after considering the gravity dual of the standard non-relativistic limit of relativistic field theories. The mass deformation of the field theory is discussed in detail for the case with 7 manifold being a round sphere.

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1 Introduction

Gauge/gravity duality with non-relativistic scale invariance has drawn considerable interest in recent years. An interesting direction of studies was based on the Schrödinger geometry [1, 2]. See also [3, 4]. Schrödinger symmetry is a scale invariant extension of the Galiean symmetry by the particle number symmetry, anisotropic scale symmetry with dynamical exponent $z = 2$ and special conformal symmetry. The geometry of [1, 2] realizes this symmetry of a QFT in $d+1$ spacetime dimension as isometries of a $d+3$ dimensional spacetime of the following form:[1]

$$ds^2 = L^2\left[-\beta^2\rho^4(dx^+)^2 + \rho^2(d\vec{x}^2 + 2dx^+dx^-) + \frac{d\rho^2}{\rho^2}\right] \quad , \quad \vec{x} \in \mathbb{R}^d \,. \tag{1.1}$$

In particular, the shift symmetry of the coordinate x^- realizes the particle number symmetry. There are many works on Schrödinger solutions in string/M-theory, including [7–12], which also discuss Schrödinger solutions with general dynamical exponent z. See also many references below that we mention in more detailed contexts.

$2+1$ dimensional non-relativistic CFT duals of 5 dimensional Schrödinger geometries (which we call Sch_5) were studied in [13–15] with string theory embedding, as deformations of 4 dimensional relativistic field theories by irrelevant operators which partly break the 4 dimensional conformal symmetry. See [16, 17] for more discussions on the holography in this context.

On the other hand, a very natural way of obtaining a non-relativistic theory is to start from a relativistic CFT living in a spacetime with *same dimension*, and to obtain the non-relativistic theory in the IR after a suitable mass deformation and performing the standard non-relativistic limit of redefining the energy and discarding 'anti-particles.' Indeed, it has been shown concretely that a class of $2+1$ dimensional mass-deformed CFT gives a non-relativistic theory with Schrödinger symmetry [18, 19]. Contrary to the other realization obtained by starting from a relativistic CFT in one higher dimension, the Schrödinger symmetry cannot be embedded into the conformal symmetry of the relativistic theory. It

[1] Such a geometry in lowest possible dimension with $d=0$ has been studied in [5, 6], prior to [1, 2].

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is thus interesting to see if there are mass-deformed CFT's admitting gravity duals which show 'holographic RG flows' from AdS_D to Sch_{D+1}.[2] The case with $D=4$ will be our main interest in this paper, although similar studies could be done in other dimensions.

In particular, [18, 19] discussed the non-relativistic superconformal field theory obtained by deforming the $\mathcal{N}=6$ SCFT [20] by a mass term preserving all Poincare supersymmetries, considering a particular vacuum in which all scalar expectation values are zero. The classical Lagrangian of this non-relativistic field theory preserves 14 supercharges, including 2 extra non-relativistic conformal supercharges. There have been some efforts to construct the gravity dual of this theory in the IR based on the geometric realization of Schrödinger symmetry [21, 22]. None of them found it, at least with all 14 Killing spinors. Soon it was argued [23, 24] with a Witten index calculation that the corresponding field theory vacuum does not exist, at least in the subset of supersymmetric vacua. The gravity dual of this theory is still ill-understood to date. See more comments about it in the discussion.

In this paper, we find a very simple class of flow solutions from AdS_4 to Sch_5 after deforming a class of relativistic CFT_3 with a mass operator. The 3 dimensional relativistic CFT_3 can be regarded as those living on parallel M2-branes in M-theory. Our construction is based on the relativistic UV CFT dual to the so-called skew-whiffed $AdS_4 \times SE_7$, where SE_7 denotes 7 dimensional Sasaki-Einstein manifold. The skew-whiffing in this context corresponds to changing the sign of M2-brane charge, generically breaking supersymmetry. For the particular case of $SE_7 = S^7$, maximal superconformal symmetry is preserved and the UV CFT is relatively well understood. In this case, we explicitly identify the mass operator, which turns out to be non-supersymmetric and invariant under an $\mathrm{SU}(4) \times \mathrm{U}(1)$ subgroup of $\mathrm{SO}(8)$.

Our finding shows how Schrödinger geometries can appear by deforming relativistic CFT's by mass operators and taking the non-realtivistic limit. It could find applications in many different gauge/gravity dual models containing massive degrees only. We briefly comment on some possible applications in the discussion.

The solution we consider in this paper can actually be understood as the zero-temperature black brane solution found and discussed in [25] from 4 dimensional gauged supergravity. The 4 dimensional metric in the Einstein frame in the 'IR regime' (with small r) is given by

$$ds^2 \sim r^2(-dt^2 + d\vec{x}^2) + \frac{dr^2}{r^{\frac{4}{3}}} \quad , \quad \vec{x} \in \mathbb{R}^2 \tag{1.2}$$

which has zero entropy. This type of geometry with Poincare symmetry and broken (isotropic) scale invariance was considered in [26], and a similar system was considered in [27]. This type of geometry is different from those considered in [28] which have approximate Lifshitz symmetry. From our model, it is clear that the broken isotropic scale invariance originates from the mass deformation of the UV CFT.

[2] What we mean by a flow solution from AdS_D to Sch_{D+1} will be clarified later, as the latter lacks $D-1$ dimensional Poincare symmetry of AdS_D. We shall obtain Sch_5 after combining an internal direction with four directions, performing the coordinate transformation (2.18) to have a time x^+ conjugate to the non-relativistic energy, and finally taking the scaling limit (2.20).

The remaining part of this paper is organized as follows. In section 2, we explain our ansatz and obtain the analytic solution in the IR in the small radius expansion. We explain the gravity dual operation of taking the non-relativistic limit of QFT, the appearance of Sch_5 geometry, and the exact solution which interpolates our IR solution with asymptotic AdS_4. In section 3, we explain the mass term of the dual field theory when the internal 7-manifold is S^7 or orbifolds thereof. Section 4 concludes with discussions.

2 Flows from AdS to Schrödinger geometry

We seek for flow solutions from $AdS_4 \times SE_7$ to 5 dimensional Schrödinger geometries in the consistent truncation ansatz discovered in [29]. It will turn out that we can realize a Schrödinger solution in the infrared if we advocate the so-called skew-whiffed truncation. This truncated theory contains an $AdS_4 \times SE_7$ vacuum, generically with broken supersymmetry. For the special case of $SE_7 = S^7$, the AdS_4 vacuum is maximally supersymmetric. For the case with $\mathbb{Z}_k$ orbifold, the AdS_4 vacuum after orbifold is non-supersymmetric with skew-whiffing. The nature of the mass term driving the flow of our solution will be discussed in section 3 after studying the gravity solutions. The full solution is actually available in an analytic form [25], but to illustrate how the solution looks like and to explain why skew-whiffed setting is necessary to have nontrivial solutions, we explain some details of the derivation.

The consistent truncation ansatz of [29] is

$$\begin{aligned} ds^2_{11} &= ds^2_4 + e^{2U} ds^2(KE_6) + e^{2V}(\eta + A_1)^2 \\ G_4 &= 6e^{-6U-V}(\epsilon + h^2 + |\chi|^2)\mathrm{vol}_4 + H_3 \wedge (\eta + A_1) + H_2 \wedge J + dh \wedge J \wedge (\eta + A_1) + 2hJ \wedge J \\ &\quad + \sqrt{3}\left[\chi(\eta + A_1) \wedge \Omega - \frac{i}{4} D\chi \wedge \Omega + c.c.\right], \end{aligned} \tag{2.1}$$

where J is the Kähler 2-form of the Kähler-Einstein base KE_6, $\eta = d\psi + \theta$ is the Reeb 1-form with $d\theta = 2J$, $D\chi \equiv d\chi - 4iA_1\chi$. A_1, H_2, H_3 are 1-form, 2-form, 3-form in 4 dimensions, while U, V, h and χ are real/complex scalars, respsectively. The truncation with $\epsilon = \pm 1$ contains supersymmetric and generically non-supersymmetric AdS_4 vacuum. The case with $\epsilon = -1$ is the skew-whiffed truncation.

We would like to study the flow solutions preserving the $2+1$ dimensional Poincare symmetry on the M2-brane world-volume. This is because the flow itself after a mass deformation does not break Poincare symmetry on the worldvolume. It is only the non-relativistic limit, discarding 'anti-particles' after suitably redefining non-relativistic energy, which breaks it. We thus consider the Poincare invariant ansatz

$$ds^2_4 = e^{2A(r)}(\eta_{\mu\nu} dx^\mu dx^\nu) + dr^2 \,. \tag{2.2}$$

From Poincare symmetry, the gauge fields H_2, F_2 are zero. At this stage, it appears that H_3 may be nonzero if proportional to the volume form $dx^0 \wedge dx^1 \wedge dx^2$. From the equations of motion and the Bianchi identities for G_4, one obtains

$$dH_3 = 0\,, \quad dH_2 = 2H_3 + H_2 \wedge F_2\,, \quad H_1 = dh \tag{2.3}$$

and

$$
\begin{aligned}
d(e^{6U-V}\star H_3) &= e^{6U+V} f F_2 - 6e^{2U+V}\star H_2 - 12hH_2 - \frac{3i}{2}D\chi\wedge D\chi^* \\
d(e^{2U+V}\star H_2) &= -2dh\wedge H_2 - 4hH_3 \\
d(e^{2U-V}\star dh) &= -e^{2U+V}\star H_2\wedge F_2 - H_2\wedge H_2 - 4h(f+4e^{-2U+V})\mathrm{vol}_4 \\
D(e^{V}\star D\chi) &= -iH_3\wedge D\chi - 4\chi(f+4e^{-V})\mathrm{vol}_4 \\
f &= 6e^{-6U-V}(\epsilon + h^2 + |\chi|^2)\,.
\end{aligned}
\tag{2.4}
$$

From the H_2 equation of motion on the second line, one should set $hH_3 = 0$. So either h or H_3 should be vanishing. As we shall see later, nonzero h is crucial to have a Schrödinger solution in the IR, similar to those in [30]. So we set $H_3 = 0$. The field χ can always be consistently set to zero, which we do from now on. Then, only the h equation of motion (third line) remains nontrivial among the above equations.

As we shall explain shortly, we want an IR solution with $U = const$, $e^V \sim e^A \to 0$ as $r\to-\infty$, to get 5 dimensional Schrödinger solutions. It turns out that one can set $U = 0$, which is the value for the AdS_4 vacuum, all along the flow. We shall later explain that this is a consistent ansatz, which actually comes from the existence of a further consistent truncation found in [25]. With $U=0$, the 11 dimensional Einstein equation becomes

$$
\begin{aligned}
R_{mn} &= \nabla_m\nabla_n V + \nabla_m V\nabla_n V + \frac{3}{2}e^{-2V}\left(\nabla_m h\nabla_n h - \frac{1}{3}g_{mn}(\nabla h)^2\right) - 2g_{mn}\left(4h^2 + \frac{f^2}{6}\right) \\
0 &= \nabla^m\nabla_m V + \nabla^m V\nabla_m V + e^{-2V}\nabla^m h\nabla_m h - 6e^{2V} - 8h^2 + \frac{f^2}{6} \\
0 &= -8 + 2e^{2V} + 8h^2 + \frac{f^2}{6}
\end{aligned}
\tag{2.5}
$$

with $m,n = 0,1,2,r$. The last equation is the U equation of motion. Note that $f = 6e^{-V}(\epsilon + h^2)$. With our 4 dimenisional metric (2.2), one obtains

$$
R_{rr} = -3(A'' + (A')^2)\,, \quad R_{\mu\nu} = -(A'' + 3(A')^2)g_{\mu\nu}\,, \quad \Gamma_{\mu\nu}^{\ \ r} = -A' g_{\mu\nu} \tag{2.6}
$$

for $\mu,\nu = 0,1,2$, where prime denotes r derivative. Therefore, the two nontrivial components (rr and $\mu\nu$) of the Einstein equations and the V,U,h equations become

$$
\begin{aligned}
0 &= 3(A'' + (A')^2) + V'' + (V')^2 + e^{-2V}(h')^2 - 8h^2 - 12e^{-2V}(\epsilon + h^2)^2 \\
0 &= A'' + 3(A')^2 + A'V' - \frac{1}{2}e^{-2V}(h')^2 - 8h^2 - 12e^{-2V}(\epsilon + h^2)^2 \\
0 &= V'' + 3A'V' + (V')^2 + e^{-2V}(h')^2 - 6e^{2V} - 8h^2 + 6e^{-2V}(\epsilon + h^2)^2 \\
0 &= -8 + 2e^{2V} + 8h^2 + 6e^{-2V}(\epsilon + h^2)^2 \\
0 &= h'' + (3A' - V')h' - 4he^{V}[4e^{V} + 6e^{-V}(\epsilon + h^2)]\,.
\end{aligned}
\tag{2.7}
$$

Apparently, there are 5 equations for 3 functions A, V, h. One more equation appears due to our Poincare invariant ansatz with r dependent functions only, and another one due to our ansatz $U=0$. The former extra equation will be eliminated from the consistency of Poincare invariant ansatz, and the second one from the consistency of $U = 0$ ansatz.

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We first explain some structures of these equations. As briefly explained above, we want $e^V \sim e^A \sim e^{nr} \to 0$ for some constant n as $r \to -\infty$. Then the terms containing A' and V' in the first three equations all yield $4n^2$, which are of order 1. Also, the last factors $\sim e^{-2V}(\epsilon + h^2)^2$ become very large due to e^{-2V}, unless $\epsilon + h^2$ is small. Especially, in the second equation, considering the signs of other terms, this term cannot be canceled at all if it is indeed large. The only way to have the desired IR solution is to take $\epsilon = -1$ with skew-whiffing and demand $h \to \pm 1$ as $r \to -\infty$. In section 3, we will also provide a group theoretical argument from UV perspective on why the desired flow solutions exist only with skew-whiffing in this truncation.

Firstly, the equation for U becomes to

$$0 = -8 + 2e^{2V} + 8h^2 + 6e^{-2V} - 12h^2 e^{-2V} + 6e^{-2V}h^4 = 2[e^V + 3(-1+h^2)e^{-V}][e^V + (-1+h^2)e^{-V}]. \tag{2.8}$$

This relates h and e^V in an algebraic manner. It will turn out that

$$e^{2V} = 1 - h^2 \tag{2.9}$$

is the solution that we want. It is a special case of an algebraic relation found in the truncation of [25] with $\chi = 0$. We are left with 4 equations for 2 variables A, V. One can easily show that the h equation is equivalent to the V equation. To show this, we multiply h to the last equation of (2.7) and obtain

$$0 = (hh')' - (h')^2 + \frac{1}{2}(3A' - V')(h^2)' + 8h^2 e^{2V} = -e^{2V}[V'' + 2(V')^2 + e^{-2V}(h')^2 + (3A' - V')V' - 8h^2] \tag{2.10}$$

after inserting (2.9). As the expression in the last parenthesis is the V equation, the two equations are equivalent as long as $h \neq 0$. In our flow solution, h will only be asymptotically zero in UV, where we can easily check separately that the UV fixed point solves all equations. So the last constraint will not obstruct the above elimination of one equation.

So far, the three equations for the two variables A, V (with (2.9) assumed) are

$$\begin{aligned} 0 &= 3(A'' + (A')^2) + V'' + (V')^2 + e^{-2V}(h')^2 - 8 - 4e^{2V} \\ 0 &= A'' + 3(A')^2 + A'V' - \frac{1}{2}e^{-2V}(h')^2 - 8 - 4e^{2V} \\ 0 &= V'' + (V')^2 + 3A'V' + e^{-2V}(h')^2 - 8(1 - e^{2V}), \end{aligned} \tag{2.11}$$

where the last equation may be replaced by

$$0 = h'' + (3A' - V')h' + 8he^{2V} = -h^{-1}e^{2V} \times [V], \tag{2.12}$$

where $[V]$ denotes the expression in the V equation of motion, on the last line of (2.11). Also, we denote by $[R_{rr}]$ and $[R_{\mu\nu}]$ the expressions on the first and second lines of (2.11). To show that one of these three equations is redundant, we find it useful to consider the following two combinations of three independent equations:

$$\frac{1}{3}[R_{rr}] - \frac{1}{3}[V] : 0 = A'' + (A')^2 - A'V' - 4e^{2V} \tag{2.13}$$

$$[R_{rr}] - [R_{\mu\nu}] : 2A'' + V'' + (V')^2 - A'V' + \frac{3}{2}e^{-2V}(h')^2 .$$

Let us take 3 combinations of equations to be $\frac{[R_{rr}]-[V]}{3} = 0$, $[R_{\mu\nu}] = 0$ and

$$\begin{aligned} 0 &= h'\left(-he^{2V}[V]\right) - 2e^{2V}A'\left([R_{rr}] - [R_{\mu\nu}]\right) - 2e^{2V}V'\left(\frac{1}{3}[R_{rr}] - \frac{1}{3}[V]\right) \\ &= e^{2V}\left[\frac{1}{2}e^{-2V}(h')^2 - 2A'V' - 2(A')^2\right]'. \end{aligned} \tag{2.14}$$

The last total derivative can be obtained after some straightforward algebra. So this equation can be integrated to yield

$$0 = \frac{1}{2}e^{-2V}(h')^2 - 2A'V' - 2(A')^2 - c = \left(\frac{1}{3}[R_{rr}] - \frac{1}{3}[V]\right) - [R_{\mu\nu}]\,,$$

where the last equation holds for $c = -8$. Thus, the first two equations guarantee this last equation with the integration constant chosen as $c = -8$. So we have two independent equations for two functions in our ansatz.

To get IR solutions near $r = -\infty$, we define $\rho \equiv e^{nr}$ with a positive number n and expand

$$h = 1 + h_1\rho^2 + \cdots\,, \quad e^V = v_0(\rho + v_1\rho^3 + \cdots)\,, \quad e^A = a_0(\rho + a_1\rho^3 + \cdots)\,. \tag{2.15}$$

We can set $v_0 = 1$ without losing generality by shifting the variable r, as all equations have translation symmetry in r. a_0 can also be eliminated to, say $a_0 = 1$, by rescaling the Poincare coordinates x^μ. We leave it as it is as we want to identify it with a mass parameter in the UV later. Plugging these in, we find $n^2 = 2$ at $\mathcal{O}(\rho^0)$ terms of the equations. Continuing the iteration, one finds the following small radius expansion in $\rho = e^{\sqrt{2}r}$

$$\begin{aligned} e^V &= \rho\left(1 - \frac{7}{16}\rho^2 + \frac{69}{256}\rho^4 - \frac{2209}{12288}\rho^6 + \cdots\right) \\ e^A &= a_0\rho\left(1 + \frac{3}{16}\rho^2 - \frac{15}{256}\rho^4 + \frac{329}{12288}\rho^6 + \cdots\right)\,, \end{aligned} \tag{2.16}$$

with $U = 0$ and $h^2 = 1 - e^{2V}$.

The solution with leading and next-to-leading terms is given by

$$\begin{aligned} ds_{11}^2 &= a_0^2\rho^2\left(1 + \frac{3}{16}\rho^2 + \mathcal{O}(\rho^4)\right)^2(-dt^2 + d\vec{x}^2) + \frac{d\rho^2}{2\rho^2} + \rho^2\left(1 - \frac{7}{16}\rho^2 + \mathcal{O}(\rho^4)\right)^2\eta^2 \\ &\quad + ds^2(KE_6) \\ G_4 &= -6a_0^3\rho^4\left(1 + \mathcal{O}(\rho^2)\right)\mathrm{vol}_3 \wedge dr - \frac{1}{2}d\left(\rho^2 + \mathcal{O}(\rho^4)\right) \wedge J \wedge \eta + \left(2 + \mathcal{O}(\rho^2)\right)J \wedge J\,, \end{aligned} \tag{2.17}$$

where $\mathrm{vol}_3 = dt \wedge dx^1 \wedge dx^2$. To obtain the Schrödinger solution, we first perform a coordinate transformation to a rotating frame. Taking $\eta = d\psi + \theta$, where $d\theta = 2J$ is twice the Kähler 2-form of KE_6, we define

$$x^- = \psi - a_0 t\,, \quad x^+ = t \tag{2.18}$$

to be the coordinates in the rotating (or boosted) frame. For later use, we note that ψ is 2π periodic for $SE_7 = S^7$ in our normalization. From the dual field theory perspective, the shift of ψ is identified as the particle number symmetry, or more precisely the symmetry conjugate the total rest mass as there could be more than one particle species. So the conserved charge for x^+ takes the form of the non-relativistic energy $E_{NR} = E_{rel} - a_0 R_b$, where E_{rel} and R_b are conserved charges for the translation of t, ψ. The above metric can be written as

$$\begin{aligned} ds_{11}^2 &= -a_0^2\rho^2\left(\frac{5}{4}\rho^2 + \mathcal{O}(\rho^4)\right)(dx^+)^2 + a_0^2\rho^2\left(1+\mathcal{O}(\rho^2)\right)^2 d\vec{x}^2 + 2a_0\rho^2(1+\mathcal{O}(\rho^2))dx^+\eta \\ &\quad + \frac{d\rho^2}{2\rho^2} + \rho^2\left(1+\mathcal{O}(\rho^2)\right)\eta^2 + ds^2(KE_6)\,. \end{aligned} \tag{2.19}$$

Note that $\rho^2(dx^+)^2$ terms cancels by the coordinate transformation, leaving the leading $\rho^4(dx^+)^2$ term. Taking the scaling limit

$$\rho \to \epsilon\rho\,, \quad x^+ \to \epsilon^{-2}x^+\,, \quad \vec{x} \to \epsilon^{-1}\vec{x}\,, \quad x^- \to x^- \tag{2.20}$$

with $\epsilon \to 0^+$, one obtains

$$\begin{aligned} ds_{11}^2 &= -\frac{5a_0^2}{4}\rho^4(dx^+)^2 + \rho^2\left(a_0^2 d\vec{x}^2 + 2a_0 dx^+(dx^- + \theta)\right) + \frac{d\rho^2}{2\rho^2} + ds^2(KE_6) \\ G_4 &= -3\sqrt{2}a_0^3\rho^3 dx^+\wedge dx^1\wedge dx^2\wedge d\rho - a_0\rho d\rho\wedge J\wedge dx^+ + 2J\wedge J\,. \end{aligned} \tag{2.21}$$

The 5 dimensional metric apart from the KE_6 part is the Schrödinger geometry dual to $2+1$ dimensional non-relativistic field theory. The coordinate transformation and scaling explained above are to be understood as the gravity dual of the non-relativistic limit.

Sch$_5$ solutions in M-theory of this type was found in [30]. However, the solution found there has different coefficients in various terms. In fact, one can find a generalized solution

$$\begin{aligned} ds_{11}^2 &= -\frac{14a_0^2 - b_0^2}{8}\rho^4(dx^+)^2 + \rho^2\left(a_0^2 d\vec{x}^2 + b_0 dx^+(dx^- + \theta)\right) + \frac{d\rho^2}{2\rho^2} + ds^2(KE_6) \\ G_4 &= -3\sqrt{2}a_0^3\rho^3 dx^+\wedge dx^1\wedge dx^2\wedge d\rho - a_0\rho d\rho\wedge J\wedge dx^+ + 2J\wedge J\,. \end{aligned} \tag{2.22}$$

which includes our solution (2.21) and that of [30] as special cases. As before, one can eliminate one of the two parameters a_0, b_0 by scaling $x^+, \vec{x}$ in same ratio, after which one parameter remains. Setting $b_0 = 2a_0$ yields our solution (2.21), while setting $b_0 = a_0$ yields that of [30].[3]

The 'leading behavior' of the IR solution (2.17) before coordinate redefinition, keeping $e^A, e^V \sim \rho$ only, might look like AdS_5 fibered over KE_6 with a compact longitudinal direction along ψ. However, this AdS_5 fibered over KE_6 *does not* satisfy the equations of motion (2.7) itself, although making the leading $\mathcal{O}(\rho^0)$ equations satisfied. Namely, there

[3] We find that their eq. (4.4) becomes a solution if we take $f_0 = \frac{13\alpha^2}{64}$ (with $c = 2$ which is our normalization), correcting a minor typo in their expression $f_0 = \frac{13\alpha}{4c^4}$. Relating their quantities with ours as $r_{\rm theirs} = \sqrt{2}a_0\rho_{\rm ours}$, $x^-_{\rm theirs} = a_0^{-1}x^-_{\rm ours}$, $\alpha = -2/a_0$, $C_1 = -a_0^{-1}\theta$ with same x^+, their eq. (4.4) takes the form of (2.22) above.

are subleading terms in the equations caused by the leading terms of the functions e^A, e^V, to be canceled by the subleading terms of the functions. There exist $AdS_5 \times KE_6$ solutions in M-theory without any fibration in the ψ angle, which is related to our 'leading behavior' by replacing $\eta \to d\psi$ with same radii for AdS_5 and KE_6 [31]. Another point to mention is that this AdS_5 cannot be obtained by a scaling limit similar to the one we explained above for Sch_5. Naively, it might appear that the scaling $\rho \to \epsilon\rho$, $t \to \epsilon^{-1}t$, $\vec{x} \to \epsilon^{-1}\vec{x}$, $\psi \to \epsilon^{-1}\psi$ with $\epsilon \to 0$ would do this job. However, the last ψ direction is compact. This scaling limit demands one to zoom out to the long wavelength along $t, \vec{x}, \psi$, which does not make sense with a compact direction with a maximal wavelength. Note that the scaling (2.20) to the Schrödinger geometry leaves x^- unchanged.

Before studying the full flow solution which interpolates the above IR solution with AdS$_4$ UV fixed point, we review the asymptotic expansion in UV [25]. We obtain

$$e^A = e^{2r}\left(1 + a_1 e^{-nr} + \cdots\right), \quad e^V = 1 - v_1 e^{-nr} + \cdots, \quad h^2 = 2v_1 e^{-nr} + \cdots, \tag{2.23}$$

with two possible solutions for the coefficients

$$n = 4, \quad v_1 = \frac{8}{3}a_1 \quad \text{and} \quad n = 8, \quad v_1 = 2a_1. \tag{2.24}$$

Defining $z \equiv e^{-2r}$, one obtains the asymptotic AdS$_4$ metric $ds_4^2 = \frac{d\tilde{x}_\mu d\tilde{x}^\mu + dz^2}{4z^2}$ upon defining $\tilde{x}^\mu = 2x^\mu$. The general asymptotic solution is

$$h \sim h_1 z^{\Delta_-} + h_2 z^{\Delta_+} + \cdots \tag{2.25}$$

with $\Delta_- = 1$ and $\Delta_+ = 2$. Both modes are normalizable with these coefficients and there are two possible dual CFT's in which one mode is the source of a boundary operator whose expectation value is given by the coefficient of the other mode. We stick to the CFT in which the field h sources the fermion mass term at the boundary as we review in detail in section 3, in which the mode with $\Delta_- = 1$ is taken to be the source. The fermion mass operator comes with dimension $\Delta_+ = 2$. From the gravity solution, one can get the precise values of the fermion and boson masses, which we explain in section 3.

We now study if the IR solution can be interpolated to the asymptotic AdS$_4$ solution at $r = \infty$. We chose the first equation of (2.13) and the h equation (2.12) for numerics. The numerical analysis of the differential equations exhibits the desired interpolation. The result for $a_0 = .01$ is shown in figure 1. One can also vary the values of a_0 and numerically relate it with the variable v_1 in the UV expansion. In doing so, we should remember that both in the IR and UV solutions we implicitly tuned a coefficient by using the translational symmetry of r in the equations. In matching the two solutions, only one tuning is allowed. So we relax our UV asymptotic behavior by admitting the r translation degrees and write the solution as

$$e^A = ce^{2r} + a_1 c^{-1} e^{-2r} + \cdots, \tag{2.26}$$

etc., where a_1 is the same coefficient as above. We fit the two coefficients a_1, c by comparing the expansion (2.26) with the numerical solution at two large values of r, which we chose as $e^r = 20, 100$, and find c, $a_1 = \frac{3v_1}{8}$ as functions of a_0. We find the relation to be

$$a_0 = \sqrt{v_1}, \quad c = a_0. \tag{2.27}$$

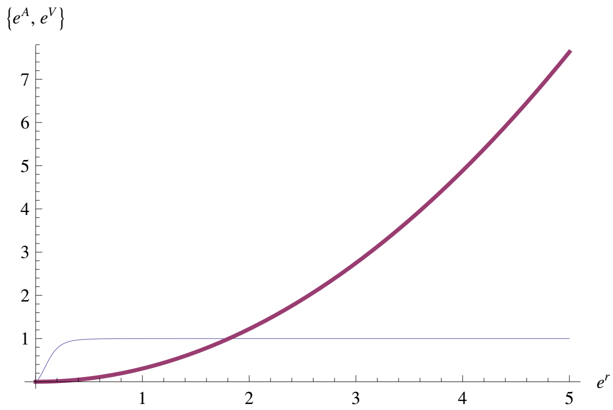

Figure 1. Plot of the numerical solutions as functions of e^r. The thick/thin lines are plots of e^A and e^V, respectively. e^V approaches 1 as $r \to \infty$ (the value for the AdS_4 vacuum).

$\sqrt{v_1}$ appearing as the leading coefficient of h in UV is the mass for fermions, consistent with the IR expectation from (2.18) that a_0 should be proportional to the mass parameter.

Surprisingly, one can find an exact analytic solution for the whole flow [25],

$$\begin{aligned} ds^2_{4E} &= = r^2(-dt^2 + d\vec{x}^2) + g^{-1}(r)dr^2 \\ h(r) &= \frac{\sqrt{3}\sqrt{\frac{6\sqrt{3}(2r^8+4r^4+1)}{\sqrt{Z}} - Z + 9} - \sqrt{3Z} + 3}{6(r^4+1)} \\ Z(r) &\equiv \frac{2\cdot 6^{2/3}(r^4+1)r^{8/3}}{(\sqrt{48r^4+81}-9)^{1/3}} - 6^{1/3}(r^4+1)\left(\sqrt{48r^4+81}-9\right)^{1/3} r^{4/3} + 3 \\ g^{-1}(r) &= \frac{\sqrt{1-h^2}}{4r^2}\left(1 - \frac{r^2(h')^2}{4(1-h^2)^2}\right), \end{aligned} \tag{2.28}$$

where the subscript E denotes the 4 dimensional metric in the Einstein frame. (Note that their r above is not our r used previously.) It is related to our 4 dimensional metric as $(g_E)_{\mu\nu} = e^V g_{\mu\nu}$. We managed to check this exact solution by a heavy use of mathematica. We also checked that our IR expansion (2.16) is completely reproduced from the above exact solution, after relating our IR radial variable ρ with their r as

$$\rho = 2^{1/3}\left(r^{2/3} + \frac{r^2}{24\cdot 2^{1/3}} - \frac{7r^{10/3}}{288\cdot 2^{2/3}} + \frac{767r^{14/3}}{82944} + \cdots\right), \tag{2.29}$$

according to $e^{V/2}\frac{d\rho}{2^{1/2}\rho} = \frac{dr}{g^{1/2}}$.

3 Mass deformation of the UV CFT

When the 7 manifold is $SE_7 = S^7$ or its $\mathbb{Z}_k$ orbifolds, we can identify the mass term which drives our flow. At $k=1$, the M2-brane worldvolume scalars and fermions transform under SO(8) as a vector $\mathbf{8}_v$ and a chiral spinor $\mathbf{8}_s$, respectively. The $\mathcal{N}=8$ supercharge as well as the conformal supercharge transform as $\mathbf{8}_c$. Our skew-whiffed truncation ansatz has a preferred $\mathrm{SU}(4)\times\mathrm{U}(1)_b$ symmetry [32, 33]: in particular, the ansatz is invariant under SU(4) isometry of the $KE_6 = \mathbb{CP}^3$ base. The three SO(8) representations decompose under $\mathrm{SU}(4)\times\mathrm{U}(1)_b$ as

$$\mathbf{8}_v \to \mathbf{4}_{-1} + \bar{\mathbf{4}}_1\,, \quad \mathbf{8}_s \to \mathbf{1}_2 + \mathbf{6}_0 + \mathbf{1}_{-2}\,, \quad \mathbf{8}_c \to \mathbf{4}_1 + \bar{\mathbf{4}}_{-1}\,. \tag{3.1}$$

The charges of $\mathrm{U}(1)_b$ are chosen so that $Q(\text{scalar}) = (\text{fermion})$ is possible, correcting a minor sign error in [33]. The M2-brane theory without skew-whiffing can be obtained by a parity transformation on $\mathbb{R}^8$, interchanging $\mathbf{8}_s$ and $\mathbf{8}_c$ representations. After this change, the supercharges in $\mathbf{6}_0$ of $\mathbf{8}_s$ are the $\mathcal{N}=6$ supersymmetries of [20]. With skew-whiffing, all the supercharges transforming in $\mathbf{4}_1 + \bar{\mathbf{4}}_{-1}$ are broken for $k > 1$. At $k = 1$, monopole operators with $\mathrm{U}(1)_b$ charge ± 1 make them gauge-invariant [34–36], enabling maximal supersymmetry.

We would like to explain the nature of the mass term triggering our flow. The mass term should be invariant under the SU(4) symmetry of the $\mathbb{CP}^3$ in our truncation, in the skew-whiffed frame. Decomposing $\mathbf{8}_s$ (matter fermions) into $\mathbf{1}_2 + \mathbf{6}_0 + \mathbf{1}_{-2}$, we call them ξ, Ψ^I, $\bar{\xi}$ with $I = 1, 2, \cdots, 6$, following [33]. The fermions Ψ^I ($I = 1, 2, \cdots, 6$) in $\mathbf{6}_0$ are Majorana spinors, proposed to be in the adjoint representation of the first $\mathrm{U}(N)$, while the complex fermion ξ in $\mathbf{1}_2$ is in the bi-fundamental representation of $\mathrm{U}(N)\times\mathrm{U}(N)$. It seems that the parity symmetry of this system is not manifest in the field theory. In our discussions below, the role of these gauge group representations will be rather minor. The mass for real fermions ψ^a, with $a = 1, 2, \cdots, 8$ in $\mathbf{8}_s$ of SO(8), is given by (at least on a single M2-brane)

$$M_{ab}\psi^a\psi^b \tag{3.2}$$

with an 8×8 symmetric mass matrix M_{ab}. The symmetric matrix has 36 real components and decomposes under SO(8) as $\mathbf{35}_+ + \mathbf{1}$, where $\mathbf{35}_+$ is the 'traceless' part with respect to the charge conjugation matrix, and $\mathbf{1}$ is the 'identity' part. The spinor bi-linear in $\mathbf{35}_+$ of SO(8) is equivalent to self-dual 4-forms in $\mathbb{R}^8$, and will turn out to be dual to the self-dual 4-form modes in the gravity dual [37] as we explain in some detail below. For now, we just mention that the last two terms on the second line of (2.1) with nonzero h provides the relevant 4-form flux.

The mass matrix decomposes under SU(4) as [32]

$$\mathbf{35}_+ \to \mathbf{1}_0 + \mathbf{1}_4 + \mathbf{1}_{-4} + \mathbf{6}_2 + \mathbf{6}_{-2} + \mathbf{20}_0\,, \tag{3.3}$$

More concretely, the possible mass terms (on a single M2-brane) look like

$$\frac{m_1}{2}\Psi^I\Psi^I + m_1'\bar{\xi}\xi + \mathrm{Re}\left[m_2\xi^2 + m_I\Psi^I\xi\right] + \frac{m_{IJ}}{2}\Psi^I\Psi^J \tag{3.4}$$

with real masses m_1 and m_1', real traceless symmetric mass matrix m_{IJ}, complex masses m_2, m_I. All but one combination of the first two mass terms proportional to m_1, m_1' belong to $\mathbf{35}_+$. A combination of the first two becomes an SO(8) invariant. This cannot be dual to the 4-form flux that we turn on in our solution, as 4-form fluxes in $\mathbb{R}^8$ always break SO(8). So we do not consider this combination. The other 'traceless' combination $m_1' = -3m_1$,

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with canonically normalized kinetic terms for the fermions, corresponds to $\mathbf{1}_0$ in the above decomposition of $\mathbf{35}_+$. This is our mass term, as our gravity solution is invariant under $\mathrm{SU}(4) \times \mathrm{U}(1)_b$. For multiple M2-branes, the first two mass terms proportional to m_1, m_1' can be made gauge invariant by taking traces, as Ψ^I is an adjoint and $\xi, \bar{\xi}$ are in conjugate representations. We emphasize that, as fermions are in $\mathbf{8}_c$ without skew-whiffing, the mass matrix, now given by anti-self-dual 4-forms, decomposes under SU(4) as $\mathbf{35}_- \to \mathbf{10}+\mathbf{10}+\mathbf{15}$ without singlets. This explains why our consistent truncation only has solutions with skew-whiffing.

The bosonic mass term is also SU(4) invariant, which is unique up to coefficient:

$$m_b^2 \bar{Z}_i Z^i \tag{3.5}$$

with $i = 1, 2, 3, 4$. This mass term is actually invariant under SO(8), at least for the single M2-brane for which we can decompose complex scalars to real scalars in $\mathbf{8}_v$. Even at $k = 1$ in which the UV CFT is supersymmetric, it is easy to see that the mass-deformed theory is always non-supersymmetric. This is simply because the mass is same for all bosonic fields, while there appear two different nonzero masses for fermions.

One can compute the precise values of the fermion/boson masses with the known form of the flux. Since we have a CFT before mass deformation, physically meaningful quantity is just the ratio m_1/m_b of the two masses. To conveniently match factors, it is useful to compare the normalization with the well known maximally supersymmetric mass deformation. As a warming up, let us first consider an illustrative example of single M2-brane coupled to a background 4-form field $G_4 = dC_3$ in (almost) flat space. We shall later explain that the same result is obtained from the large N calculation in $AdS_4 \times S^7$. The coupling to this field is given by

$$\int_{\mathbb{R}^{2,1}} \alpha [C_3]_{012} + +\beta \bar{\psi} \gamma^{IJKL} \psi [G_4]_{IJKL} + \cdots , \qquad (I,J,K,L = 1,2,\cdots,8) \tag{3.6}$$

with coefficients α, β determined in [38]. We turn on small constant self-dual G_4 on $\mathbb{R}^8$, where smallness is to keep the back reaction to the flat space to be small. The bosonic mass m_b is related to the constant flux as [38]

$${m_b}^2 = \frac{1}{8 \cdot 4!} G^2 \,. \tag{3.7}$$

This comes partly from the first term of (3.6) after the constant self-dual 4-form back-reacts to $(C_3)_{012}$, and also partly from the back-reaction to the metric. For the two cases

$$G_4 = \gamma_0 (dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4 + dx^5 \wedge dx^6 \wedge dx^7 \wedge dx^8) \tag{3.8}$$

and ($\mathbb{J}$ below is the Kahler 2-form on $\mathbb{R}^8$, not on $\mathbb{CP}^3$)

$$G_4 = \gamma \mathbb{J} \wedge \mathbb{J} \quad (\text{with } \ \mathbb{J} = dx^1 \wedge dx^2 + dx^3 \wedge dx^4 + dx^5 \wedge dx^6 + dx^7 \wedge dx^8) \tag{3.9}$$

on $\mathbb{R}^8$, one obtains

$$m_{b0}^2 = \frac{\gamma_0^2}{4} \quad \text{and} \quad m_b^2 = 3\gamma^2 \,, \tag{3.10}$$

respectively. All quantities with the subscript 0 are for the case with the maximally supersymmetric mass deformation, to be compared with the case of our interest. In each case, expanding the fermion mass term in (3.6), one obtains

$$m_{f0} = \pm 2 \cdot 4!\beta\gamma_0\,, \quad m_f = (m_\xi, m_{\Psi^i}) \equiv (\pm 12 \cdot 4!\beta\gamma,\ \pm 4 \cdot 4!\beta\gamma)\,, \tag{3.11}$$

where the last two masses are those for the complex fermion ξ in $\mathbf{1}_{\pm 2}$ and the real fermions Ψ^i in $\mathbf{6}_0$. Their mass ratio 3 is what we anticipated from group theory. In the former case, since the masses for bosons and fermions are equal due to supersymmetry, $m_{b0} = m_{f0}$ can be used to conveniently fix the normalization as $\beta = (4 \cdot 4!)^{-1}$. Inserting this back, one obtains

$$m_\xi = \sqrt{3} m_b, \quad m_{\Psi^i} = \frac{1}{\sqrt{3}} m_b\,. \tag{3.12}$$

These masses happen to satisfy a supertrace constraint $\mathrm{Str}(M^2) = 0$.[4]

Now we explain how one can obtain the same masses from the asymptotic $AdS_4 \times S^7$ gravity solution in the large N limit. Contrary to the simple case with single M2-branes, which is described by a free quantum field theory, the masses that we calculate below with the large N gravity solution is the bare mass, or the UV mass, of the field theory. Since the mass-deformed field theory preserves no supersymmetry, there can be nontrivial quantum corrections for the actual mass of particles in the IR. The latter masses will be important quantities when one discusses the non-relativistic limit from the field theory. The procedure of obtaining fermionic masses [37] is similar to the above calculation for single M2-branes. The 4-form T_4 in $\mathbb{R}^8$ defined by $4T_4 = dS_3$ and $G_4 \sim d\left[\frac{R^6}{r^6} S_3\right]$ plays the role similar to the constant 4-form which appeared in the mass matrix for M2-branes in flat space, where R is the radius of 7-sphere which we set 1 in our normalization. G_4 above is the flux on the S^7 and the radial part only. Working with the asymptotic solution of the form,

$$ds^2_{11} \sim r^4 \left(\eta_{\mu\nu} dx^\mu dx^\nu\right) + \frac{dr^2}{r^2} + ds^2(S^7)\,, \quad G_4 \sim d\left(r^6 dt \wedge dx^1 \wedge dx^2\right) + d\left(r^{-6} S_3\right)\,, \tag{3.13}$$

one obtains $T_4 = \sqrt{\frac{16a_1}{3}}(dx^{1234} + dx^{1256} + dx^{1278} + dx^{3456} + dx^{3478} + dx^{5678})$. The resulting fermion masses are

$$m_\xi = \sqrt{12a_1}\,, \quad m_{\Psi^i} = \sqrt{\frac{4a_1}{3}}\,. \tag{3.14}$$

As for the overall normalization, we matched with the convention used in [24] for the maximally supersymmetric mass deformation, in which $T_4 = 2\mu_0(dx^{1234} + dx^{5678})$ from the gravity solution leads to the boson/fermion masses μ_0.[5] The bosonic mass can be quickly computed if one takes a probe M2-brane in the mass-deformed UV solution. Using the radial coordinate r appearing in (3.13), one obtains a quadratic potential in r after partly canceling the contributions from the Nambu-Goto and Wess-Zumino terms. In the maximally supersymmetric case and for our case, we obtain $m_{b0} = \mu_0$ and $m_b = 2\sqrt{a_1}$, respectively, consistent with (3.12).

[4] This constraint turns out to hold for general self-dual flux. From the fermion mass matrix $M_f = \frac{1}{4\cdot 4!}\gamma^{IJKL} G_{IJKL}$ and the relation (3.7) to the boson mass, one can show $\mathrm{Str}(M^2) = 8m_b^2 - \mathrm{tr} M_f^2 = 0$.

[5] This relation to mass is obtained by comparing some BPS spectrum of particles or domain walls on both gauge/gravity sides. The relation corrects a minor typo of factors in v1 of [24], as explained in [39].

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Suppose we have a system with particles having various values of mass and global $\mathrm{U}(1)_b$ charge. We consider non-relativistic limit with the $\mathrm{U}(1)_b$ symmetry as we did in our gravity solution. The bosonic and two fermionic fields have charges $R_b = 1,\ 2,\ 0$, respectively from (3.1). There could be other particles coming from bound states carrying nonzero global charge. We take the non-relativistic energy to be $E_{NR} = E_{rel} - aR_b \geq 0$, where E_{rel} is the relativistic energy and a is some constant. The constant a has to be chosen such that the condition $E_{NR} \geq 0$ is met for all particles, while being saturated by a subset of them. From the gravity solution, as we made a coordinate transformation $x^- = \psi - a_0 t$, $x^+ = t$, the conserved charge for x^+ is given by $E_{NR} = E_{rel} - a_0 R_b$, with $a = a_0$. The non-relativisitic field theory only keeps a sector of particles which satisfy $E_{NR} = 0$ when they are at rest. On the other hand, it discards at low energy all the other particles with $E_{NR} > 0$ at rest, as such particles have large non-relativistic energies. This is exactly same as discarding anti-particles with non-relativistic energy $E_{NR} = 2mc^2$. So we have $\frac{R_b}{m} \leq a^{-1}$ for some constant a, where the modes which saturate the inequality survive the non-relativistic limit. Therefore, the non-relativistic theory would be keeping the particles with the largest value of $\frac{R_b}{m}$, using the IR masses of the particles.

Let us start from the theory on a single M2-brane, described by a free QFT. Considering the three elementary particles only, one obtains the ratio

$$\frac{1}{m_b} : \frac{2}{m_\xi} : \frac{0}{m_{\Psi^i}} = 1 : \frac{2}{\sqrt{3}} : 0\,. \tag{3.15}$$

Thus, on single M2-branes, the ξ mode survives in the non-relativistic theory if one uses $\mathrm{U}(1)_b$ as the particle number charge. For the case with large N, had all the UV masses been the physical masses of elementary particles without any quantum corrections, one would have obtained the same ratio as the single M2-brane. The actual values for m/R_b from our gravity solution are $\frac{m_\xi}{2} = \sqrt{\frac{9}{8}}a_0$, $m_b = \sqrt{\frac{3}{2}}a_0$ from (2.24) and (2.27), which are both larger than $a = a_0$ which we obtained from the coordinate transformation. There could be two possible reasons for this. It could be that the physical masses for these particles in IR acquire nontrivial quantum corrections causing smaller mass-to-charge ratios: at least nonzero quantum corrections are natural for our non-supersymmetric theories. And/or, there can also appear bound states of these particles with nonzero binding energies, lowering the ratio. As the Schrödinger geometry comes with an infinite tower of Kaluza-Klein states on x^-, representing infinitely many particle species [40, 41], the last issue of bound states should anyway be an essential ingredient for understanding this system. It should be interesting to have a better understanding on these quantum aspects of the spectrum.

4 Discussions

In this paper, we considered the gravity solutions for a class of mass-deformed CFT's in 3 dimensions, and showed that there exist 5 dimensional Schrödinger geometries in the IR after taking the non-relativistic limit with a global U(1) symmetry. As far as we are aware of, this is the first occasion in which non-relativistic conformal geometries are explicitly derived by taking the natural non-relativistic limit of massive relativistic systems. One could think of some generalizations and extensions of this idea to various directions.

Firstly, one may ask if a similar story holds for the maximally supersymmetric mass deformation of the same UV theory. As the 4-form flux preserves SO(4) $\times$ SO(4), the flow solution would have two round 3-spheres. We should be generalizing [21, 22] and seek for a solution with broken supersymmetry if the Witten index calculation of [23] is implying that the vacuum spontaneously breaks supersymmetry. Let us try to see if a mechanism explained in this paper could be working, namely, if the 'particle number circle' can shrink in IR to be part of the 5 dimensional Schrödinger geometry. As the U(1) symmetry is taken from the two round 3-spheres in [18, 19], any shrinking circle necessarily demands that the whole 3-spheres shrink as well, presumably making the IR geometry very singular. It may also be worth considering the possibility that this vacuum could be metastable, or does not exist at all.

As explained before, the nonrelativistic CFT of [18, 19] is based on the 'symmetric vacuum' of the supersymmetric mass-deformed theory, with zero scalar VEV. The vacuum preserves supersymmetry only when $N \leq k$. Its gravity dual will be highly curved, as the 't Hooft coupling is small. Still, formal gravity solutions with large curvature are identified in [24]. It may be possible to learn something useful from these solutions.

Let us also mention at this point that, from the viewpoint of the non-relativistic limit of mass-deformed CFT, the appearance of Schrödinger geometries as the gravity duals of non-relativistic CFT's seems to demand a strong restriction. To illustrate this point, note that there are many ways of obtaining non-relativistic theories with a given mass-deformed CFT if there exist more than one global U(1) symmetries, simply by choosing different U(1)'s as particle number symmetries. Indeed, for the maximally supersymmetric mass deformation, different non-relativistic CFT's obtained by choosing different U(1)'s in the SO(4) $\times$ SO(4) global symmetry were studied [18]. In our mass deformed theory, one can also consider using a U(1) in the SU(4) isometry as the particle number symmetry, rather than U$(1)_b$ along the ψ or x^- fiber of SE_7 as we did in this paper. Note that a crucial ingredient which allowed Sch$_5$ of this paper was the shrinking circle for U$(1)_b$ symmetry, $e^V \to 0$, in IR. This cannot happen for U(1) $\subset$ SU(4) as the KE_6 part of the metric is finite in IR. This seems to be implying that, for the geometry of [1, 2] to emerge as the gravity dual of Schrödinger invariant QFT obtained from mass-deformed relativistic CFT, there should be nontrivial conditions on the quantum dynamics. It would be interesting to further explore this issue.

Our solutions reduced down to 4 dimensional spacetime of the gauged supergravity may not look so well-behaved [25]. It should be interesting to get a low dimensional intuition for the solutions with Schrödinger symmetry to explore a more general class of such solutions, maybe using attractor mechanisms [42, 43]. Similar studies have been done in [41], which attempts to realize the particle number symmetry without using internal isometries.

Following the idea of this paper, it might be possible to find flow solutions from AdS$_5$ to Sch$_6$ by studying the consistent Kaluza-Klein truncation of [44–46] based on $AdS_5 \times S^5$. There also exists a holographic flow solution for the 4 dimensional $\mathcal{N}=4$ Yang-Mills theory with an $\mathcal{N}=1$ supersymmetric mass deformation, which is singular in the IR [47].

Perhaps it might be worth mentioning that the original motivation in [1, 2] of using this geometry to holographically study cold fermionic atoms at unitarity may be demanding alternative realizations of the Schrödinger symmetry without using the isometry of x^- as the particle number. This question has been raised due to an exotic thermodynamics of Schrödinger black holes [13–15], probably caused by the presence of an infinite tower of Kaluza-Klein particle species [40, 41]. Of course our system would be showing same phenomena, simply due to the appearance of the geometry of the form (1.1). The non-relativistic limit of [18, 19] is based on using the D0-brane charge (geometrizable to M-theory isometry) as the particle number, which is also the case for our system. The non-relativistic CFT discussed in [13–15] essentially has this x^- direction as it is obtained from a $3+1$ dimensional QFT in the UV, where the x^- isometry simply comes from the extra direction of the UV field theory. On the other hand, non-relativistic QFT's obtained by the standard non-relativistic limit of mass-deformed QFT do not necessarily have such an infinite tower of bound states. From the field theory perspective, the special example of M2-brane CFT discussed in this paper seems to be rather exceptional, containing geometrizable D0-brane global charge and exhibiting bound states for all D0-brane numbers. Thus, it could be possible to seek for a study along the line of [41], to have a gravity solution invariant under Schrödinger symmetry with non-geometrizable particle number symmetry in the context of the mass-deformed CFT.

We would also like to comment that in [25], the phases of skew-whiffed field theory with relevant deformations (i.e. coupled to nonzero h field in the gravity dual) have been studied extensively from the gravity dual, after considering more modes than what we did in this paper.[6] More precisely, a charged scalar and a massless gauge field were kept, apart from h. In particular, there is a phase transition which appears at nonzero chemical potential μ and mass, the latter one being the mass that we considered in this paper. When μ is sufficiently larger than the mass, the system is shown to be in a superconducting phase with spontaneously broken U(1) symmetry. On the other hand, if the chemical potential is small compared to the mass, the system is supposed to be in a normal phase which is gapped, without the Goldstone boson associated with spontaneously broken U(1). In particular, the line $\mu = 0$ at zero temperature on the parameter space that we have considered all belongs to this normal phase. At zero temperature in the normal phase, with all the propagating degrees being massive, one may ask if there are interesting non-relativistic solutions like what we considered on the $\mu = 0$ line. Although the numerical solution was explained to be singular at zero temperature, this could be physically acceptable. In string theory, there are examples of four-dimensional extremal black holes described by singular geometries, yet being supersymmetric [48, 49]. Furthermore if we are interested in condensed matter systems with unique ground states, the dual geometries have zero entropy, which often lead to singular geometries. Also, uplifting the 4 dimensional geometry to 11 dimensions could reveal more interesting structures, as was the case on the $\mu = 0$ line we studied in this paper.

Finally, it could be desirable to investigate the stability of the solutions discussed in this paper. An instability was reported for a solution in the skew-whiffed truncation with nonzero values of the charged field [50]. Although the solution we discussed is more

[6] The comments below are motivated by discussions with Jerome Gauntlett and Julian Sonner.

complicated than the AdS solution studied there, at least studies of the IR solution (2.21) would be doable and worthwhile. Perhaps this problem could be related to the stability analysis of a class of $AdS_5 \times KE_6$ solutions [51].

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