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THE MAXIMAL SUBGROUPS AND THE COMPLEXITY OF THE FLOW SEMIGROUP OF FINITE (DI)GRAPHS

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Abstract. The flow semigroup, introduced by John Rhodes, is an invariant for digraphs and a complete invariant for graphs. We refine and prove Rhodes's conjecture on the structure of the maximal groups in the flow semigroup for finite, antisymmetric, strongly connected graphs.

Building on this result, we investigate and fully describe the structure and actions of the maximal subgroups of the flow semigroup acting on all but k points for all finite digraphs and graphs for all $k \geq 1$. A linear algorithm is presented to determine these so-called 'defect k groups' for any finite (di)graph.

Finally, we prove that the complexity of the flow semigroup of a 2-vertex connected (and strongly connected di)graph with n vertices is $n-2$, completely confirming Rhodes's conjecture for such (di)graphs.

1. Introduction

John Rhodes in [9] introduced the *flow semigroup*, an invariant for graphs and digraphs. In the case of graphs, this is a complete invariant determining the graph up to isomorphism. The flow semigroup is the semigroup of transformations of the vertices generated by elementary collapsings corresponding to the edges of the (di)graph. (See Section 2 for precise definitions.)

A maximal subgroup of this semigroup for a finite (di)graph $D = (V_D, E_D)$ acts by permutations on all but k of its vertices ($1 \leq k \leq |V_D| - 1$) and is called a "defect k group". The set of defect k groups

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of a (di)graph is also an invariant. For each fixed k, they are all isomorphic to each other in the case of (strongly) connected (di)graphs. Rhodes formulated a conjecture on the structure of these groups for strongly connected digraphs whose edge relation is anti-symmetric in [9, Conjecture 6.51i (2)–(4)]. We show that his conjecture was correct, and we prove it here in sharper form (Theorems 1 and 27). Moreover, extending this result further, we fully determine the defect k groups for all finite graphs and digraphs (Theorem 2).

The structure of the argument is as follows. First, a maximal group in the flow semigroup of a digraph D is the direct product of maximal groups of the flow semigroups of its strongly connected components. Thus one needs only to consider strongly connected digraphs. The defect k group of D consists of elements of the flow semigroup permuting all but k vertices. It turns out, that if D is a strongly connected digraph, then the defect k group (up to isomorphism) does not depend on the choice of the vertices it acts on. Further, for a strongly connected digraph, its flow semigroup is the same as the flow semigroup of the simple graph obtained by "forgetting" the direction of the edges. This is detailed in Section 2 and is based on [9, p. 159–169]. Thus, one only needs to consider the defect k groups of the flow semigroup for simple connected graphs.

In Section 3 we list some useful lemmas and determine the defect k group of a cycle. Further, in Section 4 we lay some group theory groundwork by determining the permutation group $T_{k,l,m}$ generated by two cycles $(a_1, \dots, a_k, b_1, \dots, b_l)$ and $(a_1, \dots, a_k, c_1, \dots, c_m)$. Then in Section 5 we first determine the defect 1 group of 2-vertex connected graphs, then of arbitrary simple connected graphs by proving

Theorem 1 (Defect 1 group for simple connected graphs)**.** *Let Γ be a simple connected graph of n vertices, and let $\Gamma_1, \dots, \Gamma_m$ be its 2-vertex connected components. Then the defect 1 group of Γ is the direct product of the defect 1 groups of Γ_i $(1 \leq i \leq m)$. If Γ is 2-vertex connected, then its defect 1 group is isomorphic (as a permutation group) to*

1. *the cyclic group Z_{n-1} if Γ is a cycle;*
2. *$T_{2,2,2}$ (that is S_5 acting sharply 3-transitively on 6 points), if Γ is the exceptional graph (see Figure 1);*
3. *S_{n-1} or A_{n-1}, otherwise. Further, the defect 1 group is A_{n-1} if and only if Γ is bipartite.*

In particular, Rhodes's conjecture (as phrased in [9, Conjecture 6.51i (2)] for strongly connected, antisymmetric digraphs) about the defect 1 group holds, and more generally: the defect 1 group of the flow semigroup of a simple connected graph is indeed the product of cyclic, alternating and symmetric groups of various orders. Applying Theorem 1, a straightforward linear algorithm can be given to determine

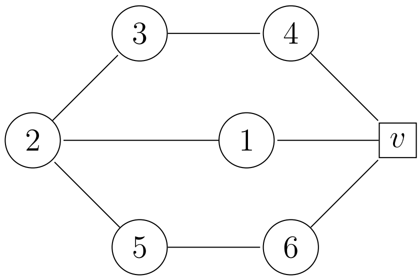


FIGURE 1. Exceptional graph

the direct components of the defect 1 group of an arbitrary connected graph (see Section 7).

Then in Section 6 we determine the defect k groups of arbitrary graphs by considering the so-called k-components (maximal subgraphs for which the defect k group is the full symmetric group) and prove

Theorem 2. *Let $k \geq 2$, Γ be a simple connected graph of n vertices, $n > k$. If Γ is a cycle, then its defect k group is the cyclic group Z_{n-k}. Otherwise, let $\Gamma_1, \ldots, \Gamma_m$ be the k-components of Γ, and let Γ_i have n_i vertices. Then the defect k group of Γ is the direct product of the defect k groups of Γ_i $(1 \leq i \leq m)$, thus it is isomorphic (as a permutation group) to*

$$S_{n_1-k} \times \cdots \times S_{n_m-k}.$$

In Section 7 we provide a linear algorithm (in the number of edges of Γ) to determine the k-components of an arbitrary connected graph.

Rhodes further conjectured [9, Conjecture 6.51i (1)] that the complexity of the flow semigroup of a strongly connected, antisymmetric digraph on n vertices is $n-2$. We confirm this conjecture in Section 8 when the digraph is 2-vertex connected:

Theorem 3. *Let $\Gamma = (V, E)$ be a 2-vertex connected graph with $n \geq 2$ vertices. Then $\#_G(S_\Gamma) = n - 2$.*

Finally, we prove some bounds on the complexity of flow semigroups in the remaining cases, and state some open problems on the complexity of flow semigroups at the end of Section 8.

2. FLOW SEMIGROUP OF DIGRAPHS

For notions in graph theory we refer to [4, 6], in group theory to [12] in permutation groups to [1, 5], in semigroup theory to [2, 3]. For a digraph $D = (V_D, E_D)$, the *flow semigroup* $S = S_D$ is defined by

$$S = S_D = \langle e_{uv} \mid uv \in E_D \rangle ,$$

where e_{uv} is the *elementary collapsing* corresponding to the directed edge $uv \in E_D$, that is, for every $x \in V_D$ we have

$$x \cdot e_{uv} = xe_{uv} = \begin{cases} v, & \text{if } x = u, \\ x, & \text{otherwise.} \end{cases}$$

Note that in this paper functions act on the right, therefore permutations are multiplied from left to right. Further, for a set $X \subseteq V_D$ and a semigroup element $s \in S_D$ we define

$$Xs = \{ xs \mid x \in X \}.$$

A maximal subgroup of S_D is a subgroup such that it is not contained properly in any other subgroup of S_D. In order to determine the maximal subgroups of S_D, one can make several reductions by [9, Proposition 6.51f]. First, one only needs to consider the maximal subgroups of S_{D_i} for the strongly connected components D_i of D. Strongly connected components are maximal induced subgraphs such that any vertex can be reached from any other vertex by a directed path.

Lemma 4 ([9, Proposition 6.51f (1)]). *Let D be a digraph, then every maximal subgroup of S_D is (isomorphic to) the direct product of maximal subgroups of S_{D_i}, where the D_i are the strongly connected components of D.*

An element $s \in S$ is *of defect k* if $|V_D s| = |V_D| - k$. Let $V_k = \{ v_1, v_2, \dots v_k \} \subseteq V_D$. The *defect k group* $G_k = G_{k,V_k}$ associated to V_k (called the *defect set*) is generated by all elements of S restricted to $V \setminus V_k$ which permute the elements of $V \setminus V_k$ and move elements of V_k to elements of $V \setminus V_k$:

$$G_k = G_{k,V_k} = \left\langle s \restriction_{V \setminus V_k} : s \in S, (V \setminus V_k)s = V \setminus V_k, V_k s \subseteq V \setminus V_k \right\rangle.$$

Now, G_k is a permutation group acting on $V \setminus V_k$. For this reason $V \setminus V_k$ is called the *permutation set* of G_k. In general, the defect k group G_k can depend on the choice of V_k. However, by [9, Proposition 6.51f (2)] it turns out that if the graph is strongly connected then the defect k group G_k is unique up to isomorphism.

Lemma 5 ([9, Proposition 6.51f (2)]). *Let D be a strongly connected digraph. Let $V_k, V_k' \subseteq D$ be subsets of nodes such that $|V_k| = |V_k'| = k$. Then $G_{k,V_k} \simeq G_{k,V_k'}$ as permutation groups.*

Further, the case of strongly connected graphs can be reduced to the case of simple graphs. Let $\Gamma = (V, E)$ be a simple (undirected) graph, we define S_Γ by considering Γ as a directed graph where every edge is directed both ways. Namely, let $D_\Gamma = (V, E_D)$ be the directed graph on vertices V such that both $uv \in E_D$ and $vu \in E_D$ if and only if the undirected edge $uv \in E$. Then let $S_\Gamma = S_{D_\Gamma}$.

Further, for every digraph $D = (V_D, E_D)$, one can associate an undirected graph Γ by "forgetting" the direction of edges in D. Precisely, let $\Gamma_D = (V_D, E)$ be the undirected graph such that $uv \in E$ if and only if $uv \in E_D$ or $vu \in E_D$. The following lemma shows that if a digraph D is strongly connected then the semigroup S_D corresponding to D and the semigroup S_{D_Γ} corresponding to the simple graph Γ_D are the same.

Lemma 6 ([9, Lemma 6.51b]). *Let D be an arbitrary digraph. Then*

$$e_{ab} \in S_D \iff \begin{cases} a \to b \text{ is an edge in } D, \text{ or} \\ b \to a \text{ is an edge in a directed cycle in } D. \end{cases}$$

In particular, if D is strongly connected then $S_D = S_{\Gamma_D}$.

Proof. Let $b \to a \to u_1 \to \cdots \to u_{n-1} \to b$ be a directed cycle in D. Then an easy calculation shows that

$$e_{ab} = \left(e_{ba} e_{u_{n-1}b} e_{u_{n-2}u_{n-1}} \dots e_{u_1u_2} e_{au_1}\right)^n .$$

For the other direction, assume $e_{ab} = e_{uv}s$ for some $s \in S_D$. Then $e_{uv}s$ moves u and v to the same vertex, while e_{ab} moves only a and b to the same vertex. Thus $\{a, b\} = \{u, v\}$. □

Therefore, in the following we only consider simple, connected, undirected graphs $\Gamma = (V, E)$, that is no self-loops or multiple edges are allowed. Further, Γ is 2-edge connected if removing any edge does not disconnect Γ. Rhodes's conjecture [9, Conjecture 6.51i (2)–(4)] is about strongly connected, antisymmetric digraphs. Note that by [11] a strongly connected antisymmetric digraph becomes a 2-edge connected graph after forgetting the directions.

Let us set some notations. By *cycle* we will mean a simple cycle, that is a closed walk with no repetition of edges or vertices except for the starting and ending vertex. A *path* is a walk with no repetition of edges or vertices. By *subgraph* $\Gamma' = (V', E')$ we mean a graph for which $V' \subseteq V$, $E' \subseteq E$. If Γ' is an *induced subgraph*, that is E' consists of all edges from E with both endpoints in V', then we explicitly indicate it. The letters k, l, m and n will denote nonnegative integers. The number of vertices of Γ is usually denoted by n, while k will denote the size of the defect set. Usually we denote the defect k group of a graph Γ by G_k or G_Γ, depending on the context. We try to heed to the convention of using u, v, w, x, y as vertices of graphs, V as the set of vertices, E as the set of edges. Further, the flow semigroup is mostly denoted by S, its elements are denoted by s, t, g, h, p, q. In Section 4 we use x and y for denoting permutations. The cyclic group of m elements is denoted by Z_m.

3. Preliminaries

Let $\Gamma = (V, E)$ be a simple, connected (undirected) graph, and for every $1 \leq k \leq |V|-1$, let G_k denote its defect k group for some $V_k \subseteq V$,

$|V_k| = k$. Let $S = S_\Gamma$ be the flow semigroup of Γ. The following is immediate.

Lemma 7 ([9, Fact 6.51c]). *Let $s \in S$ be of defect k. If se_{uv} is of defect k, as well, then $u \notin Vs$ or $v \notin Vs$.*

Further, it is not too hard to see that every defect 1 permutation arises from the permutations generated by cycles (in the graph) containing the defect point.

Lemma 8 ([9, Proposition 6.51e]). *Let Γ be a connected graph, and let G_1 denote its defect 1 group, such that the defect is $v \in V$. Then*

$$G_1 = \langle (u_1, \dots, u_k) \mid (u_1, \dots, u_k, v) \text{ is a cycle in } \Gamma \rangle .$$

These yield that the defect k group of the n-cycle graph is cyclic:

Lemma 9. *The defect k group of the n-cycle is isomorphic to Z_{n-k}.*

Proof. Let $x_1, x_2, \dots x_n$ be the (clockwise) consecutive elements of the cycle $\Gamma = (V, E)$. If $s \in S$ is an element of defect k then by Lemma 7 we have that $se_{x_i x_{i+1}}$ is of defect k if and only if $x_i \notin Vs$ or $x_{i+1} \notin Vs$. This means that if $u_1, u_2, \dots u_{n-k}$ are the (clockwise) consecutive elements of Vs in the cycle and $se_{x_i x_{i+1}}$ is of defect k, as well, then

$$u_1 e_{x_i x_{i+1}}, u_2 e_{x_i x_{i+1}}, \dots, u_{n-k} e_{x_i x_{i+1}}$$

are the (clockwise) consecutive elements of $Vse_{x_i x_{i+1}}$. Thus the cyclic ordering of these elements cannot be changed. Hence G_k is isomorphic to a subgroup of Z_{n-k}.

Now, assume that $v_1, v_2, \dots v_k, u_1, u_2, \dots u_{n-k}$ are the consecutive elements of Γ, and the defect set is $V_k = \{ v_1, \dots, v_k \}$. Let

$$\begin{aligned} s_1 &= e_{v_1 v_2} \dots e_{v_j v_{j+1}} \dots e_{v_{k-1} v_k}, \\ s_2 &= e_{u_{n-k} v_k} e_{u_{n-k-1} u_{n-k}} \dots e_{u_{j-1} u_j} \dots e_{u_1 u_2} e_{v_k u_1}, \\ s &= s_1 s_2. \end{aligned}$$

It easy to check that

$$v_i s = u_1, \quad u_1 s = u_2, \dots, u_j s = u_{j+1}, \dots, u_{n-k} s = u_1.$$

Therefore $s, s^2, \dots, s^{n-k}$ are distinct elements of G_k, hence $G_k \simeq Z_{n-k}$. □

4. Finite permutation groups generated by two cycles

In this section we investigate the group $T_{k,l,m}$ which is generated by two overlapping cycles. Let k, l, m be non-negative integers, $n = k + l + m \geq 1$, let

$$\begin{aligned} x &= (a_1, a_2, \dots, a_k, b_1, b_2, \dots, b_l), \\ y &= (a_1, a_2, \dots, a_k, c_1, c_2, \dots, c_m), \end{aligned}$$

and we define

$$T_{k,l,m} = \langle x, y \rangle \tag{1}$$

as the subgroup of the symmetric group S_n generated by the cycles x and y. The elements $a_1, \ldots, a_k, b_1, \ldots, b_l, c_1, \ldots, c_m$ are pairwise distinct. We prove that the group $T_{k,l,m}$ is either the symmetric or alternating group of degree n, apart from a few exceptions.

Theorem 10. *Let k, l, m be non-negative integers, $n = k + l + m \geq 1$, and let $T_{k,l,m}$ be the group defined in* (1)*. Then one of the following holds.*

1. *If $k = 0$ or $k + l = 1$ or $k + m = 1$ then $T_{k,l,m} \simeq Z_{k+l} \times Z_{k+m}$;*
2. *if $k \geq 1$, $k + l$ and $k + m$ are both odd, then $T_{k,l,m} = A_n$;*
3. *$T_{3,2,1} \simeq T_{2,2,2} \simeq T_{3,1,2} \simeq S_5$, and this is a sharply 3-transitive action of S_5 on 6 elements;*
4. *$T_{k,l,m} = S_n$, otherwise.*

Proof. We follow the convention of permutations acting on the right, therefore we multiply permutations from left to right. Further, the conjugation of x by y is $x^y = y^{-1}xy$.

If $k + l = 1$ then $x = id$, if $k + m = 1$ then $y = id$. If $k = 0$, then x and y are disjoint and thus $T_{0,l,m} \simeq Z_l \times Z_m$. From now on, we assume $k \geq 1$. The following technical lemma will help in handling the different cases.

Lemma 11. *Let k, l, m be non-negative integers, and assume that $k \geq 1$. Then the following three groups are isomorphic as permutation groups:*

$$T_{k,l,m} \simeq T_{l+1,k-1,m} \simeq T_{m+1,l,k-1}.$$

Proof. Now,

$$x = (a_k, b_1, b_2, \ldots, b_l, a_1, \ldots, a_{k-1}),$$

and

$$xy^{-1} = (a_k, b_1, b_2, \ldots, b_l, c_m, \ldots, c_2, c_1).$$

Therefore

$$T_{l+1,k-1,m} \simeq \left\langle x, xy^{-1} \right\rangle = \langle x, y \rangle = T_{k,l,m},$$

and $T_{m+1,l,k-1} \simeq T_{k,l,m}$ follows by exchanging the roles of x and y. □

We prove in Lemma 13 that the group $T_{k,l,m}$ is 2-transitive, therefore primitive. For the proof of Theorem 10 we are going to use Jordan's famous theorem on primitive permutation groups.

Theorem 12 (Jordan, [5, Theorem 3.3E])**.** *Let G be a primitive permutation group of degree n. If G contains a 2-cycle, or a 3-cycle, or a p-cycle for some prime $p \leq n-3$, then G is either the whole symmetric groups S_n or the alternating group A_n.*

We are going to find a 3-cycle in $T_{k,l,m}$. Then Jordan's theorem (Theorem 12) provides that $T_{k,l,m}$ is either A_n or S_n. If both x and y are even permutations (i.e. $k+l$ and $k+m$ are both odd), then $T_{k,l,m} \leq A_n$ (and thus $T_{k,l,m} = A_n$), otherwise $T_{k,l,m} \not\leq A_n$ (and hence $T_{k,l,m} = S_n$). Finally, the cases $k = l = m = 2$ or $k = 3$, $\{ l, m \} = \{ 2, 1 \}$ will be handled separately in Lemma 14. First, we prove that $T_{k,l,m}$ is 2-transitive.

Lemma 13. *If $k \geq 1$, then $T_{k,l,m}$ is a 2-transitive group on the elements*

$$\Omega = \{ a_1, a_2, \ldots, a_k, b_1, b_2, \ldots, b_l, c_1, c_2, \ldots, c_m \} .$$

Proof. Let Ω_2 be the set of the two-element subsets of Ω. To prove 2-transitivity of $T_{k,l,m}$, we show that $T_{k,l,m}$ acts transitively on Ω_2, and that there exists an element of $T_{k,l,m}$ that transposes two elements of Ω. Assume $l \geq m$, but for this lemma we *do not* assume $k \geq l$ or $k \geq m$.

First, assume $l \geq 1$, $m \geq 1$, and consider the orbit of the subset $\{ b_1, c_1 \}$. Then

$$\begin{aligned}
\{ b_i, c_j \} &= \{ b_1, c_1 \} x^{i-1} y^{j-1}, && 1 \leq i \leq l,\ 1 \leq j \leq m, \\
\{ a_i, b_j \} &= \{ b_j, c_m \} y^{i}, && 1 \leq i \leq k,\ 1 \leq j \leq l, \\
\{ a_i, c_j \} &= \{ b_l, c_j \} x^{i}, && 1 \leq i \leq k,\ 1 \leq j \leq m, \\
\{ a_i, a_j \} &= \{ a_{k+i-j+1}, b_1 \} x^{-k+j-1}, && 1 \leq i < j \leq k, \\
\{ b_i, b_j \} &= \{ b_{l+i-j+1}, a_1 \} x^{-l+j-1}, && 1 \leq i < j \leq l, \\
\{ c_i, c_j \} &= \{ c_{m+i-j+1}, a_1 \} y^{-m+j-1}, && 1 \leq i < j \leq m.
\end{aligned}$$

Further, the permutation $x^{l-1} y^{m-1} x y^k x$ transposes b_1 and c_1.

Finally, if $l \geq 1$ and $m = 0$, or if $m \geq 1$, $l = 0$, then by Lemma 11 we have $T_{k,l,0} \simeq T_{1,l,k-1}$, or $T_{k,0,m} \simeq T_{1,k-1,m}$, □

The roles of l and m are symmetric, and $T_{k,l,m} \simeq T_{k,m,l}$, thus we may assume $l \geq m$. Further, by Lemma 11 we may assume $k \geq l+1$, otherwise we consider $T_{l+1,k-1,m}$ instead. For finding a 3-cycle, we need to consider several cases.

First, assume $m = 0$. From

$$\begin{aligned}
x &= (a_1, a_2, \ldots, a_k, b_1, b_2, \ldots, b_l) , \\
y &= (a_1, a_2, \ldots, a_k) ,
\end{aligned}$$

we have

$$\begin{aligned}
x^y = y^{-1} x y &= (a_1 y, a_2 y, \ldots, a_k y, b_1 y, b_2 y, \ldots, b_l y) \\
&= (a_2, \ldots, a_k, a_1, b_1, b_2, \ldots, b_l) , \\
x^y x^{-1} &= (a_1, a_k, b_l) \in T_{k,l,m}.
\end{aligned}$$

If $l = m = 1$, then

$$\begin{aligned} x &= (a_1, a_2, \ldots, a_k, b_1)\,, \\ y &= (a_1, a_2, \ldots, a_k, c_1)\,, \end{aligned}$$

and we have

$$xy^{-1} = (a_k, b_1, c_1) \in T_{k,l,m}.$$

If neither $l = 1$, nor $m = 0$ hold, then we have $l \geq 2$, $m \geq 1$, $k \geq l+1$. Note that if $k = 3$, then $m \geq 2$, otherwise we have $(k, l, m) = (3, 2, 1)$.

Assume $(k, l, m) \neq (3, 2, 1)$. Now, we have

$$\begin{aligned} x^y &= y^{-1}xy = (a_1y, a_2y, \ldots, a_ky, b_1y, b_2y, \ldots, b_ly) \\ &= (a_2, \ldots, a_k, c_1, b_1, b_2, \ldots, b_l)\,. \end{aligned}$$

Let

$$\begin{aligned} s_1 &= x^y x^{-1} = (a_1, b_l)\,(a_k, c_1)\,, \\ s_2 &= s_1^{x^{-1}} = (b_l, b_{l-1})\,(a_{k-1}, c_1)\,, \\ s_3 &= s_2^{y^{-2}} = \begin{cases} (b_l, b_{l-1})\,(a_{k-3}, a_{k-1})\,, & \text{if } k \geq 4, \\ (b_l, b_{l-1})\,(c_m, a_{k-1})\,, & \text{if } k = 3. \end{cases} \end{aligned}.$$

If $k \geq 4$, then

$$s_2 s_3 = (a_{k-3}, a_{k-1}, c_1) \in T_{k,l,m},$$

otherwise $m \geq 2$, and

$$s_2 s_3 = (a_{k-1}, c_1, c_m) \in T_{k,l,m}.$$

Finally, we consider the remaining case $(k, l, m) = (3, 2, 1)$.

Lemma 14. *$T_{3,2,1} \simeq T_{2,2,2} \simeq T_{3,1,2} \simeq S_5$, and this is a sharply 3-transitive action of S_5 on 6 elements.*

Proof. By Lemma 11, we have $T_{3,2,1} \simeq T_{2,2,2} \simeq T_{3,1,2}$. We consider $T_{2,2,2}$ in the following. The symmetric group S_5 contains 6 Sylow 5-subgroups. Let them be

$$\begin{aligned} &P_1 = \langle (1,2,3,4,5) \rangle\,, \quad P_2 = \langle (1,2,4,5,3) \rangle\,, \quad P_3 = \langle (1,2,5,3,4) \rangle\,, \\ &P_4 = \langle (1,2,3,5,4) \rangle\,, \quad P_5 = \langle (1,2,4,3,5) \rangle\,, \quad P_6 = \langle (1,2,5,4,3) \rangle\,. \end{aligned}$$

Let $g_1 = (1,2,3,4)$, $g_2 = (5,4,3,2)$. Now,

$$\langle g_1, g_2 \rangle = \left\langle g_1, g_2^{-1} \right\rangle = \langle (2,3,4,1)\,, (2,3,4,5) \rangle \simeq T_{3,1,1} \simeq S_5$$

by the case $l = m = 1$. Let $\varphi \colon S_5 \to S_6$ be the conjugation action on these 6 Sylow subgroups. Then a straightforward calculation shows that

$$\varphi\,(g_1) = (P_1, P_2, P_3, P_4)\,, \qquad \varphi\,(g_2) = (P_1, P_2, P_5, P_6)\,.$$

Thus, $\langle \varphi(g_1), \varphi(g_2) \rangle \simeq T_{2,2,2}$, and is transitive on $\{ P_1, \dots, P_6 \}$. Further, $\ker \varphi = \{ id \}$, otherwise it would contain A_5, and then the conjugation action could not have an element of order 4. Thus,

$$S_5 \simeq \langle g_1, g_2 \rangle \simeq \langle \varphi(g_1), \varphi(g_2) \rangle \simeq T_{2,2,2}.$$

Further, the stabilizer of P_6 in $\varphi(G)$ has 20 elements, therefore contains an element of order 5, that is it contains a 5-cycle. Thus the stabilizer of P_6 is transitive on $\{ P_1, \dots, P_5 \}$. Finally, the stabilizer of P_5 and P_6 is a 4-element group containing $\varphi(g_1)$, and thus is sharply transitive on $\{ P_1, \dots, P_4 \}$. Hence $\varphi(S_5)$ is sharply 3-transitive on the 6 Sylow 5-subgroups. □

This finishes the proof of Theorem 10. □

5. Defect 1 groups

We now prove Theorem 1. Let us start with the exceptional case.

Lemma 15. *Let* Γ *be the exceptional graph (Figure 1). Then for the defect 1 group of* Γ *we have* $G_\Gamma \simeq T_{2,2,2}$.

Proof. Let us denote the vertices of Γ as in Figure 1. Let v be the defect. Now, by Lemma 8 we have

$$G_\Gamma = \langle (1,2,3,4), (1,2,5,6), (4,3,2,5,6) \rangle = \langle (1,2,3,4), (1,2,5,6) \rangle,$$

because $(4,3,2,5,6) = (1,2,5,6)(1,2,3,4)^{-1}$. Thus, $G_\Gamma \simeq T_{2,2,2}$. □

We will need the notion of open ear, and open ear decomposition.

Definition 16. Let Γ be an arbitrary graph, and let $\Gamma' \subset \Gamma$. A path $(u, c_1, \dots, c_m, v)$ is called a Γ'-*ear* (or *open ear*) with respect to Γ, if $u \neq v \in \Gamma'$, and either $m = 0$ and the edge $uv \notin \Gamma'$, or $c_1, \dots, c_m \in \Gamma \setminus \Gamma'$. An *open ear decomposition* of a graph is a partition of its set of edges into a sequence of subsets, such that the first element of the sequence is a cycle, and all other elements of the sequence are open ears of the union of the previous subsets in the sequence.

First we consider the case, where Γ is 2-vertex connected. A connected graph Γ with at least k vertices is k-*vertex connected* if removing any $k-1$ vertices does not disconnect Γ. By [14] a graph is 2-vertex connected if and only if it is a single edge or it has an open ear decomposition. This result and Theorem 10 from Section 4 play a crucial role in proving Theorem 1.

Proof of Theorem 1 if Γ *is 2-vertex connected.* Let us consider an open ear decomposition of Γ. We prove the statement by induction on the number of open ears. If Γ is a cycle, then its defect 1 group is isomorphic to Z_{n-1} by Lemma 8. Further, if Γ is the exceptional graph, then its defect 1 group is $T_{2,2,2}$ by Lemma 15.

Now, assume that Γ is not the exceptional graph and is the union of a 2-vertex connected graph Γ' and a Γ'-ear $(u, c_1, \ldots, c_m, v)$, where $u, v \in \Gamma'$, $u \neq v$, $c_1, \ldots, c_m \notin \Gamma'$. Let the defect 1 group of Γ' be denoted by $G_{\Gamma'}$, and the defect 1 group of Γ be denoted by G_Γ, where the defect is v (the defect 1 groups for different vertices are isomorphic by Lemma 5). We prove that $G_\Gamma \geq A_{n-1}$. Let $v, a_1, \ldots, a_k$ be a shortest path in Γ' from v to $u = a_k$, and let y denote the permutation $y = (a_1, \ldots, a_k, c_1, \ldots, c_m)$.

If Γ' is a cycle, then let us denote the vertices of Γ according to this cycle by $v, a_1, \ldots, a_k, b_1, \ldots, b_l$. Let $x = (a_1, \ldots, a_k, b_1, \ldots, b_l)$. Then

$$G_\Gamma \geq \langle x, y \rangle = T_{k,l,m} \geq A_{n-1},$$

by Theorem 10. Note that G_Γ is 2-transitive on $\Gamma \setminus \{ v \}$.

Assume now, that Γ' is not a cycle. Similarly as in Lemma 13, we prove that G_Γ is 2-transitive on $\Gamma \setminus \{ v \}$. By induction, $G_{\Gamma'}$ is 2-transitive on $\Gamma' \setminus \{ v \}$. It is enough to prove that G_Γ acts transitively on the two-element subsets of $\Gamma \setminus \{ v \}$, because we can transpose two elements of Γ' by the 2-transitivity of $G_{\Gamma'}$. Let $\Gamma \setminus \{ a_1, \ldots, a_k, c_1, \ldots, c_m, v \} = \{ b_1, \ldots, b_l \}$. Note that $k \geq 1$, $l \geq 1$. If $m = 0$, then $G_{\Gamma'}$ is already two-transitive on $\Gamma \setminus \{ v \}$. Otherwise, we determine the orbit of $\{ c_1, b_1 \}$. Now, we have $\{ c_1, b_1 \} y^{-1} = \{ a_k, b_1 \}$, and thus (by the 2-transitivity of $G_{\Gamma'}$) all $\{ w_1, w_2 \}$ $(w_1, w_2 \in \Gamma')$ are in the orbit of $\{ c_1, b_1 \}$. Further, $\{ c_1, b_1 \} y^{i-1} = \{ c_i, b_1 \}$ for every $1 \leq i \leq m$, and thus all $\{ c_i, w \}$ $(w \in \Gamma')$ are in the orbit of $\{ c_1, b_1 \}$ by the transitivity of $G_{\Gamma'}$. Finally, for $1 \leq i < j \leq m$, we have $\{ c_i, c_j \} = \{ a_k, c_{j-i} \} y^i$.

Now, G_Γ is 2-transitive, and $G_{\Gamma'} \subseteq G_\Gamma$ contains a 3-cycle by induction and Theorem 10, unless Γ' is the exceptional graph. Therefore, $A_{n-1} \leq G_\Gamma$ by Jordan's theorem (Theorem 12). If Γ' is the exceptional graph (see Figure 1), then note that $k \leq 3$. In particular, if $m = 0$, then y is either a 2-cycle or a 3-cycle, thus $A_{n-1} \leq G_\Gamma$ by Jordan's theorem (Theorem 12). Further, if $m \geq 2$, then $n \geq 8$, and $G_{\Gamma'}$ already has a 5-cycle, thus $A_{n-1} \leq G_\Gamma$ by Jordan's theorem (Theorem 12). Finally, if $m = 1$, then $n - 1 = 7$, and $G_\Gamma \leq S_7$ acts transitively on 7 points, hence 7 divides the order of G_Γ. Further, $|G_{\Gamma'}| = 120$ divides $|G_\Gamma|$, and therefore $|S_7 : G_\Gamma| \leq \frac{7!}{7 \cdot 120} = 6$. This yields $A_7 = A_{n-1} \leq G_\Gamma$.

Finally, note that $G_\Gamma \leq A_{n-1}$ if and only if every permutation corresponding to a cycle in Γ is even, that is the length of every cycle in Γ is even. This is equivalent to Γ being bipartite [8]. □

Finally, Theorem 1 follows by induction on the number of 2-vertex connected components from Lemma 17.

Lemma 17. *Let Γ_1 and Γ_2 be connected induced subgraphs of Γ such that $\Gamma_1 \cap \Gamma_2 = \{ v \}$, where there are no edges in Γ between $\Gamma_1 \setminus \{ v \}$ and $\Gamma_2 \setminus \{ v \}$. Then the defect 1 group of $\Gamma_1 \cup \Gamma_2$ is the direct product of the defect 1 groups of Γ_1 and Γ_2.*

Proof. Let G_{Γ_i} denote the defect 1 group of Γ_i, where the defect is v. By Lemma 8, G_Γ is generated by cyclic permutations corresponding to cycles through v in Γ. Now, $\Gamma_1 \cap \Gamma_2 = \{ v \}$, and every path between a node from Γ_1 and a node from Γ_2 must go through v, hence every cycle in Γ is either in Γ_1 or in Γ_2. Let $c_i^{(1)}, \ldots, c_i^{(m_i)}$ be the permutations corresponding to the cycles in Γ_i ($i = 1, 2$). Then $c_1^{(j_1)} c_2^{(j_2)} = c_2^{(j_2)} c_1^{(j_1)}$ for all $1 \leq j_i \leq m_i$, $i = 1, 2$, thus

$$\begin{aligned} G_\Gamma &= \left\langle c_1^{(1)}, \ldots, c_1^{(m_1)}, c_2^{(1)}, \ldots, c_2^{(m_2)} \right\rangle \\ &= \left\langle c_1^{(1)}, \ldots, c_1^{(m_1)} \right\rangle \times \left\langle c_2^{(1)}, \ldots, c_2^{(m_2)} \right\rangle = G_{\Gamma_1} \times G_{\Gamma_2}. \end{aligned}$$

□

6. Defect k groups

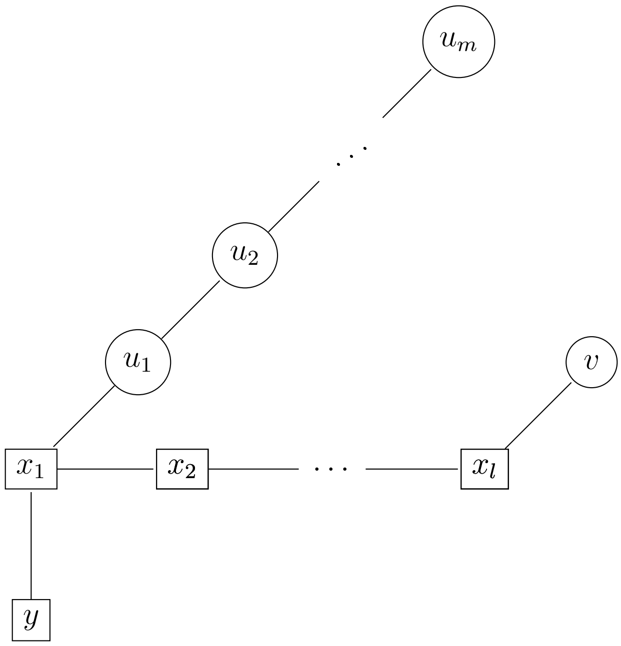
In the following we assume $k \geq 2$, and every graph Γ is assumed to be simple connected. We start with some simple observations.

Lemma 18. *Let $\Gamma' \subseteq \Gamma$ be connected graphs. Then the defect k group of Γ contains a subgroup isomorphic (as permutation group) to the defect k group of Γ'. Further, if $\Gamma \setminus \Gamma'$ contains at least one vertex, then the defect k group of Γ contains a subgroup isomorphic (as permutation group) to the defect $k - 1$ group of Γ'.*

Proof. Let G_k be the defect k group of Γ, G'_k and G'_{k-1} be the defect k group and defect $k-1$ group of Γ', respectively. Now, every elementary collapsing of Γ' is an elementary collapsing of Γ, as well. Thus $G'_k \leq G_k$ is clear.

Further, let $V_{k-1} = \{ v_1, \ldots, v_{k-1} \} \subseteq \Gamma'$, v be a vertex of $\Gamma \setminus \Gamma'$, and let $V_k = V_{k-1} \cup \{ v \}$. Let u be a neighbour of v and let $e = e_{vu}$. Then for every permutation $g \in G'_{k-1}$ with defect set V_{k-1} we have that $eg \in G_k$ with defect set V_k, where eg is identical to g on $\Gamma \setminus (\Gamma' \cup \{ v \})$, and acts exactly the same on $\Gamma' \setminus V_{k-1}$ as g. In particular, $\varphi \colon G'_{k-1} \to G_k$, $\varphi(g) = eg$ is an injective homomorphism of permutation groups. □

Lemma 19. *Let $1 \leq m \leq l < k \leq n - 2$, and assume Γ contains the following subgraph:*



If the defect set contains $y, x_1, \dots, x_l$, does not contain v, and does not contain u_i for some $1 \le i \le m$, then the defect k group contains the transposition (u_i, v).

Proof. Let

$$r = \begin{cases} ss_1 e_{yx_1} e_{x_1u_1}, & \text{if } i = 1, \\ ss_1 \dots s_i p t t_{i-1} \dots t_1 q, & \text{if } i \ge 2, \end{cases}$$

where

$$\begin{aligned} s &= e_{vx_l} e_{x_lx_{l-1}} \dots e_{x_2x_1} e_{x_1y}, \\ s_1 &= e_{u_1x_1} e_{x_1x_2} \dots e_{x_{l-1}x_l} e_{x_lv}, \\ s_j &= e_{u_ju_{j-1}} \dots e_{u_2u_1} e_{u_1x_1} e_{x_1x_2} \dots e_{x_{l-j+1}x_{l-j+2}}, && (2 \le j \le m), \\ p &= e_{yx_1} e_{x_1u_1} e_{u_1u_2} \dots e_{u_{i-1}u_i}, \\ t &= e_{x_{l-i+2}x_{l-i+1}} \dots e_{x_2x_1} e_{x_1y}, \\ t_j &= e_{x_{l-j+2}x_{l-j+1}} \dots e_{x_2x_1} e_{x_1u_1} e_{u_1u_2} \dots e_{u_{j-1}u_j}, && (2 \le j \le m), \\ t_1 &= e_{vx_l} e_{x_lx_{l-1}} \dots e_{x_2x_1} e_{x_1u_1}, \\ q &= e_{yx_1} e_{x_1x_2} \dots e_{x_{l-1}x_l} e_{x_lv}. \end{aligned}$$

Then r transposes u_i and v and fixes all other vertices of Γ outside the defect set. ☐

Note that Lemma 19 is going to be useful whenever Γ contains a node with degree at least 3.

Lemma 20. *Let $k \ge 2$, $\Gamma' = (V', E')$ be such that $|V'| > k$ and its defect k group is transitive (e.g. Γ' is a cycle with at least $k+1$ vertices). Let $\Gamma = (V' \cup \{ v \}, E' \cup \{ x_1v \})$ for a new vertex v and some $x_1 \in \Gamma'$,* *where the degree of* x_1 *in* Γ' *is at least 2. Then the defect* k *group of* Γ *is isomorphic to* S_{n-k}.

Proof. Let n be the number of vertices of Γ, then $n \geq k+2$. Let the vertices of Γ' be $y, x_1, x_2, \ldots, x_{k-1}, u_1, u_2, \ldots, u_{n-k-1}$ such that u_1 and y are neighbours of x_1 in Γ'. Let the defect set be $\{ y, x_1, \ldots, x_{k-1} \}$. Applying Lemma 19 to the subgraph with vertices $\{ x_1, v, y, u_1 \}$ we obtain that the defect k group of Γ contains the transposition (u_1, v). Since the defect k group of Γ' is transitive, the defect k group of Γ contains the transposition (u_i, v) for all $1 \leq i \leq n-k-1$. Therefore, the defect k group of Γ is isomorphic to S_{n-k}. □

Motivated by Lemma 20, we define the *k-subgraphs* and the *k-components* of a graph Γ.

Definition 21. Let Γ be a simple connected graph, $k \geq 2$. A connected subgraph $\Gamma' \subseteq \Gamma$ is called a *k-subgraph* if its defect k group is the symmetric group of degree $|\Gamma'| - k$. A k-subgraph is a *k-component* or a *maximal k-subgraph* if it has no proper extension in Γ to a k-subgraph. Finally, we say that a k-subgraph or k-component Γ' is *nontrivial* if it contains a vertex having at least 3 neighbours in Γ'.

Note that every k-component is an induced subgraph. A trivial k-subgraph is either a line on $k+1$ points or a cycle on $k+1$ or on $k+2$ points. Further, a trivial k-component cannot be a cycle by Lemma 20, unless the graph itself is a cycle. Finally, any connected subgraph of $k+1$ points is trivially a k-subgraph, thus every connected subgraph of $k+1$ points is contained in a k-component. Note that the intersection of two k-components cannot contain more than k vertices:

Lemma 22. *Let* Γ_1, Γ_2 *be k-subgraphs such that* $|\Gamma_1 \cap \Gamma_2| > k$. *Then* $\Gamma_1 \cup \Gamma_2$ *is a k-subgraph, as well.*

Proof. Choose the defect set V_k such that $V_k \subsetneq \Gamma_1 \cap \Gamma_2$, and let $v \in (\Gamma_1 \cap \Gamma_2) \setminus V_k$. Then the symmetric groups acting on $\Gamma_1 \setminus V_k$ and $\Gamma_2 \setminus V_k$ are subgroups in the defect k group of $\Gamma_1 \cup \Gamma_2$. Thus, we can transpose every member of $\Gamma_i \setminus (V_k \cup \{ v \})$ with v. Therefore, the defect k group of $\Gamma_1 \cup \Gamma_2$ is the symmetric group on $(\Gamma_1 \cup \Gamma_2) \setminus V_k$. □

Lemma 23. *Let* Γ *be a simple connected graph,* Γ' *be a k-subgraph of* Γ. *Let* $x_1 \in \Gamma'$, $v \notin \Gamma'$, *and let* $P = (x_1, x_2, \ldots, x_l, v)$ *be a shortest path between* x_1 *and* v *in* Γ *for some* $l \leq k-1$. *Assume that* x_1 *has at least 2 neighbours in* Γ' *apart from* x_2. *Then the subgraph* $\Gamma' \cup P$ *is a k-subgraph.*

Proof. First, consider the case $x_2, \ldots, x_l \in \Gamma'$. Let u, y be two neighbours of x_1 in Γ' distinct from x_2, and choose the defect set V_k such that it contains $y, x_1, \ldots, x_l$ and does not contain u. By Lemma 19 the defect k group of $\Gamma' \cup \{ v \}$ contains the transposition (u, v). Further, the defect k group of Γ' is the whole symmetric group on $\Gamma' \setminus V_k$.

Thus, the defect k group of $\Gamma' \cup \{ v \}$ is the whole symmetric group on $(\Gamma' \setminus V_k) \cup \{ v \}$.

Now, if not all of $x_2, \ldots, x_l$ are in Γ', then, by the previous argument, one can add them (and then v) to Γ' one by one, and obtain an increasing chain of k-subgraphs. □

As a corollary, we obtain that every vertex of degree at least 3 with two neighbours is contained in exactly one nontrivial k-component.

Corollary 24. *Let Γ be a simple connected graph with n vertices such that $n > k$, and let x_1 be a vertex having degree at least 3. Then there exists exactly one k-component Γ' containing x_1 such that x_1 has degree at least 2 in Γ'. Further, Γ' is a nontrivial k-component, and if Γ_{x_1} is the induced subgraph of the vertices in Γ that are of at most distance $k-1$ from x_1, then $\Gamma_{x_1} \subseteq \Gamma'$.*

Proof. Expanding x_1 and any two of its neighbours to an arbitrary connected subgraph of Γ with $k+1$ points yields to a k-subgraph. Thus there exists at least one k-component containing x_1 and two of its neighbours.

Let Γ' be a k-component containing x_1 and at least two of its neighbours. Assume that $\Gamma_{x_1} \not\subseteq \Gamma'$. Let $v \in \Gamma_{x_1} \setminus \Gamma'$ be a point such that for a shortest path $P = (x_1, \ldots, x_l, v)$ between x_1 and v we have that l is minimal. If $l = 1$, then $P = (x_1, v)$. Now x_1 has at least two neighbours in Γ' apart from v, therefore $\Gamma' \cup P$ is a k-subgraph by Lemma 23, which contradicts the maximality of Γ'. Thus $l \geq 2$, in particular all neighbours of x_1 in Γ are in Γ', as well, and thus Γ' is a nontrivial k-component. Hence x_1 has at least two neighbours in Γ' apart from x_2, therefore $\Gamma' \cup P$ is a k-subgraph by Lemma 23, which contradicts the maximality of Γ'. Thus $\Gamma_{x_1} \subseteq \Gamma'$.

Now, assume that Γ' and Γ'' are k-components containing x_1 and at least two of its neighbours. Then $\Gamma_{x_1} \subseteq \Gamma'$ and $\Gamma_{x_1} \subseteq \Gamma''$. Note that either $\Gamma_{x_1} = \Gamma$ (and hence $|\Gamma_{x_1}| = n > k$), or there exists a vertex $v \in \Gamma$ which is of distance exactly k from x_1. Let $P = (x_1, \ldots, x_k, v)$ be a shortest path between x_1 and v, and let u and y be two neighbours of x_1 distinct from x_2. Then $\{ x_1, \ldots, x_k, y, u \} \subseteq \Gamma_{x_1}$, thus $|\Gamma_{x_1}| > k$. Therefore $|\Gamma' \cap \Gamma''| \geq |\Gamma_{x_1}| > k$, yielding $\Gamma' = \Gamma''$ by Lemma 22. □

Lemma 25. *Let Γ' be a nontrivial k-subgraph of Γ, P be a Γ'-ear. Then $\Gamma' \cup P$ is a (nontrivial) k-subgraph of Γ.*

Proof. Let Γ, Γ' and $P = (w_0, w_1, \ldots w_i, w_{i+1})$ be a counterexample, where i is minimal. There exists a shortest path $(w_0, y_1, \ldots, y_l, w_{i+1})$ in Γ' among those where the degree of some y_j or of w_0 or of w_{i+1} is at least 3 in Γ'. For easier notation, let $y_0 = w_0$, $y_{l+1} = w_{i+1}$. Let $y' \in \Gamma' \setminus \{ y_0, y_1, \ldots, y_l, y_{l+1} \}$ be a neighbour of y_j; this exists, otherwise a shorter path would exist between w_0 and w_{i+1}.

If $j+1 \leq k-1$ (that is $j \leq k-2$), then by Lemma 23 the induced subgraph on $\Gamma' \cup \{ w_1 \}$ is a k-subgraph, thus $\Gamma' \cup \{ w_1 \}$ with the ear $(w_1, \dots, w_i, w_{i+1})$ is a counterexample with a shorter ear.

Similarly, if $l-j+2 \leq k-1$ (that is $l+3-k \leq j$), then by Lemma 23 the induced subgraph on $\Gamma' \cup \{ w_i \}$ is a k-subgraph, thus $\Gamma' \cup \{ w_i \}$ with the ear $(w_0, w_1, \dots, w_i)$ is a counterexample with a shorter ear.

Finally, if $k-1 \leq j \leq l+2-k$, then $2k-3 \leq l$. Let Γ'' be the cycle $P \cup (y_0, y_1, \dots, y_l, y_{l+1})$ together with y' and the edge $y_j y'$. Then Γ'' is a k-subgraph by Lemma 20, $|\Gamma' \cap \Gamma''| = l+2 \geq 2k-1 > k$, hence $\Gamma' \cup \Gamma'' = \Gamma' \cup P$ is a k-subgraph by Lemma 22. □

Corollary 26. *Let Γ be a simple connected graph with n vertices such that $n > k$, and assume that Γ is not a cycle. Suppose uv is an edge contained in a cycle of Γ. Then there exists exactly one k-component Γ' containing the edge uv. Further, Γ' is a nontrivial k-component, and if Γ_{uv} is the 2-edge connected component containing uv, then $\Gamma_{uv} \subseteq \Gamma'$.*

Proof. Expanding the edge uv to an arbitrary connected subgraph of Γ with $k+1$ points yields to a k-subgraph. Thus there exists at least one k-component Γ' containing the edge uv. We prove first that Γ' is a nontrivial k-component, then prove $\Gamma_{uv} \subseteq \Gamma'$, and only after that do we prove that Γ' is unique.

Assume first that Γ' is a trivial k-component. If Γ' were a cycle, then $\Gamma \setminus \Gamma'$ contains at least one vertex, because Γ' is an induced subgraph of Γ. Then Lemma 20 contradicts the maximality of Γ'. Thus Γ' is a line of $k+1$ vertices. Let Γ_2 be a shortest cycle containing uv. Now, there must exist a vertex in $\Gamma \setminus \Gamma_2$, otherwise either $\Gamma = \Gamma_2$ would be a cycle, or there would exist an edge in $\Gamma \setminus \Gamma_2$ yielding a shorter cycle than Γ_2 containing the edge uv. Let $x_2 \in \Gamma \setminus \Gamma_2$ be a neighbour of a vertex in Γ_2. By Lemma 20 the induced subgraph on $\Gamma_2 \cup \{ x_2 \}$ is a k-subgraph. Thus $\Gamma' \not\subseteq \Gamma_2$, otherwise Γ' would not be a maximal k-subgraph. Let $x_1 \in \Gamma' \cap \Gamma_2$ be a vertex such that two of its neighbours are in Γ_2 and its third neighbour is some $x_2 \in \Gamma' \setminus \Gamma_2$. Note that every vertex in Γ' is of distance at most $k-1$ from x_1, because $u, v \in \Gamma' \cap \Gamma_2$. Thus, if $|\Gamma_2| \geq k+1$, then Γ_2 together with x_2 and the edge $x_1 x_2$ is a k-subgraph by Lemma 20, and hence $\Gamma_2 \cup \Gamma'$ is a k-subgraph by Lemma 23, contradicting the maximality of Γ'. Otherwise, if $|\Gamma_2| \leq k$, then every vertex in Γ_2 is of distance at most $k-1$ from x_1, and hence $\Gamma_2 \cup \Gamma'$ is a k-subgraph by Lemma 23, contradicting the maximality of Γ'. Therefore Γ' is a nontrivial k-component.

Now we show that the two-edge connected component $\Gamma_{uv} \subseteq \Gamma'$. Let Γ, Γ' be a counterexample to this such that the number of vertices of Γ_{uv} is minimal, and among these counterexamples choose one where the number of edges of Γ_{uv} is minimal. Using an ear-decomposition [11], Γ_{uv} is either a cycle, or there exists a 2-edge connected subgraph $\Gamma_1 \subseteq \Gamma_{uv}$ and there exists

(1) either a Γ_1-ear P such that $\Gamma_{uv} = \Gamma_1 \cup P$,
(2) or a cycle Γ_2 such that $|\Gamma_1 \cap \Gamma_2| = 1$ and $\Gamma_{uv} = \Gamma_1 \cup \Gamma_2$.

If Γ_{uv} is a cycle containing the edge uv, and $\Gamma_{uv} \not\subseteq \Gamma'$, then going along the edges of Γ_{uv}, one can find a Γ'-ear $P \subseteq \Gamma_{uv}$. Then $\Gamma' \cup P$ is a k-subgraph by Lemma 25, contradicting the maximality of Γ'. Thus Γ_{uv} is not a cycle. Let us choose Γ_1 from cases (1) and (2) so that it would have the least number of vertices.

Assume first that case (1) holds. By minimality of the counterexample, $\Gamma_1 \subseteq \Gamma'$. If $P \not\subseteq \Gamma'$, then going along the edges of P one can find a Γ'-ear $P' \subseteq P$. But then $\Gamma' \cup P'$ is a k-subgraph by Lemma 25, contradicting the maximality of Γ'.

Assume now that case (2) holds. Again, by induction, $\Gamma_1 \subseteq \Gamma'$. If $\Gamma_2 \not\subseteq \Gamma'$, then either $|\Gamma' \cap \Gamma_2| = 1$ or going along the edges of Γ_2 one can find a Γ'-ear $P' \subseteq \Gamma_2$. The latter case cannot happen, because then $\Gamma' \cup P'$ is a k-subgraph by Lemma 25, contradicting the maximality of Γ'. Thus $|\Gamma' \cap \Gamma_2| = 1$, and hence $\Gamma' \cap \Gamma_2 = \Gamma_1 \cap \Gamma_2$. Let $\Gamma_1 \cap \Gamma_2 = \{ x_1 \}$, and let v_1 be a neighbour of x_1 in $\Gamma_1 \setminus \Gamma_2$, and let v_2 be a neighbour of x_1 in $\Gamma_2 \setminus \Gamma_1$. If $|\Gamma_2| \leq k$, then Γ_2 can be extended to a connected subgraph of Γ having exactly $k+1$ vertices, which is a k-subgraph. If $|\Gamma_2| \geq k+1$, then $\Gamma_2 \cup \{ v_1 \}$ is a k-subgraph by Lemma 20. In any case, there exists a k-component $\Gamma_2' \supseteq \Gamma_2$. For notational convenience, let Γ_1' denote the k-component Γ' containing Γ_1. We prove that $\Gamma_2' = \Gamma_1' = \Gamma'$, thus Γ' contains Γ_2, contradicting that we chose a counterexample.

Now, both Γ_1 and Γ_2 contain at least two neighbours of x_1. Let $V_i \subseteq \Gamma_i$ be the set of vertices with distance at most $k-1$ from x_1 ($i \in \{ 1, 2 \}$). If $|\Gamma_i| \leq k$, then V_i contains all vertices of Γ_i, otherwise $|V_i| \geq k$ ($i \in \{ 1, 2 \}$). By Lemma 23, the induced subgraph on V_1 is contained in Γ_2'. Thus, if V_1 contains all vertices of Γ_1, then $\Gamma_1 \subseteq \Gamma_2'$, hence we have $\Gamma_1' = \Gamma_2'$. Similarly, the induced subgraph on V_2 is contained in Γ_1'. Thus, if V_2 contains all vertices of Γ_2, then $\Gamma_2 \subseteq \Gamma_1'$, hence we have $\Gamma_1' = \Gamma_2'$. Otherwise, $|\Gamma_1' \cap \Gamma_2'| \geq |V_1| + |V_2| - |\{ x_1 \}| \geq 2k - 1 > k$, hence by Lemma 22 we have $\Gamma_1' = \Gamma_2'$.

Finally, we prove uniqueness. Let Γ' and Γ'' be two k-components containing the edge uv. Then both Γ' and Γ'' contain Γ_{uv}. If $\Gamma = \Gamma_{uv}$, then $\Gamma' = \Gamma_{uv} = \Gamma''$. Otherwise, there exists a vertex $x_2 \in \Gamma \setminus \Gamma_{uv}$ such that it has a neighbour $x_1 \in \Gamma_{uv}$. Note that x_1 has degree at least 3 in Γ. Let V_1 be the vertices of Γ of distance at most $k-1$ from x_1. Note that if V_1 does not contain all vertices of Γ, then $|V_1| > k$. By 2-edge connectivity, $\Gamma_{uv} \subseteq \Gamma'$ contains at least two neighbours of x_1, thus $V_1 \subseteq \Gamma'$ by Lemma 23. Similarly, $\Gamma_{uv} \subseteq \Gamma''$ contains at least two neighbours of x_1, thus $V_1 \subseteq \Gamma''$ by Lemma 23. If V_1 contains all vertices of Γ, then $\Gamma' = \Gamma = \Gamma''$. Otherwise, $|\Gamma' \cap \Gamma''| \geq |V_1| > k$, and $\Gamma' = \Gamma''$ by Lemma 22. □

Recall that by [11] a strongly connected antisymmetric digraph becomes a 2-edge connected graph after forgetting the directions. Thus Rhodes's conjecture about strongly connected, antisymmetric digraphs [9, Conjecture 6.51i (3)–(4)] follows immediately from the following theorem on 2-edge connected graphs:

Theorem 27. *Let $n > k \geq 2$, Γ be a 2-edge connected simple graph having n vertices. If Γ is a cycle, then the defect k group is Z_{n-k}. If Γ is not a cycle, then the defect k group is S_{n-k}.*

Proof. If Γ is a cycle, then its defect k group is Z_{n-k} by Lemma 9. If Γ is not a cycle, then the defect k group is S_{n-k} by Corollary 26. □

The final part of this section is devoted to prove Theorem 2.

Lemma 28. *Let $\Gamma_1 \neq \Gamma_2$ be k-components of the connected simple graph Γ. Assume that Γ is not a cycle. Then $\Gamma_1 \cap \Gamma_2$ is either empty, or is a path $(x_1, \ldots, x_l)$ such that*

(1) *$l \leq k$,*
(2) *the degree of x_i is 2 in Γ $(2 \leq i \leq l-1)$,*
(3) *if $l \geq 2$ and $\Gamma_i \setminus \{ x_1, \ldots, x_l \}$ $(i \in \{ 1, 2 \})$ contains a neighbour of x_1 (resp. x_l), then Γ_i contains all neighbours of x_1 (resp. x_l),*
(4) *if $l \geq 2$ then $\Gamma \setminus \{ x_j x_{j+1} \}$ is disconnected for all $1 \leq j \leq l-1$.*

Proof. Note that Γ_1 and Γ_2 are induced subgraphs of Γ, thus so is $\Gamma_1 \cap \Gamma_2$.

We prove first that $\Gamma_1 \cap \Gamma_2$ is connected (or empty) if Γ_1 is a nontrivial k-component. Suppose that $u, v \in \Gamma_1 \cap \Gamma_2$ are in different components of $\Gamma_1 \cap \Gamma_2$ such that the distance between u and v is minimal in Γ_2. Due to the minimality, there exists a path $(u, x_1, \ldots, x_l, v)$ such that $x_1, \ldots, x_l \in \Gamma_2 \setminus \Gamma_1$. Then $P = (u, x_1, \ldots, x_l, v)$ is a Γ_1-ear, and $\Gamma_1 \cup P$ would be a k-subgraph by Lemma 25, contradicting the maximality of Γ_1. Thus $\Gamma_1 \cap \Gamma_2$ is connected. One can prove similarly that $\Gamma_1 \cap \Gamma_2$ is connected if Γ_2 is a nontrivial k-component.

Now we prove that $\Gamma_1 \cap \Gamma_2$ is connected, even if both Γ_1 and Γ_2 are trivial k-components. As $\Gamma_1 \subsetneq \Gamma$, Γ_1 cannot be a cycle hence must be a line $(x_1, \ldots, x_{k+1})$. Note that the degree of x_i in Γ for $2 \leq i \leq k$ must be 2, otherwise a nontrivial k-component would contain x_i, and thus also Γ_1 by Corollary 24. In particular, if $\Gamma_1 \cap \Gamma_2$ is not connected, then $x_1, x_{k+1} \in \Gamma_1 \cap \Gamma_2$, $x_i \notin \Gamma_1 \cap \Gamma_2$ for some $2 \leq i \leq k$, and $\Gamma_1 \cup \Gamma_2$ would be a cycle. However, by Corollary 26, the edge $x_1 x_2$ is contained in a unique nontrivial k-component, contradicting that it is also contained in the trivial k-component Γ_1.

Now, we prove (1–4). By Corollary 26, $\Gamma_1 \cap \Gamma_2$ cannot contain any edge uv which is contained in a cycle. As $\Gamma_1 \cap \Gamma_2$ is connected, it must be a tree. However, $\Gamma_1 \cap \Gamma_2$ cannot contain any vertex of degree at least 3 in $\Gamma_1 \cap \Gamma_2$, otherwise that vertex would be contained in a unique k-component by Corollary 24. Thus $\Gamma_1 \cap \Gamma_2$ is a path $(x_1, \ldots, x_l)$.

Now, $l \le k$ by Lemma 22, proving (1). For (2) note that if any x_i $(2 \le i \le l-1)$ is of degree at least 3 in Γ, then $\{ x_{i-1}, x_i, x_{i+1} \}$ is contained in a unique k-component by Corollary 24, a contradiction. For (3) observe that at least two neighbours of x_1 (resp. x_l) are in Γ_i, and thus all its neighbours must be in Γ_i by Corollary 24. Finally, (4) follows immediately from Corollary 26 and the fact that any edge that is not contained in any cycle disconnects the graph Γ. □

Motivated by the structure of intersections of k-components, we define bridges in Γ:

Definition 29. A path $(x_1, \dots, x_l)$ in a connected graph Γ for some $l \ge 2$ is called a *bridge* if the degree of x_i in Γ is 2 for all $2 \le i \le l-1$, and if $\Gamma \setminus \{ x_j x_{j+1} \}$ is disconnected for all $1 \le j \le l-1$. The *length* of the bridge $(x_1, \dots, x_l)$ is l.

Edges of short bridges (having length at most $k-1$) are contained in a unique k-component:

Lemma 30. *Let Γ be a simple connected graph with n vertices such that $n > k$, and let uv be an edge which is not contained in any cycle. Let $(x_1, \dots, x_l)$ be a longest bridge containing the edge uv. If $l \le k-1$, then uv is contained in a unique k-component Γ', and furthermore, Γ' is a nontrivial k-component.*

Proof. As uv is not part of any cycle in Γ, uv is a bridge of length 2. Note that a longest bridge $(x_1, \dots, x_l)$ containing uv is unique, because as long as the degree of at least one of the path's end vertices is 2 in Γ, the path can be extended in that direction. The obtained path is the unique longest bridge containing uv.

Let Γ' be a k-component containing uv, and assume $l \le k-1$. Note that the distance of x_1 and x_l is $l-1 \le k-2$. As $|\Gamma| \ge k+1$, at least one of x_1 and x_l has degree at least 3 in Γ, say x_1. We distinguish two cases according to the degree of x_l.

Assume first that x_l is of degree 1. As Γ' is a connected subgraph having at least $k+1$ vertices, Γ' must contain x_1 and at least two of its neighbours. Then by Corollary 24 it contains all vertices of Γ of distance at most $k-1$ from x_1. In particular, Γ' must contain the bridge $(x_1, \dots, x_l)$. However, there is a unique (nontrivial) k-component Γ_1' containing x_1 and two of its neighbours by Corollary 24, and thus $\Gamma' = \Gamma_1'$ is that unique k-component.

Assume now that x_l is of degree at least 3. As Γ' is a connected subgraph having at least $k+1$ vertices, Γ' must contain x_1 and at least two of its neighbours, or x_l and at least two of its neighbours. If Γ' contains x_1 and at least two of its neighbours, then by Corollary 24 it contains all vertices of Γ of distance at most $k-1$ from x_1. In particular, Γ' must contain the bridge $(x_1, \dots, x_l)$ and all of the neighbours of x_l. Similarly, one can prove that if Γ' contains x_l and two of its neighbours, then it also contains the bridge $(x_1, \dots, x_l)$ and all of the neighbours of x_1. However, there is a unique (nontrivial) k-component Γ_1' containing x_1 and two of its neighbours by Corollary 24, and also a unique (nontrivial) k-component Γ_l' containing x_l and two of its neighbours by Corollary 24. Therefore Γ' must equal to both Γ_1' and Γ_l', and hence is unique. □

In particular, in non-cycle graphs trivial k-components or intersections of two different k-components consist of edges that are contained in long bridges (having length at least k). The key observation in proving Theorem 2 is that a defect k group cannot move a vertex across a bridge of length at least k:

Lemma 31. *Let $2 \le k \le l$, Γ_1 and Γ_2 be disjoint connected subgraphs of the connected graph Γ, and $(x_1, x_2, \dots, x_l)$ be a bridge in Γ such that $x_1 \dots, x_l \notin \Gamma_1 \cup \Gamma_2$, x_1 has only neighbours in Γ_1 (except for x_2), x_l has only neighbours in Γ_2 (except for x_{l-1}). Assume Γ has no more vertices than $\Gamma_1 \cup \Gamma_2 \cup (x_1, \dots, x_l)$. Let the defect set be $V_k = \{ x_1, \dots, x_k \}$, let the defect k group corresponding to V_k be G_k. Then for any $u \in \Gamma_1$ and $v \in \Gamma_2$ there does not exist any permutation in G_k which moves u to v.*

Proof. Let $S = S_\Gamma$. Assume that there exists $u \in \Gamma_1$, $v \in \Gamma_2$, and a transformation $g \in S$ of defect V_k such that $g \restriction_{V \setminus V_k} \in G_k$ and $ug = v$. Let $s_0 \in G_k$ be the unique idempotent power of g, that is s_0 is a transformation of defect V_k that acts as the identity on $\Gamma \setminus V_k$. Then there exists a series of elementary collapsings $e_1, \dots, e_m$ such that $g = e_1 \dots e_m$. For every $1 \le d \le m$ let $s_d = s_0 e_1 \dots e_d$. Now, $s_m = s_0 e_1 \dots e_m = s_0 g = g s_0 = g$. In particular, both s_m and s_0 are of defect k, hence s_d is of defect k for all $1 \le d \le m$. Consequently, $|\Gamma_1 s_d| = |\Gamma_1|$, $|\Gamma_2 s_d| = |\Gamma_2|$ and $\Gamma_1 s_d \cap \Gamma_2 s_d = \emptyset$ for all $1 \le d \le m$.

For an arbitrary $s \in S$, let

$$i(s) = \begin{cases} 0, & \text{if } \Gamma_1 s \subseteq \Gamma_1, \\ l+1, & \text{if } \Gamma_1 s \not\subseteq \Gamma_1 \cup \{ x_1, \dots, x_l \}, \\ \min\limits_{1 \le i \le l} \{ \Gamma_1 s \subseteq \Gamma_1 \cup \{ x_1, \dots, x_i \} \}, & \text{otherwise.} \end{cases}$$

Similarly, let

$$j(s) = \begin{cases} l+1, & \text{if } \Gamma_2 s \subseteq \Gamma_2, \\ 0, & \text{if } \Gamma_2 s \not\subseteq \Gamma_2 \cup \{ x_1, \dots, x_l \}, \\ \max\limits_{1 \le j \le l} \{ \Gamma_2 s \subseteq \Gamma_2 \cup \{ x_j, \dots, x_l \} \}, & \text{otherwise.} \end{cases}$$

Note that for arbitrary $s \in S$ and elementary collapsing e, we have $|i(s) - i(se)| \le 1$, $|j(s) - j(se)| \le 1$. Further, both $|i(s_d) - i(s_d e)| = 1$ and $|j(s_d) - j(s_d e)| = 1$ cannot happen at the same time for any $1 \le d \le m$, because that would contradict $\Gamma_1 s_d \cap \Gamma_2 s_d \neq \emptyset$.

For s_0 we have $i(s_0) = 0 < l + 1 = j(s_0)$, for s_m we have $i(s_m) = l + 1 \geq j(s_m)$. Let $1 \leq d \leq m$ be minimal such that $i(s_d) \geq j(s_d)$. Then $i(s_{d-1}) < j(s_{d-1})$. From s_{d-1} to s_d either i or j can change and by at most 1, thus $i(s_d) = j(s_d)$. If $i(s_d) = j(s_d) \in \{ 1, \dots, l \}$, then $x_{i(s_d)} \in \Gamma_1 s_d \cap \Gamma_2 s_d$, contradicting $\Gamma_1 s_d \cap \Gamma_2 s_d = \emptyset$. Thus $i(s_d) = j(s_d) \notin \{ 1, \dots, l \}$. Assume $i(s_d) = j(s_d) = l + 1$, the case $i(s_d) = j(s_d) = 0$ can be handled similarly.

Now, $j(s_d) = l + 1$ yields $\Gamma_2 s_d \subseteq \Gamma_2$. Further, $|\Gamma_2 s_d| = |\Gamma_2|$, thus $\Gamma_2 s_d = \Gamma_2$. From $i(s_d) = l+1$ we have $\Gamma_1 s_d \cap \Gamma_2 \neq \emptyset$. Thus $\Gamma_1 s_d \cap \Gamma_2 s_d = \Gamma_1 s_d \cap \Gamma_2 \neq \emptyset$, a contradiction. □

Corollary 32. *Let Γ_1 and Γ_2 be connected subgraphs of Γ such that $\Gamma_1 \cap \Gamma_2$ is a length k bridge in Γ. Let $V_k = \Gamma_1 \cap \Gamma_2$ be the defect set. Let G_i be the defect k group of Γ_i, G be the defect k group of $\Gamma_1 \cup \Gamma_2$. Then*

$$G = G_1 \times G_2.$$

Proof. By Lemma 18 we have $G_1, G_2 \leq G$. Since G_1 and G_2 act on disjoint vertices, their elements commute. Thus $G_1 \times G_2 \leq G$. Now, V_k is a bridge of length k, thus by Lemma 31 (applied to the disjoint subgraphs $\Gamma_1 \setminus V_k$ and $\Gamma_2 \setminus V_k$) there exists no element of G moving a vertex from Γ_1 to Γ_2 or vice versa. Therefore $G \leq G_1 \times G_2$. □

Finally, we are ready to prove Theorem 2.

Proof of Theorem 2. If Γ is a cycle, then its defect k group is Z_{n-k} by Lemma 9. Otherwise, we prove the theorem by induction on the number of k-components of Γ. If Γ is a k-component, then the theorem holds, and the defect k group of Γ is S_{n-k}.

Otherwise, we consider two cases. Assume first that there exists a degree 1 vertex $x_1 \in \Gamma$, such that there exists a path $(x_1, \dots, x_{k+1})$ which is a bridge. Let Γ_1 be the path $(x_1, \dots, x_{k+1})$, and let Γ_2 be $\Gamma \setminus \{ x_1 \}$. Now, Γ_1 is a trivial k-component, hence Γ_2 contains one less k-component than Γ. Further, Γ_2 is connected, and cannot be a cycle because the degree of x_2 in Γ_2 is 1. Thus induction and Corollary 32 finishes the proof in this case.

In the second case, no degree 1 vertex x_1 is in a path $(x_1, \dots, x_{k+1})$ which is a bridge. Then any maximal bridge $(x_1, \dots, x_l)$ with a degree 1 vertex x_1 has length $l \leq k$, and, as the bridge cannot be extended, x_l must have degree at least 3. Moreover, $(x_1, ..., x_l)$ lies in a k-component containing x_l and all its neighbours by Lemma 30 and Corollary 24. In particular every bridge in Γ of length at least $k + 1$ occurs between nodes of degree at least 3. Hence every bridge of length at least $k + 1$ occurs between two nontrivial k-components by Corollary 24. For every vertex v having degree at least 3 in Γ, let Γ_v be the unique k-component containing v and all its neighbours (Corollary 24). By definition, these are all the nontrivial k-components of Γ.

Let Γ^k be the graph whose vertices are the nontrivial k-components, and $\Gamma_u\Gamma_v$ is an edge in Γ^k (for $\Gamma_u \neq \Gamma_v$) if and only if there exists a bridge in Γ between a vertex $u' \in \Gamma_u$ of degree at least 3 in Γ_u and a vertex $v' \in \Gamma_v$ of degree at least 3 in Γ_v. By Corollary 26, $\Gamma_u = \Gamma_v$ if u and v are in the same 2-edge connected component. As the 2-edge connected components of Γ form a tree, the graph Γ^k is a tree.

Assume Γ^k has m vertices. Let Γ_1 be a leaf in Γ^k, and let Γ_m be its unique neighbour in Γ^k. Let $x_1 \in \Gamma_1$ and $x_l \in \Gamma_m$ be the unique vertices of degree at least 3 in Γ_i ($i \in \{1, l\}$) such that there exists a bridge $P = (x_1, \ldots, x_l)$ in Γ. Note that the length of P is at least k, otherwise $\Gamma_1 = \Gamma_m$ would follow by Lemma 30. Further, any other bridge having an endpoint in Γ_1 must be of length at most k, because every degree 1 vertex is of distance at most $k-1$ from a vertex of degree at least 3. Thus every bridge other than P and having an endpoint in Γ_1 is a subset of Γ_1 by Corollary 24.

Let $\Gamma_2 = (\Gamma \setminus \Gamma_1) \cup P$. Now, Γ_1 is a k-component, Γ_2 has one less k-component than Γ. Further, Γ_2 is connected, because every bridge other than P and having an endpoint in Γ_1 is a subset of Γ_1. Finally, Γ_2 is not a cycle, because it contains the vertex x_1 which is of degree 1 in Γ_2. Thus induction and Corollary 32 finishes the proof in this case. □

7. An algorithm to calculate the defect k group

Note that by Theorem 1 the defect 1 group can be trivially computed in $O(|E|)$ time by first determining the 2-vertex connected components [7], and whether each is a cycle, the exceptional graph (Figure 1) or if not, whether or not is bipartite.

For $k \geq 2$ one can check first if Γ is a cycle (and then the defect group is Z_{n-k}) or a path (and then the defect group is trivial). In the following, we give a linear algorithm (running in $O(|E|)$ time) to determine the k-components ($k \geq 2$) of a connected graph Γ having n vertices, $|E|$ edges where at least one vertex is of degree at least 3.

During the algorithm we color the vertices. Let us call a maximal subgraph with vertices having the same color a *monochromatic component*. First, one finds all 2-edge connected components and the tree of two-edge connected components in $O(|E|)$ time using e.g. [13]. Color the vertices of the 2-edge connected components such that two vertices have the same color if and only if they are in the same 2-edge connected component. Further, color the uncolored vertices having degree at least 3 by different colors from each other and from the colors of the 2-edge connected components. Then the monochromatic components are each contained in a unique nontrivial k-component by Corollaries 24 and 26 (a nontrivial k-component may contain more than one of these monochromatic components). Further, the monochromatic components and the degree 1 vertices are connected by bridges. If any of the bridges connecting two monochromatic components is of length at most $k-1$, then recolor the two monochromatic components at the ends of the bridge and the vertices of the bridge by the same color, because these are contained in the same k-component by Corollary 24. Similarly, if any of the bridges connecting a monochromatic component and a degree 1 vertex is of length at most $k-1$, then recolor the monochromatic component and the vertices of the bridge by the same color, because these are contained in the same k-component by Lemma 30. Repeat recoloring along all bridges of length at most $k-1$ in $O(|E|)$ time. Then we obtain monochromatic components $\Gamma_1, \ldots, \Gamma_l$ connected by long bridges (i.e. bridges of length at least k), and possibly some long bridges to degree 1 vertices. Now, we have finished coloring.

For every $1 \le i \le l$, let Γ'_i be the induced subgraph having all vertices of distance at most $k-1$ from Γ_i, which can be obtained in $O(|E|)$ time by adding the appropriate $k-1$ vertices of the long bridges to the appropriate monochromatic component. Note that the obtained induced subgraphs are not necessarily disjoint. Then $\Gamma'_1, \ldots, \Gamma'_l$ are the nontrivial k-components of Γ by Lemma 30. Again, by Lemma 30, the trivial k-components of Γ are the paths containing exactly $k+1$ vertices in a long bridge. These can also be computed in $O(|E|)$ time by going through all long bridges. By Theorem 2, the defect k group of Γ as a permutation group is the direct product of the defect k groups of $\Gamma'_1, \ldots \Gamma'_l$, and the defect k groups of the trivial k-components.

8. Complexity of the flow semigroup of (di)graphs

In this section we apply our results and the complexity lower bounds of [10] to verify [9, Conjecture 6.51i (1)] for 2-vertex connected graphs. That is, we prove that the Krohn–Rhodes (or group-) complexity of the flow semigroup of a 2-vertex connected graph with n vertices is $n-2$. Then we derive further consequences of our results, and finish by stating some open problems.

For standard definitions on wreath product of semigroups, we refer the reader to e.g. [9, Definition 2.2]. A finite semigroup S is called *combinatorial* if and only if every maximal subgroup of S has one element. Recall that the *Krohn–Rhodes (or group-) complexity of a finite semigroup S* (denoted by $\#_G(S)$) is the smallest non-negative integer n such that S is a homomorphic image of a subsemigroup of the iterated wreath product

$$C_n \wr G_n \wr \cdots \wr C_1 \wr G_1 \wr C_0,$$

where $G_1, \ldots, G_n$ are finite groups, $C_0, \ldots, C_n$ are finite combinatorial semigroups, and $\wr$ denotes the wreath product (for the precise definition, see e.g. [9, Definition 3.13]). The definition immediately implies that if a finite semigroup S is the homomorphic image of a subsemigroup of T, then $\#_G(S) \leq \#_G(T)$. More can be found on the complexity of semigroups in e.g. [9, Chapter 3]. We need the following results on the complexity of semigroups.

Lemma 33 ([9, Prop. 6.49(b)]). *The flow semigroup K_n of the complete graph on $n \geq 2$ vertices has $\#_G(K_n) = n - 2$.*

Lemma 34 ([10, Sec. 3.7]). *The complexity of the full transformation semigroup F_n on n points is $\#_G(F_n) = n - 1$.*

The well-known $\mathcal{L}$-*order* is a pre-order, i.e. a transitive and reflexive binary relation, on the elements of a semigroup S given by $s_1 \succeq_{\mathcal{L}} s_2$ if $s_1 = s_2$ or $ss_1 = s_2$ for some $s \in S$. The $\mathcal{L}$-*classes* are the equivalence classes of the $\mathcal{L}$-order. We say that a finite semigroup S is a T_1-*semigroup* if it is generated by some $\mathcal{L}$-chain of subsets of its $\mathcal{L}$-classes $L_1 \succeq_{\mathcal{L}} \cdots \succeq_{\mathcal{L}} L_m$, where $L_i \succeq_{\mathcal{L}} L_{i+1}$ if and only if $SL_i \cup L_i \supseteq SL_{i+1} \cup L_{i+1}$ $(1 \leq i \leq m-1)$.

Lemma 35 ([10, Lemma 3.5(b)]). *Let S be a noncombinatorial T_1-semigroup. Then*

$$\#_G(S) \geq 1 + \#_G(EG(S)),$$

where $EG(S)$ is the subsemigroup of S generated by all its idempotents.

Now we prove [9, Conjecture 6.51i (1)] for 2-vertex connected graphs.

Proof of Theorem 3. Let K_n denote the flow semigroup of the complete graph on vertices V, where $|V| = n$. Then $\#_G(S_\Gamma) \leq \#_G(K_n) = n-2$ by Lemma 33. We proceed by induction on n. If $n \leq 3$, then Γ is a complete graph, and $\#_G(S_\Gamma) = n - 2$ by Lemma 33. From now on we assume $n > 3$.

Case 1. Assume first that Γ is not a cycle. Let (u, v) and (x, y) be two disjoint edges in Γ. Let G_1 be the defect 1 group with defect set $V \setminus \{u\}$ and idempotent e_{uv} as its identity element. Then $e_{uv} \succeq_{\mathcal{L}} e_{xy}e_{uv} = e_{uv}e_{xy}$. Let T be $\langle G_1 \cup \{e_{uv}e_{xy}\}\rangle$. Since $G_1 \succeq_{\mathcal{L}} \{e_{uv}e_{xy}\}$ is an $\mathcal{L}$-chain in T, T is a T_1-semigroup. Further, T is noncombinatorial since G_1 is nontrivial. Thus, by Lemma 35

$$\#_G(T) \geq 1 + \#_G(EG(T)). \tag{2}$$

Let Γ' be the complete graph on $V \setminus \{u\}$. Let $a, b \in V \setminus \{u\}$ be arbitrary distinct vertices. By Theorem 1, G_1 is 2-transitive. Let $\pi \in G_1$ be such that $\pi(x) = a$ and $\pi(y) = b$. There is a positive integer $\omega > 1$, with $\pi^\omega = e_{uv}$. In particular, e_{uv} commutes with π. Observe that

$$\pi^{\omega-1}e_{uv}e_{xy}\pi = e_{uv}\left(\pi^{\omega-1}e_{xy}\pi\right) = e_{uv}e_{ab}, \text{ and thus}$$
$$\left(\pi^{\omega-1}e_{xy}e_{uv}\pi\right)\restriction_{V\setminus\{u\}} = e_{ab}.$$

That is, we obtain the generators e_{ab} of $S_{\Gamma'}$ by restricting the idempotents $e_{uv}e_{ab} \in T$ to $V \setminus \{ u \}$. Therefore, $S_{\Gamma'}$ is a homomorphic image of a subsemigroup of $EG(T)$, yielding

$$\#_G (EG(T)) \geq \#_G (S_{\Gamma'}) .$$

By induction, $\#_G (S_{\Gamma'}) = n - 3$. Applying (2), we obtain $\#_G (T) \geq n-2$. Since T is a subsemigroup of S_Γ, we obtain $\#_G (S_\Gamma) \geq \#_G (T) \geq n - 2$.

Case 2. Assume now that Γ is the n-node cycle $(u, v_1, \ldots, v_{n-1})$. Then (u, v_1) and (v_2, v_3) are disjoint edges. Let $G_1 \simeq Z_{n-1}$ be the defect 1 group with defect set $V \setminus \{ u \}$ and idempotent e_{uv_1} as its identity element. Let π be a generator of G_1 with cycle structure $(v_1, \ldots, v_{n-1})$. Then $e_{uv_1} \succeq_{\mathcal{L}} e_{v_2v_3}e_{uv_1} = e_{uv_1}e_{v_2v_3}$. Let T be $\langle G_1 \cup \{ e_{uv_1}e_{v_2v_3} \}\rangle$. Since $G_1 \succeq_{\mathcal{L}} \{ e_{uv_1}e_{v_2v_3} \}$ is an $\mathcal{L}$-chain in T, T is a T_1-semigroup. Further, T is noncombinatorial since G_1 is nontrivial. Thus, by Lemma 35

$$\#_G (T) \geq 1 + \#_G (EG(T)) . \tag{3}$$

Let Γ' be an $(n-1)$-node cycle with nodes $V \setminus \{ u \} = \{ v_1, \ldots, v_{n-1} \}$. Note that $e_{uv_1} = \pi^{n-1}$, and therefore e_{uv_1} commutes with π. Let $v_{i-1}, v_i, v_{i+1} \in V \setminus \{ u \}$ be three neighboring nodes in Γ', where the indices are in $\{ 1, \ldots, n-1 \}$ taken modulo $n-1$. Observe that

$$\pi^{n-2}e_{uv_1}e_{v_{i-1}v_i}\pi = e_{uv_1}\left(\pi^{n-2}e_{v_{i-1}v_i}\pi\right) = e_{uv_1}e_{v_iv_{i+1}}, \text{ and thus}$$
$$\left(\pi^{n-2}e_{uv_1}e_{v_{i-1}v_i}\pi\right) \restriction_{V\setminus\{ u \}} = e_{v_iv_{i+1}}.$$

That is, we obtain the generators $e_{v_iv_{i+1}}$ of $S_{\Gamma'}$ by restricting the idempotents $e_{uv_1}e_{v_iv_{i+1}} \in T$ to $V \setminus \{ u \}$. Therefore, $S_{\Gamma'}$ is a homomorphic image of a subsemigroup of $EG(T)$, yielding

$$\#_G (EG(T)) \geq \#_G (S_{\Gamma'}) .$$

By induction, $\#_G (S_{\Gamma'}) = n - 3$. Applying (3), we obtain $\#_G (T) \geq n - 2$. Since T is a subsemigroup of S_Γ, we have $\#_G (S_\Gamma) \geq \#_G (T) \geq n - 2$. ☐

Note that by Lemma 6 a strongly connected digraph has the same flow semigroup as the corresponding graph. Thus, Theorem 3 proves Rhodes's conjecture [9, Conjecture 6.51i (1)] for 2-vertex connected strongly connected digraphs, as well. The following lemma bounds the complexity in the remaining cases.

Lemma 36. *Let k be the smallest positive integer such that for a graph Γ the flow semigroup S_Γ has defect k group S_{n-k}. Then $\#_G (S_\Gamma) \geq n - 1 - k$.*

Proof. Assume first $k = n - 1$. Then the lemma holds trivially. From now on, assume $k \leq n - 2$. Let uv be an edge in Γ. Let V_k be an arbitrary k-element subset of the vertex set V disjoint from $\{ u, v \}$. Let G_k be the defect k group with defect set V_k. Let S be the subsemigroup of S_Γ generated by G_k and e_{uv}. As $G_k \simeq S_{n-k}$, we have that S is the semigroup of all transformations on $V \setminus V_k$. Hence, $\#_G(S) = \#_G(F_{n-k}) = n-k-1$ by Lemma 34. Whence, $\#_G(S_\Gamma) \geq \#_G(S) = n-k-1$. □

Rhodes's conjecture [9, Conjecture 6.51i (1)] is about strongly connected, antisymmetric digraphs. By [11] a strongly connected antisymmetric digraph becomes a 2-edge connected graph after forgetting the directions. By Theorem 27, it immediately follows that the complexity of the flow semigroup of a 2-edge connected graph is at least $n-3$.

Corollary 37. *Let Γ be a 2-edge connected graph with $n \geq 3$ vertices. Then $n-2 \geq \#_G(S_\Gamma) \geq n-3$.*

This leaves some questions open. To completely settle the last remaining part of Rhodes's conjecture [9, Conjecture 6.51i (1)], one should find the complexity of the flow semigroups for the rest of the 2-edge connected graphs.

Problem 1. Determine the complexity of S_Γ for a 2-edge connected graph Γ which is not 2-vertex connected.

The smallest such graph is the "bowtie" graph:

Problem 2. Let Γ be the graph with vertex set $\{u, v, w, x, y\}$ and edge set $\{uv, vw, wu, wx, xy, yw\}$. Determine the complexity of S_Γ.

Ultimately, the goal is the determine the complexity for all flow semigroups.

Problem 3. Determine the complexity of S_Γ for an arbitrary finite graph (or digraph) Γ.

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