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Transport of congestion in two-phase compressible/incompressible flows

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ABSTRACT

We study the existence of weak solutions to the two-phase fluid model with congestion constraint. The model encompasses the flow in the uncongested regime (compressible) and the congested one (incompressible) with the free boundary separating the two phases. The congested regime appears when the density in the uncongested regime $\varrho(t,x)$ achieves a threshold value $\varrho^*(t,x)$ that describes the comfort zone of individuals. This quantity is prescribed initially and transported along with the flow. We prove that this system can be approximated by the fully compressible Navier–Stokes system with a singular pressure, supplemented with transport equation for the congestion density. We also present the application of this approximation for the purposes of numerical simulations in the one-dimensional domain.

1. Introduction

Our aim is to analyse the free-boundary two-phase fluid system that could be used to model the congestions in the large group of individuals in a bounded area. Individuals are just the agents that have their own preferences for how close they let the closest neighbour to approach and they carry this information with them in the course of motion. They do not follow any neighbour trying to align their velocities, nor they are trying to reach a certain target, as for example, the evacuation point. We simply prescribe their initial velocity that determines their direction of motion and check how the individual preferences as well as the initial distribution of the agents determines creation of congestions. Such model could be used as a building block of more involved crowds modelling [1], some progress in this direction has been made in our recent numerical work [2].

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Crowd modelling is a problem of strategic importance for safety reasons. It has been studied in many parallel approaches. We can distinguish, for example, the mean-field game models [3,4], in which the individuals behave as the players following some strategy, or optimizing certain cost; the microscopic models which describe precise position and velocity of an individual (Individual-Based-Models) using Newtonian framework [5–8]; or the macroscopic models formulated in the language developed for the fluids [9–13]. The behaviour of the crowd in the later is characterized by some averaged quantities such as the number density or mean velocity. The macroscopic models, although less precise than the microscopic ones, are computationally more affordable. Moreover, they allow for asymptotic studies that proved to be useful for understanding various aspects like: swarming or pattern formation observed in the experiments.

It is an extensive field of research to develop continuum models that are able to exhibit features of the kinetic approach. Although it would be desirable to use computationally cheap fluid approach to describe the crowd dynamics, the nowadays models are not developed enough to recreate behaviour observed in real world. In this paper we present, as far as we know, the first mathematical result for the fluid model that incorporates various sizes of the individuals/particles and their inhomogeneities. Similar approach has been recently applied in [14] in the context of granular media flow with memory effects. In order to use this model for more specific applications, its current version needs to be supplemented with agents specific features and we postpone this topic to further research.

Our system writes as follows:

$$\partial_t \varrho + \mathrm{div}(\varrho \mathbf{u}) = 0, \tag{1a}$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla \pi + \nabla p\left(\frac{\varrho}{\varrho^*}\right) - \mathrm{div}\mathbf{S}(\mathbf{u}) = \mathbf{0}, \tag{1b}$$

$$\partial_t \varrho^* + \mathbf{u} \cdot \nabla \varrho^* = 0, \tag{1c}$$

$$0 \le \varrho \le \varrho^*, \tag{1d}$$

$$\mathrm{div}\mathbf{u} = 0 \text{ in } \{\varrho = \varrho^*\}, \tag{1e}$$

$$\pi \ge 0 \text{ in } \{\varrho = \varrho^*\}, \quad \pi = 0 \text{ in } \{\varrho < \varrho^*\}. \tag{1f}$$

with the unknowns: $\varrho = \varrho(t,x)$ – the mass density, $\mathbf{u} = \mathbf{u}(t,x)$ – the velocity vector field, $\varrho^* = \varrho^*(t,x)$ – the congestion density, also referred to as the barrier or the threshold density, and π – the congestion pressure, that appears only when $\varrho = \varrho^*$.

The barotropic pressure is an explicit function of $\frac{\varrho}{\varrho^*}$

$$p\left(\frac{\varrho}{\varrho^*}\right) = \left(\frac{\varrho}{\varrho^*}\right)^\gamma, \quad \gamma > 1, \tag{2}$$

and plays the role of the background pressure.

The stress tensor $\mathbf{S}$ is a known function of $\mathbf{u}$, characteristic for the Newtonian fluid, namely

$$\mathbf{S} = \mathbf{S}(\mathbf{u}) = 2\mu\,\mathrm{D}(\mathbf{u}) + \lambda \mathrm{div}\mathbf{u}\,\mathrm{I}, \quad \mu > 0,\ 2\mu + \lambda > 0, \tag{3}$$

where $\mathrm{D}(\mathbf{u}) = (\nabla \mathbf{u} + \nabla^T \mathbf{u})/2$ denotes the symmetric part of the gradient of $\mathbf{u}$, and $\mathrm{I} = \mathrm{I}_3$ is the identity matrix.

In the system (1) variable ϱ^* models preferences of the individuals, it is given initially and then transported with the flow. Therefore, ϱ^* depends on time and position, but more importantly it depends on initial configuration ϱ_0^*. The form of ϱ^* relaxes the restrictions from the models studied in [15,16], where the threshold density ϱ^* was either assumed to be constant or independent of time. This allows to cover more physical applications. Including the transport of the congestion density ϱ^* allows also to study the system (8) with the contribution from the pressure in the form of the pure gradient, without factor ϱ^* as it was done in [16].

It is justified to call the above system the two-phase system because for $\varrho(t,x) < \varrho^*(t,x)$ it behaves as the compressible Navier–Stokes system with the barotropic pressure, while when the congestion is achieved, i.e. for $\varrho(t,x) = \varrho^*(t,x)$, the system behaves like the incompressible Navier–Stokes equations. We thus observe a switching between two phases: compressible and incompressible depending on the size of the density ratio $\frac{\varrho}{\varrho^*}$. The fluid systems with congestion constraints have been recently intensively studied, especially in the hyperbolic regime [17–19]. The first analytical result for system (1) with $\varrho^* = 1$ is due to P.-L. Lions and N. Masmoudi [20], who showed that it can be obtained as a limit of compressible Navier–Stokes equations with barotropic pressure ϱ^γ with $\gamma \to \infty$, similar studies were performed recently for the model of tumour growth [21,22].

We will consider the system (1) in the 3-dimensional domain Ω with the smooth boundary $\partial\Omega$, and the Dirichlet boundary conditions for the velocity vector field

$$\mathbf{u}|_{\partial\Omega} = \mathbf{0}. \tag{4}$$

The initial conditions are given by:

$$\varrho(0,x) = \varrho_0(x), \quad (\varrho\mathbf{u})(0,x) = \mathbf{m}_0(x), \quad \varrho^*(0,x) = \varrho_0^*(x), \tag{5}$$

and we assume that they satisfy:

$$\begin{gathered}\varrho_0 \geq 0, \quad \int_\Omega \varrho_0 \,\mathrm{d}x > 0,\\ \mathbf{m}_0 = \mathbf{0} \text{ a.e. in } \{\varrho_0 = 0\}, \quad \frac{\mathbf{m}_0}{\varrho_0}\mathbf{1}_{\{\varrho_0>0\}} \in L^2(\Omega),\\ \varrho_0 \leq \varrho_0^*, \text{ a.e. in } \Omega, \quad \varrho_0 \not\equiv \varrho_0^*, \quad \varrho_0^* \in L^\infty(\Omega).\end{gathered} \tag{6}$$

Moreover, we assume that in the region of the absence of the individuals $\varrho_0(x) = 0$, the congestion density is equal to a constant value, being the characteristic mean preference of the group:

$$\varrho_0^*\big|_{\{\varrho_0=0\}} = \tilde{\varrho^*} > 0. \tag{7}$$

The main result of this paper is the existence of solutions to the system (1) under the aforementioned assumptions on the constitutive relations and the initial condition, in the sense of the following definition.

Definition 1 (*Weak Solution*)**.** A quadruple $(\varrho, \mathbf{u}, \varrho^*, \pi)$ is called a global finite energy weak solution to (1), (4), with the initial data (5), (6), (7) if for any $T > 0$:

(i) There holds:

$$0 \leq \varrho \leq \varrho^* \quad a.e.\ in\ (0,T)\times\Omega, \qquad \mathbf{u}|_{(0,T)\times\Omega} = \mathbf{0},$$

$$\operatorname{div}\mathbf{u} = 0 \quad a.e.\ in\ \{\varrho = \varrho^*\}, \qquad (\varrho^* - \varrho)\pi = 0,$$

and

$$\begin{aligned}&\varrho \in C_w([0,T]; L^\infty(\Omega)),\\ &\varrho^* \in C_w([0,T]; L^\infty(\Omega)),\\ &\mathbf{u} \in L^2(0,T; W_0^{1,2}(\Omega,\mathbb{R}^3)), \quad \varrho|\mathbf{u}|^2 \in L^\infty(0,T;L^1(\Omega)),\\ &\pi \in \mathcal{M}^+((0,T)\times\Omega).\end{aligned}$$

(ii) For any $0 \le \tau \le T$, Eqs. (1a), (1b), (1c) are satisfied in the weak sense, more precisely:

$$\int_\Omega \varrho(\tau,\cdot)\varphi(\tau,\cdot)\,\mathrm{d}x - \int_\Omega \varrho_0\varphi(0,\cdot)\,\mathrm{d}x = \int_0^\tau\!\!\int_\Omega \Big(\varrho\partial_t\varphi + \varrho\mathbf{u}\cdot\nabla\varphi\Big)\,\mathrm{d}x\,\mathrm{d}t,$$

holds for all $\varphi \in C^1([0,T]\times\overline{\Omega})$,

$$-\int_\Omega \mathbf{m}_0\cdot\boldsymbol{\psi}(0,\cdot)\,\mathrm{d}x = \int_0^\tau\!\!\int_\Omega \left(\varrho\mathbf{u}\cdot\partial_t\boldsymbol{\psi} + \varrho\mathbf{u}\otimes\mathbf{u} : \nabla\boldsymbol{\psi}\right)\,\mathrm{d}x\,\mathrm{d}t$$
$$+ \int_0^\tau\!\!\int_\Omega \left(\pi\mathrm{div}\boldsymbol{\psi} + p\left(\frac{\varrho}{\varrho^*}\right)\mathrm{div}\boldsymbol{\psi} - \mathbf{S}(\mathbf{u}):\nabla\boldsymbol{\psi}\right)\,\mathrm{d}x\,\mathrm{d}t,$$

holds for all $\boldsymbol{\psi} \in C_c^1([0,\tau)\times\Omega,\mathbb{R}^3)$,

$$\int_\Omega \varrho^*(\tau,\cdot)\varphi(\tau,\cdot)\,\mathrm{d}x - \int_\Omega \varrho_0^*\varphi(0,\cdot)\,\mathrm{d}x = \int_0^\tau\!\!\int_\Omega \Big(\varrho^*\partial_t\varphi + \varrho^*\mathrm{div}(\mathbf{u}\varphi)\Big)\,\mathrm{d}x\,\mathrm{d}t,$$

holds for all $\varphi \in C^1([0,T]\times\overline{\Omega})$.

(iii) For *a.e.* $\tau \in (0,T)$, there holds the energy inequality

$$\mathcal{E}(\varrho,\mathbf{u},\varrho^*)(\tau) + \int_0^\tau\!\!\int_\Omega \Big(\mu|\nabla\mathbf{u}|^2 + (\mu+\lambda)(\mathrm{div}\mathbf{u})^2\Big)\,\mathrm{d}x\,\mathrm{d}t \le \mathcal{E}(\varrho_0,\frac{\mathbf{m}_0}{\varrho_0},\varrho_0^*),$$

where

$$\mathcal{E}(\varrho,\mathbf{u},\varrho^*)(\tau) = \int_\Omega \Big(\frac{1}{2}\varrho|\mathbf{u}|^2 + \left(\frac{\varrho}{\varrho^*}\right)\Gamma\left(\frac{\varrho}{\varrho^*}\right)\Big)(\tau)\,\mathrm{d}x,$$
$$\Gamma\left(\frac{\varrho}{\varrho^*}\right) = \int_0^{\frac{\varrho}{\varrho^*}} \frac{p(s)}{s^2}\,\mathrm{d}s.$$

Remark 1. The condition $(\varrho^*-\varrho)\pi = 0$ is not satisfied in the pointwise sense, its validity is justified using further information about the regularity of weak solution in the sense of Lions–Masmoudi [20], see (54).

Remark 2. Introduction of the value of the weak solution at the final time for ϱ and ϱ^* implies that the initial condition for both of them is fulfilled, i.e. $\varrho(0,\cdot) = \varrho_0(\cdot)$, and $\varrho^*(0,\cdot) = \varrho_0^*(\cdot)$. The initial condition for the momentum is fulfilled only if the test function is divergence-free, i.e. $\mathrm{div}\psi = 0$, due to the fact that π is only a measure.

The main theorem of the paper states as follows.

Theorem 1. *Let the initial conditions ϱ_0, $\mathbf{m_0}$, ϱ_0^* satisfy the conditions above. Then the system (1) with p and $\mathbf{S}$ given by (2), (3) respectively, has a weak solution in the sense of Definition 1.*

Remark 3. The same result holds for the lower dimensions, $d = 1, 2$.

Theorem 1 is the first mathematical result on congested fluid system with time and space variable congestion barrier ϱ^*. In the literature devoted to fluid models with constraint ϱ^* is almost always equal to a constant. This is very severe mathematical simplification tacitly assumed in most of hyperbolic as well as parabolic models. Actually, the only other earlier result in which this assumption is not imposed is the previous work [16], where, however, only special dependence on the space variable was covered.

Including the space and time dependent congestion barrier ϱ^* leads to a sequence of mathematical difficulties already at the level of derivation of a-priori estimates, and definition of a weak solution. If, in addition, ϱ^* satisfies transport equation, it may lose its initial regularity due to low regularity of the velocity vector field ($\mathbf{u} \in L^2(0,T;H^1(\Omega))$). Moreover, the fact that the most nonlinear term in the approximate compressible Navier–Stokes system – the pressure – depends on two quantities that satisfy only hyperbolic PDEs causes further difficulties with identification of the limit. In fact, the area of more general than isentropic pressure laws, depending for example on more than one variables or in non-monotone way, is currently intensively investigated [23–26], but so far this field is still at its infancy.

The core of the proof of Theorem 1 is to show that the system (1) can be obtained as a limit when $\varepsilon \to 0$ of the following approximation

$$\partial_t \varrho + \operatorname{div}(\varrho \mathbf{u}) = 0, \tag{8a}$$

$$\partial_t(\varrho \mathbf{u}) + \operatorname{div}(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla \pi_\varepsilon \left(\frac{\varrho}{\varrho^*}\right) + \nabla p \left(\frac{\varrho}{\varrho^*}\right) - \operatorname{div}\mathbf{S}(\mathbf{u}) = \mathbf{0}, \tag{8b}$$

$$\partial_t \varrho^* + \mathbf{u} \cdot \nabla \varrho^* = 0, \tag{8c}$$

where the π_ε stands for the singular pressure of the form

$$\pi_\varepsilon \left(\frac{\varrho}{\varrho^*}\right) = \varepsilon \frac{\left(\frac{\varrho}{\varrho^*}\right)^\alpha}{\left(1 - \frac{\varrho}{\varrho^*}\right)^\beta}, \quad \alpha \geq 0,\ \beta > 0. \tag{9}$$

A similar form of the pressure

$$\varepsilon \nabla \frac{1}{\left(\frac{1}{\varrho} - \frac{1}{\varrho^*}\right)^\beta}$$

was proposed in [27] or [28]. Singularities of the pressure of this type were also previously studied in the context of traffic models [27,29,30], collective dynamics [28,31], or granular flow [32,33,14].

Note that for $\varepsilon > 0$ fixed and $\varrho^* = const.$ system (8) is purely compressible Navier–Stokes system with singular pressure term. A similar system was studied in [34,35]. From this perspective, the new ingredients covered by this paper are the constitutive relation depending on more than one transported quantities, and the singular limit $\varepsilon \to 0$ leading to the two-phase free boundary problem. In the limit ε, uncongested (compressible) flow changes to incompressible when the density ϱ hits the value ϱ^*. When this happens the dynamics of the system is modified abruptly, meaning that the transition from the uncongested motion ($\varrho < \varrho^*$) to the congested motion ($\varrho = \varrho^*$) is very sudden.

The first goal is to prove the existence of solutions to a certain reformulation of the system (8). Our choice of approximation of the singular pressure (9) allows us to use some of ideas developed in the previous work [36], see also the stability result from [37], and the low Mach number analysis [38], in the context of geophysical flow model. In particular, we essentially use the formulation involving a new unknown Z, that for ϱ, ϱ^* smooth enough can be identified with the density fraction $Z = \frac{\varrho}{\varrho^*}$. We can formally check, that dividing (8a) by ϱ^*, multiplying (8c) by $-\frac{\varrho}{\varrho^{*2}}$, and summing the resulting expressions, the system (8) can be transformed to the following one

$$\partial_t \varrho + \operatorname{div}(\varrho \mathbf{u}) = 0, \tag{10a}$$

$$\partial_t(\varrho \mathbf{u}) + \operatorname{div}(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla \pi_\varepsilon(Z) + \nabla p(Z) - \operatorname{div}\mathbf{S}(\mathbf{u}) = \mathbf{0}, \tag{10b}$$

$$\partial_t Z + \operatorname{div}(Z \mathbf{u}) = 0, \tag{10c}$$

with the initial data

$$\varrho(0,x) = \varrho_0(x), \quad (\varrho \mathbf{u})(0,x) = \mathbf{m}_0(x), \quad Z(0,x) = Z_0(x). \tag{11}$$

In consistency with the assumptions on $\varrho_0,\ \mathbf{m}_0,\ \varrho_0^*$ from the previous section, we postulate that $\varrho_0,\ \mathbf{m}_0,\ Z_0$ satisfy

$$\begin{gathered} 0 \le c_\star \varrho_0 \le Z_0 \le c^\star \varrho_0 \text{ a.e. in } \Omega, \quad \text{for} \quad 0 < c_\star \le c^\star < \infty, \\ 0 < \int_\Omega \varrho_0 \, \mathrm{d}x, \quad Z_0 \le 1, \quad \int_\Omega Z_0 \, \mathrm{d}x < |\Omega|, \quad \frac{\varrho_0}{Z_0}\Big|_{\{\varrho_0 = 0\}} = \tilde{\varrho^*}, \\ \mathbf{m}_0 = \mathbf{0} \text{ a.e. in } \{\varrho_0 = 0\}, \quad \frac{\mathbf{m}_0}{\varrho_0} \mathbf{1}_{\{\varrho_0 > 0\}} \in L^2(\Omega). \end{gathered} \tag{12}$$

The first condition from (12) should be understood as the restriction of the initial congestion density, having in mind our notation $Z = \frac{\varrho}{\varrho^*}$ we see that the condition above means that ϱ_0^* cannot be zero on the regions where the density ϱ_0 is positive. The left condition means that the congestion density is initially bounded. The restriction on the integral $\int_\Omega Z \, \mathrm{d}x < |\Omega|$ means that the assumption $\varrho_0 \neq \varrho_0^*$ holds on a set of non-zero Lebesgue measure.

Existence of solutions to system (10) with singular pressure replaced by the barotropic one i.e. $p(Z) = Z^\gamma$ was studied in the recent paper [36]. In the present work we show that a similar result holds even with a very low a-priori integrability of the pressure. Indeed, the classical energy estimate does not provide a bound for the pressure, as it is for the barotropic case. This can be somehow compensated by better estimate of the pressure argument Z. As a result, the definition of the weak solution to (10) is very similar to the one from [36], namely:

Definition 2 (*Weak Solution of the Approximate System with Z*)**.** A triplet $(\varrho, \mathbf{u}, Z)$ is called a global finite energy weak solution to (10), (4), with the initial data (11), (12), if for any $T > 0$:

(i) There holds:

$$0 \le c_\star \varrho \le Z \le c^\star \varrho \text{ a.e. in } (0,T)\times\Omega, \quad \text{for} \quad 0 < c_\star \le c^\star < \infty,$$

$$Z \le 1 \text{ a.e. in } (0,T)\times\Omega, \quad \mathbf{u}|_{(0,T)\times\partial\Omega} = \mathbf{0},$$

and

$$\begin{aligned} &\varrho \in C_w([0,T]; L^\infty(\Omega)), \\ &\varrho\mathbf{u} \in C_w([0,T]; L^2(\Omega,\mathbb{R}^3)),\ \mathbf{u} \in L^2(0,T; W_0^{1,2}(\Omega,\mathbb{R}^3)),\ \varrho|\mathbf{u}|^2 \in L^\infty(0,T; L^1(\Omega)), \\ &Z \in C_w([0,T]; L^\infty(\Omega)). \end{aligned}$$

(ii) For any $0 \le \tau \le T$, Eqs. (10a), (10b), (10c) are satisfied in the weak sense, more precisely:

$$\int_\Omega \varrho(\tau,\cdot)\varphi(\tau,\cdot) \, \mathrm{d}x - \int_\Omega \varrho_0 \varphi(0,\cdot) \, \mathrm{d}x = \int_0^\tau \!\! \int_\Omega \Big(\varrho \partial_t \varphi + \varrho \mathbf{u} \cdot \nabla \varphi \Big) \, \mathrm{d}x \, \mathrm{d}t,$$

holds for all $\varphi \in C^1([0,T] \times \overline{\Omega})$,

$$\begin{aligned} &\int_\Omega (\varrho\mathbf{u})(\tau,\cdot)\cdot\boldsymbol{\psi}(\tau,\cdot) \, \mathrm{d}x - \int_\Omega \mathbf{m}_0 \cdot \boldsymbol{\psi}(0,\cdot) \, \mathrm{d}x \\ &\quad = \int_0^\tau \!\! \int_\Omega (\varrho\mathbf{u}\cdot\partial_t\boldsymbol{\psi} + \varrho\mathbf{u}\otimes\mathbf{u} : \nabla\boldsymbol{\psi}) \, \mathrm{d}x \, \mathrm{d}t \\ &\qquad + \int_0^\tau \!\! \int_\Omega (\pi_\varepsilon(Z)\mathrm{div}\boldsymbol{\psi} + p(Z)\mathrm{div}\boldsymbol{\psi} - \mathbf{S}(\mathbf{u}) : \nabla\boldsymbol{\psi}) \, \mathrm{d}x \, \mathrm{d}t, \end{aligned}$$

holds for all $\boldsymbol{\psi} \in C_c^1([0,T] \times \Omega, \mathbb{R}^3)$,

$$\int_\Omega Z(\tau,\cdot)\varphi(\tau,\cdot) \, \mathrm{d}x - \int_\Omega Z_0 \varphi(0,\cdot) \, \mathrm{d}x = \int_0^\tau \!\! \int_\Omega \Big(Z \partial_t \varphi + Z \mathbf{u} \cdot \nabla \varphi \Big) \, \mathrm{d}x \, \mathrm{d}t,$$

holds for all $\varphi \in C^1([0,T] \times \overline{\Omega})$.

(iii) For a.e. $\tau \in (0,T)$, there holds the energy inequality

$$\mathcal{E}(\varrho, \mathbf{u}, Z)(\tau) + \int_0^\tau\int_\Omega \left(\mu|\nabla \mathbf{u}|^2 + (\mu+\lambda)(\mathrm{div}\mathbf{u})^2\right)\,\mathrm{d}x\,\mathrm{d}t \le \mathcal{E}\left(\varrho_0, \frac{\mathbf{m}_0}{\varrho_0}, Z_0\right) \tag{13}$$

where

$$\mathcal{E}(\varrho, \mathbf{u}, Z)(\tau) = \int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + Z\Gamma(Z)\right)(\tau)\,\mathrm{d}x, \tag{14}$$
$$\Gamma(Z) = \int_0^Z \frac{\pi_\varepsilon(s) + p(s)}{s^2}\,\mathrm{d}s.$$

Our second goal is to show the convergence of the weak solutions to the system (10), to the solutions of the limit system:

$$\partial_t\varrho + \mathrm{div}(\varrho\mathbf{u}) = 0, \tag{15a}$$

$$\partial_t(\varrho\mathbf{u}) + \mathrm{div}(\varrho\mathbf{u}\otimes\mathbf{u}) + \nabla\pi + \nabla p(Z) - \mathrm{div}\mathbf{S}(\mathbf{u}) = \mathbf{0}, \tag{15b}$$

$$\partial_t Z + \mathrm{div}(Z\mathbf{u}) = 0, \tag{15c}$$

$$0 \le Z \le 1, \quad c_\star\varrho \le Z \le c^\star\varrho, \tag{15d}$$

$$\mathrm{div}\mathbf{u} = 0 \text{ in } \{Z = 1\}, \tag{15e}$$

$$\pi \ge 0 \text{ in } \{Z = 1\}, \quad \pi = 0 \text{ in } \{Z < 1\}. \tag{15f}$$

The weak solutions to the limit system are defined below.

Definition 3 (*Weak Solution of the Limit System with Z*)**.** A quadruple $(\varrho, \mathbf{u}, Z, \pi)$ is called a global finite energy weak solution to (15), (4), with the initial data (11), (12), if for any $T > 0$:

(i) There holds:

$$0 \le c_\star\varrho \le Z \le c^\star\varrho \text{ a.e. in } (0,T)\times\Omega, \quad \text{for} \quad 0 < c_\star \le c^\star < \infty,$$

$$Z \le 1 \text{ a.e. in } (0,T)\times\Omega, \quad \mathbf{u}|_{(0,T)\times\Omega} = \mathbf{0},$$

$$\mathrm{div}\mathbf{u} = 0 \quad \textit{a.e. in } \{\varrho = \varrho^*\}, \qquad (\varrho^* - \varrho)\pi = 0,$$

and

$$\begin{aligned}
&\varrho \in C_w([0,T]; L^\infty(\Omega)),\\
&Z \in C_w([0,T]; L^\infty(\Omega)),\\
&\mathbf{u} \in L^2(0,T; W_0^{1,2}(\Omega,\mathbb{R}^3)), \quad \varrho|\mathbf{u}|^2 \in L^\infty(0,T; L^1(\Omega)),\\
&\pi \in \mathcal{M}^+((0,T)\times\Omega).
\end{aligned}$$

(ii) Eqs. (15a), (15b) are satisfied in the weak sense as in Definition 2, and (15b) is satisfied in the weak sense as in Definition 1 (with $p\left(\frac{\varrho}{\varrho^*}\right)$ replaced by $p(Z)$).

(iii) The energy inequality from Definition 2 holds for $\Gamma(Z) = \int_0^Z \frac{p(s)}{s^2}\,\mathrm{d}s$.

The convergence result reads as follows.

Theorem 2. *Let $(\varrho_\varepsilon, \mathbf{u}_\varepsilon, Z_\varepsilon)_{\{\varepsilon>0\}}$ be a sequence of weak solutions to the approximate system (10), (9). Then, for $\varepsilon \to 0$, the sequence $(\varrho_\varepsilon, \mathbf{u}_\varepsilon, Z_\varepsilon)$ converges to the weak solution of (15) in the sense of Definition 3.*

More precisely,

$$\varrho_\varepsilon \to \varrho \quad in \ C_w([0,T]; L^\infty(\Omega)), \quad and\ weakly\ in\ L^p((0,T)\times\Omega),$$

$$Z_\varepsilon \to Z \quad in \ C_w([0,T]; L^\infty(\Omega)), \quad and\ strongly\ in\ L^p((0,T)\times\Omega),$$

for any $p < \infty$, and

$$\mathbf{u}_\varepsilon \to \mathbf{u} \qquad weakly\ in\ L^2(0,T; W^{1,2}(\Omega, \mathbb{R}^3)).$$

Moreover,

$$\pi_\varepsilon(Z_\varepsilon) \longrightarrow \pi \quad weakly\ in \quad \mathcal{M}^+((0,T)\times\Omega).$$

We see that rewriting the system in terms of the conserved quantities $(\varrho, \varrho\mathbf{u}, Z)$ causes that the limit passage $\varepsilon \to 0$ leads to the switching relation $(1-Z)\pi = 0$. This resembles the homogeneous condition from the works [17,15,20]. The novelty of this paper is the proof of the fact that the same relation can be obtained for system with two densities, and that the final relation $(1-Z)\pi = 0$ can be identified with $(\varrho^* - \varrho)\pi = 0$. For this we need to prove that various weak formulations of the limit system are equivalent, and that the formal derivation of (8a) by ϱ^* leading to equation for Z can be inverted and made rigorous.

The paper is organized in the following manner. In Section 2, we present details of approximation and prove the existence of solutions to the system (10) for ε fixed. Then, in Section 3, we recover the two-phase system (15) by letting $\varepsilon \to 0$. After this, in Section 4 we recover the solution to the original two-phase system (1). Finally, in Section 5 we briefly describe the numerical scheme and present computational examples that illustrate the behaviour of approximate solutions to the system (1).

2. The existence of solution for ε fixed

When ε is fixed, say $\varepsilon = 1$, system (10) resembles the system considered in [36]. The difference is the presence of the singular pressure. It provides uniform L^∞ estimate for Z, which, however, does not imply the higher integrability of the pressure itself. Indeed, the energy estimate provides uniform bound for a corresponding potential energy whose singularity at $Z = 1$ is one order weaker. Therefore, in order to use the result from [36] to prove the existence of approximate solution to (10), we need to approximate the singular pressure by a monotone non-singular function of Z. This is obtained in our work by introducing the truncation parameter δ and a parameter κ that improves integrability of Z at the first approximation level. Approximation like this was considered in the previous paper [16], where ϱ^* was assumed to be a function dependent on x only. The difference with respect to [16] is that now, the uniform estimates as well as the effective viscus flux technique (see, for example, [39]) can be applied to the variable Z only, but it does not imply strong convergence of the sequences approximating ϱ nor ϱ^*.

In the following proof of existence of solutions we will recall some elements of the two approaches from [16] and [36] in order to avoid repetitions.

2.1. Formulation of the approximate problem

The first level of approximation introduces truncation parameter δ in the singular pressure and artificial pressure κZ^K, with K sufficiently large to be determined in the course of the proof. We consider

$$\partial_t \varrho_\delta + \operatorname{div}(\varrho_\delta \mathbf{u}_\delta) = 0, \tag{16a}$$

$$\partial_t(\varrho_\delta \mathbf{u}_\delta) + \operatorname{div}(\varrho_\delta \mathbf{u}_\delta \otimes \mathbf{u}_\delta) + \nabla \pi_\delta(Z_\delta) + \nabla p_\kappa(Z_\delta) - \operatorname{div}\mathbf{S}(\mathbf{u}_\delta) = \mathbf{0}, \tag{16b}$$

$$\partial_t Z_\delta + \operatorname{div}(Z_\delta \mathbf{u}_\delta) = 0, \tag{16c}$$

with $\kappa,\ \delta > 0$, and π_δ, p_κ given by

$$\pi_\delta(Z_\delta) = \begin{cases} \dfrac{Z_\delta^\alpha}{(1-Z_\delta)^\beta} & \text{if } Z_\delta < 1-\delta, \\ \dfrac{Z_\delta^\alpha}{\delta^\beta} & \text{if } Z_\delta \geq 1-\delta, \end{cases} \qquad p_\kappa(Z_\delta) = \kappa Z_\delta^K + Z_\delta^\gamma. \tag{17}$$

We drop the subindex δ when no confusion can arise, and we introduce the notion of weak solution for the system (16).

Definition 4. A triplet $(\varrho, \mathbf{u}, Z)$ is called a global finite energy weak solution to (16), (4), with the initial data (11), (12), if for any $T > 0$:

(i) There holds:

$$0 \leq \varrho, \quad 0 \leq Z \text{ a.e. in } (0,T) \times \Omega, \quad \mathbf{u}|_{(0,T)\times\Omega} = \mathbf{0},$$

and

$$\varrho \in C_w([0,T]; L^K(\Omega)),$$
$$\varrho\mathbf{u} \in C_w([0,T]; L^{\frac{2K}{K+1}}(\Omega,\mathbb{R}^3)),\ \mathbf{u} \in L^2(0,T; W_0^{1,2}(\Omega,\mathbb{R}^3)),\ \varrho|\mathbf{u}|^2 \in L^\infty(0,T;L^1(\Omega)),$$
$$Z \in C_w([0,T]; L^K(\Omega)).$$

(ii) Eqs. (16a), (16b), (16b) are satisfied in the weak sense as in Definition 2 (with π_ε replaced by π_δ, and p replaced by p_k).

(iii) The energy inequality from Definition 2 holds for

$$\Gamma(Z) = \int_0^Z \frac{\pi_\delta(s) + p_\kappa(s)}{s^2}\,\mathrm{d}s. \tag{18}$$

We have the following existence result for solutions defined by Definition 4 (see also [36], Theorem 2).

Theorem 3. *Let $\mathbf{S}$ satisfy (3), $K > 6$, $\beta > 5/2$, $\alpha \geq 0$, $\kappa, \delta, \varepsilon$ be fixed and positive, and the initial data $(\varrho_0, \mathbf{m}_0, Z_0)$ satisfy (12).*

Then there exists a weak solution $(\varrho, \mathbf{u}, Z)$ to problem (16), (17) with boundary conditions (4), in the sense of Definition 4.

Moreover, $(Z, \mathbf{u})$ solves (16c) in the renormalized sense, i.e. $(Z, \mathbf{u})$, extended by zero outside of Ω, satisfies

$$\partial_t b(Z) + \operatorname{div}(b(Z)\mathbf{u}) + \big(b'(Z)Z - b(Z)\big)\operatorname{div}\mathbf{u} = 0, \tag{19}$$

in the sense of distributions on $(0,T) \times \mathbb{R}^3$, where

$$b \in C^1(\mathbb{R}), \quad b'(z) = 0, \quad \forall z \in \mathbb{R} \text{ large enough}. \tag{20}$$

In addition,

$$0 \leq c_\star \varrho \leq Z \leq c^\star \varrho \quad \text{a.e. in } (0,T) \times \Omega. \tag{21}$$

The proof of Theorem 3 requires further modification of the system (16) with two additional approximation levels involving the parabolic regularization of two continuity equations and the Galerkin approximation of the velocity. This approximation allows, in particular, to deduce inequalities (21). At this point existence of regular solution can be obtained following the arguments presented in [36]. Also the compactness arguments needed to recover system (16) are analogous, therefore we skip this part and focus only on the a-priori estimates needed to perform the limit passages $\delta \to 0$, $\kappa \to 0$.

Having the existence of solutions to the system (16)–(17), we show that this solution can be used to recover the weak solution to the system (10), where only the parameter ε is present.

Theorem 4. *Let ε be fixed and let $(\varrho_{\kappa,\delta}, \mathbf{u}_{\kappa,\delta}, Z_{\kappa,\delta})$ be a weak solution to the approximate system (16)–(17) established in Theorem 3. Then, for $\delta, \kappa \to 0$, the sequence $(\varrho_{\kappa,\delta}, \mathbf{u}_{\kappa,\delta}, Z_{\kappa,\delta})_{\kappa,\delta>0}$ converges to the weak solution of (10) in the sense of Definition 4.*

More precisely,

$$\varrho_{\kappa,\delta} \to \varrho \quad \text{in} \;\; C_w([0,T]; L^\infty(\Omega)), \quad \text{and weakly in } L^p((0,T)\times\Omega),$$

$$Z_{\kappa,\delta} \to Z \quad \text{in} \;\; C_w([0,T]; L^\infty(\Omega)), \quad \text{and strongly in } L^p((0,T)\times\Omega),$$

for any $p < \infty$, and

$$\mathbf{u}_{\kappa,\delta} \to \mathbf{u} \qquad \text{weakly in } L^2(0,T; W^{1,2}(\Omega,\mathbb{R}^3)).$$

Moreover,

$$\pi_\delta(Z_{\kappa,\delta}) \longrightarrow \pi_\varepsilon(Z) \quad \text{strongly in} \quad L^1((0,T)\times\Omega).$$

In the next two sections we prove Theorem 4. Starting from the weak solution to the system (16), we first derive uniform bounds and then we let $\delta \to 0$. The modification of the reasoning needed to perform the limit passage $\kappa \to 0$ is explained at the end.

2.2. Uniform estimates

In this section we obtain the uniform estimates for the weak solutions to the system (16)–(17). This involves three a-priori estimates: the standard energy estimate, and two estimates involving the application of the Bogovskii operator, one of which gives us the uniform bound for the singular pressure and the other the uniform integrability of the pressure. All of the aforementioned estimates are essential for performing the limit passage $\delta, \kappa \to 0$, as well as the last limit passage $\varepsilon \to 0$. However, as we shall see later on, the uniform integrability of the pressure will no longer be valid in this case.

2.2.1. The energy estimate

We present a formal computation that can be made rigorous at the level of the Galerkin approximation of the velocity. Multiplying the momentum equation (16b) by $\mathbf{u}$ and integrating by parts with respect to space, yield the energy equality

$$\frac{d}{dt}\int_\Omega \frac{1}{2}\varrho|\mathbf{u}|^2 \,\mathrm{d}x + \int_\Omega \nabla\left(\pi_\delta(Z) + p_\kappa(Z)\right)\cdot\mathbf{u}\,\mathrm{d}x + \int_\Omega \mathbf{S}(\mathbf{u}) : \nabla\mathbf{u}\,\mathrm{d}x = 0. \tag{22}$$

Note that the second term in our case is different than in [16], there is no additional ϱ^* in front of the gradient. This however allows us to proceed straightforwardly, for reader's convenience we repeat here the derivation

$$\begin{aligned}\int_\Omega \nabla\left(\pi_\delta(Z) + p_\kappa(Z)\right)\cdot\mathbf{u}\,\mathrm{d}x &= \int_\Omega \frac{\pi_\delta'(Z) + p_\kappa'(Z)}{Z}\nabla Z\cdot(Z\mathbf{u})\,\mathrm{d}x \\ &= -\int_\Omega Q_{\kappa,\delta}(Z)\mathrm{div}(Z\mathbf{u})\,\mathrm{d}x = \int_\Omega Q_{\kappa,\delta}(Z)\partial_t Z\,\mathrm{d}x = \frac{d}{dt}\int_\Omega Z\Gamma_{\kappa,\delta}(Z)\,\mathrm{d}x\end{aligned}$$

where we used Eq. (16c), we denoted $Q'_{\kappa,\delta}(Z) = \frac{\pi'_\delta(Z)+p'_\kappa(Z)}{Z}$, and $\Gamma_{\kappa,\delta}$ is a solution of the following ODE:

$$\Gamma_{\kappa,\delta}(Z) + Z\Gamma'_{\kappa,\delta}(Z) = Q_{\kappa,\delta}(Z).$$

Using the definition of $Q_{\kappa,\delta}$ and $\pi_{\kappa,\delta}$ we can express $\Gamma_{\kappa,\delta}$ as in (18). Integrating (22) with respect to time, and using the definition of the stress tensor (3) we get the following uniform estimates

$$\begin{aligned} &\sup_{t\in[0,T]} \left(\|\sqrt{\varrho_\delta}\mathbf{u}_\delta(t)\|_{L^2(\Omega)} + \|Z_\delta \Gamma_{\kappa,\delta}(Z_\delta)(t)\|_{L^1(\Omega)} \right) \leq C, \\ &\int_0^T \|\mathbf{u}_\delta\|^2_{W^{1,2}(\Omega,\mathbb{R}^3)} \,\mathrm{d}t \leq C. \end{aligned} \tag{23}$$

2.2.2. The integrability of the pressure

The energy estimate obtained above is insufficient to control the L^1 norm of the pressure, because the singularity appearing in $\Gamma_{\kappa,\delta}(Z)$ at $Z = 1$ is of lower order than the singularity for $\pi_{\kappa,\delta}(Z)$ (take f.i. $\alpha = 2$, $\beta = 1$ in (9) and use (18)). Therefore, further estimates are needed. The first of them is obtained by testing the momentum equation by the function

$$\boldsymbol{\psi} = \phi(t)\mathcal{B}\left(Z - \frac{1}{|\Omega|}\int_\Omega Z(t,y)\,\mathrm{d}y \right), \tag{24}$$

where ϕ is smooth and compactly supported in the interval $(0,T)$, and $\mathcal{B}$ is the Bogovskii operator, i.e. a solution operator $\mathcal{B} : \{f \in L^p(\Omega); \int_\Omega f(y)\mathrm{d}y = 0\} \to W_0^{1,p}(\Omega,\mathbb{R}^3)$ to the following problem

$$\mathrm{div}\,\Phi = f, \quad \Phi|_{\partial\Omega} = \mathbf{0}.$$

The main properties of $\mathcal{B}$ can be found, for example, in [39, Lemma 3.17], and [16, Appendix]. In particular, for $\Phi = \mathcal{B}(f)$ we have the following estimate

$$\|\nabla\Phi\|_{L^p(\Omega,\mathbb{R}^3\times\mathbb{R}^3)} \leq c(p,\Omega)\|f\|_{L^p(\Omega)}, \quad 1 < p < \infty.$$

Moreover, if $f = \mathrm{div} g$, with $g \in L^q(\Omega,\mathbb{R}^3)$, $\mathrm{div} g \in L^p(\Omega)$, $1 < q < \infty$, then

$$\|\Phi\|_{L^q(\Omega,\mathbb{R}^3)} \leq c(q,\Omega)\|g\|_{L^q(\Omega,\mathbb{R}^3)}.$$

Remark 4. Note that, alike for ϱ, the integral of Z over the space is an invariant of the motion, therefore we have

$$\int_\Omega Z(t,x)\,\mathrm{d}x = \int_\Omega Z_0(x)\,\mathrm{d}x < |\Omega|, \tag{25}$$

according to (12), at least for sufficiently regular solutions. In what follows we denote $\int_\Omega Z_0(x)\,\mathrm{d}x = M_Z$.

Remark 5. Function φ must be of a certain regularity in order to use it as a test function in the weak formulation of the momentum equation (16b), see Definition 4. In fact, it follows from the construction of the solution in [36], that taking K sufficiently large guarantees admissibility of this function.

Using in the weak formulation of (16b) the test function (24) results in the following equality

$$\begin{aligned} &\int_0^T\!\!\int_\Omega \phi\left(\pi_\delta(Z_\delta) + p_\kappa(Z_\delta)\right)\left(Z_\delta - \frac{1}{|\Omega|}\int_\Omega Z_\delta\,\mathrm{d}y \right)\,\mathrm{d}x\,\mathrm{d}t \\ &= -\int_0^T\!\!\int_\Omega \varrho_\delta\mathbf{u}_\delta\cdot\partial_t\boldsymbol{\psi}\,\mathrm{d}x\,\mathrm{d}t - \int_0^T\!\!\int_\Omega \varrho_\delta\mathbf{u}_\delta\otimes\mathbf{u}_\delta : \nabla\boldsymbol{\psi}\,\mathrm{d}x\,\mathrm{d}t \\ &+ \int_0^T\!\!\int_\Omega \mathbf{S}(\mathbf{u}_\delta) : \nabla\boldsymbol{\psi}\,\mathrm{d}x\,\mathrm{d}t, \end{aligned}$$

whose r.h.s. can be bounded using the uniform estimates (23) together with (21), provided that K is sufficiently large, say $K > 4$. Therefore, one gets the following estimate, which is now uniform with respect to κ and δ

$$\int_0^T\int_\Omega \phi\left(\pi_\delta(Z_\delta) + p_\kappa(Z_\delta)\right)\left(Z_\delta - \frac{1}{|\Omega|}\int_\Omega Z_\delta \,\mathrm{d}y\right)\,\mathrm{d}x\,\mathrm{d}t \le C. \tag{26}$$

We then consider two complementary subsets of $(0,T)\times\Omega$: $\Sigma_1 = \{Z_\delta(t,x) < Z^*\}$ and $\Sigma_2 = \{Z_\delta(t,x) \ge Z^*\}$, for $\frac{M_Z}{|\Omega|} < Z^* < 1$. The l.h.s. of (26) can be easily controlled on Σ_1, because on this subset Z_δ stays far away from the singularity, for Σ_2 we have

$$\begin{aligned}\int_0^T\int_\Omega \phi\pi_\delta(Z_\delta)\left(Z_\delta - \frac{1}{|\Omega|}\int_\Omega Z_\delta\,\mathrm{d}y\right)\mathbf{1}_{\Sigma_2}\,\mathrm{d}x\,\mathrm{d}t \\ \ge \left(Z^* - \frac{M_Z}{|\Omega|}\right)\int_0^T\int_\Omega \phi\pi_\delta(Z_\delta)\mathbf{1}_{\Sigma_2}\,\mathrm{d}x\,\mathrm{d}t,\end{aligned}$$

and so, from (26) it follows that

$$\|\pi_\delta(Z_\delta)\|_{L^1((0,T)\times\Omega)} + \|Z_\delta\pi_\delta(Z_\delta)\|_{L^1((0,T)\times\Omega)} \le C, \tag{27}$$

as well as

$$\kappa\|Z_\delta\|^{K+1}_{L^{K+1}((0,T)\times\Omega)} \le C. \tag{28}$$

The estimate (26) implies that the L^{K+1} norm of Z_δ is bounded uniformly in δ, but not uniformly in κ, which suggests that the passage to the limit $\delta \to 0$ should be performed as first.

2.2.3. The equi-integrability of the singular pressure

Because π_δ is a nonlinear function of Z, identification of the limit in this term, after letting $\delta \to 0$ cannot be justified only by the uniform L^1 bound. Following the idea from [35], we can prove that the pressure π_δ enjoys some additional estimate near to the singularity $Z = 1$. Indeed, for $K > 6$, $\beta > 5/2$ and

$$\eta_\delta(s) = \begin{cases} -\log(1-s) & \text{if } s \le 1-\delta, \\ -\log(\delta) & \text{if } s > 1-\delta, \end{cases} \tag{29}$$

uniformly with respect to δ one has

$$\int_0^T\int_\Omega \pi_\delta(Z)\eta_\delta(Z)\,\mathrm{d}x\,\mathrm{d}t \le C. \tag{30}$$

The proof follows by testing the momentum equation (16b) by the function of the form

$$\boldsymbol{\psi} = \phi(t)\mathcal{B}\left(\eta_\delta(Z_\delta) - \frac{1}{|\Omega|}\int_\Omega \eta_\delta(Z_\delta)\,\mathrm{d}y\right), \tag{31}$$

where ϕ is smooth and compactly supported in the interval $(0,T)$. For the details of this estimate we refer to [16], for $\beta > 3$ and to [34] for $\beta > \frac{5}{2}$. One of the difficulties in the proof of analogue of (30) presented in [16] concerned the renormalization of the equation for $\frac{\varrho}{\varrho^*}$. Here this problem does not appear anymore, since $(Z_\delta, \mathbf{u}_\delta)$ is by definition a distributional solution of the renormalized transport equation (19).

2.3. Passage to the limit $\delta \to 0$

2.3.1. Convergences following from the uniform estimates

Using the uniform estimates (21), (23), and the Hölder inequality, we can deduce that up to a subsequence

$$\begin{aligned} Z_\delta \to Z \quad & \text{weakly-* in } L^\infty(0,T;L^K(\Omega)),\\ \varrho_\delta \to \varrho \quad & \text{weakly-* in } L^\infty(0,T;L^K(\Omega)),\\ \mathbf{u}_\delta \to \mathbf{u} \quad & \text{weakly in } L^2(0,T;W^{1,2}(\Omega,\mathbb{R}^3)). \end{aligned} \tag{32}$$

Using the continuity equation (16a) and (16c), the first two convergences can be strengthened to

$$\begin{aligned} Z_\delta \to Z \quad & \text{in } C_w([0,T];L^K(\Omega)),\\ \varrho_\delta \to \varrho \quad & \text{in } C_w([0,T];L^K(\Omega)), \end{aligned} \tag{33}$$

which together with the weak convergence of the velocity gradient, after using the momentum equation (16b) implies that

$$\varrho_\delta \mathbf{u}_\delta \to \varrho \mathbf{u} \quad \text{in } C_w([0,T];L^{\frac{2K}{K+1}}(\Omega,\mathbb{R}^3)). \tag{34}$$

This in turn, assuming that K is sufficiently large so that the imbedding of $L^{\frac{2K}{K+1}}(\Omega,\mathbb{R}^3)$ to $W^{-1,2}(\Omega,\mathbb{R}^3)$ is compact, implies that

$$\varrho_\delta \mathbf{u}_\delta \otimes \mathbf{u}_\delta \to \varrho \mathbf{u} \otimes \mathbf{u} \quad \text{weakly in } L^q((0,T)\times\Omega,\mathbb{R}^3\times\mathbb{R}^3), \tag{35}$$

for some $q > 1$. Moreover, the uniform bound (27) together with the growth condition $\beta > 5/2$ in (17) implies that

$$Z \le 1 \quad \text{a.a. } (t,x) \in (0,T)\times\Omega, \tag{36}$$

and thus also

$$\varrho \le \frac{1}{c_\star} \quad \text{a.a.} (t,x) \in (0,T)\times\Omega,$$

on the account of (21). Finally, from the uniform bounds (27) we can extract the subsequence such that

$$\begin{aligned} \pi_\delta(Z_\delta) \to \overline{\pi(Z)} \quad & \text{weakly in} \quad \mathcal{M}^+((0,T)\times\Omega),\\ Z_\delta \pi_\delta(Z_\delta) \to \overline{Z\pi(Z)} \quad & \text{weakly in} \quad \mathcal{M}^+((0,T)\times\Omega), \end{aligned} \tag{37}$$

for some $\overline{\pi(Z)}$, $\overline{Z\pi(Z)}$ that need to be determined. Note, however, that (30) together with De La Vallée-Poussin criterion allow us to deduce the equi-integrability of the pressure, therefore the first limit can be strengthened to

$$\pi_\delta(Z_\delta) \to \overline{\pi(Z)} \quad \text{weakly in} \quad L^1((0,T)\times\Omega). \tag{38}$$

As for the second convergence, we cannot say that much immediately. However, using the fact that $\pi_{\delta^1}(\cdot) \ge \pi_{\delta^2}(\cdot)$ provided $\delta^1 \le \delta^2$, we can estimate for any smooth, nonnegative, compactly supported function $\phi(x,t)$

$$\begin{aligned} & \liminf_{\delta\to 0} \int_0^T\!\!\int_\Omega \phi\left(\pi_\delta(Z_\delta)Z_\delta - \overline{\pi(Z)}Z\right)\,\mathrm{d}x\,\mathrm{d}t\\ & \ge \liminf_{\delta\to 0} \int_0^T\!\!\int_\Omega \phi\left(\pi_{\delta^*}(Z_\delta)Z_\delta - \overline{\pi(Z)}Z\right)\,\mathrm{d}x\,\mathrm{d}t\\ & \ge \liminf_{\delta\to 0} \int_0^T\!\!\int_\Omega \phi\left(\overline{\pi_{\delta^*}(Z)} - \overline{\pi(Z)}\right) Z\,\mathrm{d}x\,\mathrm{d}t \end{aligned} \tag{39}$$

where the last inequality is a consequence of the fact that $\cdot \mapsto \pi_{\delta^*}(\cdot)$ is non-decreasing function for fixed δ^*. Using again equi-integrability of the pressure and letting $\delta^* \to 0$ we verify that the r.h.s. of (39) vanishes and thus

$$\overline{\pi(Z)} \geq Z\overline{\pi(Z)} \tag{40}$$

in the sense of distributions. In order to say something more, we need to investigate the strong convergence of the sequence Z_δ, which is the purpose of the next section.

2.3.2. Strong convergence of Z_δ

It was observed in [36], that the strong convergence of Z_δ does not imply the strong convergence of ϱ_δ. The proof of the strong convergence of Z_δ requires an analogue of the effective flux equality for the barotropic compressible Navier–Stokes equations. In our case it can be written as follows.

Lemma 5. *Let $\varrho_\delta, \mathbf{u}_\delta, Z_\delta$ be the sequence of approximate solutions enjoying the properties from above. Then, at least for a subsequence*

$$\begin{aligned}\lim_{\delta\to 0^+} \int_0^T\int_\Omega \psi\phi\big(\pi_\delta(Z_\delta) + p_\kappa(Z_\delta) - (\lambda+2\mu)\mathrm{div}\mathbf{u}_\delta\big)Z_\delta \,\mathrm{d}x\,\mathrm{d}t \\ = \int_0^T\int_\Omega \psi\phi\big(\overline{\pi(Z)} + \overline{p_\kappa(Z)} - (\lambda+2\mu)\mathrm{div}\mathbf{u}\big)Z \,\mathrm{d}x\,\mathrm{d}t\end{aligned} \tag{41}$$

for any $\psi \in C_c^\infty((0,T))$ and $\phi \in C_c^\infty(\Omega)$.

The proof of this fact can be seen as a special case of an analogous result proven in [36, cf. Lemma 11]. On the account of (40) and the monotonicity of $p_\kappa(\cdot)$ we obtain from (41) that

$$\lim_{\delta\to 0^+} \int_0^T\int_\Omega \psi\phi Z_\delta \,\mathrm{div}\mathbf{u}_\delta \,\mathrm{d}x\,\mathrm{d}t - \int_0^T\int_\Omega \psi\phi Z \,\mathrm{div}\mathbf{u} \,\mathrm{d}x\,\mathrm{d}t \geq 0, \tag{42}$$

for any $\psi\phi \geq 0$. Recall that Z_δ satisfies the renormalized continuity equation (19) with b specified in (20). By density argument and standard approximation technique, we may extend the validity of (19) to functions $b \in C([0,\infty)) \cap C^1((0,\infty))$ such that

$$|b'(t)| \leq Ct^{-\lambda_0}, \qquad \lambda_0 < -1, \quad t \in (0,1],$$

$$|b'(t)| \leq Ct^{\lambda_1}, \qquad -1 < \lambda_1 \leq \frac{q}{2} - 1, \quad t \geq 1.$$

The renormalization technique applied to the barotropic Navier–Stokes system is due to DiPerna and Lions [40], and the above extension can be found, for example, in [41].

We can now write the renormalized continuity equation with $b(Z_\delta) = Z_\delta \log Z_\delta$:

$$\partial_t(Z_\delta \log Z_\delta) + \mathrm{div}(Z_\delta \log Z_\delta \mathbf{u}_\delta) = -Z_\delta \mathrm{div}\mathbf{u}_\delta \quad \text{in} \quad \mathcal{D}'((0,T)\times\mathbb{R}^3).$$

Passing to the limit $\delta \to 0^+$, we hence obtain

$$\partial_t(\overline{Z\log Z}) + \mathrm{div}(\overline{Z\log Z}\mathbf{u}) = -\overline{Z\mathrm{div}\mathbf{u}} \quad \text{in} \quad \mathcal{D}'((0,T)\times\mathbb{R}^3).$$

Writing an analogous equation for the limit function Z and subtracting it from the above, we obtain

$$\partial_t(\overline{Z\log Z} - Z\log Z) + \mathrm{div}[(\overline{Z\log Z} - Z\log Z)\mathbf{u}] = Z\mathrm{div}\mathbf{u} - \overline{Z\mathrm{div}\mathbf{u}}, \tag{43}$$

satisfied in the sense of distributions on $(0,T)\times\mathbb{R}^3$. Integrating the above equation with respect to time and space, and using the convexity of the function $s\mapsto s\log s$ we get from (43) that

$$\int_0^T\int_\Omega \overline{Z\mathrm{div}\mathbf{u}}\,\mathrm{d}x\,\mathrm{d}t \le \int_0^T\int_\Omega Z\mathrm{div}\mathbf{u}\,\mathrm{d}x\,\mathrm{d}t,$$

which is opposite to (42). Therefore, recalling (43) we see that $\overline{Z\log Z}(t,x) = Z\log Z(t,x)$ almost everywhere in $(t,x)\in(0,T)\times\Omega$, which yields the strong convergence of Z_δ in $L^p((0,T)\times\Omega)$ for any $p<K+1$. With this at hand, we verify that (38) can be replaced by

$$\pi_\delta(Z_\delta)\to\pi(Z)\quad\text{strongly in}\quad L^1((0,T)\times\Omega),$$

similarly

$$p_\kappa(Z_\delta)\to p_\kappa(Z)\quad\text{strongly in}\quad L^1((0,T)\times\Omega).$$

In order to complete the proof of Theorem 4, we have to show that we are allowed to let $\kappa\to 0$. Note that all the uniform estimates obtained above stay in force independently of κ. Indeed, the only issue here is to see that the uniform L^{K+1} bound on Z_κ, that was previously obtained from (27) follows now directly from (36). On the account of (21), the same is true also for the sequence ϱ_κ. The proof of Theorem 4 is now complete. □

3. Passage to the limit $\varepsilon\to 0$

The purpose of this section is to prove our main Theorem 1. For technical reasons we first perform the limit $\varepsilon\to 0$ in the auxiliary system (10) proving Theorem 2 and then we prove equivalence between systems (15) and (1) in the certain class of solutions.

3.1. Convergence following from the uniform estimates

The estimates performed in the previous section give rise to several estimates that are uniform with respect to ε. Indeed, passing to the limit in the energy estimate we obtain

$$\begin{gathered}\sup_{t\in[0,T]}\Big(\|\sqrt{\varrho_\varepsilon}\mathbf{u}_\varepsilon(t)\|_{L^2(\Omega)} + \|Z_\varepsilon\Gamma_\varepsilon(Z_\varepsilon)(t)\|_{L^1(\Omega)}\Big)\le C,\\ \int_0^T \|\mathbf{u}_\varepsilon\|^2_{W^{1,2}(\Omega,\mathbb{R}^3)}\,\mathrm{d}t\le C.\end{gathered}\tag{44}$$

Passing to the limit in (36) and in (21) we obtain

$$0\le Z_\varepsilon\le 1,\quad 0\le c_\star\varrho_\varepsilon\le Z_\varepsilon\le c^\star\varrho_\varepsilon,\tag{45}$$

in particular both sequences Z_ε, ϱ_ε are uniformly bounded in $L^p((0,T)\times\Omega)$ for $p\le\infty$. Therefore, by means of the arguments from the previous section we get, up to the subsequence, that

$$\begin{aligned}\mathbf{u}_\varepsilon\to\mathbf{u}\qquad&\text{weakly in } L^2(0,T;W^{1,2}(\Omega,\mathbb{R}^3)),\\ Z_\varepsilon\to Z\qquad&\text{in } C_w([0,T];L^\infty(\Omega)),\\ \varrho_\varepsilon\to\varrho\qquad&\text{in } C_w([0,T];L^\infty(\Omega)).\end{aligned}\tag{46}$$

This information allows us to pass to the limit in all terms of the system (10), apart from the nonlinear pressure terms $p(Z)$ and $\pi_\varepsilon(Z)$. Repeating the Bogovskii type estimate with the test function (24) we obtain

$$\|Z_\varepsilon p(Z_\varepsilon)\|_{L^1((0,T)\times\Omega)} + \|\pi_\varepsilon(Z_\varepsilon)\|_{L^1((0,T)\times\Omega)} + \|Z_\varepsilon\pi_\varepsilon(Z_\varepsilon)\|_{L^1((0,T)\times\Omega)}\le C,\tag{47}$$

however, the estimate (30) does not hold anymore. Therefore the convergence in the sense of measures is the most we can hope for, we have

$$\begin{aligned} &\pi_\varepsilon(Z_\varepsilon) \to \pi \quad \text{weakly in} \quad \mathcal{M}^+((0,T)\times\Omega),\\ &Z_\varepsilon\pi_\varepsilon(Z_\varepsilon) \to \pi_1 \quad \text{weakly in} \quad \mathcal{M}^+((0,T)\times\Omega). \end{aligned} \tag{48}$$

For the background pressure, due to (45), we have

$$p(Z_\varepsilon) \to \overline{p(Z)} \quad \text{weakly in} \quad L^p((0,T)\times\Omega),$$

for any $p<\infty$. At this point, we can identify the second limit in (48) using the explicit form of the pressure (9). We have

$$Z_\varepsilon\pi_\varepsilon(Z_\varepsilon) = \varepsilon\frac{1}{(1-Z_\varepsilon)^\beta} = \pi_\varepsilon(Z_\varepsilon) - \varepsilon\frac{1}{(1-Z_\varepsilon)^{\beta-1}}, \tag{49}$$

thus letting $\varepsilon\to 0$ and observing that the last term converges to zero strongly, we obtain the relation

$$\pi_1 = \pi \tag{50}$$

in the sense of the measures. The recovery of the constraint condition $(1-Z)\pi = 0$ and the identification of the limit $\overline{p(Z)} = p(Z)$ require stronger information about the convergence of Z_ε.

3.2. Strong convergence of Z_ε

The nowadays well known technique of proving the strong convergence of the density in the compressible barotropic Navier–Stokes equations involves the study of propagation of the so called oscillation defect measure [42]. Such level of precision is not needed in our case, because, due to the singularity of the pressure argument Z, we have sufficiently high integrability of the pressure in order to apply the DiPerna–Lions technique [40]. Nevertheless, a variant of effective viscous flux equality is still needed. It can be derived the same way as in Lemma 5, after observing that the inverse divergence operator $\nabla\Delta^{-1}[1_\Omega Z]$ is regular enough to be used as a test function in the limiting momentum equation.

With this information the statement of Lemma 5 adapted to the ε-labelled sequences gives rise to the equality

$$\begin{aligned} &\lim_{\varepsilon\to 0^+}\int_0^T\!\!\int_\Omega \psi\phi\big(\pi_\varepsilon(Z_\varepsilon) + p(Z_\varepsilon) - (\lambda+2\mu)\mathrm{div}\mathbf{u}_\varepsilon\big)Z_\varepsilon \,\mathrm{d}x\,\mathrm{d}t\\ &\quad = \int_0^T\!\!\int_\Omega \psi\phi\big(\pi + \overline{p(Z)} - (\lambda+2\mu)\mathrm{div}\mathbf{u}\big)Z \,\mathrm{d}x\,\mathrm{d}t, \end{aligned} \tag{51}$$

for any $\psi\in C_c^\infty((0,T))$ and $\phi\in C_c^\infty(\Omega)$.

Let us now explain the meaning of the product $Z\pi$ on the r.h.s. of (51). To this end, one needs to come back to the limiting momentum equation

$$\partial_t(\varrho\mathbf{u}) + \mathrm{div}(\varrho\mathbf{u}\otimes\mathbf{u}) + \nabla\pi + \nabla\overline{p(Z)} - \mathrm{div}\mathbf{S}(\mathbf{u}) = \mathbf{0},$$

and use the bounds (44), (45) to justify that π is in fact more regular than it follows just from (48). Indeed, we have

$$\pi \in W^{-1,\infty}(0,T;W^{1,2}(\Omega)) \cup L^p(0,T;L^q(\Omega)) \quad p,q>1. \tag{52}$$

Moreover from the equation for Z, we easily get

$$Z \in C_w([0,T];L^\infty(\Omega)) \cap C^1([0,T];W^{-1,2}(\Omega)). \tag{53}$$

Regularizing in space and time the limits Z and π using the standard multipliers ω_n, $Z_n = Z*\omega_n$, $\pi_n = \pi*\omega_n$, we can clarify the meaning of $Z\pi$ by writing

$$Z\pi = Z_n\pi_n + (Z - Z_n)\pi_n + Z(\pi - \pi_n), \tag{54}$$

and by passing to the limit with the support of mollifying kernel, see [16] for more details.

From (51) it follows that

$$\begin{aligned}
&(\lambda + 2\mu)\int_0^T\!\!\int_\Omega \psi\phi\left(\overline{Z\mathrm{div}\mathbf{u}} - Z\mathrm{div}\mathbf{u}\right)\,\mathrm{d}x\,\mathrm{d}t \\
&= \int_0^T\!\!\int_\Omega \psi\phi\left(\pi_1 - Z\pi\right)\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\!\!\int_\Omega \psi\phi\left(\overline{Zp(Z)} - Z\overline{p(Z)}\right)\,\mathrm{d}x\,\mathrm{d}t \\
&\geq \int_0^T\!\!\int_\Omega \psi\phi\,(1 - Z)\,\pi\,\mathrm{d}x\,\mathrm{d}t \geq 0,
\end{aligned} \tag{55}$$

where to get to the r.h.s. of the above we have used subsequently: monotonicity of $p(\cdot)$, (50), and the limit of (45). Since both pairs $(Z_\varepsilon, \mathbf{u}_\varepsilon)$ and $(Z, \mathbf{u})$ satisfy the renormalized continuity equation, we can use the renormalization in the form $b(z) = z\log z$ to justify that

$$Z_\varepsilon \to Z \quad \text{strongly in } L^p((0,T)\times\Omega), \quad \forall p < \infty.$$

Note however, that this property is not transferred to the sequence ϱ_ε, for which we only have (46). Nevertheless, using this information and formula (54) we can justify that

$$\pi_1 = Z\pi,$$

which together with (50) implies (15f).

It remains to show the condition (15e), or rather its compatibility with the other conditions in system (15). This follows from the following lemma proven by Lions and Masmoudi in [20], that we recall here without the proof.

Lemma 6 (*[20, Lemma 2.1]*)**.** *Let* $\mathbf{u} \in L^2(0,T; W_0^{1,2}(\Omega, \mathbb{R}^3))$ *and* $f \in L^2((0,T)\times\Omega)$ *such that*

$$\partial_t f + \mathrm{div}(f\mathbf{u}) = 0 \quad \textit{in } (0,T)\times\Omega, \quad f(0,x) = f_0(x) \quad \textit{in } \Omega,$$

$$0 \leq f_0 \leq 1, \quad f_0 \not\equiv 0,\ f_0 \not\equiv 1,$$

then the following two assertions are equivalent

(i) $\mathrm{div}\mathbf{u} = 0$ *a.e. on* $\{f = 1\}$,
(ii) $0 \leq f(t,x) \leq 1$.

Applying this lemma for $f = Z$, we conclude the proof of Theorem 2. □

4. Recovery of the original system

In Section 3 we proved that system (15) possesses a weak solution in the sense of Definition 3. Our next aim is to prove that this solution can be identified with the solution to the original problem (1). In other words, we need to deduce the existence of $\varrho^* \in L^\infty((0,T)\times\Omega)$ satisfying the transport equation, such that the measure in the momentum equation vanishes for $\varrho = \varrho^*$.

We proceed similarly to [36]. Note that

$$\frac{\varrho_0}{Z_0}\Big|_{\{\varrho_0=0\}} = \frac{\varrho_0}{Z_0}\Big|_{\{Z_0=0\}} = \tilde{\varrho^*} > 0.$$

When extended by 0 outside Ω the couples $(\varrho, \mathbf{u})$ and $(Z, \mathbf{u})$ satisfy the renormalized continuity equations, due to uniform L^∞ bounds for both ϱ_ε and Z_ε. Therefore, we may test Eqs. (15a), (15c) by $\omega_n(x - \cdot)$, where ω_n is a standard mollifier, which leads to

$$\partial_t \varrho_n + \operatorname{div}(\varrho_n \mathbf{u}) = r_n^1, \tag{56}$$

$$\partial_t Z_n + \operatorname{div}(Z_n \mathbf{u}) = r_n^2, \tag{57}$$

satisfied a.e. in $(0,T)\times\mathbb{R}^3$, where by a_n we denoted $a*\omega_n$. It follows from the Friedrichs commutator lemma, see e.g. [41, Lemma 10.12], that r_n^1, and r_n^2 converge to 0 strongly in $L^1((0,T)\times\mathbb{R}^3)$ as $n\to\infty$.

We now multiply (56) by $\frac{1}{Z_n+\lambda}$, and (57) by $-\frac{\varrho_n+\lambda\tilde{\varrho^*}}{(Z_n+\lambda)^2}$, with $\lambda > 0$, and obtain, after some algebraic transformations, that

$$\partial_t\left(\frac{\varrho_n+\lambda\tilde{\varrho^*}}{Z_n+\lambda}\right) + \operatorname{div}\left[\left(\frac{\varrho_n+\lambda\tilde{\varrho^*}}{Z_n+\lambda}\right)\mathbf{u}\right] - \left[\frac{(\varrho_n+\lambda\tilde{\varrho^*})Z_n}{(Z_n+\lambda)^2} + \frac{\lambda\tilde{\varrho^*}}{Z_n+\lambda}\right]\operatorname{div}\mathbf{u}$$
$$= r_n^1\frac{1}{Z_n+\lambda} - r_n^2\frac{\varrho_n+\lambda\tilde{\varrho^*}}{(Z_n+\lambda)^2}.$$

By passing with $n\to\infty$, we get

$$\partial_t\left(\frac{\varrho+\lambda\tilde{\varrho^*}}{Z+\lambda}\right) + \operatorname{div}\left[\left(\frac{\varrho+\lambda\tilde{\varrho^*}}{Z+\lambda}\right)\mathbf{u}\right] - \left[\frac{(\varrho+\lambda\tilde{\varrho^*})Z}{(Z+\lambda)^2} + \frac{\lambda\tilde{\varrho^*}}{Z+\lambda}\right]\operatorname{div}\mathbf{u} = 0. \tag{58}$$

We distinguish two cases:

Case 1. For $Z = 0$, from (15d) it follows that $\varrho = 0$ and therefore $\frac{\varrho+\lambda\tilde{\varrho^*}}{Z+\lambda} = \tilde{\varrho^*}$, and $\frac{(\varrho+\lambda\tilde{\varrho^*})Z}{(Z+\lambda)^2} + \frac{\lambda\tilde{\varrho^*}}{Z+\lambda} = \tilde{\varrho^*}$, thus (58) becomes trivial.

Case 2. For $Z > 0$, we notice that $\frac{\varrho+\lambda\tilde{\varrho^*}}{Z+\lambda} \le \max\{\tilde{\varrho^*}, \frac{1}{c_\star}\}$. By means of the strong convergence of $\varrho_\lambda = \varrho+\lambda$ and $Z_\lambda = Z+\lambda$ to ϱ and Z, respectively, we can now let $\lambda\to 0$ in (58) to obtain

$$\partial_t\left(\frac{\varrho}{Z}\right) + \operatorname{div}\left(\frac{\varrho}{Z}\mathbf{u}\right) - \frac{\varrho}{Z}\operatorname{div}\mathbf{u} = 0.$$

Obviously, ϱ^* defined as $\frac{\varrho}{Z}$ satisfies $\varrho^* \in \left\{\min\{(c^\star)^{-1}, \tilde{\varrho^*}\}, \max\{(c_\star)^{-1}, \tilde{\varrho^*}\}\right\}$ a.e. in $(0,T)\times\Omega$, and thus $Z = \frac{\varrho}{\varrho^*}$ a.e. in $(0,T)\times\Omega$. This leads to the conclusion that the condition $(1-Z)\pi = 0$ can be replaced by $\left(1-\frac{\varrho}{\varrho^*}\right)\pi = 0$, or, equivalently by $(\varrho^*-\varrho)\pi = 0$, where the product is defined as in (54). This finishes the proof of Theorem 1. □

5. Numerical scheme

Numerical simulation of two-phase flows with free boundary requires to design a method that captures phase transition and the limit behaviour. In our case, the main difficulty is to propose a scheme that is independent of singular pressure parameter ε (9). This property is referred to as the Asymptotic Preserving (AP) property, see e.g. [43]. The passage with $\varepsilon\to 0$ resembles the low Mach number limit problem, where one observes incompressible behaviour in regions where the Mach number approaches 0. However, in the contrary to the low Mach number limit, in our model the singularity is embedded in the definition of the singular pressure π.

We adapt numerical method from [28] where the Euler system with constant maximal density constraint has been studied. One dimensional version of *the Direct method* is modified and extended to capture variable density constraint and the viscosity term in the momentum equation. In what follows we focus only on the new elements of our approach, for the detailed description of the other parts we refer to [28] and references therein. Following this rule, we present our technique on the time semi-discrete level, the discretization in space is omitted for brevity.

5.1. Discretization scheme

To find an approximate solution to the system (8) we propose a splitting algorithm. Time is discretized by one step finite difference (with fixed time step Δt) and a finite volume method is used in space. At each time step the set of the equations is decomposed into three parts which are solved subsequently in three sub-steps.

5.1.1. Step 1: hyperbolic part

The numerical solution of the Euler part of the system follows the strategy presented in [28]. The flux in the mass balance and the singular pressure are treated implicitly:

$$\frac{\varrho^{n+1}-\varrho^n}{\Delta t}+\operatorname{div}(\varrho^{n+1}\mathbf{u}^*)=0, \tag{59a}$$

$$\frac{(\varrho^{n+1}\mathbf{u}^*)-(\varrho^n\mathbf{u}^n)}{\Delta t}+\operatorname{div}(\varrho^n\mathbf{u}^n\otimes\mathbf{u}^n)+\nabla\pi_\varepsilon\left(\frac{\varrho^{n+1}}{\varrho^{*n}}\right)+\nabla p\left(\frac{\varrho^n}{\varrho^{*n}}\right)=\mathbf{0}. \tag{59b}$$

The system (59) is reformulated on a discrete level in terms of singular pressure. Into (59a) we substitute implicit mass flux from (59b) and obtain an elliptic equation for the singular pressure

$$\varrho^{n+1}(\pi_\varepsilon)-(\Delta t)^2\Delta\pi_\varepsilon\left(\frac{\varrho^{n+1}}{\varrho^{*n}}\right)=\phi(\varrho^n,\varrho^{*n},\mathbf{u}^n), \tag{60}$$

where the right hand side of (60) reads

$$\phi(\varrho^n,\varrho^{*n},\mathbf{u}^n)=\rho^n-(\Delta t)\operatorname{div}(\rho^n\mathbf{u}^n)+(\Delta t)^2\operatorname{div}\left(\operatorname{div}(\rho^n\mathbf{u}^n\otimes\mathbf{u}^n)+\nabla p\left(\frac{\varrho^n}{\varrho^{*n}}\right)\right).$$

The singular pressure π_ε is computed by solving (60) by means of the Newton method with numerical Jacobian. In the next step we invert singular pressure to get the density. The purpose of this approach is to ensure that the density constrain is satisfied ($\varrho\le\varrho^*$), which now follows from the definition of π_ε.

After the new density is obtained we directly update the momentum. This approach is called *the Direct method*, see [28, Section 4.1]. The second approach presented in the literature is referred to as *the Gauge method* [43] that is based on the decomposition of the momentum $\varrho\mathbf{u}=\mathbf{a}+\nabla\varphi$, $\operatorname{div}\mathbf{a}=0$ into a divergence free part $\mathbf{a}$, and the irrotational part φ. As reported in [28] the Direct method indicates oscillations of the velocity in congested part, while the Gauge method is diffusive in uncongested region. Since the first method does not introduce any additional numerical dissipation, we adapt it for this work. For detailed description of the space discretization we refer to [28].

We would like to emphasize that (59) is a strictly hyperbolic problem, with characteristic wave speeds $\lambda_{1,2}=\mathbf{u}\pm\sqrt{\frac{\partial p}{\partial(\varrho/\varrho^*)}}$. By the definition, the Courant–Friedrichs–Lewy (CFL) condition for the explicit part is equal to $\max(|\lambda_{1,2}|)\le\sigma\frac{\Delta x}{\Delta t}$, with the Courant number σ. Splitting for implicit singular and explicit background pressure (59b) provides that CFL condition is satisfied uniformly in ε.

5.1.2. Step 2: diffusion

For the sake of numerical simulations, we consider (3) in a simplified form $\mathbf{S}(\mathbf{u}) = 2\mu\Delta\mathbf{u}$. We treat the diffusion term implicitly to avoid additional stability restrictions:

$$\frac{(\varrho^{n+1}\mathbf{u}^{n+1}) - (\varrho^{n+1}\mathbf{u}^{*})}{\Delta t} + 2\mu\Delta\mathbf{u}^{n+1} = 0. \tag{61}$$

The presence of the diffusion term is important from analytical reasons only. In fact, the presented numerical scheme has been designed to solve the Euler system, therefore one can take arbitrary viscosity, such that $\mu \geq 0$. The viscosity coefficient is fixed independently of ε and small enough to recover compressible/incompressible transition. Eq. (61) is discretized in space by cell-centred finite volume scheme.

5.1.3. Step 3: congestion transport

The transport of the congested density is undoubtedly a main new feature of the model (8) and so of the presented numerical scheme. Having the new velocity $\mathbf{u}^{n+1}$ we compute the congested density as follows

$$\frac{\varrho^{*n+1} - \varrho^{*n}}{\Delta t} + \mathbf{u}^{n+1}\nabla\varrho^{*n} = 0,$$

where a cell-centred finite volume scheme together with upwind is used in space.

5.2. Numerical results

In this section we present four numerical examples that demonstrate behaviour of the proposed model in one-dimensional periodic setting. As a consequence of finite volume framework the proposed scheme conserves mass. As for domain we take the unit interval with the mesh size $\Delta x = 10^{-3}$ and the time-step $\Delta t = 10^{-4}$. In the following we choose singular pressure parameter $\varepsilon = 10^{-4}$ with the exponents $\alpha = \beta = 2$, (9), and background pressure (2) with the exponent $\gamma = 2$, if not stated differently.

The test cases are:

- Case 1 (constant congestion):

$$\begin{cases} \varrho(x,0) & = 0.7, \\ \varrho^*(x,0) & = 1.0, \\ \mathbf{u}(x,0) & = \begin{cases} 0.8 \text{ if } 0.2 < x < 0.6 \\ -0.8 \text{ otherwise} \end{cases}, \end{cases}$$

- Case 2:

$$\begin{cases} \varrho(x,0) & = 0.7, \\ \varrho^*(x,0) & = 0.8 + 0.15\left(\tanh(50(x-0.4)) - \tanh(50(x-0.6))\right), \\ \mathbf{u}(x,0) & = \begin{cases} 0.8 \text{ if } 0.25 < x < 0.5 \\ -0.8 \text{ if } 0.5 < x < 0.75 \\ 0.0 \text{ otherwise} \end{cases}, \end{cases}$$

- Case 3:

$$\begin{cases} \varrho(x,0) & = \begin{cases} 0.8 \text{ if } 0.3 < x < 0.7 \\ 0.1 \text{ otherwise} \end{cases}, \\ \varrho^*(x,0) & = 0.34 + 0.3(\tanh(50(x-0.275)) - \tanh(50(x-0.725))), \\ \mathbf{u}(x,0) & = \begin{cases} 0.8 \text{ if } 0.1 < x < 0.7 \\ 0.0 \text{ otherwise} \end{cases}, \end{cases}$$

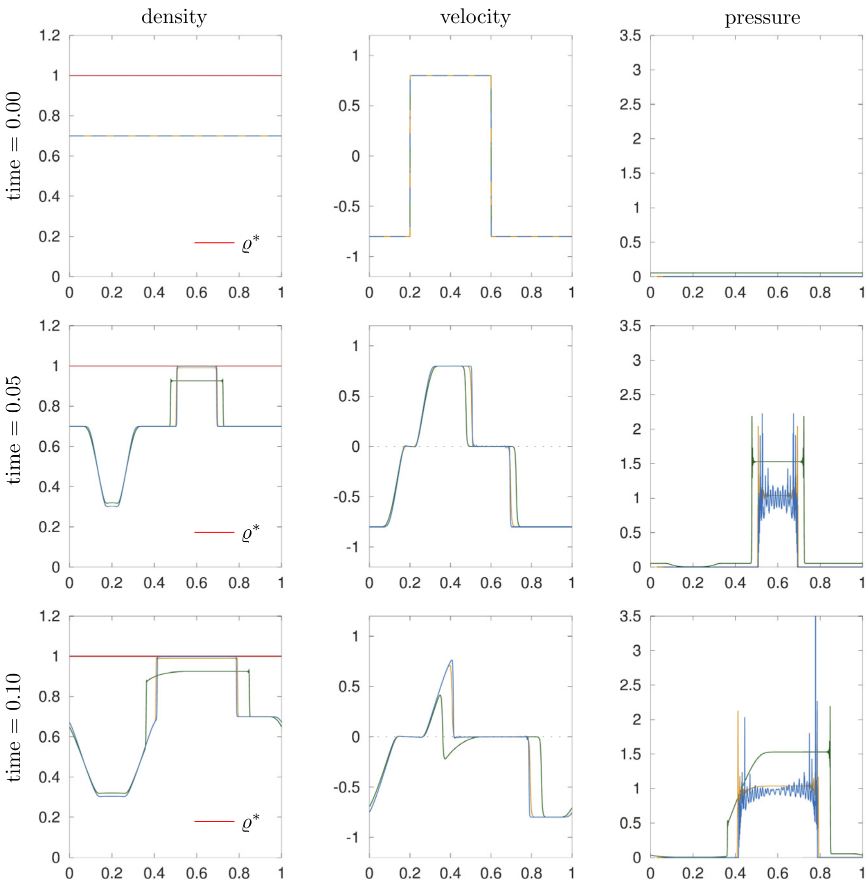

Fig. 1. Case 1: density, velocity, and singular pressure for $\varepsilon = 10^{-2}$ (green), $\varepsilon = 10^{-4}$ (yellow), $\varepsilon = 10^{-6}$ (blue). (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

- Case 4:

$$\begin{cases} \varrho(x,0) &= 0.6, \\ \varrho^*(x,0) &= 0.9 + 0.05(\cos(10\pi x) - \cos(6\pi x) + \cos(134\pi x) + \cos(24\pi x)), \\ \mathbf{u}(x,0) &= \begin{cases} 0.8 \text{ if } 0.3 < x < 0.7 \\ -0.8 \text{ otherwise} \end{cases}. \end{cases}$$

Case 1 illustrates shock and rarefaction of the density for constant initial congestion density. The initial value of the congestion density stays the same for all times, due to transport. The congestions and rarefactions are created solely due to opposite initial velocities, exactly as in the analogous case from [28]. In Fig. 1 we moreover present the behaviour of unknowns for different values of parameter in the singular pressure: $\varepsilon \in \{10^{-2}, 10^{-4}, 10^{-6}\}$. These numerical results show that the algorithm indeed satisfies the Asymptotic Preserving property. Note that with ε decreasing to zero we approach incompressible limit more thoroughly. However, the exact value of the maximal density constraint can never be reached by the numerically computed density.

Case 2 and Case 3 show the main feature of presented model, namely variable congestion density. For both cases the initial maximal density is set to “smooth hat”. In the first of them the initial velocity describes the velocities of two groups of individuals that want to go in the opposite directions. The individuals close to the contact line are more willing to compress (as the congestion density is higher). We see that initially the individuals at the rear of the groups press so intensively onto the front members, so that the whole “hat” is

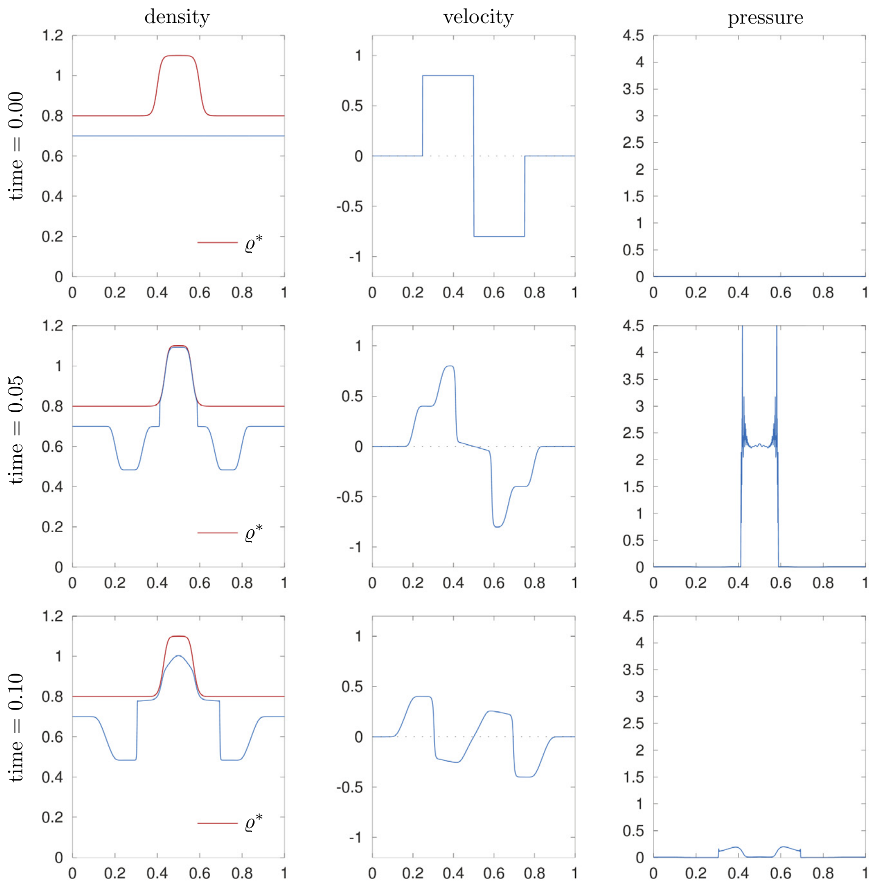

Fig. 2. Case 2: density, velocity and singular pressure.

tightly filled. When this happens, we see a similar effect as for elastic collision: part of individuals start to move in the opposite direction to initially intended (see Figs. 2 and 3).

Case 3 describes a situation, when the well organized crowd moving in one direction with the same velocity approaches a barrier ahead, being the group of individuals that move much slower and prefer to keep bigger distances between each other. We observe how the faster individuals behind push the slower group to speed up, by filling the all available gaps between the individuals (this is where the congestion occurs at position $x \approx 0.8$). This kind of behaviour could be observed, for example, at airports or in the groups of marathon runners.

Case 4 illustrates shock and rarefaction of the density when the maximal density constrain consists of a sum of cosines (periodic setting) with different frequencies. This example mimics randomness in the individual preferences of the members of population. We observe that congested regions "freeze" maximal density due to the zero velocity, which is consistent with the theoretical prediction. As expected from the properties of the limiting system (1f), for $\varepsilon \ll 1$, the singular pressure π_ε is activated only in the congested region. In the theoretical part of the paper, this pressure is merely a nonnegative measure in the limit and our simulations seem to confirm this lack of regularity.

Another interesting feature observed in this case is the travelling wave-like behaviour of the density of the crowd. Note that taking time derivative of (1a) and substituting $\partial_t(\rho\mathbf{u})$ from (1b) we obtain wave-like equation for density. We observe this effect on Fig. 4 between time $t = 0.25$ and $t = 0.5$ at $x = 0.2$, where

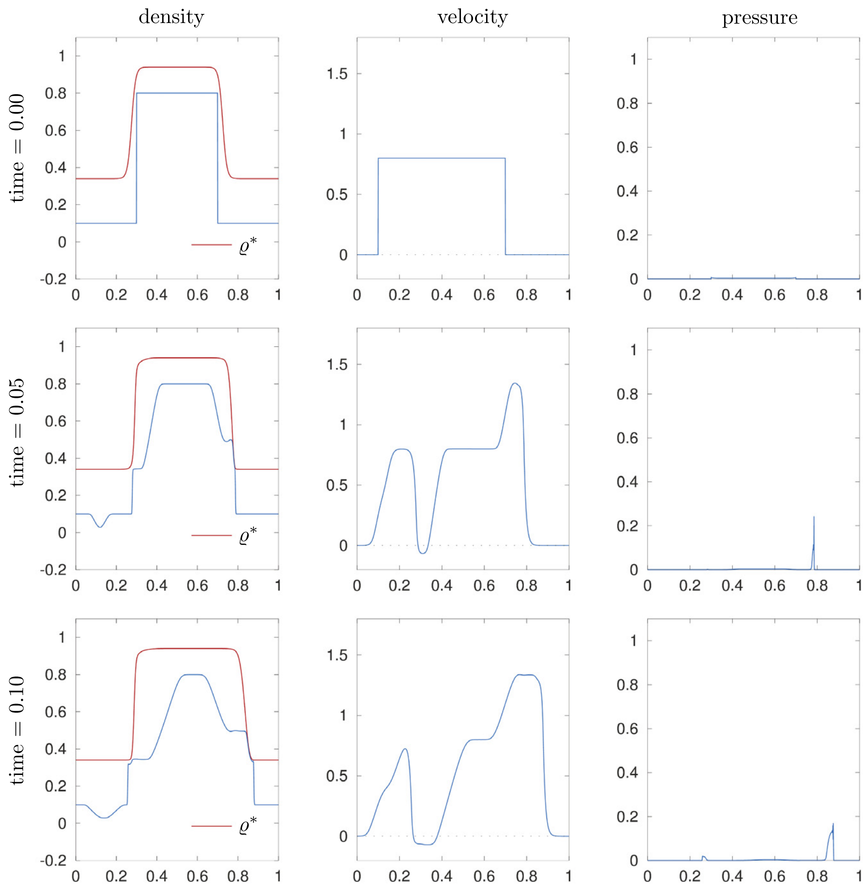

Fig. 3. Case 3: density, velocity and singular pressure.

two "crowds" interfere with each other. This leads to reaching the congestion density and propagation of the congestion in the opposite directions.

Looking at Fig. 4 it seems that the scheme for the transport of the congestion density is quite diffusive. The high spatial frequency oscillations present at the beginning are very quickly washed away and there only subsists the small frequency components. Thus, the numerical examples presented above should be treated just as the illustration of the behaviour of solutions to the approximation (8).

The thorough numerical discussion as well as the study of two-dimensional case, has been addressed in [2]. It has been shown that the presented model, however without diffusion, exhibits typical crowd behaviour like: stop-and-go waves and faster goes slower effect.

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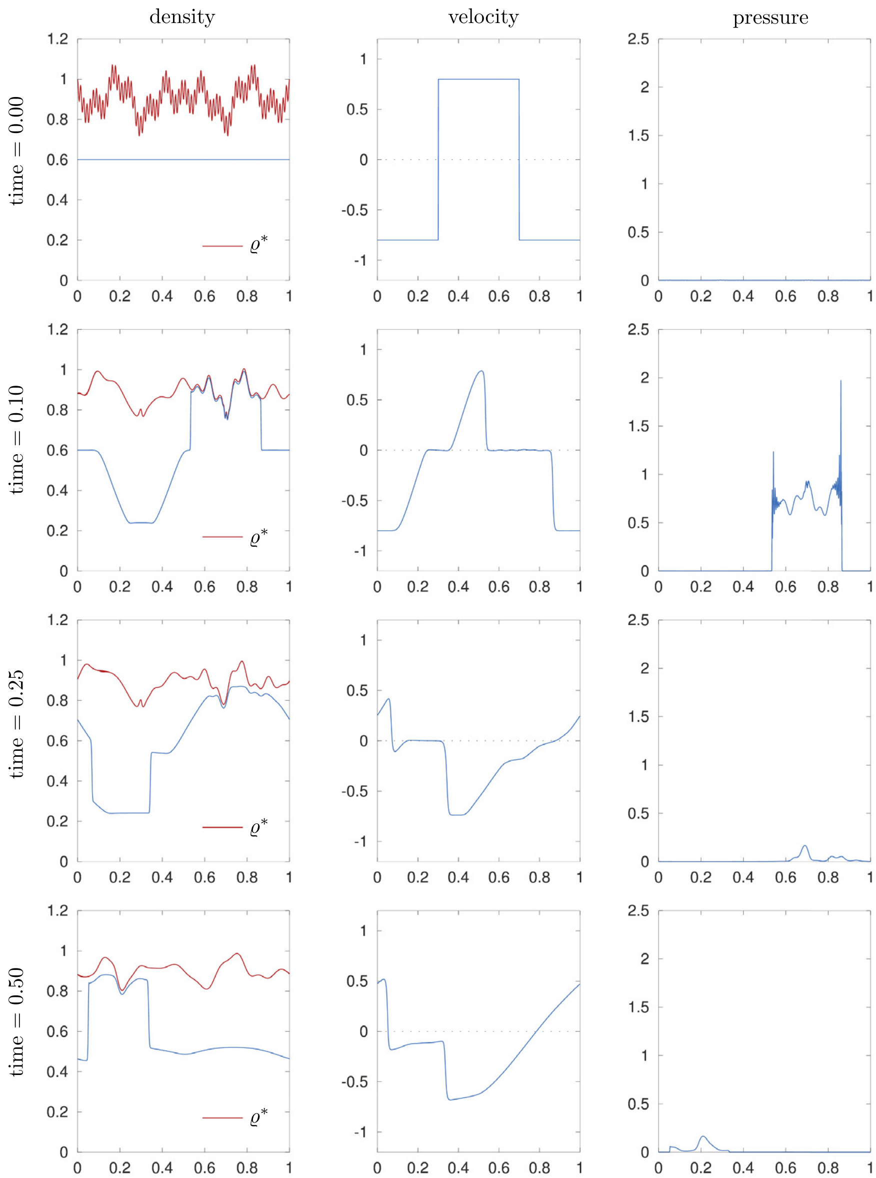

Fig. 4. Case 4: density, velocity and singular pressure.

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