Centre for Doctoral Training in Controlled Quantum Dynamics
Department of Physics, Imperial College London

A process theoretic triptych

Two roads to the emergence of classicality	:	Reconstructing quantum theory from diagrams	:	Looking for post-quantum theories

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Except when otherwise acknowledged and referenced, the following thesis is my own original work that took place during my PhD at Imperial College. It has not been already submitted to satisfy any degree requirement at this or any other university.

John Selby
September 2017

Abstract

This thesis asks what can be learnt about quantum theory by investigating it from the perspective of process theories. This is based on the diagrammatic compositional structure of Categorical Quantum Mechanics, leading to a very general framework to describe alternate theories of nature. In particular this framework is well suited to understanding the relationship between different theories.

In the first part of the thesis we investigate the relationship between quantum and classical theory, showing how an abstract description of decoherence in terms of leaking information leads to emergent classicality. Moreover, this process theoretic notion of a 'leak' allows us to capture the distinction between quantum and classical theory in a particularly simple way, highlighting how the quantum and classical worlds diverge.

In the second part we look at how to reconstruct quantum theory from diagrammatic principles showing that i) the existence of a classical interface with the theory plus ii) standard notions of composition and iii) a time symmetric form of purification are sufficient to reconstruct the standard quantum formalism. Thereby demonstrating that the standard tools of Categorical Quantum Mechanics come very close to capturing the essence of quantum theory.

In the third part we abstract the key features of this emergence of classicality to define a notion of 'hyperdecoherence' whereby some post-quantum theory might appear quantum due to an uncontrolled interaction with an environment. We prove a no-go theorem which states that any operational post-quantum theory must violate the purification principle, and so must radically challenge our understanding of how information behaves.

To summarise, we use the framework of process theories to gain a better understanding of quantum theory, its sub-theories, and its potential super-theories.

Publications

The majority of this thesis has been adapted from the following papers:

John H Selby and Bob Coecke, *A diagrammatic derivation of the Hermitian adjoint,* Foundations of Physics, pages 1–17 (2017).

John H Selby and Bob Coecke, *Leaks: quantum, classical, intermediate and more,* Entropy, 19(4):174 (2017).

Bob Coecke, John H Selby, and Sean Tull, *Two roads to classicality,* arXiv:1701.07400 (2017).

John H Selby, Carlo Maria Scandolo, and Bob Coecke, *A diagrammatic reconstruction of quantum theory,* forthcoming.

Ciarán M Lee and John H Selby, *A no-go theorem for theories that decohere to quantum mechanics,* arXiv:1701.07449 (2017).

Acknowledgements

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Next I must thank Terry Rudolph, it was his course on quantum information that drove me towards quantum foundations in the first place, and our meetings have always pushed me to look for the bigger picture and to ask (if not answer) the deep questions in physics. Thanks also to Lucien Hardy, my supervisor during my masters at PI, he introduced me to so many new ideas and ways of looking at things, and, more to the point, he can be held responsible for my current obsession with drawing diagrams rather than doing any 'proper' maths.

Ciarán Lee deserves a special mention, I've learnt as much from him as I have any of my supervisors and he's provided as much moral support as anyone else. He's been a great friend and a fantastic person to work with over the last five years. I hope it continues for many more. I am also indebted to all of my other collaborators, Carlo Maria, Howard, Jon, Sabri and Sean, for both the work that has gone into this thesis and for everything else that didn't. They have all done a huge amount of work that has been instrumental in my PhD, and I've learnt a great deal from each of them.

I would also like to thank everyone in the group in Oxford, in particular Jon Barrett and the Lobsters, for creating a great working environment. In particular I should thank Matty, Nathan, Niel and many others for all of the pub trips and Carlo Maria, John-Mark, Robin, Subhayan and everyone else in my office for providing great distractions from work, the PhD would have been a miserable experience without all of them around. Similarly, I must thank my cohort at Imperial, we might be (i.e. definitely are) the worst cohort, but I've had a fantastic time being part of it with them, I wouldn't have it any other way.

Thanks must also go to my examiners, Miguel Navascués and David Jennings, for their thorough reading of this thesis, their insightful comments (and criticisms), and the interesting (if extremely tiring) viva.

Away from the academic side of things I'd like to thank all of my friends for their existence and their company, and in particular my music playing friends who have provided me with a distraction from work. In particular my guitar teacher Rafael, who somehow manages to make counting to twelve harder than category theory, and Cliff, we'll properly learn generator one day....

Finally, and most importantly, it goes without saying that my Mum my Dad and my Sister have been wonderful both throughout the PhD and generally throughout my life. They might not have a clue what I've spent the last four years doing, but they have always supported and encouraged me in doing it nonetheless. Thanks for keeping me (somewhat) grounded in the real world!

Kilgore Trout once wrote a short story which was a dialogue between two pieces of yeast. They were discussing the possible purposes of life as they ate sugar and suffocated in their own excrement. Because of their limited intelligence, they never came close to guessing that they were making champagne.

– Kurt Vonnegut

Contents

Chapter 1

Introduction

For nearly a century quantum theory has fascinated and frustrated in equal measure. The beguiling simplicity of the mathematical formulation foiled by its abstract presentation, resulting in the abject inability for any consensus to be formed as to what it all ultimately means for physical reality. Hence, we still have papers and books written, conferences and workshops organised, and many an argument held over a few pints regarding the "correct" way to interpret the theory. However, is this difficulty something intrinsic to nature; the mathematical theory; our limited human perception of the world; or perhaps, could this instead be just due to the *language* that we use to describe the theory? Are we in a situation akin to explaining Shakespeare via Morse code, computer code with abstract art [1] or linear algebra through interpretive dance? Is there a more suitable "higher-level" language which could be used to describe quantum theory in which some or all of the quantum weirdness is easily explained?

Somewhat ironically, it seems like category theory [109, 59, 97] —aka generalised abstract nonsense [36]— may provide such a language. Indeed, this is what the research programme of Categorical Quantum Mechanics (CQM) [4, 49] has been developing for the last 14 years, using tools developed for monoidal category theory to provide a diagrammatic representation of quantum processes. In this language certain odd features of quantum theory seem like natural, almost inevitable, features of a physical theory. This viewpoint has admittedly not yet provided a compelling account of *all* of the peculiarities of quantum theory, but it seems plausible that this —by providing a simpler description in a more powerful intuitive language— is a step in the right direction. It therefore seems pertinent to ask whether or not there is anything of foundational interest to be learnt from this categorical approach.

There are three particular foundational questions that this thesis aims to address:

1. How are classical and quantum theory related? The relationship between

quantum and classical theory is an odd one. On one level we expect classical theory to be some limiting case of quantum theory, and yet, we rely on classical theory in the formulation of quantum theory[1]. How can we best understand the emergence of classical theory? Can we find some clear physical principle that distinguishes the two theories?

2. Can we find a more compelling axiomatisation of quantum theory? The standard Hilbert space formulation of quantum theory takes mathematical statements as axioms, and so given any particular quirk of quantum theory it is difficult to explain *why* that particular behaviour occurs. In contrast, special relativity has many "paradoxes" but these can be explained by the invariance of the speed of light, or the equivalence of inertial reference frames, and so find satisfactory resolution. Can we therefore find some better motivated axioms which reconstruct the standard formalism and so provide a more satisfactory explanation of quantum phenomena?

3. What can we learn about any theory that could one day supersede quantum theory? It would be exceedingly arrogant to assume that quantum theory is the fundamental theory of nature, even putting aside the philosophical issues, there is the more immediate problem as to how to unify quantum theory and general relativity. As —given the difficulties in quantizing gravity— it seems likely that both quantum theory as well as general relativity will need to be modified to achieve such a unification. We therefore seek to answer the question as to whether it is possible to go beyond quantum theory. Is there some deeper theory of nature yet to be discovered which could be as radically different from quantum theory as quantum is from classical? If so, how should we go about looking for such a theory, what constraints can we place on it, and what features should we expect it to have?

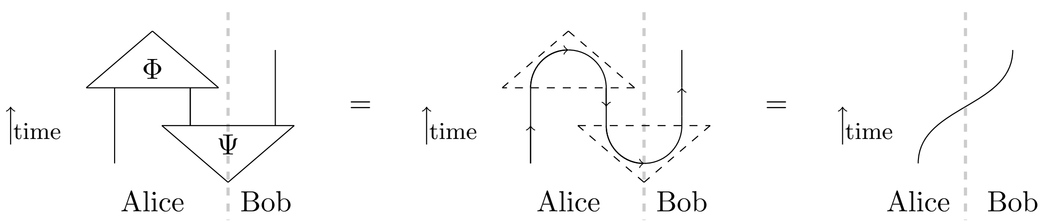
An offshoot of the development of Categorical Quantum Mechanics is the framework of generalised process theories [55]. This provides a broad framework for describing potential theories of nature in which quantum and classical theory are two particular examples. Studying physics "from the outside" using such a framework allows one to gain an understand of why physics is the way it is by considering what the alternatives would be like. For example, do all other possible theories have particularly problematic features such as faster than light propagation of information [123]? Moreover, this approach gives insight into how different features of a theory interact with each other from a theory independent perspective, irrespec-

[1] At least if one wants a description of measurements then one must introduce classical agents as part of the theory.

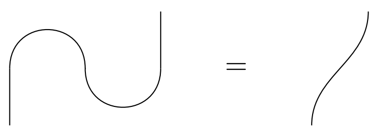
tive of the mathematical formulation of the theory. For example, can we understand how thermodynamic constraints [15, 47, 48, 44, 46] or computational features [99, 24, 98, 2, 14, 101, 100, 104, 20] arise directly from physical principles?

The fundamental underpinning of the process theory framework is compositionality. The central tenet being that nature is best understood not via the traditional reductionist approach of breaking things apart and studying isolated irreducible components, but instead, by studying the relations and interactions between systems. It is this idea that leads to the aforementioned diagrammatic representation of these theories. The existence of such a representation has important conceptual and practical consequences. On the practical side this diagrammatic representation lends itself to automation and computational reasoning [93], whilst on the conceptual side it provides an intuitive description of certain 'strange' quantum phenomena.

The classic motivating example of this is the teleportation protocol [49]. Which is diagrammatically represented as follows:



On the left we have a schematic drawing of what two agents, Alice and Bob, do to achieve quantum teleportation. They share some state Ψ and then Alice performs some measurement getting outcome Φ. If the state and measurement are chosen correctly[2], and the right outcome occurs, then we can draw the 'information flow' of these various diagrammatic features as is done in the middle part of the above diagram. This can then be rewritten as the right hand side, as in these diagrams the only relevant data is the *connectivity* rather than the specific layout on the page so we automatically have the rewrite rule:



This is just the identity channel from Alice to Bob, and so we can see that any state that Alice inputs on her side will end up with Bob. This is clearly not the complete story of quantum teleportation [28] as we have made no mention of the necessary classical communication between the two parties. However, this forms the (post-selective) core of the protocol as will be properly explained once we have

[2] I.e. are a Bell state and measurement.

formally introduce the notation and framework in section 2.6. The simplicity of this diagrammatic description motivates the idea that perhaps this is the correct language to describe quantum theory, or at very least, provides a novel perspective under which certain phenomena are more readily understood.

As mentioned earlier, the process theoretic approach to quantum foundations is built on the mathematics of category theory. This has a rich literature which has (so far) not been widely exploited in theoretical physics research[3], and so has significant potential for providing new tools for performing concrete calculations as well as new ways of viewing the world. A historic precedent for this can be seen in the influence of group theory in physics, which, not only is a useful tool for performing calculations, but provides deep insights into nature. In particular, via Noether's theorem [116] revealing a remarkable connection between symmetries and conservation laws.

Studying of quantum physics from the perspective of a broad framework of alternate theories has a long history stretching back to various approaches to quantum logic [30, 122, 107, 110] through to modern forms of generalised probabilistic theories building on [107] such as [76, 23]. The connections between some of these frameworks will be discussed in detail in chapter 4, but broadly speaking much of the recent work on generalised probabilistic theories can be seen as examples of process theories or at the very least as inspired by the diagrammatic approach. In particular the work of Hardy has taken the compositional approach as its core for reconstructing quantum theory [79], a general principle for formulating physical theories [81], and most recently, for general relativity [82]. The work of Chiribella et al. can also be viewed as taking process theories as the basis for an axiomatisation of quantum theory [41, 42, 43] and more recently their approach has been used to explore thermodynamics [45, 46] and computation [98, 24, 99, 101] in generalised theories. The work of Barnum et al. [18] on a unification of real, complex and quaternionic quantum theory has also been heavily influenced by the categorical viewpoint in particular focusing on trying to find suitable composites of real, complex and quaternionic systems. More directly within the process theory framework there has been work connecting 'terminality' and no-signalling [53], a general framework for resource theories [54, 71], a diagrammatic representation of Bayesian inference [62], representing quantum theory as a quasi-stochastic process theory [140], and now substantial progress into fields as diverse as linguistics and cognition [60, 12, 57].

[3] Although it has fairly recently been used in condensed matter [106] and topological quantum field theory [25].

1.1 Summary of the thesis

In the first chapter of the thesis we introduce the framework that will be used throughout the thesis, that is the framework of generalised process theories. We introduce the basic compositional structure as well as notions of causality and purity for process theories. We show how the former is closely related to compatibility with a *classical interface* for a theory, which can be thought of as how we interact with the world by classically choosing which experiment to perform and by getting classical data as an output of an experiment. The later requires that we introduce the notion of a *leak* for a system, which can be thought of as the process describing how information can freely escape particular systems in certain process theories. We then consider how we can relax the constraints on the composition of processes to obtain commonly used structures for the process theory framework. Finally, we introduce the notion of a *sharp dagger*, which can be seen as a refinement of the standard notion of a dagger such that it has a clear operational interpretation. This chapter contains generic expository material adapted from a paper with Carlo Maria Scandolo and Bob Coecke [131], some results from the same paper, as well as two further papers with Bob Coecke [130] and [129].

In the second chapter we demonstrate that, given any process theory, it is natural, from an operational perspective, to add in extra systems which are those that arise from the *leak construction*. Indeed, we show that one example of this leak construction provides a process-theoretic account of the emergence of classicality within quantum theory. More generally, we show that the leak construction leads to all C*-algebraic systems and no more. Therefore, from this perspective one should describe *Operational Quantum Theory* as a process theory of C*-algebras and completely positive maps. Moreover, we then can ask, how do we distinguish the specifically quantum or classical C*-algebras. We show that there are two equivalent ways do this. We show that quantum systems are those for which two equivalent conditions hold. The first, is that the maximally correlated state is pure, essentially, that there is a maximally entangled state. The second, is that there are only trivial leaks for a theory, essentially, that losing information necessarily disturbs a system. The main results of this chapter are first presented paper with Sean Tull and Bob Coecke [61] however they have been presented here in a different style along with some further results from [130].

In chapter three we consider how close the process-theoretic structures introduced in the first chapter come to reconstructing the standard quantum formalism. We demonstrate that we need just a single extra postulate to obtain Operational Quantum Theory. This extra postulate is the notion of *time-symmetric purification*, based on the notion of purification of [41] but with three benefits. Firstly, it more naturally fits in the process theory framework as it is defined for all processes rather than just for states. Secondly, it is a time symmetric notion and so this allows us to more clearly pin down where in this reconstruction time asymmetry is introduced. Thirdly, it applies equally well to quantum and classical theory and so we can get a more refined notion as to what, on an axiomatic level, distinguishes these two theories. This chapter presents the main result of the paper with Carlo Maria Scandolo and Bob Coecke [131].

In the final chapter we ask: can quantum theory be seen to emerge from some deeper post-quantum theory? We formalise this in an analogous way to the emergence of classicality presented in chapter two, and prove a no-go theorem that states: there is no such post-quantum theory that satisfies the purification postulate of [41]. As such, any post-quantum theory will necessarily challenge our fundamental understanding of how information behaves. This chapter is based a paper with Ciarán Lee [103] which has been modified slightly to fit more closely with the process theory framework.

Chapter 2

Process theories

The original objective of the Categorical Quantum Mechanics research programme [4] was to provide a high-level language to describe quantum processes, giving an abstract account of information flow in quantum information protocols such as entanglement sharing, state teleportation, and gate teleportation. This was achieved by describing quantum theory in the language of category theory, specifically, as a compact-closed symmetric monoidal category with biproducts. The upshot of this was a pictorial representation of quantum processes [49, 51] which provides an intuitive description of the abstract mathematical notions[1], and hence, an intuitive understanding of information flow in the aforementioned protocols. Of particular note is that, unlike in many traditional presentations of quantum theory, this categorical description also handles classical information 'internally' to the theory, and so, hybrid quantum-classical protocols can also be reasoned about in an intuitive but mathematically rigorous way.

It is worth mentioning that this diagrammatic representation of quantum processes has more recently been extended such that it forms a sound, complete, and universal representation of quantum processes known as the ZX-calculus [114, 88, 75, 11, 10]. That is, there is not only a representation of each quantum process as some diagram, but that anything that can be calculated in quantum theory can be calculated via a set of diagrammatic rewrite rules. In other words, anything that can be proved in the standard formalism using standard linear-algebraic techniques can instead be proved (in principle at least) with this diagrammatic representation. Moreover, this representation proves to be amenable to computer automation [93, 73, 37, 68] such that novel results can be found that would be impossible to even check by hand. Indeed, the ZX-calculus has in recent years found practical applications in the field of quantum computation and error correction [66, 87, 67, 69].

[1] I.e. of "compact-closed symmetric monoidal category with biproducts".

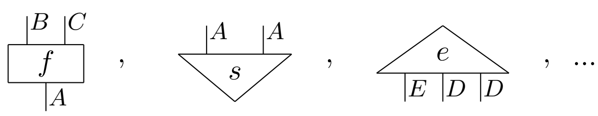
In this thesis however, we are not interested in the practical applications of the categorical approach, but instead, with whether we can use this tool to understand foundational aspects of quantum theory. One of the primary ways that questions in quantum foundations are tackled is by considering quantum theory not as a theory in isolation, but, by considering it as one particular theory in a wide landscape of potential theories. There have been many frameworks proposed for exploring these alternate theories, from early work in quantum logic [30, 72, 108] to more recent operational approaches [76, 23, 3]. Despite the wide range of the conceptual motivation and mathematical formulation of these approaches, the focus of all of them has been in describing physics via the properties of single perfectly isolated systems. Dealing with composite systems was therefore traditionally a significant technical challenge. In contrast, process theories are based on the diagrammatic representation of Categorical Quantum Mechanics, and so, take composition of systems, how they interact with each other, and with the environment as primitive notions. This framework therefore provides a new perspective on quantum foundations, and, along with the new perspective, a new set of mathematical tools allowing us to avoid many of the pitfalls of the earlier approaches.

In this chapter we begin by introducing the process theory framework, illustrating it with some key examples that will be used throughout the thesis. We then introduce some of the basic tools that are commonly used in the framework, notably, classical interfaces, sums, compact structure, leaks and sharp-daggers. Finally we demonstrate how these can be used to understand some fundamental concepts in quantum physics, namely, causality and purity of processes.

2.1 Defining process theories

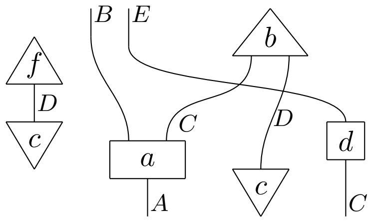
Process theories [55] are defined by a collection of *systems* – denoted as wires where multiple wires represent *composite systems*, for example:

and a collection of *processes* – denoted as boxes with input systems at the bottom and output systems at the top[2], for example:



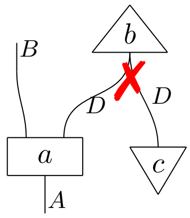
[2]That processes have fixed input and output systems may seem to be a limitation of the framework, however (as we show in chapter 3), an indefinitely typed input or output can just be seen as a new system type.

along with a notion of composition of processes – they can be wired together to form *diagrams* such as:

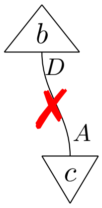


where any such diagram corresponds to another process in the theory, in this case a process with inputs A and C and outputs B and E. This wiring together of processes is not completely free but is subject to certain constraints:

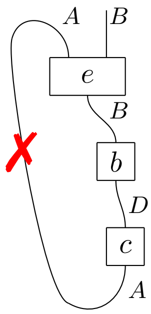
1. each output can be connected to at most one input (and vice versa),



2. such that system types match,



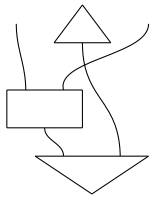
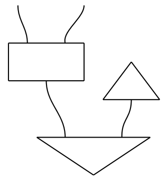
3. and this wiring does not create cycles[3],



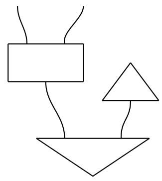
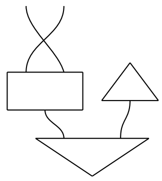
The only relevant data provided by the diagram is the *connectivity*: which outputs are connected to which inputs, which inputs and outputs are disconnected/free and

[3]We will introduce a way to construct cycles in certain process theories in section 2.6. This is closely related to the existence of a (post-selected) teleportation protocol in a theory.

the ordering of the free inputs and outputs. For example,

 and 

are "the same diagram" as —despite not being drawn in precisely the same way— the connectivity is identical for each. On the other hand,

 and 

are different diagrams as the ordering of the free outputs is swapped.

In the above diagrams we have dropped system and process labels for convenience as we will regularly do throughout the thesis when they are either irrelevant —such as in the above examples— or clear from the context.

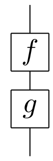
This is the complete description of the process theory framework, and so it is clearly very broad in scope – see [55] for a wide range of examples from cooking breakfast to data structures. However, to gain a better understanding of this framework it is useful to consider how it connects to more standard 'symbolic' notation, in particular for making connections with standard presentations of physical theories. There is a straightforward procedure to do just that, however, as we will now show, this procedure does not result in a unique symbolic expression for each diagram. Hence these equivalent symbolic representations must then be equated by introducing a number of additional axiomatic equations. It is therefore generally much simpler to remain on the diagrammatic level so as to avoid the ambiguity introduced by the symbolic notation, and so the diagrammatic description seems to be the natural choice of notation[4].

The procedure is as follows. The first step is to specify the inputs and outputs of each process in a diagram, for example we denote a process f with an input A and an output B by $f : A \to B$. We then define two primitive notions of composition,

[4]Hardy [81, 79] uses an alternative symbolic representation of the diagrams which does not suffer from such a great degree of ambiguity, however this notation is more difficult to connect to standard descriptions of physics and is less clear to read compared to the notation we will now introduce.

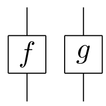
sequential composition, denoted $\circ$ defined as:

$$f \circ g \quad :=$$



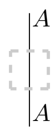
and *parallel* composition denoted $\otimes$ where:

$$f \otimes g \quad :=$$



we similarly denote the parallel composition of systems A and B as $A \otimes B$. Next, for each system we must introduce an *identity* process denoted as:

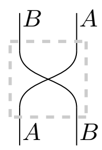
$$\mathbb{1}_A \quad :=$$



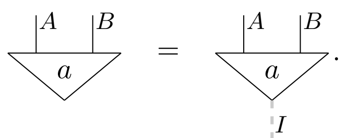
Note that the grey dashed line here does not form part of the diagrammatic language itself, but is just used for illustrative purposes, for example, to indicate grouping a collection of processes together.

We must also introduce, for each pair of systems, a swap process:

$$\text{SWAP}_{AB} \quad :=$$

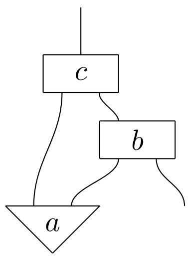


Additionally, it is convenient to introduce the concept of the *trivial* system, I, such that,

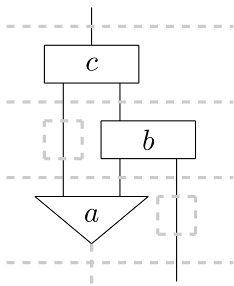


This allows one to describe all processes as having both an input and an output system, in this case $a : I \to A \otimes B$. As this system represents 'nothing' then composing it with other systems must leave them invariant, $I \otimes A = A = A \otimes I$.

We can then observe that any diagram can be built up out of these primitive elements. For example:



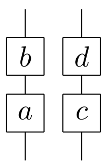
$=$



$$= c \circ (\mathbb{1} \otimes b) \circ (a \otimes \mathbb{1})$$

This however introduces the unnecessary baggage of the identity transformations and trivial system and is far less easy to read, in particular, all of the system type information must be specified separately. Moreover, as mentioned above, this procedure does not lead to a unique symbolic expression, and so there is always ambiguity when switching from the diagrammatic notation to this more standard symbolic notation. For example:

$$(b \circ a) \otimes (d \circ c) \quad =$$

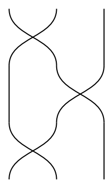


$$= \quad (b \otimes d) \circ (a \otimes c)$$

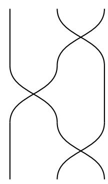
In this case it is fairly easy to understand why the two symbolic expressions should be equal, but this is not always the case. For example the Yang-Baxter equation:

$$(\mathrm{SWAP} \otimes \mathbb{1}) \circ (\mathbb{1} \otimes \mathrm{SWAP}) \circ (\mathrm{SWAP} \otimes \mathbb{1}) = (\mathbb{1} \otimes \mathrm{SWAP}) \circ (\mathrm{SWAP} \otimes \mathbb{1}) \circ (\mathbb{1} \otimes \mathrm{SWAP})$$

seems like a non-trivial equation, however when written as a diagram it becomes an obvious statement:



$=$

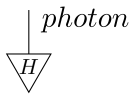


as it is immediately clear that the connectivity of these two diagrams is the same. To account for this ambiguity in the symbolic notation we must introduce a large number of axiomatic equations for the 'artificial' features we have introduced in the above procedure, i.e. for $\circ$, $\otimes$, $\mathbb{1}_A$, SWAP_{AB}, and I. This is essentially what is provided in the standard textbook definition of a symmetric monoidal category. Luckily however, we can use the diagrammatic notation and neatly avoid all of these issues by noting that two symbolic expressions are automatically equal if they correspond to the same diagram.

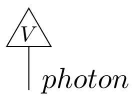
Simply put, if a theory has a process-theoretic description, then the diagrammatic representation immediately provides a huge simplification over its symbolic counterpart, eliminating the need for a large number of equations which must be introduced to deal with the ambiguity in the symbolic description. However, this does leave open the question, are process theories a useful way to describe physical theories?

To describe physical theories as process theories, we interpret systems as corresponding to some physical degree of freedom, for example, the polarisation of a photon, the temperature of a thermal bath or the phase space of a classical particle. Whilst processes are interpreted as something that can happen to that degree of freedom, or something that transforms one type of degree of freedom into another, for example, encoding the polarisation degree of freedom of a photon in the energy level of an atom.

Given this interpretation then certain processes then have particular physical significance. For example, processes with no inputs are processes that prepare a particular system in some state, for simplicity we call these *states*, for example:

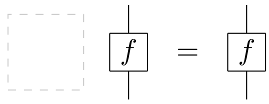


could correspond to the preparation of a photon with horizontal polarisation. Processes with no outputs correspond to the outcome of some measurement, or *effects* for short, for example:

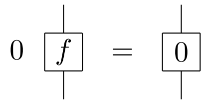


would be the process associated to measuring the polarisation of a photon and obtaining 'vertical' as the outcome. Finally processes with neither inputs nor outputs are known as *scalars* or *numbers*. They are often taken to be probabilities representing the probability of some outcome in an experiment occurring:

for example, note that if we wire together a state and an effect then we get a scalar, this would correspond to the probability of obtaining the particular effect given the particular state that was prepared. We are particularly interested in two particular numbers which have a special diagrammatic representation as they are fully characterised by their compositionality properties. The first is 'certainty', which is either written as 1 or by the empty diagram. This can be defined diagrammatically by the fact that it leaves process invariant:



Another one is 'impossible', written as 0, which 'eats' all other diagrams, in the sense that for each set of input and output wires there is a 0-process, again simply denoted by 0, and when composing any process with the 0-scalar we obtain the corresponding 0-process:



Given this interpretation of the processes as physical transformations of some physical systems, it then is clear that compositionality should be a feature of any reasonable physical theory, it is simply what we do when building experiments. Wiring together two processes is simply taking the output of one piece of apparatus and feeding it into the input of another. In terms of the symbolic notation, sequential composition is just what happens when one first applies one transformation and then the other, whilst parallel composition is applying two transformations to two distinct systems at the same time. It therefore seems natural to expect that a physical theory should be a process theory, and so, as mentioned earlier, the process theoretic description should be much simpler on a notational level if nothing else.

2.1.1 Example process theories

For our purposes we are interested in process theories representing descriptions of physical theories, as such, we describe some key examples in this section. However, this should not be taken as exhaustive of the uses of process theories, indeed as mentioned earlier they have also been used for linguistics [60, 12], cognition [6, 33, 57], and the categorical approach has very wide applicability in computer science and mathematics [50].

Note that in the following examples for now will not talk about measurements, just states and how states evolve. We will address this apparent limitation in the next section.

Example 2.1.1 (Classical probability theory)**.** *We consider a model of classical probabilities over finite sets as a process theory. In particular, systems correspond to natural numbers* $n \in \mathbb{N}$ *corresponding to the size of the set, composite systems are given by the product of natural numbers, i.e.* $n \otimes m = nm$*. Hence, the 'nothing object' is represented by* $n = 1$*.*

Processes $f : n \to m$ then correspond to $n \times m$ stochastic matrices (i.e. matrices with non-negative real entries such that each column sums to 1). Sequential and parallel composition are provided by standard matrix multiplication and tensor product respectively.

To provide some more intuition for this process theory note that states are $1 \times m$ stochastic matrices, and so correspond to m dimensional vectors with elements p_i satisfying $p_i \geq 0$ and $\sum_i p_i = 1$. In other words, states of m are probability distributions over an m element set. Parallel composition of a pair of states then corresponds to the product distribution, however, note that not every state on the composite system is of this form as correlated states are also part of the theory.

General processes in this theory then can be seen as stochastic evolution of the states. Linearity of the evolution ensures that convex mixtures are preserved by processes, and stochasticity ensures that probability distributions are mapped to probability distributions.

Example 2.1.2 (Quantum theory)**.** *Quantum theory can be described as a process theory where we take systems to be finite dimensional Hilbert spaces $\mathcal{H}, \mathcal{K}, \ldots$ with the trivial system as $\mathbb{C}$ and composition of systems by the standard tensor product $\mathcal{H} \otimes \mathcal{K}$.*

Processes with input $\mathcal{H}$ and output $\mathcal{K}$ correspond to completely positive trace preserving (CPTP) maps

$$\xi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$$

where sequential and parallel composition are respectively given by sequential composition and the tensor product of linear maps.

States in this theory are therefore CPTP maps $\rho : \mathcal{B}(\mathbb{C}) \to \mathcal{B}(\mathcal{H})$ which are in one-to-one correspondence with density operators on $\mathcal{H}$ i.e. quantum states. Linearity of these processes ensures that convex decompositions of density matrices are preserved, and trace-preservation and complete-positivity ensure that states (including composite states) are mapped to states.

Example 2.1.3 (Classical possibilistic theory)**.** *Possibilistic classical theory can be described in the same way as the probabilistic case, but where we take the elements of the matrices to be Boolean valued. States are therefore possibility distributions over the finite sets, with a Boolean value of 1 indicating possible and 0 impossible. General processes are then maps that preserve possibilistic mixtures and map possibility distributions to possibility distributions.*

Example 2.1.4 (Modal quantum theory)**.** *Modal quantum theory is another possibilistic theory described in [127, 128] and provides another useful example of a* *process theory. Modal quantum theory is defined for any particular prime number p such that systems correspond to vector spaces over prime finite fields,* $\mathbb{Z}_p^n$ *where n is the dimension of the vector space. Composition of systems is given by the vector space tensor product and the trivial object is* $\mathbb{Z}_p$*.*

Processes are then maps between the associated subspace lattices

$$\xi : \mathcal{L}(\mathbb{Z}_p^n) \to \mathcal{L}(\mathbb{Z}_p^m)$$

such that ξ *preserves the join* $\xi(x \vee y) = \xi(x) \vee \xi(y)$ *of the lattice and respects the bottom element (i.e. the zero-dimensional subspace)* $\xi(x) = \perp \iff x = \perp$*.*

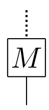
2.2 Classical interface

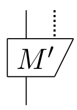
To discuss measurements for a theory we will now introduce a classical interface which allow us to describe how we interact with the world and perform 'experiments' within the process theory. We say that a theory has a classical interface if three conditions are satisfied:

1. there is a classical sub-theory,
2. all classically controlled processes exist,
3. and processes can be characterised by finite tomography.

Definition 2.2.1 (Classical sub-theory)**.** *To begin discussing a classical interface we must have a process theory that contains classical theory as a sub-theory, where we define classical theory in the sense of example 2.1.1. By a sub-theory we mean that there exists a subset of systems such that when we restrict to processes with inputs and outputs in this subset we obtain classical theory. For convenience of notation we distinguish these classical systems by denoting them with dotted wires, e.g.:*

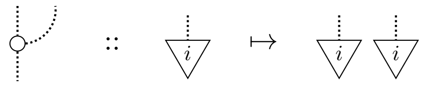
Given this classical sub-theory we can now describe measurements as processes that have a classical output, which can be thought of as the degree of freedom associated to some classical pointer on some experimental measurement apparatus. Destructive and non-destructive measurements are therefore represented by the processes



and


respectively.

Examples 2.2.2. *Classical probabilistic theory (example 2.1.1) clearly satisfies this assumption, the sub-theory is just the theory itself. Measurements in classical theory are then really just the encoding of the state into a new system. For example the map,*



(2.1)

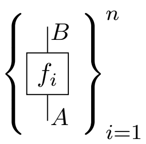
where i are the classical point distributions forming a basis for the system, can be seen as measuring the state of the ingoing system in a non-destructive way, and encoding this information in the new system.

In quantum theory as defined in example 2.1.2 there is not a classical sub-theory, and so to define measurements we must first adjoin on some extra classical systems. We will show how this can be achieved in chapter 3, but the end result of this is that destructive measurements correspond to POVMs where the outcome of the measurement is encoded in the outgoing classical system.

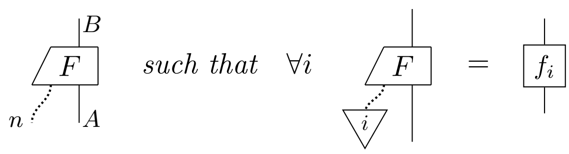
The two possibilistic theories do not have a classical sub-theory as described above, but if rather than demanding that the sub-theory is probabilistic classical theory we allowed for possibilistic classical theory then we can – similarly to the quantum case – adjoin on classical possibilistic systems to Modal Quantum Theory to allow for a description of measurements.

The next two parts to the classical interface are standard operational assumptions about how we do physics. Classical control describes our influence of the world, it formalises the idea that we can use some classical randomness to choose which process to implement, say by rolling a die and then choosing which piece of experimental apparatus to use as a consequence.

Definition 2.2.3 (Classically controlled processes)**.** *A process theory has classically controlled processes if for any set of processes*



there exists a process

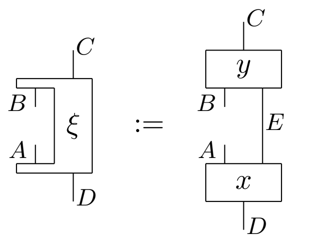


(2.2)

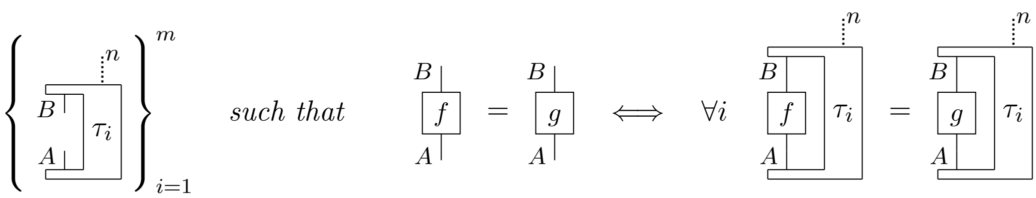
where again these classical states i are the point distributions forming a basis for the *system.*

Finite tomography describes how we can learn about the world, stating that processes can be characterised by the probabilities obtained in experiments. The finiteness condition demands that it is never necessary to perform an infinite number of distinct experiments to characterise a process, and so, the set of probabilities needed to describe a process is finite. Note that this would not be true of the (standard presentation of) infinite dimensional quantum theory or of quantum field theory. Yet, in practice, we can also never perform an infinite number of distinct experiments, and so rather than viewing this as a limitation on the theory we can view this as an experimental limitation, our operational description of the theory should reflect this by taking appropriate equivalence classes of the fundamental processes.

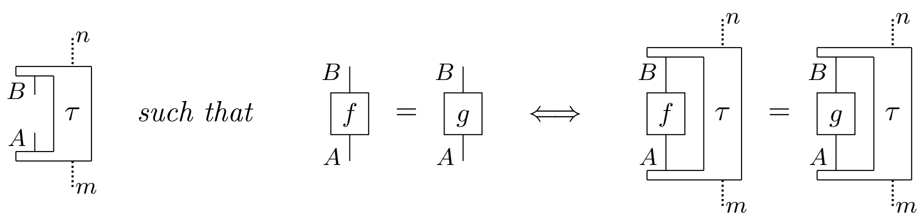
Before formally defining finite tomography we introduce some shorthand notation, that is, we denote a *circuit fragment* as:



Definition 2.2.4 (Finite tomography)**.** *For all pairs of systems* (A, B) *there exists a finite set of tests*



Note that if we have classically controlled processes then we can write this in a cleaner form, that there exists a controlled test

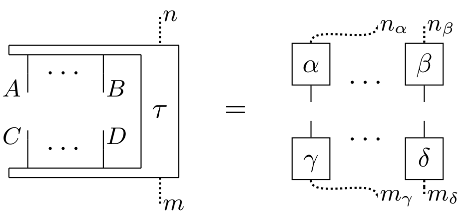


(2.3)

where finiteness demands that n *and* m *are finite.*

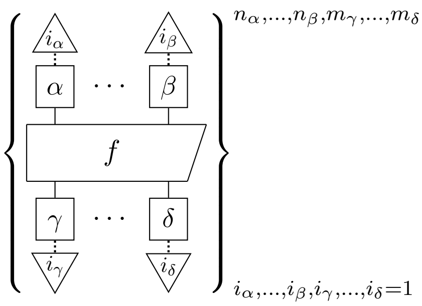
There is a commonly used strengthening of this notion of tomography which requires that it can be performed *locally.* This expresses the idea that, although we know the world to be non-local (in the sense of [27]), that there are still no holistic degrees of freedom, and that the description of two distinct regions of space can be formulated entirely in terms of their individual properties and the correlations between them. Alternatively, one can consider the ability to be characterised locally as the defining feature of what we mean by a system, it is something that we can study in its own right independent of the rest of the world. The notion of tomographic locality was first used in reconstructions by Hardy in [76], but was identified as a property of quantum theory much earlier [7, 29].

Definition 2.2.5 (Tomographic locality)**.** *Locality implies that τ in the definition of process tomography can be chosen such that it factorises over parallel composition:*



(2.4)

where $n = n_\alpha ... n_\beta$ and $m = m_\gamma ... m_\delta$. Note that this implies that we can characterise any process f by the set of scalars[5]:



and so, for example, if we wanted to characterise a bipartite state A, B and we needed k_A and k_B scalars to characterise the single system states then we would need $k_A k_B$ scalars to characterise the bipartite state [79].

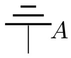
Note that we will in general not take this stronger notion of tomography as part of our classical interface and will make sure to specify when it is being used. In particular, tomographic locality is not necessary for the following key consequence of this classical interface (discussed in the next section), that is, any theory with a classical interface must be *causal.*

[5]Note that because we are using the non-causal classical effects in this diagram the scalars will similarly be non-causal and so will be positive real numbers.

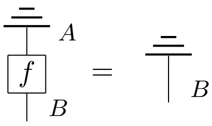
2.3 Causality

A key feature of a process theory is whether or not it is *causal*. A causal process theory satisfies the no-signalling principle [53] and implies that information can only propagate from present to future [42]. The notion of causality that we use was introduced in [41], although we adopt a modified form that applies to the general process theories of [56].

Definition 2.3.1 (Causal process theories)**.** *A process theory often comes with a discarding effect for each system which provides a way to 'throw away' systems. We denote this by:*



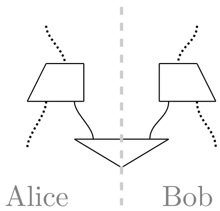
A process, f, in such a theory is said to be causal *[41, 42, 55] if it satisfies:*



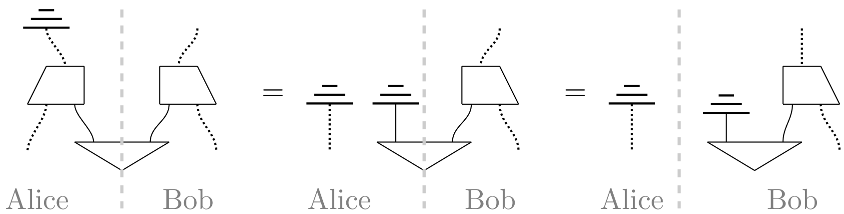
(2.5)

A theory with such discarding effects is then said to be causal if all processes in the theory are causal. Note that this automatically implies that in causal theories the only effects are the discarding effects themselves.

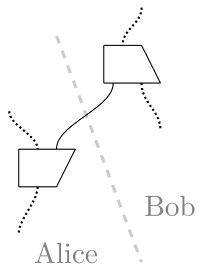
This seems like a somewhat odd definition of causality so to understand this in more detail we derive some consequences of this condition. Consider the scenario where Alice and Bob each share a system from a bipartite state, and each locally perform some measurement dependent on some classical input.



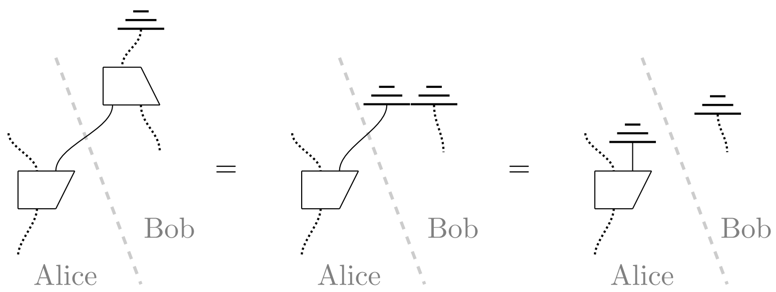
Then, if Bob does not have access to Alice's input or output then there can be no signal transmitted as the diagram separates:



Theories satisfying causality therefore satisfy a no-signalling theorem [53]. Similarly if we consider the scenario where Bob is in the causal future of Alice, such that there is some system transmitted between them, i.e.:



Then Bob cannot signal backwards in time to Alice as we again find that the diagram separates:



In general we can see that causality implies that the causal structure implicit in the input-output structure of the diagrams becomes an explicit constraint on the observable probability distributions such that there is no signalling without the transfer of a physical system, and such that there is no signalling backwards in time. See [91] for a more formal (and general) proof of this result.

Examples 2.3.2. *It is simple to show that all of our example process theories are causal, they all have a unique effect for each system which must be deterministic as the only scalar in each theory is* 1*. In classical probabilistic theory the discarding effects are given by the* $n \times 1$ *matrix (i.e. covector) where every element is* 1 *and so applying this to half of a bipartite state corresponds to marginalisation, causality of the processes is then implied by stochasticity. In quantum theory the discarding effects are the (partial) traces, and so when applied to half of a bipartite state we obtain the reduced density matrix, causality of processes is then guaranteed by trace-preservation. In classical possibilistic theory the discarding effects are the same as the probabilistic case (just with* 1 *from the Booleans rather than reals), causality is guaranteed by the demand that possibility distributions are mapped to possibility distributions, i.e. the zero state is not a valid state. Finally, in Modal Quantum theory the discarding effects maps from the subspace lattice to the two element lattice such that the bottom element is mapped to the bottom element and everything else is* *mapped to the top, causality of processes then corresponds to the fact that* $\xi(x) = \perp \implies x = \perp$.

We mentioned earlier that we want to interpret effects as measurement outcomes, and so clearly in most theories we expect to have more than one measurement outcome – we just do not expect these other outcomes to happen deterministically. If a process theory includes these effects then it cannot be a causal process theory. This is why we, when introducing measurements in the previous section, discussed measurements as a whole rather than the individual outcomes. We did this by describing measurements as processes with a classical output. These can be causal processes, indeed, in the quantum case, the causality constraint is then nothing more than the requirement that POVM elements sum to the identity. It is however in practice very convenient to be able to work directly with effects, we show how this can be done in the following section.

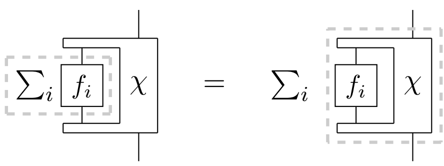
2.3.1 Acausal process theories and sums

It is often inconvenient to work with causal process theories and instead is often simpler to work with an 'acausal' extension of the theory. We will however generally still want to have a designated discarding map for each system as this allows us to return to the operationally meaningful causal theory by restricting ourselves to the processes satisfying equation 2.5.

Examples 2.3.3. *We can extend classical probabilistic theory by dropping the stochasticity requirement, such that processes are matrices valued in the non-negative reals. Quantum theory can be extended by dropping the trace-preservation requirement allowing for arbitrary completely positive maps. Classical possibilistic theory can be extended by allowing for zero-processes i.e. matrices where every element is a zero. Finally, modal quantum theory can be extended by weakening the constraint on the bottom element to be* $\xi(x) = \perp \Leftarrow x = \perp$.

One of the main benefits of allowing for acausal processes is that it allows us to define *sums* of processes.

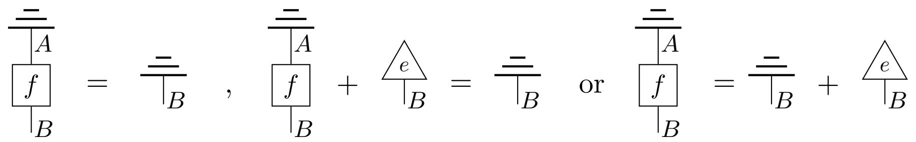
Definition 2.3.4. *A sum of processes is a binary operator* $+$ *such that for any pair of processes* $f, g : A \to B$ *we obtain a new process* $f + g : A \to B$ *such that* $+$ *is associative, commutative, has a* 0 *element, and, moreover, it must satisfy:*



(2.6)

for any circuit fragment χ. That is, the sum distributes over diagrams.

Given a theory with discarding maps and these sums we can then classify processes in this theory according to whether they are causal, sub-causal or super-causal respectively as



note that in some process theories it is also possible for processes to satisfy none of these.

Examples 2.3.5. *In (the acausal extension of) classical probability theory, quantum theory and classical possibilistic theory a sum of processes is provided by the standard sum of linear maps. Whilst in modal quantum theory we define $f + g$ by its action on states $(f + g)(x) := f(x) \vee g(x)$.*

Throughout this thesis we therefore will often be working with these acausal extensions both for the sake of performing calculations and for understanding the compositional structure, but as mentioned above, we can always return to the causal theory to understand the operationally realisable part of the theory. In fact, as we will now show, a theory must be causal for it to be compatible with the operationally motivated classical interface that we introduced previously.

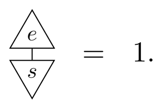
2.3.2 Causality from a classical interface

We are now in a position to understand the connection between causality and the classical interface.

Proposition 2.3.6. *If a theory has a classical interface, that is, has all classically controlled processes and all processes can be characterised via finite tomography then the theory is causal.*

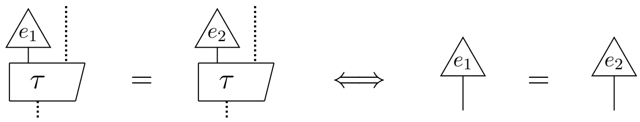
Proof (adapted from lemma 7 in [41]). In this proof we show that by demanding causality of the classical processes that we obtain a notion of causality for *all* processes.

Firstly, note that scalars are always classical (as the trivial system I must also be the trivial system for the classical sub-theory) and hence, the only causal scalar is 1. Therefore for all states s and effects e we have:

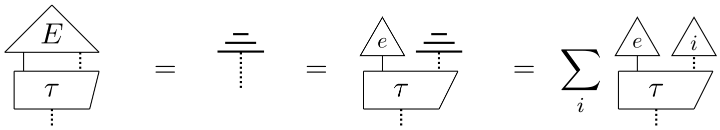


(2.7)

In other words, we know that all effects e in the theory are deterministic, to prove that the theory is causal it just remains to check that there is a *unique* effect for each system. We check for uniqueness via tomography of effects. Finite Tomography (definition 2.2.4) implies that there is some test τ such that:

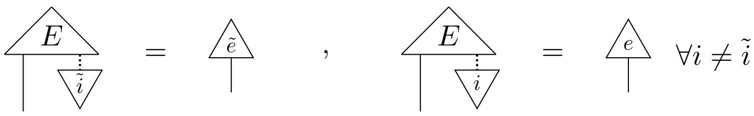


Now, note that as all classical processes are causal the only classical effect is the discarding map and so we find:

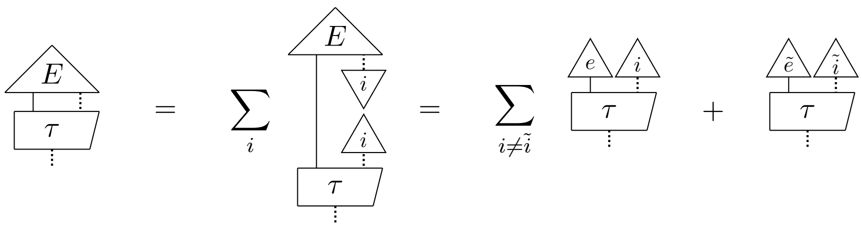


(2.8)

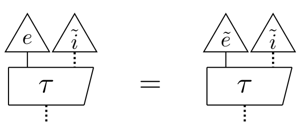
where the last equality is given by decomposing the classical discarding map into a sum over basis elements. This is true for any effects e and E, but now let us pick a specific choice of E which exists due to Classical Control (definition 2.2.3):



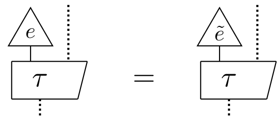
Now, by decomposing the classical identity as a sum over rank-1 projectors and using the definition of E we obtain:



Therefore, by comparing this to equation 2.8 we find that:



There is nothing special about $\tilde{i}$ so we could equally prove this for any i just by considering different E, therefore, we obtain:



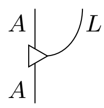
and so, by the definition of finite tomography this means that $e = \tilde{e}$. As these could be arbitrary effects we therefore have a unique effect for each system, which, by equation 2.7 must be deterministic. Hence, by definition 2.5 the theory is causal. □

It is clear that what is happening in this proof is that we are using the fact that probabilistic classical theory is causal, along with the classical interface to give us a notion of causality for the whole theory. However, in classical theory we often work with the non-causal extension, we can likewise do the same thing here. We can then see that processes are sub-causal if they give rise to sub-causal classical processes under tomography, or similarly super-causal if they give super-causal classical processes under tomography. In particular this gives the only causal scalar as 1, the sub-causal scalars as $[0, 1)$ and the super-causal scalars as $(1, \infty]$.

2.4 Leaks

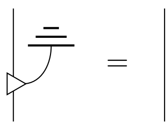
Another important feature of a process theory are the leaks for a theory [130], in particular these are necessary for defining a process-theoretic notion of purity and also can be used to distinguish between quantum theory and classical theory as we show later.

Definition 2.4.1. *A* leak *is a process:*



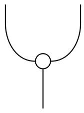
(2.9)

which has discarding as a right counit, that is:

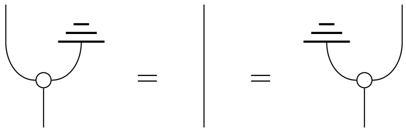


(2.10)

This can be seen as one half of the defining equations for a broadcasting map, or conversely, a broadcasting map is a map that leaks in both directions. That is, a broadcasting map, denoted as:



is one that satisfies:



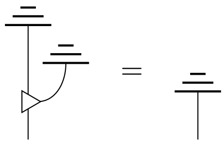
(2.11)

Note that classical theory has a broadcasting map as defined in equation 2.1, and so has leaks.

We now consider some basic properties of leaks.

Proposition 2.4.2. *All leaks are causal.*

Proof. Causality of a leak means:

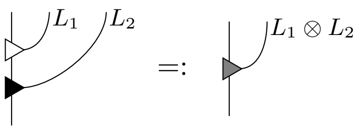


and this equation is obtained by discarding the outputs in (2.10). □

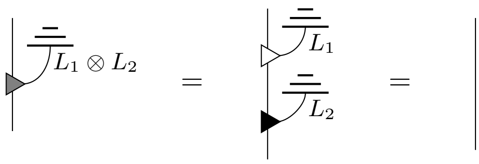
When we have multiple leaks around we may often represent them with different colours to distinguish them.

Proposition 2.4.3. *Leaks compose to give leaks.*

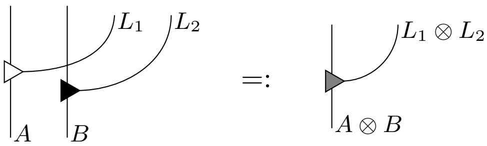
Proof. Sequential composition of leaks is again a leak:



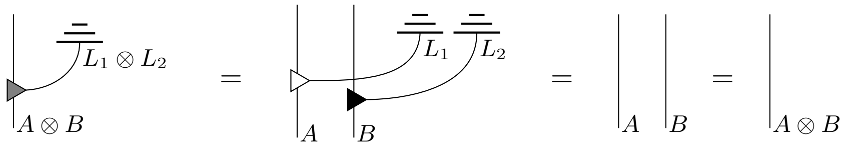
since we have:



and the same goes for parallel composition:

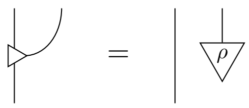


since we have:

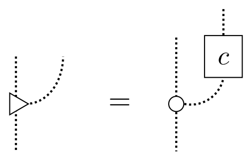


□

Examples 2.4.4. *In quantum theory and modal quantum theory all leaks are trivial, that is we find that:*



where ρ is an arbitrary causal state. Whilst in classical theory (either probabilistic or possibilistic) all leaks can be written in the form:



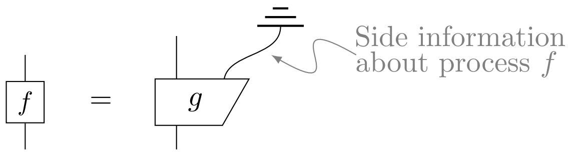
where c is an arbitrary causal process and the white dot is the broadcasting map defined in equation 2.1.

This provides a clear separation between quantum and classical theory, as we can see —at least qualitatively— that quantum theory is minimally leaking whilst classical theory is maximally leaking. This can be made a more qualitative statement by defining the 'quality' of a leak see [130]. However, for our purposes here we are more interested in the role that leaks play in defining process-purity.

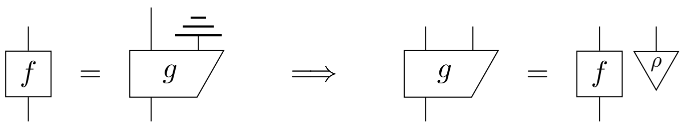
2.5 Purity of processes

In this section we consider how leaks relate to purity in process theories [130]. The purity (or lack of purity) of a state is a fundamental concept in quantum theory and is equally important in most approaches to generalised physical theories. However, there is no reason to consider this as solely a property for states but should be considered for all processes in a theory. Indeed, lack of knowledge about a process, noisiness of a channel and detection errors on an POVM-element all correspond to process-impurities. We will show that defining such a property for general theories, and classical theory in particular, requires leaks.

In [43] Chiribella et al. introduce the notion of side-information, this can be thought of as information that is lost during a process which —in principle— could be possessed by some other agent. The use of this in cryptographic scenarios is clear, where the side-information can be thought of as being possessed by an eavesdropper attempting to influence or gain information about some cryptographic protocol. Diagrammatically this side information is depicted as:



Lack of side-information for a process, would imply that g must separate such that the side-information is independent of the process f. Indeed, this must be the case for any such g, i.e.:

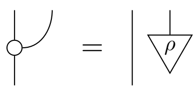


(2.12)

Or in other words, all dilations of f must separate. Separability of dilations, has been proposed as a definition of process-purity. Indeed for the case of quantum theory this corresponds to the expected notion of purity, that is, that the CP map must be Kraus rank 1. Remarkably however, in the form (2.12) this definition doesn't extend to general processes of classical probability theory. In fact, nor does it do so for any theory that has broadcasting:

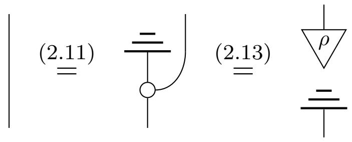
Proposition 2.5.1. *If a non-trivial theory has broadcasting, and one defines purity by means of (2.12), then plain wires (i.e. identity processes) aren't pure.*

Proof. Assuming identities are pure, and applying (2.12) to the defining equation of a broadcasting map (2.11) we obtain:

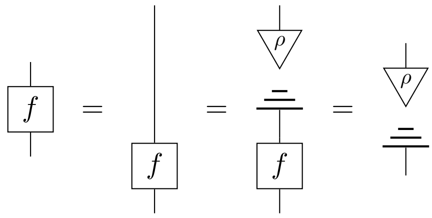


(2.13)

that is, it is a constant leak. But then from the second defining equation of broadcasting we obtain:

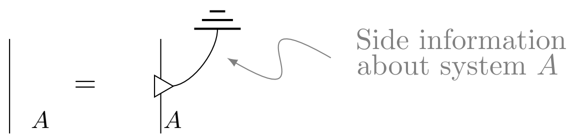


that is, each plain wire is a constant process, and hence the theory is trivial since as a consequence all processes must then be constant since for (causal) processes we have:

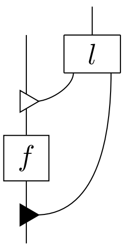


Hence, in a non-trivial theory with broadcasting identities cannot be pure in the sense of (2.12). □

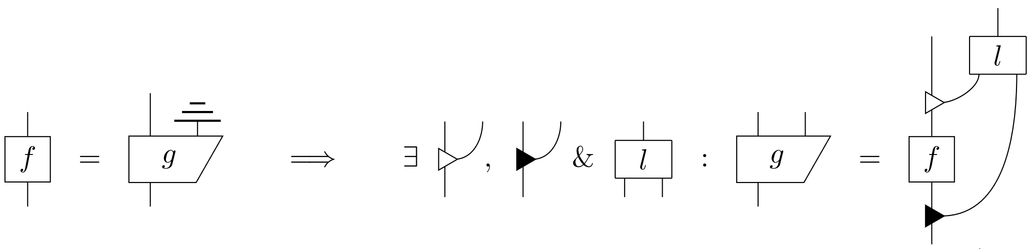
From the first part of this proof, namely that this definition of purity implies that leaks must be constant, it follows that this issue arises in any theories with non-constant leaks. We can think of this as the fact that, if a system has a leak, then there is irreducible side-information contained within the system itself:



Fortunately, leaks also allow us to fix this problem. Firstly, let us suppose that a theory has leaks, and also has a pure process f. Then, clearly the following is a dilation of f:

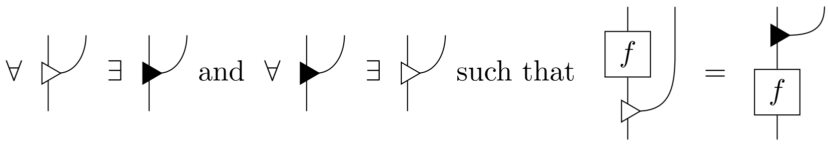


where l is causal. One may therefore consider explicitly bringing leaks into play in the definition of purity. A first step in this direction is to weaken (2.12) as follows:



(2.14)

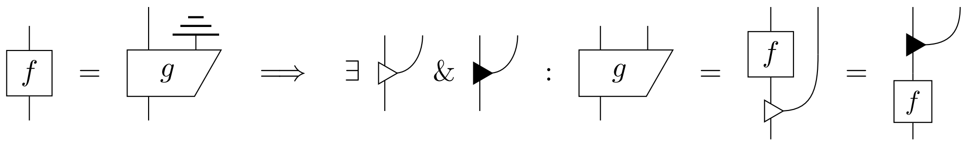
However, now we have the opposite problem: all classical processes, including all states, are pure! It is clear that we are missing a constraint. The original idea was that for a process to be pure it should have no side-information that some eavesdropper could take advantage of. However, we have shown that for some systems there is irreducible side-information represented by leakage. Therefore to ensure that the eavesdropper cannot gain information or influence the process we must demand that the process does not interact with this irreducible side-information, such that leaking before or after are equivalent:



(2.15)

Hence, we propose the following definition of process-purity which packages these two conditions, 2.14 and 2.15, into a neat form:

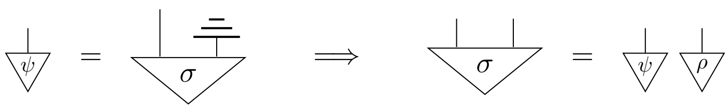
Definition 2.5.2. *f is pure if and only if*



(2.16)

This, ensures that the only side-information is this irreducible kind i.e. system leakage, and moreover, that pure processes do not interact with this irreducible side information. To further motivate this definition we will show that it provides a sensible definition for quantum, classical and composite systems. But first, note that for states this definition reduces to:

Example 2.5.3. *A state ψ is* pure *if we have:*



This is the same as the previously proposed definition based on equation 2.12, and so we see that it is only for general processes that we must take leaks into account and that this new definition is necessary.

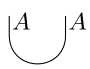
Examples 2.5.4. *In quantum theory as we mentioned earlier all leaks are trivial and hence this definition reduces to equation 2.12 which is equivalent to the processes having Kraus rank* 1*. In classical theory (probabilistic or possibilistic) this definition implies that the matrices have at most a single non-zero element in each row and in each column.*

Remark 2.5.5. *A closely related definition is given in [64] based on the categorical notion of a 'weak factorisation system' and taking inspiration from the Stinespring* *dilation theorem. The precise connection between these two definitions is a subject of ongoing work, however, the main distinction between these two approaches is whether or not we demand that pure processes are closed under composition. This is not guaranteed by the definition provided here but is in [64]. Whether or not this should be the case is a matter for debate, but it is worth noting that this is closely related to the 'purity preservation' postulate of [45] which is a non-trivial constraint that requires that the pure processes are closed under composition.*

2.6 Relaxing constraints on composition

In section 2.1 we introduced how processes can be composed to form diagrams, however, this composition of processes was subject to certain constraints. In this section we explore what happens when one of these constraints is relaxed, relaxing the other constraints is a matter for future work. Specifically, we want to remove the assumption that outputs must be connected to inputs to allow for a freer notion of composition. This is clearly going to be in conflict with the notion of causality that we introduced in the previous section, as such, we will for now consider the acausal extension of our example theories.

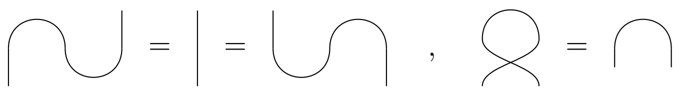
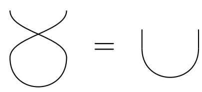
Definition 2.6.1 (String diagrams)**.** *To connect inputs to inputs and outputs to outputs we require a wires of the shape:*



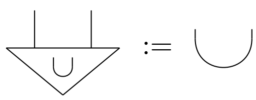
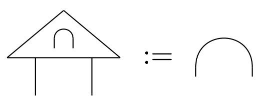
and


respectively.

as only the connectivity matters these satisfy the obvious equations of:


and


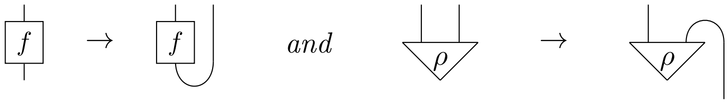
Remark 2.6.2. *Note that rather than thinking of this as relaxing the constraints on composition for process theories, we can instead 'internalise' these wires into a process theory as a particular bipartite state (called a* cup*) and effect (called a* cap*) for each system:*


and


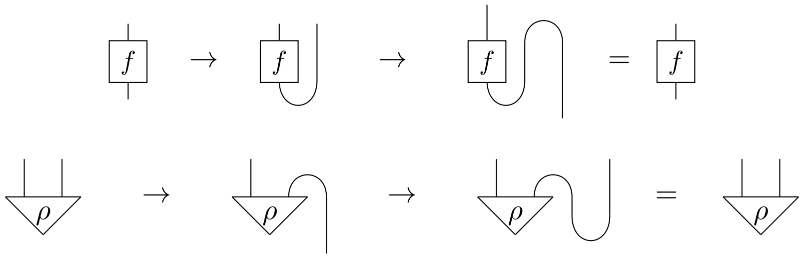
which also must satisfy the equations presented in the above definition. This is the notion of compact structure that has been part of the CQM since its conception [4].

Examples 2.6.3. *In quantum theory this looser form of composition is closely re-*

lated to the Choi-Jamiolkowski isomorphism between completely positive maps and bipartite (acausal) states. More precisely, this isomorphism is a special case of these string diagrams, the two directions of the isomorphism are provided by:

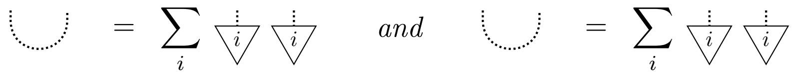


where it is simple to check that this is indeed an isomorphism:



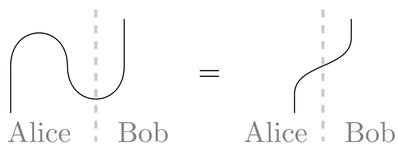
More concretely, we can take the cup and the cap to each be super-normalised Bell states and effects respectively, in Dirac notation represented as $\sum_{ij} |ii\rangle\langle jj|$.

In classical theory these correspond to the super-normalised perfectly correlated state and effect, i.e.

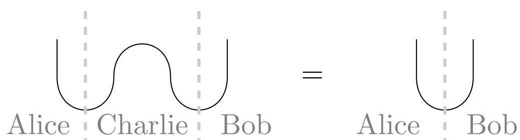


where the states i correspond to classical point probability/possibility distributions and the associated effects. It is simple to verify that these satisfy the required equations.

It is these string diagrams that really capture the flow of information in quantum information protocols, for example, the teleportation protocol mentioned in the introduction:



where this diagram shows that Alice and Bob sharing a Bell state followed by Alice post-selectively obtaining an Bell outcome in a measurement, is equivalent to Alice and Bob sharing an identity channel. Another simple example is entanglement entanglement sharing:

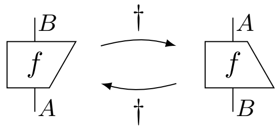


where Alice and Bob both share a Bell state with Charlie, who then post-selectively obtains a Bell outcome in a measurement leading to Alice and Bob sharing a Bell state. These are clearly not the full protocol as this involves measurements, classical communication and corrections, but, this post-selected version captures the core of these schemes in a concise and intuitive way.

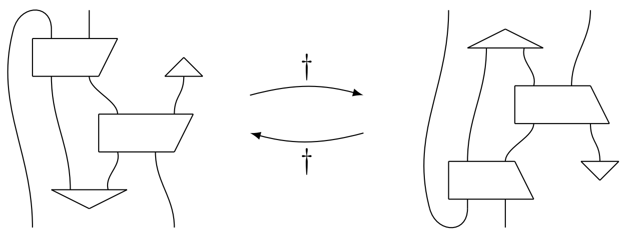
2.7 Sharpening the dagger

Another key feature of many process theories of interest is the existence of a *dagger*, like the string diagrams/compact structure of the previous section this has been part of the categorical formulation of quantum theory from the beginning, at least, in the sense that besides transposition there has to be another involution that captures conjugation. Initially conjugation was taken as the primitive [4], and later it became the dagger [132]. Again, Like the string diagrams of the previous section, this weakens the distinction between input and outputs for a process.

Definition 2.7.1 (Dagger)**.** *A dagger provides a way to interchange forwards and backwards in time propagation, swapping inputs and outputs. It is defined as a reflection of processes:*



where the asymmetry of the processes has been introduced to make the reflection clear. Which moreover acts on diagrams as a whole:

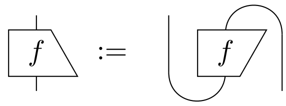


(2.17)

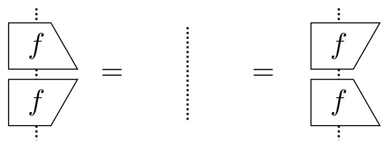
Intuitively, the dagger is acting as a time-reversal operator, swapping forwards in time propagating processes for backwards in time. However, as we will discuss below, the above definition is not sufficient to fully capture this notion.

Remark 2.7.2. *Note that if we are interpreting the compact structure as a particular state and effect within the theory, then we need to make explicit that the dagger maps the cup to the cap. Similarly, if we consider the wires as identity transformations, and a crossing of wires as a swap transformation, then these are taken to be invariant under the dagger.*

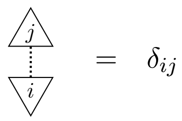
Note that we do not represent the dagger in terms of processes in the theory, but, have it as an 'external' structure. There are some daggers that can indeed be represented internally, but, as we will discuss, these are not always the ones that we are interested in. One way to internally provide a dagger is via the string diagrams defining:



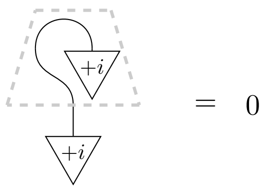
and in the case of classical theory this has many desirable properties for a 'time-reversal operation'. In particular, if a process $f : n \to n$ is reversible then this dagger provides the inverse process:



Moreover, for point distributions this dagger provides the effects that perfectly distinguish them:



However these properties don't hold more generally, in particular in the case of quantum theory we find that:

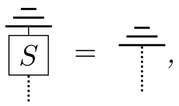


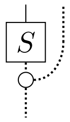
where we take the cap to be the Bell effect $\sum_{ij} |ii\rangle\langle jj|$ and the $+i$ state defined as $|{+i}\rangle\langle{+i}|$ where $|{+i}\rangle := \frac{1}{\sqrt{2}} |0\rangle + i\,|1\rangle$[6]. Similarly it does not provide the inverse of reversible processes.

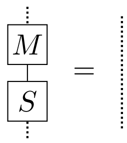
Luckily though, in quantum theory there are other possible daggers for the theory which are more suitable as being interpreted as giving the time-reverse of processes. We therefore need to find some extra constraints to identify the relevant dagger, that is, we must *sharpen* the dagger. This is closely related to the logical sharpness axiom of Hardy [80] and related works [46, 129]. To define this sharpness condition we must first define the *testability* of a state preparation procedure.

[6]Essentially, this is because quantum theory has no perfectly correlating state, for example, the Bell state correlates in the X and Z directions but anticorrelates in Y.

Definition 2.7.3 (Testability)**.** *A state preparation $S : n \to A$ is testable, if it satisfies three conditions:*

1. 

2.  *is pure*[7]*, and*

3. *there exists M such that* 

such a testable state preparation is maximal *if n is the largest value such that a testable state preparation exists.*

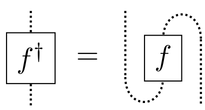
In quantum theory a testable state preparation corresponds to a set of pure, normalised and perfectly distinguishable states, where maximality then means that they form a basis for the underlying Hilbert space.

Given this notion of testability we can define a sharp dagger for theories with a classical interface as follows.

Definition 2.7.4 (Sharp dagger)**.**

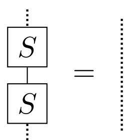
1. *Is a dagger in the sense of definition 2.7.1*

2. *Compatibility with the classical dagger,*



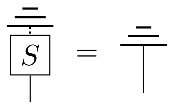
(2.18)

3. *Sharpness, if a state preparation S is testable (definition 2.7.3) then*



[7] One might assume that we should simply demand that S is pure, however, there are no pure processes with only a classical input and only a quantum output [130]. Instead, demanding purity of this process guarantees that each state of the classically controlled state preparation is a pure state.

moreover, if S is maximal (see definition 2.7.3) then



Examples 2.7.5. *The dagger defined in terms of the string diagrams can easily be seen to be the sharp dagger for classical theory. Whilst for quantum theory the sharp dagger is provided by the Hermitian adjoint arising from the trace inner product on states.*

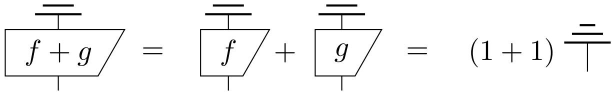
This, for the case of quantum theory, manages to capture what it means for a dagger to be a time-reversal operator, however, it seems like there should be a more general principle underpinning this. In particular, there should be a more general principle that holds beyond theories with a classical interface. What this principle is remains ongoing work but, hopefully, will provide a more intuitive definition than the somewhat ad-hoc one given above.

Remark 2.7.6. *Note that this 'sharpening' of the dagger is closely related to the definition that we gave in [129] but here, rather than considering the sharpness condition for just single states, we consider more general state preparation procedures instead.*

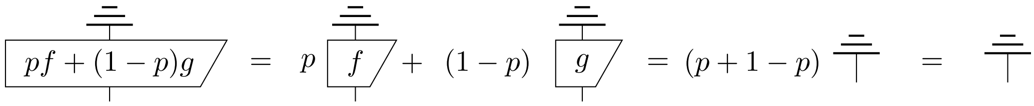
2.8 Tension with causality

We have now seen three important features of process theories that are in conflict with the notion of causality, and hence, by proposition 2.3.6, in conflict with having a classical interface for the theory.

Firstly, we have sums of processes, see definition 2.3.4. It is clear that the sum of the two causal processes will not (in general) be causal:

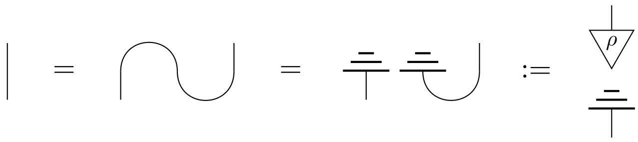


Therefore, as in most theories $1 + 1 \neq 1$, we find that $f + g$ is usually not causal. Note that this is not actually the case in possibilistic theories and so in possibilistic theories it is often the case that the sum of causal processes is again causal. However, even in probabilistic theories we see a remnant of this structure in the causal part of the theory in the form of *convexity*. That is, if we have a causal process f and a causal process g then a convex combination of these $pf + (1-p)g$ will be another causal process in the theory as:



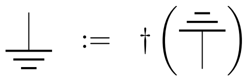
this can be seen as a probabilistic mixture of the process f and the process g with probabilistic weights p and $1-p$ respectively.

Secondly we had string diagrams. It seems obvious that these would not interact well with causality as this allows for a free notion of composition where the distinction between inputs and outputs is lost. Indeed, if we assume that the cap is causal then we can derive the following:



that is, the identity wire separates and so there can be no information flow in the theory and the theory must be trivial. Therefore, for interesting process theories and in particular for the cases of quantum and classical theory, the cap cannot be causal. Like in the case of sums however, there is an impact of this structure at the operational level, both the cup and the cap appear in a sub-normalised form in some branch of a measurement process. Taking into account this branching structure is what ultimately leads to the full teleportation protocol.

Finally, we have the sharp dagger. Again it seems obvious that this should not interact well with causality as it can be seen as reversing the arrow of time. If we did try to make the dagger compatible with causality this would mean that the dagger was a map between causal processes, as such, if we only have a single causal effect, then we also have only a single causal state:



again, we find that demanding compatibility between the dagger and causality gives us a trivial theory. As before though, we do find some sign of this process theoretic structure in the operational theory, here providing an isomorphism between the causal states and a subset of the effects in branches of measurements as well as in providing the inverse of unitary transformations.

We therefore have a tension between the classical interface for a theory giving rise to a notion of causality, with the compositional structures that we use as standard tools in Categorical Quantum Mechanics. In the generalised probabilistic theory framework it is the operational structure which tends to be given precedence with these compositional features taken as only convenient mathematical tools. However, from the process-theoretic perspective this is not so clear cut, and is suggestive of the idea that it maybe this compositional structure which is more fundamental. There are two reasons to think this, firstly, we expect that classical theory should be emergent from quantum theory and so taking classical theory and the resultant causality as a starting point seems to be pre-empting this, perhaps to gain a deeper understanding of quantum theory we need to discard this operational starting point and work towards a purely compositional framework. Secondly, foundational research in quantum gravity suggests that nature may have fundamental indefiniteness of causal structure, this freer notion of composition seems to capture this to some extent, and so again this adherence to the operational viewpoint could be a hindrance to our understanding of this deeper theory.

2.9 Conclusion

We have now introduced some of structure that is commonly use within the process theory framework providing the abstract definitions as well as examples of how this relates to quantum theory and classical theory. In particular introducing the basic compositional framework, the classical interface for a theory, the notions of causality and purity for processes. We then look to relaxing the notion of composition and how this leads to a standard part of categorical quantum mechanics known as string diagrams, it is the subject of ongoing work as to how relaxing other constraints on composition leads to new structures for the process theory. Finally, we consider another standard part of the categorical formulation of quantum theory, the dagger, and show that to be able to have an interpretation of this either in terms of providing tests for states, or in terms of time reversal symmetry, requires a *sharpening* which provides an additional set of constraints that uniquely single out the Hermitian adjoint in the case of quantum theory.

One of the odd features of our current presentation of quantum theory —both as a process theory and more generally— is that it is built on top of classical theory. Fundamentally we describe our quantum processes via tomography and hence in terms of classical probability distributions, where these measurements are best described as processes with quantum inputs and classical outputs. However, this does not mean that there are necessarily fundamentally distinct systems that are classical, instead we expect 'classical-like' systems to emerge in certain regimes. We explore a process theoretic view of this emergent classicality in the following chapter.

Another question that is posed by what we have described so far is, given this process theoretic machinery, how close are we to quantum theory? Are there any other theories with a classical interface, string diagrams and a sharp dagger? We answer this in chapter 4 showing how given two further assumptions of the theory (time-symmetric purification and tomographic locality) that one can reconstruct quantum theory from diagrammatic postulates.

A final question that we address is whether it is possible to find a process-theory that could one day supersede quantum theory, a theory that is to quantum theory what quantum theory is to classical. We explore this in chapter 5.

Chapter 3

Two roads to classicality

The goal of this chapter is to gain a process theoretic understanding of the emergence of classicality in quantum theory, to do so however, we must first complete the full description of operational quantum theory that we have brushed over so far. In example 2.2.2 we remarked that we needed classical systems to encode the results of quantum measurements, but we did not concretely define such a process theory for quantum theory. To do so we need a process theory in which we have both quantum systems as presented in example 2.1.2 as well as classical systems as presented in example 2.1.1 such that they interact in a suitable way to provide a classical interface for quantum theory. The notion of a finite dimensional C*-algebra is suitable to capture this.

Definition 3.0.1 (C*-algebra)**.** *A finite dimensional C*-algebra $\mathcal{A}$ is the direct sum of finite dimensional complex matrix algebras A_k,*

$$\mathcal{A} = \bigoplus_k A_k = \bigoplus_k \mathcal{B}(\mathcal{H}_k)$$

where $\mathcal{H}_k$ are finite dimensional Hilbert spaces. Note that there is a standard way to embed this into a quantum system as

$$e : \bigoplus_k \mathcal{B}(\mathcal{H}_k) \to \mathcal{B}\left(\bigoplus_k \mathcal{H}_k\right)$$

embedding the elements of the C-algebra as block diagonal density matrices in a quantum system, note that $e^\dagger \circ e = \mathbb{1}_{\mathcal{A}}$.*

This defines the systems for a process theory that allows us to describe quantum theory with a classical interface. Note that when there is just a single branch in the direct sum then we have a quantum system, and when we have for all k that

$\mathcal{H}_k = \mathbb{C}$ then this is embedded by e as diagonal density matrices which correspond to classical probability distributions. However, we want to not only consider systems but also processes as well. We call this process theory *Operational Quantum Theory* (OQT) which will (hopefully) be satisfactorily justified by the end of this chapter.

Example 3.0.2 (Operational Quantum Theory)**.** *Systems are finite dimensional C*-algebras (definition 3.0.1) and processes are completely positive*[1] *linear maps between these systems. If we consider the embedding of these systems into quantum systems then the processes are completely positive maps that preserve the block diagonal structure. Parallel composition is given by the tensor product of the matrix algebras, and sequential composition by composition of linear maps. The trivial system is given by* $\mathcal{B}(\mathbb{C})$*.*

Restricting to quantum systems we find that this theory is equivalent to quantum theory (example 2.1.2) and restricting to classical systems we obtain classical probabilistic theory (example 2.1.1). If we simply take these systems and their composites, ignoring all others, then this would describe quantum theory with a classical interface. Indeed, processes with a quantum input and a classical output would then correspond to POVMs; those with quantum inputs and both quantum and classical outputs to quantum instruments; and those with both quantum and classical inputs and outputs to classically controlled quantum instruments. However, as we will see in the remainder of this chapter, if we view these classical systems as emergent from the quantum systems, then it is equally natural to include all of the other C*-algebraic systems as well.

3.1 Introduction

Mixing and decoherence are both manifestations of classicality within quantum theory, each of which admit a very general process theoretic construction. In this chapter we show under which conditions these two 'roads to classicality' coincide. This is indeed the case for (finite-dimensional) quantum theory, where each construction yields Operational Quantum Theory (example 3.0.2).

We have seen in section 2.8 having sums of processes provides a way to describe probabilistic mixtures of processes by considering convex combinations. On the other hand, decoherence, which in the quantum formalism is a process that sets all off-diagonal entries of a density matrix to zero, is generalised to any causal

[1] Which if trace-preserving will be part of the causal theory otherwise they are part of the acausal theory.

idempotent[2]. That this generalisation makes sense follows from the fact that any idempotent can arise from leaking some information into the environment [130]. In the case of quantum theory, by considering more general idempotents, one not only recovers classical theory, but also intermediate ones described by C*-algebras, and, remarkably, C*-algebras only.

Each of these manifestations of classicality can be used to construct new systems to describe classical data. These are perfectly embodied by two standard universal constructions for categories namely the *biproduct completion* [109] and the *Karoubi envelope* [34] (i.e. splitting of all idempotents or *Cauchy completion*). The biproduct completion —corresponding to the case of mixing— generates classical set-like systems with the biproduct playing the role of the set-union, whilst the Karoubi envelope —corresponding to the case of decoherence— generates systems equipped with a decoherence map which 'classicise' their processes.

In this chapter we present the coincidence of these two roads for the case of quantum theory. In [61] we present a more general abstract result, the technical core of which is both a strengthening and generalisation of a theorem by Heunen, Kissinger and Selinger [86]. The key strengthening of the result is that, by drawing on a result of Blume-Kohout et al. [31], we no longer need to assume that the idempotents (i.e. generalised decoherence maps) are self-adjoint. While the passage from self-adjoint idempotents to general idempotents might seem minor, it is precisely this relaxation that allows for a clear physical interpretation which applies to any process theory.

3.2 The two roads

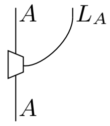
In this section we introduce the two 'universal constructions' which adjoin classical systems to quantum theory, showing that they both lead to the process theory of 'Operational Quantum Theory' i.e. C*-algebras and completely positive maps. We also consider these two constructions for our other example theories.

3.2.1 The leak construction and Karoubi envelope

Firstly we show how one can construct new process theories from old ones by creating leaks. To do this there must be processes in the theory that called *pre-leaks*:

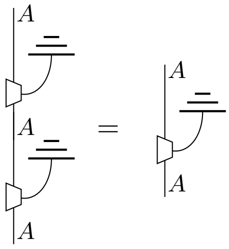
[2] That is, a causal process f such that $f \circ f = f$, setting the off-diagonal elements to zero twice is the same as doing it once.

Definition 3.2.1. *A* pre-leak *is a causal process*

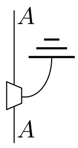


(3.1)

such that



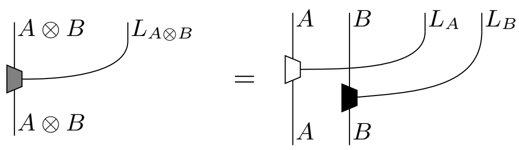
In other words, a pre-leak is a causal process for which



(3.2)

is idempotent. By inserting particular pre-leaks of the old theory on all of the wires we obtain a new theory in which the pre-leaks are leaks. Hence, the leak construction turns pre-leaks into leaks.

Theorem 3.2.2. *Given any process theory and a pre-leak for each system which are chosen* coherently *for composite systems, that is, such that for all A and B:*

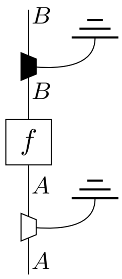


(3.3)

we can construct a new process theory in which each pre-leak becomes a leak for the associated system. This construction goes as follows:

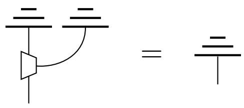
1. *systems stay the same;*

2. one restricts processes to those of the form:



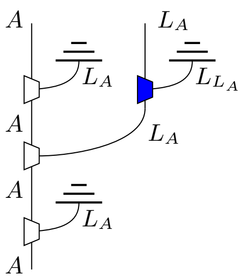
(3.4)

Proof. By causality of (3.1):

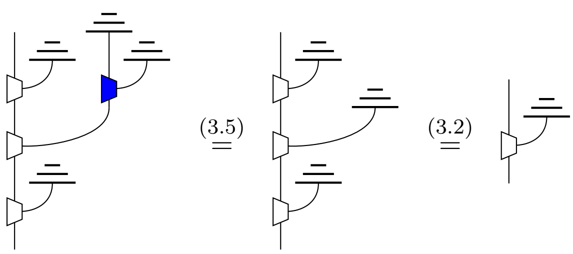


(3.5)

discarding is preserved by the leak-construction. Given the form (3.4) of the processes in the theory and due to idempotence of (3.2), plain wires have taken the form (3.2), so the defining equation of a leak (2.10) is satisfied. To consider the pre-leak in the new theory we must apply the leak construction 3.4 and using the condition for composites (3.3) we get the following process in the new theory:



which is indeed a leak in the new theory:



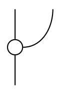
which is the form of a plain wire in the new theory, and so this construction does indeed turn pre-leaks into leaks. It is moreover straightforward to see that we again obtain a process theory. □

Sometimes the leak-construction does nothing, in particular, when the pre-leaks are already leaks:

Example 3.2.3 (Trivial)**.** *A simple example of the leak construction is the one where the pre-leaks are taken to already be leaks, since then (3.4) will reduce to the processes f themselves.*

The main motivating example for this construction is of course the following:

Example 3.2.4 (Decoherence)**.** *The leak construction for the pre-leak:*



$$: \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H} \otimes \mathcal{H}) :: |i\rangle\langle i| \mapsto |i\rangle\langle i| \otimes |i\rangle\langle i|$$

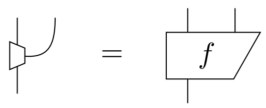
applied to the process theory of quantum processes (i.e. Example 2.1.2) we obtain classical probability theory (i.e. Example 2.1.1).

There is no good reason to limit our consideration to just a single idempotent per system, rather, we can modify construction in theorem 3.2.2, by not fixing a pre-leak for each type but rather considering all pairs of a system and a corresponding pre-leak. This is known as the *Karoubi envelope*, or *Cauchy completion*, or *splitting of idempotents*. This provides our first universal construction.

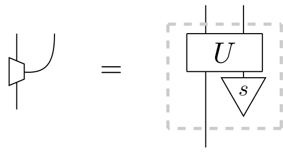
Definition 3.2.5 (Karoubi envelope)**.** *Given a process theory we can define a new process theory where the systems correspond to pairs (A, p) where A is a system in the original theory and p is an idempotent on that system. Processes in the new theory between (A, p) and (B, q) are simply given by $q \circ f \circ p$ where $f : A \to B$ is a process in the original theory.*

In the above construction it is really the idempotents rather than the specific pre-leaks which determines the theory that is obtained. We can therefore have several different perspectives on the 'cause' of this idempotent, by considering the different pre-leaks from which it could have been obtained. Firstly, we can always take the trivial case, where the pre-leak is just the idempotent itself, i.e. taking the leaked system as the empty system. There are however three alternate forms that always exist in quantum theory, and which are more insightful.

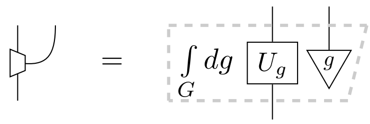
Examples 3.2.6. *Firstly we can consider the* purification *of the idempotent (in the sense of [41]) that is, we can always take the pre-leak to be some pure process f:*



This corresponds to the idea that information can never be fundamentally destroyed, only discarded, and so we can see this leaking of information into some causally separated system as the cause of decoherence. Another standard way to represent a general process is —via Stinespring dilation [136]— as a reversible interaction with an environment:



and so, we can equivalently view decoherence as arising due to a reversible interaction with some uncontrolled environment [148]. A final example, suggested to us by Rob Spekkens, is that the idempotent can be viewed as describing a system which lacks a reference frame [26] Section IVB, the leaked system would then correspond to the reference system itself:



where G is a group associated with a reference frame for a particular degree of freedom, U_g is the representation of G on the system of interest and g a basis state of the reference system that is in one-to-one correspondence with the elements of the group. Note, however, that making sense of this integral for general symmetry groups requires the reference be an infinite dimensional quantum system and so is beyond the scope of this thesis. One could replace, at least for compact groups, the integral by a finite convex mixture[3], this could then be thought of as there only being a finite set of possible orientations for the reference frame. However, a comprehensive understanding of the connections here demands further justification or consideration of the infinite dimensional case.

3.2.2 The biproduct completion

We now introduce the second way to add classicality to quantum theory. This can be seen as describing the branching structure that arises from a quantum measurement. In particular we can consider performing a measurement and then preparing a different system conditioned on the outcome of the measurement. To represent this we construct a new process-theory via the *biproduct completion.*

Definition 3.2.7 (Biproduct completion)**.** *We take systems to be lists of systems from the original theory, $(A_1, ..., A_n)$ where we can think of each system A_i as being the system prepared given a particular measurement outcome. Processes in this new*

[3]For example by using the results of [41] Corollary 33 from Carathéodory's theorem

theory are described as matrices of processes $M : (A_1, ..., A_n) \to (B_1, ..., B_m)$ *defined by:*

$$\left(f_i^j : A_i \to B_j\right)_{i=1,...,n}^{j=1,...,m}$$

where these compose sequentially by standard matrix multiplication and in parallel via the Kronecker product.

We will denote the system $(A_1, ..., A_n)$ as $A_1 \oplus ... \oplus A_n = \bigoplus_{k=1}^n A_k$ and moreover define $\oplus$ for processes as $\bigoplus_{k=1}^n f_k = (f_i \delta_{ij})_{i=1,...,n}^{j=1,...,n}$.

Note in particular that a measure and prepare set up as described above would be a map from $M : A \to \bigoplus_k B_k$ where the systems B_k correspond to the system prepared conditioned on obtaining outcome k in a measurement. More general processes $f : \bigoplus_k A_k \to \bigoplus_l B_l$ in this theory can be seen as processes that allow for an 'indeterminate' input system as well as indeterminate outputs. Therefore, just as the sum $+$ allows for mixing of processes, we find that $\oplus$ allows for 'mixing' of systems such that we can describe classical systems within the theory.

3.2.3 Comparison

We can now compare the two constructions, for a more general abstract view of this see [61], we present the result for quantum theory below.

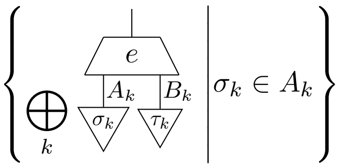
Theorem 3.2.8. *Both the Karoubi envelope and the biproduct completion applied to quantum theory (i.e. example 2.1.2) gives the theory of Operational Quantum Theory (example 3.0.2).*

Proof sketch. A complete proof is provided [61] however we will provide a sketch of the proof here.

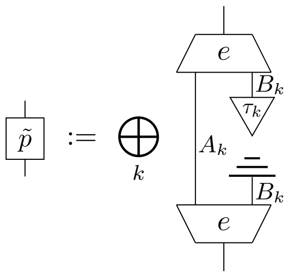
That the biproduct completion of Quantum Theory is Operational Quantum Theory immediately follows from the definition of finite dimensional C*-algebras 3.0.1 as they are direct sums of quantum systems, see Example 3.4 of [86] for details.

The Karoubi envelope is more work, firstly, note that the Karoubi envelope is nothing but the splitting of all causal idempotents, and so, the relevant data is the system over which the idempotent splits. A splitting of $p : A \to A$ over α is a decomposition $p = x \circ y$ where $y : A \to \alpha$, $x : \alpha \to A$ and $x \circ y = \mathbb{1}_\alpha$. If we have the Karoubi envelope of a theory in which $p : A \to A$ splits over α and $q : B \to B$ splits over β then the processes $(A, p) \to (B, q)$ are in one-to-one correspondence with processes $\alpha \to \beta$. Therefore, to prove our result we just need to show that every causal quantum idempotent splits over a C*-algebra such that the processes in the Karoubi envelope of quantum theory are in one-to-one correspondence with processes between C*-algebras.

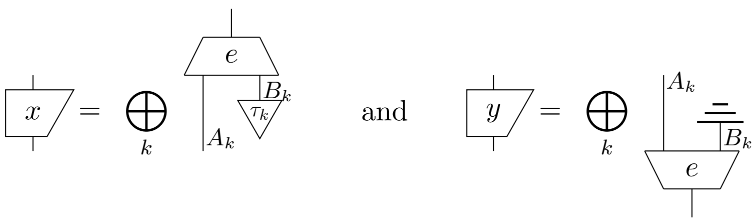
Firstly note that if we have two idempotents p and $\tilde{p}$ where $\mathrm{Im}(p) = \mathrm{Im}(\tilde{p})$ then $\tilde{p} \circ p = p$ as idempotence of $\tilde{p}$ guarantees that it must be identity on its image. Secondly, from Theorem 5 of [31], we know that the image of any causal quantum idempotent p is a 'distorted C*-Algebra', that is, a set of states of the form:



where τ_k is fixed causal state of B_k and e the embedding described in definition 3.0.1. We can define another idempotent $\tilde{p}$ with the same image:



which clearly splits through the C*-algebra $\bigoplus_k A_k$ as can be checked by defining $\tilde{x}$ and $\tilde{y}$ as:



Hence as we know that $p = \tilde{p} \circ p = (\tilde{x} \circ \tilde{y}) \circ p$ as they have the same image, then it is simple to check that p also splits as $p = x \circ y$ where $x := \tilde{x}$ and $y := \tilde{y} \circ p$. We have therefore shown that any causal idempotent splits over a C*-algebra which completes the proof. □

So despite the weak structure of a leak, for the specific case of quantum theory we obtain precisely the C*-algebras via the leak construction. This leads one to contemplate the view that the operational essence of (finite dimensional) C*-algebras is entirely captured by leaks, and that the additional structure of C*-algebras is merely an artefact of the Hilbert space representation.

Based on these two constructions it therefore seems natural not just to describe operational quantum theory as the combination of quantum and classical theory, but, to also include more general finite dimensional C*-algebraic systems. Having said that, it is still interesting to ask, are there any simple principles that single out the 'pure' quantum systems within this theory. Before turning to this question we will first consider our other example process theories, see [61] for the proofs.

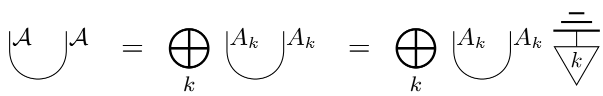
Examples 3.2.9. *For classical probabilistic theory we again find that the two constructions coincide, and moreover, leave the theory invariant. In contrast, for possibilistic classical theory and for modal quantum theory we find that the two do not coincide. In particular for classical possibilistic theory the biproduct completion leaves the theory invariant whilst the leak construction adds some extra systems.*

3.3 Separating quantum and classical theory

In this section we ask how we can characterise particular systems in Operational Quantum Theory as being strictly 'quantum' or 'classical'? We could simply take the mathematical assumption that we want to consider the commutative C*-algebras for the classical systems, or the irreducible C*-algebras as the quantum systems. But, there are two more insightful principles that also achieve this.

Proposition 3.3.1 (The existence of a pure cup restricts to quantum systems)**.**

Proof. It is simple to check that for a C*-algebra $\mathcal{A} = \bigoplus_k A_k$ that we can define the cup and cap as the direct sum of the cups and caps for the A_k, it is then simple to see that this has the following dilation:



where $\{k\}$ are some set of perfectly distinguishable states. This can only be represented as a leak on the trivial input system (necessary for purity) if k takes a single value. Hence this is only pure when the C*-algebra has only a single branch, that is, when it is a quantum system. □

This provides an axiomatic account of the following statement of Schrödinger:

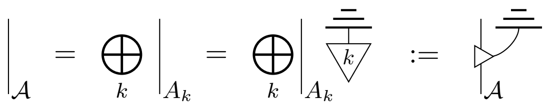
> *"When two systems, of which we know the states by their respective representatives, enter into temporary physical interaction due to known*

> *forces between them, and when after a time of mutual influence the systems separate again, then they can no longer be described in the same way as before, viz. by endowing each of them with a representative of its own. I would not call that one but rather the characteristic trait of quantum mechanics, the one that enforces its entire departure from classical lines of thought."*

That is, if known states interact in a known way then the resulting state should be pure. So it is really the fact that we have a non-separable pure state in the form of a cup, which forces us to depart the classical world.

Proposition 3.3.2 (Lack of non-trivial leaks restricts to quantum systems)**.**

Proof. For quantum theory all leaks are trivial, and for any other C*-algebra we can construct a non trivial leak by leaking the 'which branch' information. Essentially,



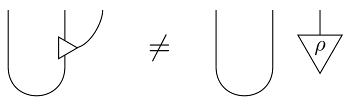
is only trivial when k takes a single value and the system is quantum. □

This can be seen as equivalent to the statement that information gain causes disturbance, a result about quantum theory that is known to explain many of its non-classical features. To make this precise, note that if a non-trivial leak exists, then the leaked system must contain some information about the ingoing system, and so by measuring it we can learn something in a way that does not disturb the original system. Conversely, if we had a measurement that did not disturb the system then this would be a leak where the outgoing system were classical.

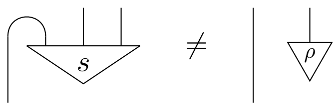
The fact that we have these two ways to characterise quantum systems is not surprising as it can generally be shown that:

Proposition 3.3.3. *Existence of non-trivial leaks* $\iff$ *Cup is mixed*

Proof. If we have a non trivial leak, then this defines a non-trivial dilation of the cup:



and so the cup must be mixed. Similarly, if the cup is mixed it has a non-trivial dilation s, which allows us to define a non-trivial leak:



□

One could ask how the classical systems can be singled out from the C*-algebras. These can be seen as those that are maximally leaking, and which have the minimally pure compact structure. Similarly, one could ask whether there is some principle that picks out the quantum systems, classical systems and their composites. One such principle is *transitivity*, that there is a reversible transformation between any two pure states of a system. This principle is used —in some form or other— in every modern reconstructions of quantum theory. However, there seems to be no good justification for including classical systems but not the other C*-algebras, and so, assuming transitivity in an operational reconstruction quantum theory seems unfounded[4].

3.4 Conclusion

We have seen that there are two natural ways to adjoin classical systems onto quantum theory. Thus allowing for a complete description of operational quantum theory, with both the usual quantum systems as well as with a classical interface. Moreover, from each of these constructions we also obtain some 'intermediate' systems corresponding to more general C*-algebraic systems. These too however have a clear operational interpretation as: the types of systems that arise from the branching structure of quantum measurements; allowing for classical indeterminacy as to which system is prepared; arising from decoherence due to interaction with an environment; leaking of information; or from the loss of some quantum reference frame. These systems should therefore also be included in our operational description of quantum theory along with the strictly quantum or classical systems.

[4]Specifically, the generalised probabilistic theory framework allows for probabilistic mixtures of states, we include states which correspond to: 'flip a coin, if heads prepare system A in state s_1 if tails prepare system A in state s_2'. If we can do this as a fundamental part of the framework then we should also allow the state which corresponds to: 'flip a coin, if heads prepare system A in state s_1 if tails prepare system B in state s_2'. This will usually, and in particular in the case of quantum theory, lead to non-transitive state spaces. Therefore assuming transitivity as a postulate in a reconstruction is contrary to the conceptual underpinnings of framework in which they take place.

Given this process-theoretic understanding of the emergence of classicality, we can imagine that quantum theory might equally well just be an effective description of the world. Can quantum theory emerge from some deeper theory in an analogous way? We return to this question in the final chapter.

First however, having seen how classical systems can emerge from quantum via these two roads to classicality, we use these classical systems, along with the compositional features discussed in chapter 2, to see how Operational Quantum Theory can be reconstructed from diagrammatic postulates.

Chapter 4

Quantum from diagrams

In this chapter we reconstruct quantum theory from diagrammatic postulates. But, why would one want to reconstruct quantum theory in the first place? The fundamental motivation for this is a dissatisfaction with the standard axioms, as their mathematical nature makes it difficult to understand *why* quantum theory is the way it is. The aim of a reconstruction is therefore to start from conceptually justified structures and axioms, and, from this, to derive the standard Hilbert space formalism. The hope being that this will allow us to derive more applications of quantum theory, and to understand how it can, and should, be modified to reconcile the theory with general relativity. Furthermore, reconstructions enable us to build toy theories other than quantum theory, which provide concrete points of comparison, contributing to the understanding of why nature is quantum.

This dissatisfaction with the standard axiomatisation is not at all new. Indeed, the first to contribute to the reconstruction endeavour was John von Neumann. A mere three years after the publication of his book [142] that cemented the mathematical formalism of quantum theory, he wrote to the mathematician Garrett Birkhoff stating that he was no longer satisfied with the Hilbert space formalism [125]. His original goal however was not so much to simply reconstruct quantum theory, but, to find a different formalism that may also allow for new physics. The actual reconstruction programme building on this was outlined by George Mackey [110], and mostly completed by Constantin Piron [121] and Günther Ludwig [107].

There have been many subsequent attempts (and successes) at reconstructing quantum theory, these have had a wide range of conceptual starting points, however, broadly there have been two kinds of approaches. The first wave of reconstructions [142, 110, 121, 107] (surveyed in [58]) took place in the previous century, where assumptions were taken to be axioms in the mathematical sense. Nothing was left implicit, and there were no givens. As a result, the mathematical sophistication of these reconstructions was substantial. For example, Piron's Theorem [121] was actually only finalised in 1995 by Solèr [134], while arguably it already took off from von Neumann's book [142] in 1932. In this first wave of reconstructions the focus was on single systems under observation. Indeed, the Achilles heel of these approaches was the description of composite systems at a general axiomatic level: while these approaches were able to recover Hilbert space, they all failed to reproduce the tensor product as part of the formalism.

The second wave of reconstructions [76, 65, 111, 42, 79] was driven —under the impetus of Lucien Hardy's [76]— by modern developments in quantum information theory. These developments had two important consequences, firstly, they showed that there was much to be learnt from finite-dimensional quantum theory, and so the target of these reconstructions shifted to a mathematically much simpler objective. The second change was that composite systems could no longer be ignored, as they are essential to the description of quantum information theory, and so, the failure of the earlier approaches to derive the tensor product structure needed to be overcome. Moreover, there was also a broader shift: the mathematical axiomatic flavour was relaxed, leaning more towards a 'physicist's approach' (like Einstein's conception of the principles underpinning relativity theory – the limited speed of light and the invariance of laws for different observers). This relaxation means that many pieces of structure are often sneaking into the reconstructions (again like in Einstein's derivation of relativity theory), for example, the structure of real numbers. To distinguish the assumptions used in these approaches from the mathematical axioms of the first wave, the term 'postulates' was adopted. Now the goal was to focus more on the conceptual grounding of each of these postulates, and in casting them as much as possible within some interpretational school. For example, principles should only make reference to measurement [76], should only refer to single systems [21], or, should be information-theoretic [42]. But, despite their success, these postulate-based approaches generally lacked the mathematical clarity of the earlier attempts, and so, in many cases, the proofs were not so insightful.

In response to the issue of dealing with composite systems, learning from the failures of the earlier axiomatic reconstructions, a rigorous focus on composition of quantum systems and processes has been the subject of Categorical Quantum Mechanics [4, 55] for well over ten years. While the first reconstructions of the second wave focused on states of physical systems and their geometry, the categorical approach changed the paradigm completely, from states to *processes*. In this way the attention switched naturally to composition [52]. Specifically, as we have seen, the particular nature of the categorical structures found in CQM has the great upshot that it admits a full and faithful diagrammatic representation [49, 51] (a.k.a. *quantum picturalism*). Meanwhile, borrowing from CQM, the reconstructionists of the second wave had adopted 'diagrams of processes' as their starting point [78, 41, 42, 79], hence embracing composition of process as the core ingredient of quantum theory. More recently, a third wave of reconstruction attempts took off [17, 21, 146], which can be seen as resurrecting the mathematical spirit of the first wave while still embracing the principled underpinning that guides the second wave.

This chapter provides a reconstruction that is conceptually grounded whilst still being based on crisp mathematical axioms. This is achieved by exploiting the correspondence:

$$\text{diagrams} \simeq \text{category theory} \simeq \text{process theory}$$

our postulates are now entirely diagrammatic, which means entirely category-theoretic, that is entirely process-theoretic – providing them with an intuitive, elegant as well as principled underpinning.

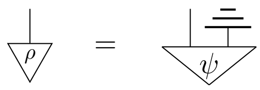
Essentially, what we prove, is that if we have a process theory with all of the 'standard' process-theoretic tools (i.e. those introduced in chapter 2), then a single additional postulate *time-symmetric purification* suffices to be able to reconstruct Operational Quantum Theory. We will introduce this time-symmetric purification postulate in detail in the next section. More specifically, we prove that it is possible to reconstruct Operational Quantum Theory from the following postulates:

1. The theory is a process theory,
2. that has a classical interface comprising of,
    (a) all classically controlled processes, and
    (b) tests to perform finite local tomography,
3. moreover, the theory admits string diagrams,
4. a sharp dagger,
5. and all processes have time-symmetric purifications.

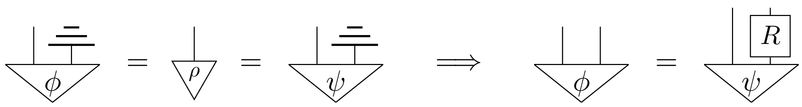
These are formally presented in section 4.2. The proof of this result is remarkably simple in contrast to many other reconstructions, largely owing to the use of the diagrammatic notation for proving the results. Indeed, the entire reconstruction can be presented in a simple flowchart (figure 4.1) showing that the structure of the reconstruction consists of few lemmas and a standard result, the Koecher-Vinberg theorem [96, 141] (introduced to the authors by the works of Barnum and Wilce [19, 18, 22, 146, 16, 17]).

4.1 Time-symmetric purification

The final postulate for the reconstruction is based on the notion of purification introduced in [41, 42, 39, 43] as an operational generalisation of the Stinespring dilation theorem [136]. The purification principle states that for all states ρ there exists a pure bipartite state ψ such that:



where this purification is 'essentially unique' [43], that is, if there are two such purifications ψ and ϕ with the same purifying system, then they are related via a reversible transformation R:

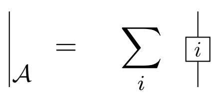


This notion of purification is problematic for us for three reasons. Firstly, it is not compatible with the classical interface – classical theory does not satisfy this principle. Secondly, it is explicitly time asymmetric, whereas we want to be able to pin down the asymmetry as purely a consequence of the classical interface. Thirdly, it is formulated specifically in terms of states, whilst we aim to treat all processes on an equal footing taking the process-theoretic view that processes are the fundamental entities, states being special instances thereof. In this section we therefore introduce the postulate of *time-symmetric purification* which resolves these issues: stipulating that every process arises from a pure process by 'discarding' a system to both the future and the past. However, before we can get to defining such a notion of purification we first need to understand what the pure processes are in the process theory of Operational Quantum Theory.

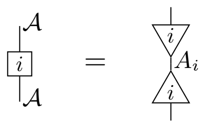
4.1.1 Purity and leaks in Operational Quantum Theory

We first recall the following results about C*-algebras discussed in chapter 3 that we will need to understand process purity for operational quantum theory.

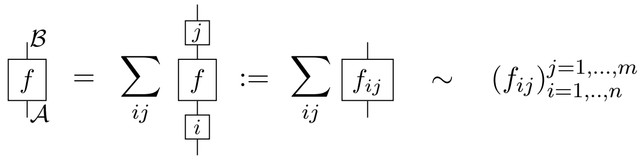
1. Given a C*-algebra $\mathcal{A} = \bigoplus_i A_i$ there is a decomposition of the identity into orthogonal projectors



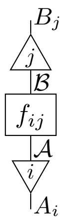
2. such that each of these projectors splits through an irreducible C*-algebra



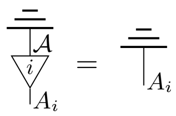
3. this decomposition of the identity provides us with a matrix representation of these processes with input $\mathcal{A} = \bigoplus_{i=1}^{n} A_i$ and output $\mathcal{B} = \bigoplus_{j=1}^{m} B_j$



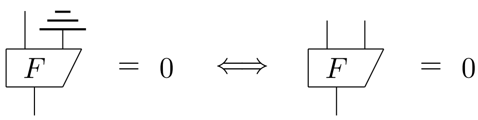
4. where each f_{ij} defines a map between the irreducible C*-algebras A_i and B_j as



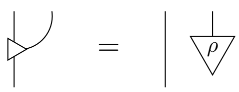
5. such that the quantum processes defined by the discarding map for the C*-algebras are the discarding maps for the quantum systems,



6. finally, we note that,

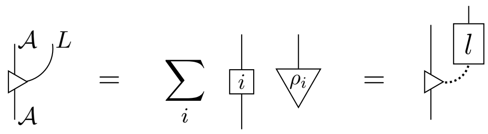


Recalling the definition of pure processes 2.5.2 we see that to understand what the pure processes are we must understand what the leaks are for these systems. In quantum theory, as mentioned in example 2.4.4 all leaks are trivial, that is, any leak separates as:

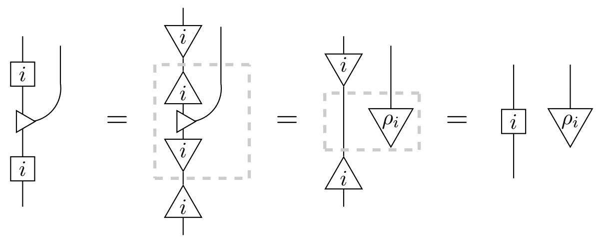


however, in classical theory, and more general C*-algebras the leaks are more interesting as we will now demonstrate.

Proposition 4.1.1. *Given a C*-algebra, $\mathcal{A} = \bigoplus_i A_i$ any leak can be written as:*

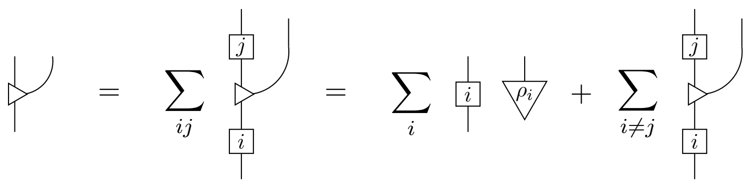


Proof. Note first that any leak for $\mathcal{A}$ defines a leak for each of the A_i by pre- and post- composing with the relevant projector, hence, as the A_i are quantum systems these leaks must be constant:

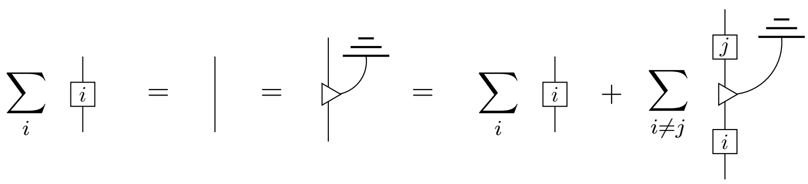


(4.1)

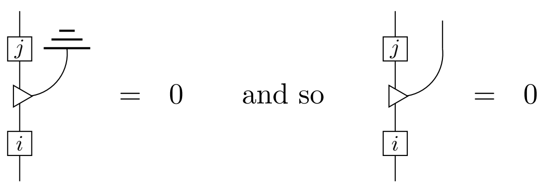
We therefore have



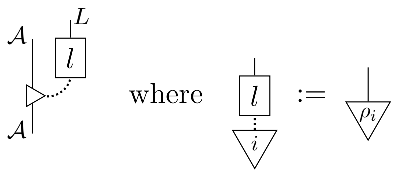
However, by decomposing the identity as a sum of the projectors and using the defining equation of a leak we find



and hence if $i \neq j$



Combining this with equation 4.1 provides us with the result. Using classical control then allows us to write this in the form

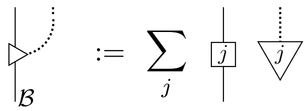


which completes the proof. □

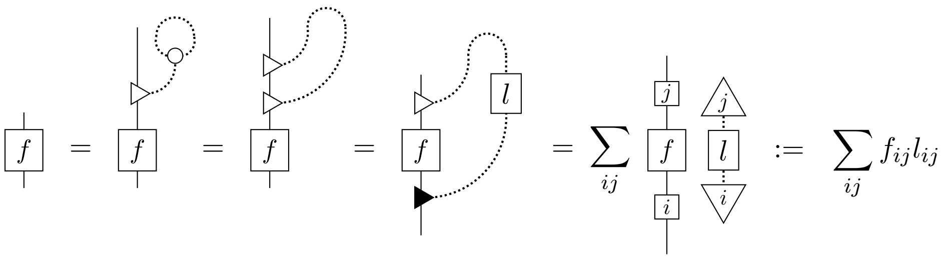
In example 2.5.4 we noted that our definition of purity for quantum processes reduces to the standard notion and hence, pure quantum processes are those with Kraus rank 1. We now consider what our definition means for more general C*-algebras.

Proposition 4.1.2. *Pure processes between C*-algebras are processes whose matrix representation has i) pure processes as matrix elements, and ii) at most a single non-zero process in each row and in each column.*

Proof. Consider a pure process $f : \mathcal{A} \to \mathcal{B}$ where $\mathcal{A} = \bigoplus_i A_i$ and $\mathcal{B} = \bigoplus_j B_j$, and its matrix representation $f = \sum_{ij} f_{ij}$. Now given the leak of system $\mathcal{B}$ which leaks the 'which branch' information to a classical system

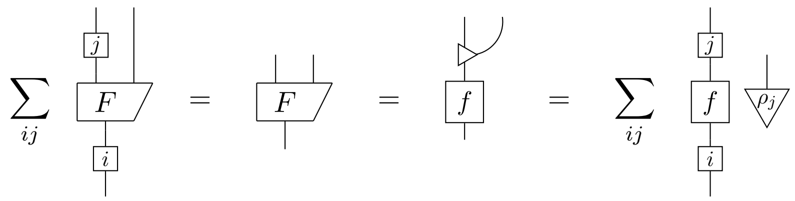


we find that

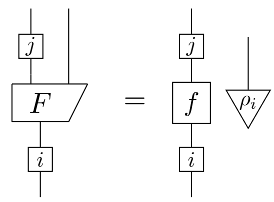


Hence we find that $f_{ij}l_{ij} = f_{ij}$ and so either $f_{ij} = 0$ or $l_{ij} = 1$. However, as we know that l is causal this means that $\sum_j l_{ij} = 1$ for all i and so for each i there is only a single value of j such that $l_{ij} = 1$. Therefore for that i all other values of j must result in $f_{ij} = 0$. That is, the matrix representation of f has at most a single non-zero element in each row. We can make an equivalent argument starting with the leak at the bottom pushing it through to the top, in this case we find that there is at most one non-zero element in each column.

Now note that every dilation F of f must be given by some leak, which means that



therefore

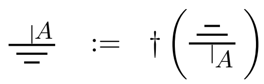


and so every dilation of each of the f_{ij} must separate. In other words, the f_{ij} are pure quantum processes.

To summarise, from the 'commutativity' conditions we find that a pure process f maps each branch of $\mathcal{A}$ to at most one branch of $\mathcal{B}$ and vice versa, and, from the 'all dilations are leaks' condition we find that these maps between branches are themselves pure. $\square$

4.1.2 Defining time-symmetric purification

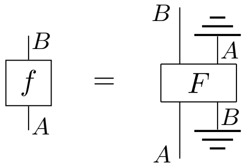
To produce a time-symmetric version of purification we need to have a notion of 'discarding' systems in the past. We can think of the standard discarding effect as an operational way to describe a scenario where we have no knowledge about, control over, or interaction with the future of a system. However, we can also imagine a scenario where we have no information about, no control over, and no interaction with the *past* of a system, to represent this we define:



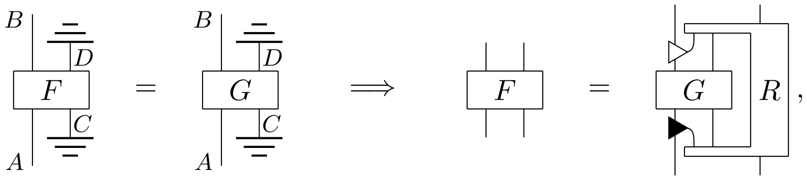
(4.2)

That is, as the time reverse of the discarding (to the future) map. In the cases of quantum and classical theory, the dagger of the discarding map is the unnormalised maximally mixed state. One may worry that this is not actually a valid state (i.e. is not normalised/causal), however, discarding to the past is not something that we can 'do in the lab' and so should not correspond to a state that we can prepare. Given this 'state', we can then define a symmetric version of purification as follows.

Definition 4.1.3 (Symmetric purification)**.** *A theory has* symmetric purifications *if every process* $f : A \to B$ *can be dilated to a* pure *process* $F : A \otimes B \to B \otimes A$ *as* *follows:*

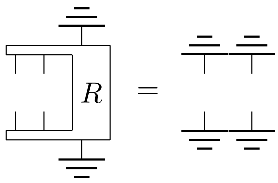


we say that F is a symmetric purification *of f. Moreover, a theory has* essentially unique *symmetric purifications if any two pure processes $F, G : A \otimes C \to B \otimes D$ with the same 'marginals' are connected by a circuit fragment R, that is:*



(4.3)

where the 'backwards leak' is provided by the dagger of a leak and where R must satisfy:



(4.4)

Symmetric purification expresses the requirement that all processes of the theory are fundamentally pure, and the apparent lack of purity should arise from lacking information about and control over the past and/or future of some environment systems. Here the word "symmetric" emphasizes the fact that both the discarding map and its dagger appear in the definition of the purification of a process as well as the fact that the input and output systems are now the same. Indeed, Operational Quantum Theory actually satisfies a stronger form of time-symmetric purification where we can demand also that $F = F^\dagger$, however, this stronger version is not necessary for our reconstruction and so understanding the implications of this are left to future work.

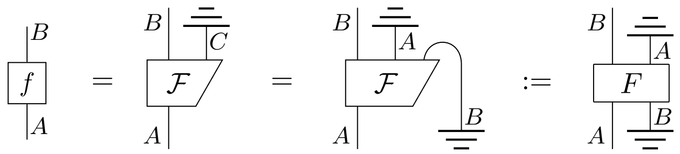
The 'essential uniqueness' part of this postulate is not exactly an intuitive postulate, here it is postulated simply as the natural generalisation of the essential uniqueness part of the standard purification postulate (see [41] or the start of section 4.1). Understanding this condition is (like the sharp-dagger) the subject of ongoing work, however, it seems clear that it should be seen as a separate postulate to purification. One direction to pursue, is to view it as a corollary of a much more general principle regarding equivalence of processes under arbitrary decoherence maps.

Remark 4.1.4. *Given the symmetric nature of this postulate it may well turn out* *to be useful in causally neutral scenarios [120, 105], time symmetric formulations of quantum theory [5, 118] or for theories with indefinite causal order [77, 9, 92, 35, 38].*

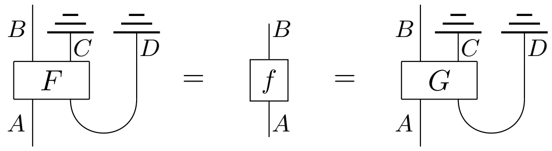
We now want to check that this postulate is actually satisfied by the theories we are considering:

Theorem 4.1.5. *Operational Quantum Theory has symmetric purifications.*

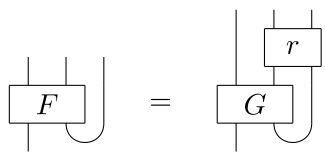
Proof. First let us consider the quantum case. We know that any process $f : A \to B$ can be purified to a process $\mathcal{F} : A \to B \otimes C$ where $C = A \otimes B$, and so it is simple to see that it can also be purified in a symmetric way $F : A \otimes B \to B \otimes A$:



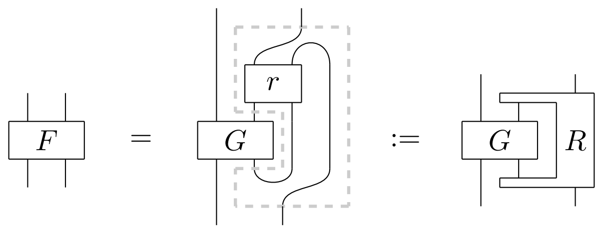
using the fact that the compact structure is pure for quantum theory and that the composite of pure processes is pure. It therefore just remains to check that any two such purifications are related in the correct way. Consider two such purifications F and G, then we can use the compact structure – as it is pure for quantum theory – to define two more purifications as:



which must be related by a reversible and hence causal transformation r:



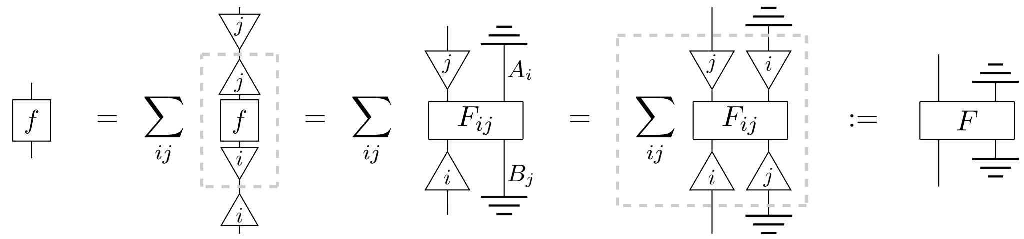
Using the compact structure for a second time therefore means that:



(4.5)

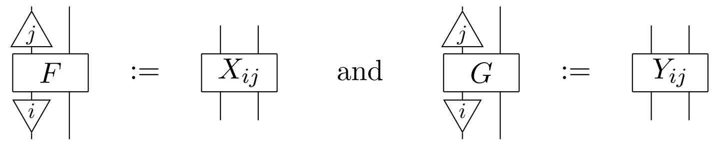
where it is simple to check that causality of r implies that R satisfies equation 4.4 as required.

Next we turn to the general C*-algebraic case. Consider a process $f : \mathcal{A} \to \mathcal{B}$ where $\mathcal{A} = \oplus_i A_i$ and $\mathcal{B} = \oplus_j B_j$ with $\{A_i\}$ and $\{B_j\}$ irreducible C*-algebras, i.e. quantum systems. It is simple to check that this does have a symmetric purification by noting that we can define a dilation of a process f by symmetrically purifying the quantum maps of matrix representation

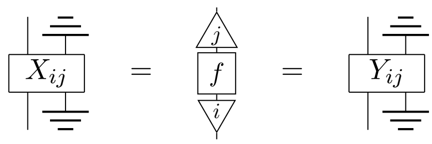


It is then simple to check that this dilation is moreover pure and hence a symmetric purification of f. Like for the quantum case, the more interesting part to check is equation. 4.3.

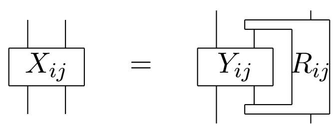
Given two purifications F and G let us define:



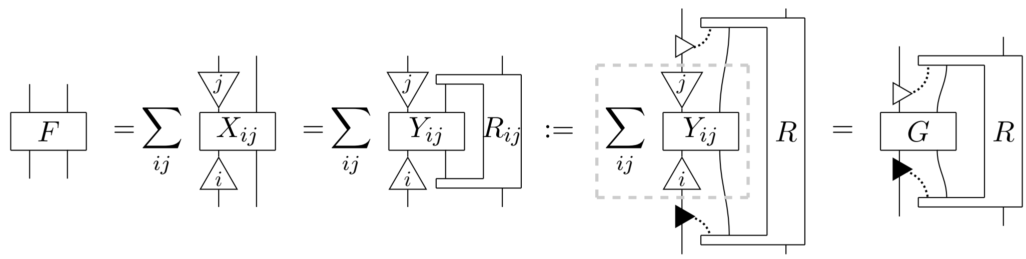
Noting that these are purifications of the same quantum process, i.e.



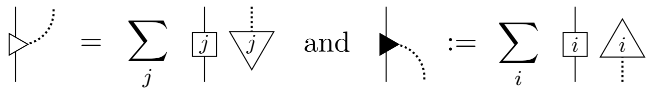
and so using our result for quantum systems we obtain:



and therefore:



where in the last step we have used classical control to construct R and defined the forwards and backwards leaks as



respectively.

That R satisfies equation 4.4 follows directly from the quantum case. □

4.2 The postulates

We now formally present the full list of postulates that we will use in the reconstruction. Firstly, we set up the basic framework that we work in.

Postulate 1. *The theory is a process theory as described in section 2.1.*

Next, two postulates that provide a classical interface see section 2.2.

Postulate 2a. *All classically controlled processes exist (definition 2.2.3).*

Postulate 2b. *Processes are characterised by finite local tomography (definitions 2.2.4 and 2.4).*

Next we introduce two compositional postulates.

Postulate 3. *The theory has string diagrams (definition 2.6.1).*

Postulate 4. *The theory has a sharp dagger (definition 2.7.4).*

Finally, the symmetric purification postulate discussed earlier in this chapter.

Postulate 5. *All processes have essentially unique symmetric purifications (definition 4.1.3).*

We can now show how these postulates take us to operational quantum theory.

4.3 The reconstruction

Figure 4.1 shows a high-level view of the structure of the reconstruction showing how the different postulates are used to provide various results that combine to reconstruct quantum theory. This demonstrates the simplicity of this reconstruction when compared to earlier reconstructions of quantum theory some of which involve hundreds of lemmas.

To begin reconstructing quantum theory we first demonstrate how we can obtain a notion of *sums* from our postulates. In particular, showing that the state spaces have

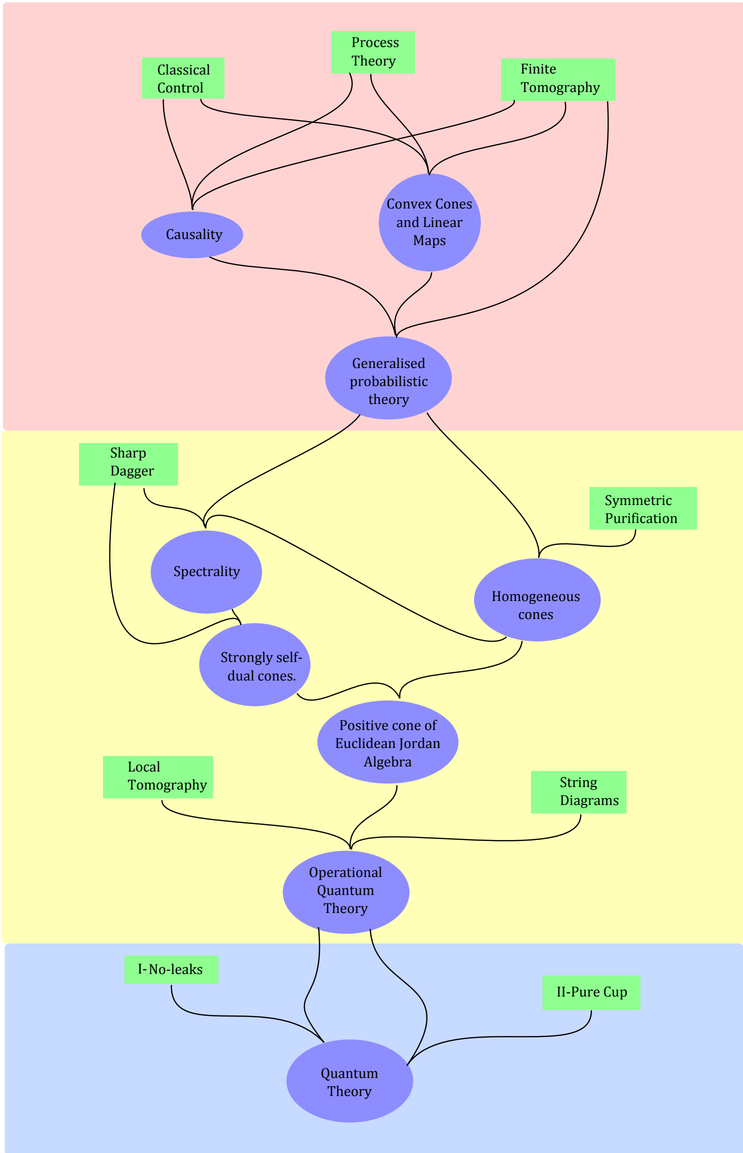
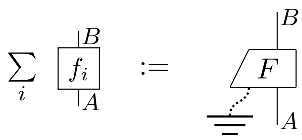


Figure 4.1: Flowchart outlining the structure of the reconstruction, the top section obtains something similar to the generalised probabilistic theory framework, the middle section takes us to Operational Quantum Theory and the final section gives two routes to quantum theory as described in section 3.3. The green rectangles correspond to postulates and the blue ellipses to the lemmas and theorems constituting the proof of the reconstruction.

the structure of convex cones, bringing us close to the structure that is typically used in the Generalised Probabilistic Theory framework. To prove many of the results in this section it is convenient to first consider the acausal extension of the theory —essentially, by allowing for acausal processes in the classical interface— and then to restrict to the physical, causal, theory using Prop. 2.3.6.

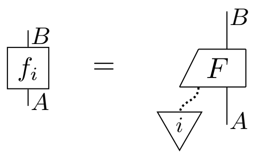
Lemma 4.3.1. *In a theory satisfying classical control (Post. 2a) we can define a sum of processes.*

Proof. First, let us show that we can introduce a notion of sum for processes by using classical control and exploiting the properties of classical systems. We define the sum of any finite set of processes as



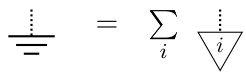
(4.6)

where F is the classically controlled process satisfying

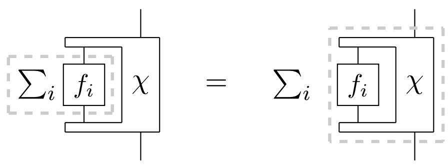


(4.7)

and the 'dagger' (c.f. equation 4.2) of the classical discarding map is the (super-normalised) maximally mixed state:

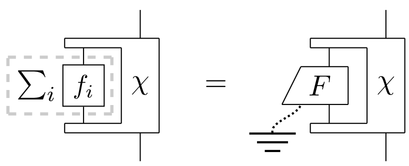


where we are exploiting the sum present in classical theory and where i are the classical point distributions used in the definition of classical control (c.f. definition 2.2.3). Given this definition we need to check that this sum is actually well defined, specifically, that they distribute over diagrams. Formally we need to show that:

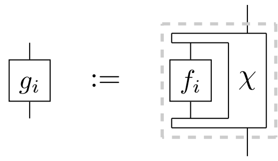


(4.8)

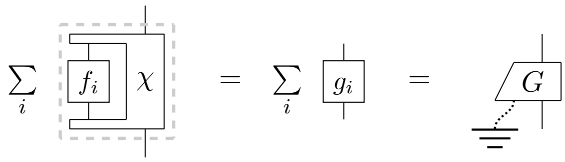
holds for any circuit fragment χ. The LHS of this is defined through equation (4.6) as:



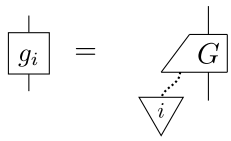
On the other hand, to understand the RHS let us first define:



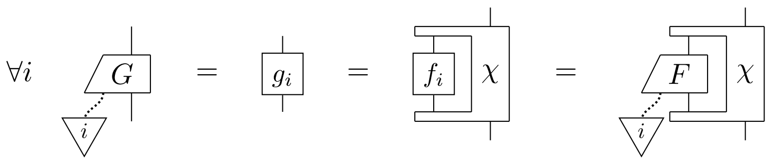
then using equation (4.6) again the RHS is equal to:



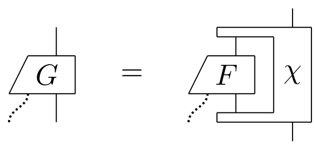
where we are again using classical control to define G such that it satisfies:



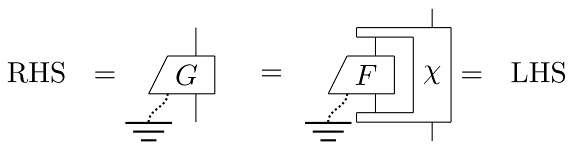
Then it is simple to see that:



which using local tomography of classical theory implies that:

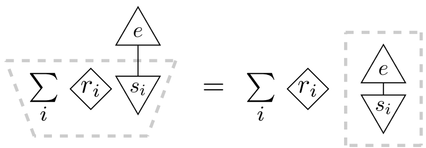


We therefore find that:



and so, equation 4.8 is satisfied for all circuit fragments χ, and hence, that the sums are free to move around diagrams. The other constraints on the sum (e.g. commutativity etc.) are immediately inherited from the equivalent property of the classical sum. □

Remark 4.3.2. *Note that this immediately provides the standard interpretation of* *these sums as probabilistic mixtures if we provide suitable weights, in the form of scalars r_i, as for all effects e we have:*



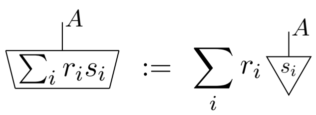
i.e. $$\mathsf{Prob}\left(e, \sum_i r_i s_i\right) = \sum_i r_i \mathsf{Prob}(e, s_i)$$

Noting that tomography implies that states are characterised by these probabilities, it is then simple to check that this sum is associative, commutative and has a zero as expected.

Given this notion of summation, we can prove some basic properties regarding the state spaces of systems and maps between them.

Lemma 4.3.3. *In a theory satisfying classical control (Post. 2.2.3) and finite tomography (Post. 2.2.4), the states form a finite dimensional pointed cone. Processes then induce positive linear maps between these cones. The causal states are defined by an intersecting hyperplane and causal processes preserve this hyperplane.*

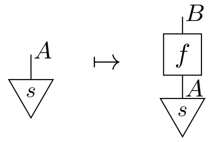
Proof. Firstly note that the scalars in the theory are going to be non-negative real numbers (scalars are just processes $s : I \to I$ as there is a classical subtheory then I must be classical, hence, these scalars are classical and are non-negative real numbers). Given the above definition of summation it is then clear that the state space of a given system A is a convex cone C_A as it is closed under linear combinations with non-negative real coefficients. Defined by:



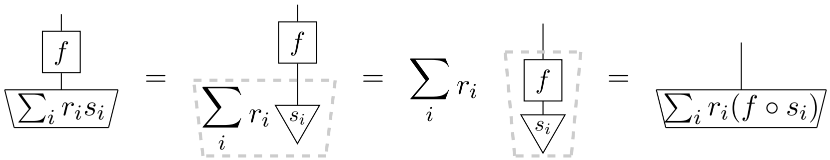
Allowing the coefficients of linear combination to be negative, the cone extends naturally to a real ordered vector space, spanned and ordered by the cone itself. By construction the cone is full dimensional (i.e. spans the vector space) and it is simple to show that it is pointed (i.e. the zero-vector is in the cone and is the unique vector for which v and $-v$ are in the cone). Moreover the cones are finite dimensional, this immediately follows from finite tomography as it implies that a state is characterised by a finite number of real numbers.

Note also that the hyperplane defined by $\bar{\top} \circ s = 1$ intersects the cone not through the origin, we first consider $\bar{\top} \circ s = 0$ and note that $\bar{\top} \circ s = 0 \implies \forall e \;\; e \circ s = 0 \implies s = 0$. Therefore, for any state $s \neq 0$ there is some scalar r_s such that $\bar{\top} \circ (r_s s) = 1$ hence the hyperplane intersects the cone.

Thanks to distributivity of sums (Eq. 4.8), any process $f : A \to B$ induces a positive linear map between the vector space spanned by the states of system A and the vector space spanned by the states of system B. Defined by



where linearity follows immediately as a special case of Eq. 4.8 as



Moreover, the map is positive because it maps states to states. □

Remark 4.3.4. *The systems therefore have the same structure as in the "convex-cones" framework [23, 76] apart from the fact that we have made no mention of the necessity of the convex sets to be* closed *(see [131] for a discussion of the consequences of this). For a more categorical approach to connecting these two frameworks see [143, 74, 137, 16, 145, 144].*

Remark 4.3.5. *Note that many of the above results can easily be extended to arbitrary processes, for example, the set of processes from A to B will form a finite dimensional proper cone with a convex subset of causal processes.*

We now consider how some key features of the quantum state space arise from our axioms, namely, homogeneity, spectrality and strong self duality. We begin with homogeneity which is defined as follows.

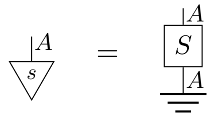
Definition 4.3.6 (Homogeneous cone)**.** *A convex cone C is homogeneous if for every pair of vectors v_1, v_2 internal to C, there exists a cone automorphism T such that $T(v_1) = v_2$.*

By considering the implication of the time-symmetric purification postulate for states we can obtain homogeneity of the state-cone.

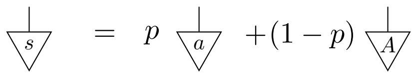
Lemma 4.3.7. *If in addition to the classical interface (postulate 2a and postulate 2b) the theory satisfies* symmetric purification *(postulate 5) then the state cone is* homogeneous *(definition 4.3.6).*

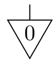
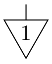
Proof. Homogeneity is the statement that, for any pair of internal states s_1 and s_2 there exists a cone automorphism T such that $T(s_1) = s_2$. As T is reversible this is equivalent to the statement that, there is a cone automorphism between a particular chosen internal state and any other. For this proof we take the particular internal state to be the 'discarding' state. We proceed in two steps, firstly, we show that there is a process that maps the discarding state to any other internal state, then secondly, we show that this map is surjective and so is a cone automorphism as it is automatically linear due to lemma 4.3.3.

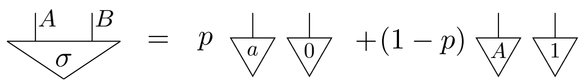
The first part is a simple corollary of symmetric purification. Consider an arbitrary internal state s, then its purification S gives a map from the discarding state to s:



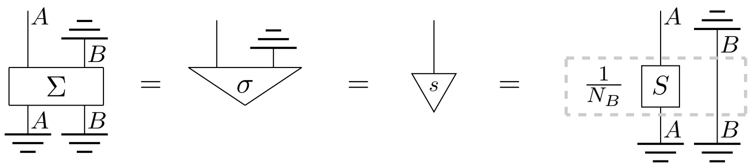
For the second part, we adapt [43, Proposition 7] to show that S is surjective. Note that as s is internal, then any state a is in some decomposition of s:

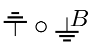


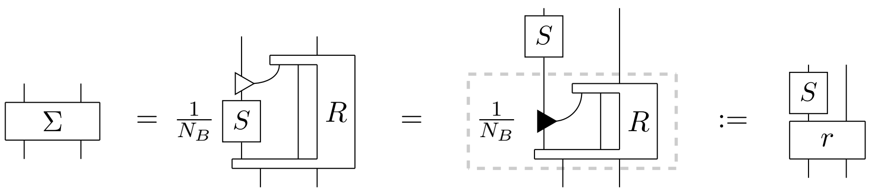
where $p \neq 0$. Therefore we can construct the following dilation, σ, of s where  and  are causal, perfectly distinguishable states of some system B:



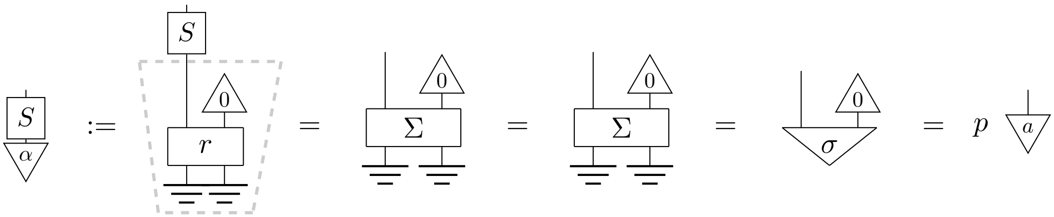
which has a purification Σ, which is moreover a purification of s, hence we can construct two purifications of s with the same input and output systems:



where $N_B :=$  . Then, from the definition of purification are related by some R as:



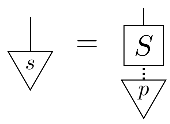
where in the second step we use that S is pure and so we can replace the leak afterwards with a leak before. It is then clear that there is a state, $\frac{1}{p}\alpha$ that is mapped to a by S:



Hence, as a is an arbitrary state S is surjective. □

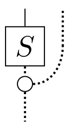
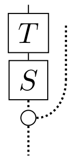
Strong self duality is really the key property we need to prove along with homogeneity, however to get there we will first show a spectrality result.

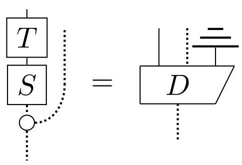
Definition 4.3.8 (Spectrality). *Any (causal) state can be written as*



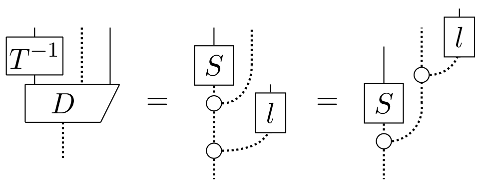
where p is a (causal) classical state and S is a maximal testable state preparation (definition 2.7.3).

Lemma 4.3.9. *If in addition to the classical interface (postulates 2a and 2b), the Homogeneity from lemma 4.3.7, and the theory additionally has a* sharp-dagger *(postulate 4) then the state-cone is* spectral *(definition 4.3.8).*

Proof. Firstly note that if  is pure then  is pure for any reversible transformation T. To see this we must consider the possible dilations of this process:

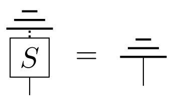


which gives a dilation of the original process (pure by assumption) by composing this with T^{-1} so we have:

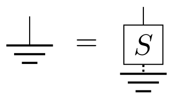


and hence, by composing this with T we see that the same is true of D and so we obtain the result.

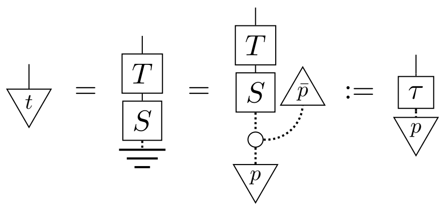
Now to obtain spectrality, note that every system A must have a (generally non-unique) maximal testable state preparation $S : n \to A$ (def. 2.7.3) although it could be trivial (i.e. $n = 0$). For such a maximal testable state preparation and the definition of the sharp dagger (def. 2.7.4) we therefore have:



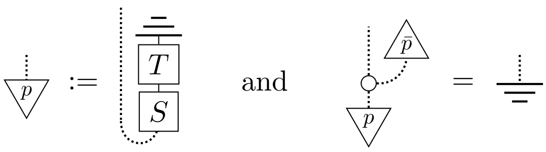
and so taking the dagger of this equation we find that:



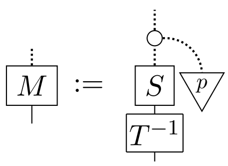
which means that, by homogeneity we have, for any internal state t, a reversible map T such that:



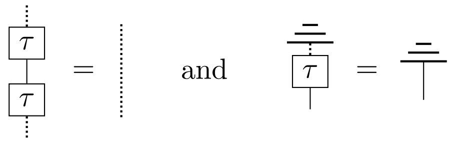
where p and $\bar{p}$ are defined by:



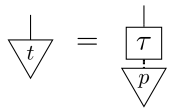
We can then also define a measurement:



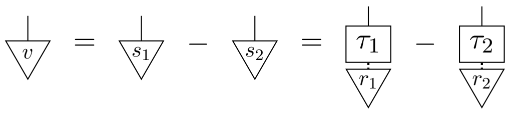
One may worry that T^{-1} is not guaranteed to be a physical transformation, however, regardless, this measurement is well defined as each of the effects that make it up must be physical (as T induces an automorphism on the effect cone so does T^{-1}). Hence, this measurement can then be defined as a classically controlled process. It is then simple to check that the pair τ, M satisfy the conditions for the sharp dagger such that:



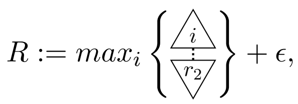
Therefore,



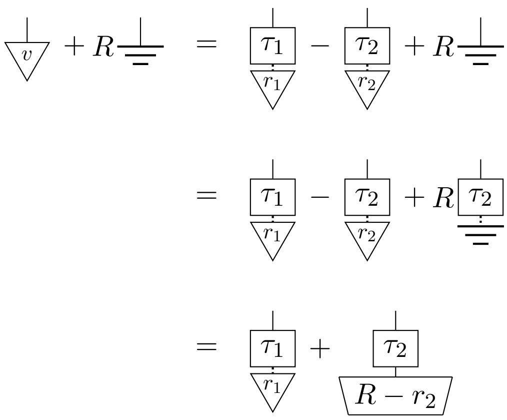
satisfies the conditions of spectrality. What we have so far proved is that there is a spectral decomposition of any internal state, we want to extend this to all states, and, to the vector space in which the convex cone is embedded. Note that any vector can be written as the difference of two internal states which can each be spectrally decomposed:



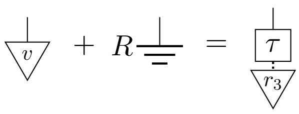
Then define,



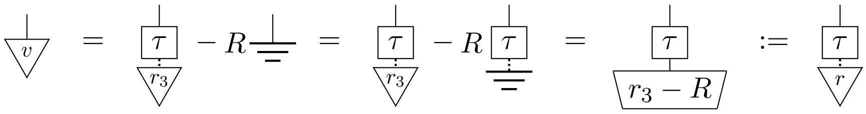
where $\epsilon > 0$. Then



is an internal state as all of the elements of the classical vectors are strictly positive thanks to the definition of R. As this is an internal state it therefore has a spectral decomposition



and so we can write



hence we have spectral decompositions for arbitrary vectors. □

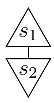
Given this spectrality result we can show that the state cone is strongly self dual in a fairly straightforward way.

Definition 4.3.10 (Strongly self dual cone)**.** *A convex cone C is strongly self dual if there exists an inner product on the vector space spanned by C, $\langle\ ,\ \rangle$ such that,*

$$x \in C \iff \langle x, c\rangle \geq 0 \quad \forall\ c \in C.$$

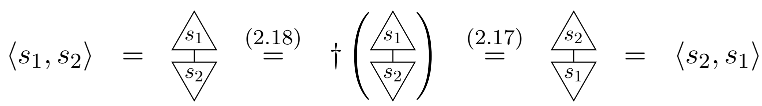
Lemma 4.3.11. *If in addition to the classical interface (postulates 2a and 2b), the Spectrality from lemma 4.3.9, and the theory additionally has a* sharp-dagger *(postulate 4) then the state-cone is* strongly self dual *(definition 4.3.10).*

Proof. First we will show that the sharp-dagger provides an inner product defined as:

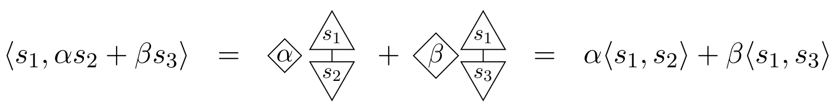
$\langle s_1, s_2\rangle :=$  (4.9)

and secondly we show that the state cone is strongly self dual with respect to this inner product.

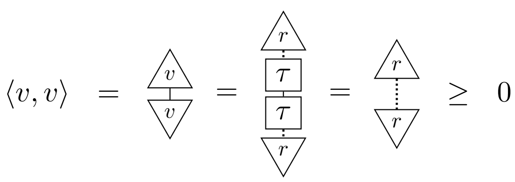
To show that this is a valid inner product, firstly, we check that this is symmetric:



Secondly that it is linear follows immediately from linearity of effects given by Lem. 4.3.3:

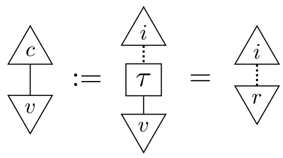
 (4.10)

Finally, we can check positivity using the previous spectrality result for the vector space:



where equality implies that $r = 0$ and hence $v = 0$. Hence Eq. 4.9 defines a valid inner product.

Note that if all elements of r are strictly positive we have an internal state, if they are non-negative then we have a state and if any are negative then the vector cannot be a state as it would give a negative probability for some effect. It is then simple to check strong self-duality. Firstly, if s is an element of the state cone C then $\langle s, c\rangle \geq 0$ for all $c \in C$ as $\langle s, \cdot\rangle = s^\dagger$ is an effect and so evaluates to a positive real number on the cone of states. Conversely, if $v \notin C$ then there is a negative coefficient in the spectral decomposition, without loss of generality we label this element i. There then exists some $c \in C$ such that $\langle c, v\rangle < 0$, that is:

$\langle c, v\rangle \quad =$  $< \quad 0$

The state cone is therefore strongly self dual with respect to the inner product defined by the sharp dagger. □

These properties, in particular homogeneity and strong self duality, are well known to get us close to quantum theory, specifically, by using the Koecher-Vinberg theorem [96, 141] we get that the state cones correspond to Euclidean Jordan Algebras.

Theorem 4.3.12 (Koecher-Vinberg theorem)**.** *There is a one-to-one correspondence between Euclidean Jordan Algebras and symmetric cones i.e. convex cones that are closed, pointed, homogeneous, and self-dual.*

Theorem 4.3.13. *Systems in our theory correspond to finite dimensional Euclidean Jordan Algebras.*

Proof. By using the Koecher-Vinberg theorem above we must simply demonstrate that our state-cones are indeed symmetric cone and hence also correspond to EJAs.

First note that given a cone C in a vector space V we define the dual cone by

$$C' := \{v | v \in V \text{ s.t. } \langle v, u\rangle \geq 0 \ \ \forall\, u \in C\}$$

This implies that the dual cone is closed as it is the intersection of closed half spaces, one for each $u \in C$ defined by $\langle v, u\rangle \geq 0$. Hence, strong self-duality implies that the state cone must be closed too. This therefore follows for our systems from lemma 4.3.11. Lemma 4.3.1 then implies that the cones are pointed, finite dimensional and lemma 4.3.7 that they are homogeneous. Hence the state spaces are finite dimensional symmetric cones. It therefore immediately follows that each system in our theory corresponds to a finite dimensional EJA. □

There is a well known classification result for finite dimensional EJAs [90], they correspond to direct sums of five types of simple EJAs. There are two important properties of each of these, their *rank* which corresponds to the maximum number of pure and perfectly distinguishable states and their dimension, the number of fiducial effects.

i. $\mathbb{C}_n$, algebra of self adjoint $n \times n$ complex matrices, which have rank n and dimension n^2

ii. $\mathbb{R}_n$, algebra of self-adjoint $n \times n$ real matrices, with rank n and dimension $\frac{n(n+1)}{2}$

iii. $\mathbb{H}_n$, algebra of self-adjoint $n \times n$ quaternionic matrices, with rank n and dimension $n(2n-1)$

iv. $\mathbb{O}_3$, algebra of self adjoint 3×3 octonionic matrices, with rank 3 and dimension 3^3, and finally,

v. $\mathbf{Spin}_K$, spin factors with rank 2 and dimension K.

Note that for the spin factors in the case of $K = 3$ coincides with $\mathbb{R}_2$, $K = 4$ with $\mathbb{C}_2$ and $K = 6$ with $\mathbb{H}_2$. Given this classification we are in a position to ask which of these EJAs are compatible with our compositional structure.

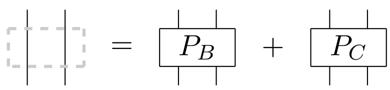
Theorem 4.3.14. *Given that state-cones correspond to EJAs, then* local tomography *(postulate 2b) and* string diagrams *(postulate 3) imply that the theory must be* Operational Quantum Theory *(i.e. the theory of finite dimensional C*-algebras and completely positive maps as presented in example 3.0.2).*

Proof. We start by making two assumptions that need to be checked. Firstly that if a Euclidean Jordan Algebra A is simple, then the composite $A \otimes A$ is also simple. Secondly, that $\mathsf{Rank}(A \otimes A) = \mathsf{Rank}(A)^2$.

Firstly, we consider the composite $\mathbb{H}_n \otimes \mathbb{H}_n$ this must be a simple-EJA with $\mathsf{Rank} = n^2$ and $\mathsf{Dim} = n^2(2n-1)^2$, it is straight forwards to check that this does not exist for $n > 1$. Therefore, the quaternionic case is ruled out. Next we turn to the real case, $\mathbb{R}_n \otimes \mathbb{R}_n$ and see that this requires a simple-EJA with $\mathsf{Rank} = n^2$ and $\mathsf{Dim} = \frac{n^2(n+1)^2}{4}$ we already know that the quaternionic case is ruled out, and it is again straight forwards to rule out the other options as well. Considering the octonionic case we have for $\mathbb{O}_3 \otimes \mathbb{O}_3$ that $\mathsf{Rank} = 3^2$ and $\mathsf{Dim} = 3^6$ again having ruled out the quaternionic and real cases then it is simple to check that this is not satisfied by the complex case for which if $\mathsf{Rank} = 3^2$ then $\mathsf{Dim} = 3^4$ and rule out the spin factors as they all have $\mathsf{Rank} = 2$. Finally we consider the spin factors $\mathbf{Spin}_K \otimes \mathbf{Spin}_K$ which requires that $\mathsf{Rank} = 4$ and $\mathsf{Dim} = K^2$ we only have the complex case to check now which implies that $K = 4$ which is the situation when $\mathbf{Spin}_4 = \mathbb{C}_2$ i.e. the cone is the Bloch Ball. Hence, given our two assumptions, it is only the complex case which have valid self composition. We can then check that the standard quantum tensor product i.e. $\mathbb{C}_n \otimes \mathbb{C}_m := \mathbb{C}_{nm}$ is the only choice consistent with our constraints as it has $\mathsf{Rank} = nm$ and $\mathsf{Dim} = n^2m^2$ as required.

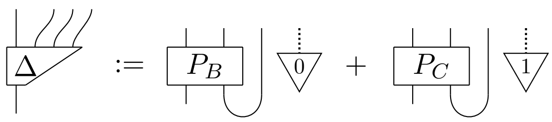
To extend this result to non-simple cases we note that $\otimes$ is bilinear and so distributes over $\oplus$. Therefore our above result rules out any EJA with a non-complex component in its decomposition. This leaves only the C*-algebras as valid systems, with their standard tensor product.

Now let us check the two assumptions we made. We intend to show that if $A \otimes A$ is not simple, then A is not simple either. Consider $A \otimes A = B \oplus C$ i.e.:

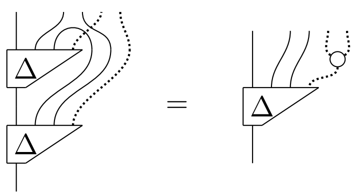


where $P_I \circ P_J = \delta_{IJ} P_I$ for $I, J \in \{A, B\}$ i.e. they are orthogonal projectors.

We can use this to define a leak for A as:

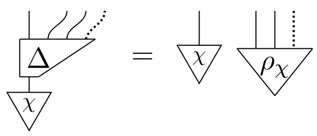


It is straightforward to check that this is indeed a leak, and moreover, using the orthogonality of P_B and P_C we can show that:

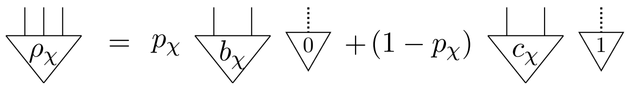


(4.11)

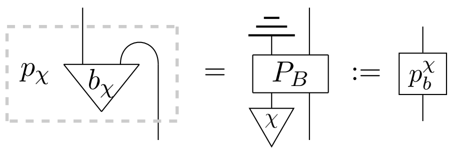
Now consider the effect of the leak on pure states χ, the definition of purity immediately implies that:



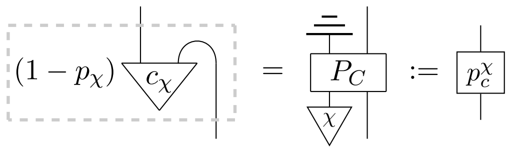
where



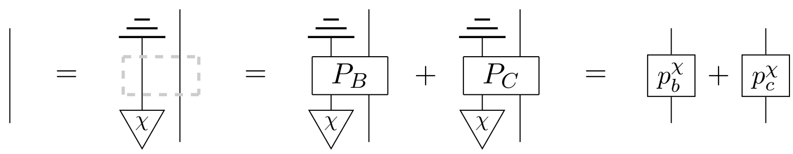
Now by considering this along with Eq 4.11 implies that:



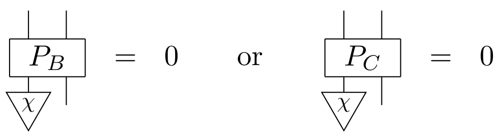
and



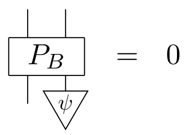
are orthogonal projectors, $p_i \circ p_j = \delta_{ij} p_i$ where $i, j \in \{a, b\}$ and therefore:



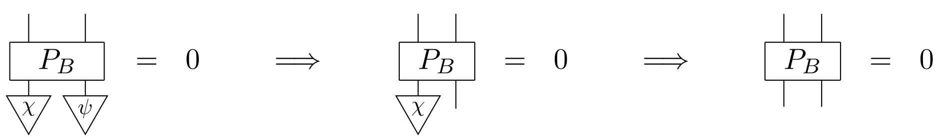
provides a decomposition of $A = b \oplus c$ unless either p_b^χ or p_c^χ is zero. This means that for each χ either



The same argument can be made for the other input. Therefore, consider some ψ such that:



this means that for all χ we have:

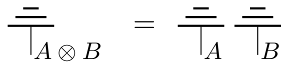


Hence, $A \otimes A$ was not decomposable which is in contradiction to our starting assumption and so if A is simple then so is $A \otimes A$ as we assumed.

Now let us consider the second assumption. It is clear that

$$\mathsf{Rank}(A \otimes B) \geq \mathsf{Rank}(A)\mathsf{Rank}(B)$$

as given a set of pure and perfectly distinguishable states for A and for B we can form a set of the composite as just all possible product of these. We then know that (using tomographic locality):



and as the discarding map is an internal effect we have a spectral decomposition of this effect given by the composite of the spectral decomposition of the two individual discarding maps. Hence,

$$\mathsf{Rank}(A \otimes B) \leq \mathsf{Rank}(A)\mathsf{Rank}(B)$$

the conjunction of these two inequalities then gives the required result. □

4.4 Conclusion

We have therefore reconstructed Operational Quantum Theory from our five postulates. It is then possible, should one be interested, to single out just the pure quantum systems by introducing an additional postulate. There are many potential ways to distinguish the quantum systems from the general C*-algebras, and so, many different postulates that could be chosen. However, of particular interest for us are those that we discussed in detail in section 3.3, that is:

1. the cups and/or caps of the string diagrams are pure, or equivalently,
2. all leaks are trivial.

4.4.1 Comparison to generalised probabilistic theories

Most recent reconstructions of quantum theory have been in the generalised probabilistic theory framework. These typically describe systems as finite dimensional, regular, closed cones, along with an intersecting hyperplane picking out the normalised states. Transformations are then described as linear maps and in particular effects are linear functionals on the state space. It is interesting to contrast this to our reconstruction where we see that we obtain most of this structure from our classical interface. In fact our classical interface can be interpreted as 'internalising' the generalised probabilistic framework within the process theory itself.

Yet, we do not obtain all of the structure of the GPT framework. In particular, we do not obtain that the state space is closed. Closure is —from an operational perspective— a very natural assumption, as, up to any finite error, a state space and its closure would be indistinguishable. We obtain closure as a corollary of strong self duality, it would however be interesting to know if we can obtain this more directly from a process-theoretic viewpoint.

In this reconstruction to make use of results in the GPT framework, in particular to allow the use of the Koecher-Vinberg theorem we focus on the cone of states. However, our postulates apply equally well to arbitrary processes, and so many of our results can immediately be extended to apply to general processes. These generalisations however have been omitted to try to keep the reconstruction as simple as possible. It seems plausible that by using the full generality of these postulates could show that some of them are redundant, or provide a more direct reconstruction that does not use the Koecher-Vinberg theorem.

4.4.2 Sharpening the dagger

As explained in the introduction, the notion of a dagger has been part of CQM since its conception. However, the standard definition of the dagger does not uniquely pick out the Hermitian adjoint as the dagger in the case of quantum theory, instead, this is an ad-hoc assumption that must be made. For example, the transpose also provides a valid dagger, but, the transpose however is not suitable for how we use the dagger in practice. For example, we use the dagger to give the inverse to unitary evolution, and to provide 'tests' for states. It is therefore clear that some further constraints on the dagger are necessary.

In [129] we introduce one way to do this, by considering the dagger as an operation that associates to each pure state a test for that state. This is closely related to sharpness conditions used, for example, in [80] and [46]. It is this sharpening of the dagger that allows it to fill the role that the Hermitian adjoint does in quantum theory, indeed we show in [129] that this singles out the Hermitian adjoint in the case of quantum theory. We have developed this idea further here, extending sharpness from a property about single states and effects to a property of general state preparations and measurements. This however remains a somewhat inelegant postulate and it seems like there should be a more general principle behind this that extends this to arbitrary processes.

4.4.3 Leaks and symmetric purification

In [130] we showed that —if one expects reversible transformations to be pure— that leaks must be taken into account when defining purity for general processes in general process theories. In particular, in the case of classical theory this is essential if the identity transformation is taken to be pure.

We show here that this new notion of purity allows for arbitrary classical processes to be purified. However, unlike the standard notion of purification that introduces an extra output that is then discarded we must introduce both an extra output and an extra input both of which are 'discarded'. It is interesting to note that this form of symmetric purification is now compatible with both the classical interface that we introduce for the theory, but also the freer notion of composition that we introduce which removes the distinction between inputs and outputs.

It is simple to see that we can return to the standard notion of purification in the case where we have a pure cup for each system, as we can then turn the discarded input into a discarded output instead. Purification in the reconstruction of [42] was used as a postulate that ruled out classical theory leaving just quantum theory, we can now see that this postulate can be thought of as comprising of two parts. The first, being symmetric purification which is satisfied by both classical and quantum theory, and the second that the cup is pure (or in the language of [41] that there is a pure dynamically faithful state) which rules out classical theory. From this perspective it is therefore the ability to turn uncertainty about the past into uncertainty about the future which is the defining characteristic of quantum theory.

There are many results in quantum foundations [101, 100, 103, 45, 46, 40] which use the standard notion of purification in the derivation. However, in many cases, the result is satisfied by classical theory but this case is not covered by the proof. It therefore seems plausible that these results might instead be derivable using the symmetric purification postulate (perhaps along with the existence of a dynamically faithful —not necessarily pure— state) to give a more general result. There has also been recent interest in formulating quantum theory in a causally neutral [105, 120] or time symmetric [118, 5] or with indefinite causal order [77, 9, 92, 35, 38], for which this notion of purification may be more applicable.

4.4.4 Comparison to other reconstructions

There have been several other reconstructions of quantum theory from various different perspectives, many results of which have been adapted for this work. It is therefore worth highlighting the features of this particular reconstruction which distinguish it from the others.

Firstly, the postulates that we impose are entirely diagrammatic. Moreover, we do not pick out states as being special, instead the postulates apply equally to all processes. As such, they fit with the spirit of Categorical Quantum Mechanics and the process-theoretic understanding of the world, that is, as being about processes and composition rather than being about states of isolated systems.

Secondly, the reconstruction is relatively simple. In particular, the structure of the reconstruction is clear (see figure 4.1) allowing for a high-level view of how the different postulates relate to each other and how they are used in each step. This will hopefully make it simpler to understand how relaxing or altering any given postulate will impact on the theory. In contrast, in many other reconstructions it is difficult to know precisely which postulates, and which of the assumptions going into the generalised probabilistic theory framework, are necessary to obtain each result.

Finally, we reconstruct Operational Quantum Theory. Rather than aiming to reconstruct quantum theory in its standard presentation we instead first reconstruct the process theory of C*-algebras and CPTP maps. This allows for a unified way to describe both quantum and classical systems as well as some other operationally meaningful systems. We argue that this is the correct operational account of quantum theory, and moreover, allows us to clearly see what it is that separates quantum systems from the others—specifically that is the lack of leaks, or equivalently, purity of the cups. In contrast, the majority of other reconstructions take *transitivity* as a postulate[1]. Transitivity is the property that there is a reversible transformation between any pair of pure states. This does not hold for general C*-algebras and so other reconstructions rule out Operational Quantum Theory from the start.

4.4.5 Future work

Whilst the postulates of this reconstruction are entirely diagrammatic the proofs ultimately rely on standard linear algebraic techniques. Moreover, whilst the postulates are defined at the level of processes we often only use them in the context of states. Both of these are against the spirit of process theories, as such, it would be interesting in future work to try to make the proof of the reconstruction process-theoretic along with postulates. Indeed, it seems plausible that by using the full strength of the postulates that there may be a much simpler and direct way to go about the reconstruction.

As with all reconstructions we assume the existence of a classical interface for the theory representing the action of experimenters in a lab deciding the experiments to perform and getting classical data as outputs. In our reconstruction this leads to the postulates of classical control and finite tomography. However, we expect this to be an emergent feature of quantum theory (as we discuss in chapter 3), and so it would be interesting to see if we can move beyond this operational approach. Can we

[1] Or have it as a direct corollary of a stronger postulate such as purification or strong symmetry.

instead express everything in terms of diagrams and find some replacement for the classical interface? That is, can we find a diagrammatic axiomatisation of classical theory that does not presuppose a probabilistic structure? The process theoretic notion of 'spiders' [55] seems to capture some aspects of classicality, namely, the copyability and deletability of classical information, but there are also many non-classical theories that have this structure and so we clearly need something more.

We presented the notion string diagrams, as a relaxation of the constraints on compositionality built into the process-theory framework. That is, it can be seen as relaxing the constraint that inputs cannot be connected to inputs and that outputs cannot be connected to outputs. It is interesting to ask what happens if we relax the other constraints on compositionality as well. For example, relaxing the constraint that only pairs of inputs/outputs are connected leads to the aforementioned 'spiders'. It therefore seems like many common tools of Categorical Quantum Mechanics would naturally be recovered from this approach, this will be discussed in more detail in a future paper.

4.4.6 Summary

In this chapter we have shown how the process theories provide a suitable framework and set of tools to reconstruct quantum theory. But, what if we want to go beyond quantum theory —as von Neumann originally aimed when he tried to move away from the Hilbert space formalism— can we use the process theory framework to find new physics? This will be the subject of the next chapter.

Chapter 5

Post-quantum theories

In 1903 Michelson wrote

> *"The more important fundamental laws and facts of physical science have all been discovered, and these are so firmly established that the possibility of their ever being supplanted in consequence of new discoveries is exceedingly remote"* [112].

Within two years Einstein had proposed the photoelectric effect [70] and within thirty quantum theory was an established field of scientific research. This new science revolutionised our understanding of the physical world and brought with it a litany of classically counter-intuitive features such as superposition, entanglement, and fundamental uncertainty.

Today, quantum theory is the most accurately tested theory of nature in the history of science. Yet, just as for Michelson, it may turn out to be the case that quantum theory is only an effective description of our world. There may be some more fundamental theory yet to be discovered that is as radical a departure from quantum theory as quantum was from classical. If such a theory exists, there should be some mechanism by which effects of this theory are suppressed, explaining why quantum theory is a good effective description of nature. This would be analogous to decoherence, which both suppresses quantum effects and gives rise to the classical world [89, 147][1]. As such, this mechanism is called *hyperdecoherence*. To the best of our knowledge, the notion of hyperdecoherence was first discussed in [149] and has commonly been considered as a mechanism to explain why we do not observed post-quantum effects, such as in [65], and, in particular, in the context of higher-order interference [135, 102, 101, 100, 21, 32, 115, 85, 138, 133].

[1] Given a process theoretic description as the leak construction in chapter 3.

We formalise such a hyperdecoherence mechanism within a broad framework of operationally-defined physical theories[2] by a key feature of quantum to classical decoherence. Using this we prove a no-go result: there is no operationally-defined theory that satisfies the natural physical principle of *purification*, and which reduces to quantum theory via a hyperdecoherence mechanism. Here, purification formalises the idea that each state of incomplete information arises in an essentially unique way due to a lack of information about some larger environment system. In a sense, purification can be thought of as a statement of "information conservation"; any missing information about the state of a given system can always be accounted for by considering it as part of a larger system. Our result can either be viewed as a justification of why the fundamental theory of nature is quantum, or as highlighting the ways in which any post-quantum theory must radically depart from a quantum description of the world.

5.1 Decoherence

One of the standard descriptions of the quantum to classical transition is environment-induced decoherence [147]. In this description, a quantum system interacts deterministically with some environment system, after which the environment is discarded, leading to a loss of information. This procedure formalises the idea of a quantum system irretrievably losing information to an environment, leading to an effective classical description of the decohered system. The decoherence process can be viewed as inducing a completely positive trace preserving map on the original quantum system, which is termed the *decoherence map*.

In chapter 3 we abstractly captured this notion of decoherence via the leak construction, but we will now further develop example 3.2.4 to illustrate the key features. Consider the following reversible interaction with an environment: $U = \sum_i |i\rangle\langle i| \otimes \pi_i$, where $\{|i\rangle\}$ is the computational basis and π_i is a unitary which acts on the environment system as $\pi_i |0\rangle = |i\rangle, \forall\ i$. Switching to the density matrix formalism, the decoherence map arising from the above interaction corresponds to

$$\mathcal{D}(\rho) = \mathrm{Tr}_E \left(U \rho \otimes |0\rangle\langle 0|_E U^\dagger \right) = \sum_i \langle i| \rho |i\rangle |i\rangle\langle i| ,$$

where ρ is the input state. Hence, in this concrete setting, the decoherence map $\mathcal{D}$ is a de-phasing map.

It is clear that $\mathcal{D}(\rho)$ will always be diagonal in the $\{|i\rangle\}$ basis, regardless of the input. Hence, as they have no coherences between distinct elements of $\{|i\rangle\}$, the states $\mathcal{D}(\rho)$

[2]That is, process theories with a classical interface.

correspond to classical probability distributions. In fact, the entirety of classical probability theory can be seen to arise from quantum theory by applying $\mathcal{D}$ to density matrices ρ as $\mathcal{D}(\rho)$, completely positive trace preserving maps $\mathcal{E}$ as $\mathcal{D}\left(\mathcal{E}\left(\mathcal{D}(\_)\right)\right)$, and POVM elements M as $\mathrm{Tr}\left(M\mathcal{D}(\_)\right)$. In this manner, one can consider classical probability theory to be a sub-theory of quantum theory —meaning that applying stochastic maps to probability distributions results in probability distributions— where $\mathcal{D}$ is the map restricting quantum theory to the classical sub-theory. This is precisely what is captured by the leak construction in example 3.2.4.

There are three key features of the decoherence map that we will use to define our hyper-decoherence map in section 5.3.

1. It is trace preserving, corresponding to the fact that it is a deterministic process.
2. It is idempotent, meaning

$$\mathcal{D}\left(\mathcal{D}(\rho)\right) = \mathcal{D}(\rho), \text{ for all } \rho.$$

   This corresponds to the intuitive fact that classical systems have no more coherence 'to lose'.
3. If $\mathcal{D}(\rho)$ is a pure classical state, i.e. $\mathcal{D}(\rho) = |i\rangle\langle i|$ for some i, then it is clearly also a pure quantum state. This is a consequence of the fact that decoherence arises from an irretrievable loss of information to an environment and if the state that results from this procedure is a state of maximal information, then no information can have been lost to the environment.

Note that the first two of those are captured by the leak construction whilst the third is an additional feature of decoherence which we have not discussed earlier.

5.2 Generalised theories

In this chapter we work with operationally defined theories, that is:

Definition 5.2.1 (Operational theory)**.** *An operational theory is a process theory (defined in section 2.1) with a classical interface (defined in section 2.2), that is, it has classically controlled processes (definition 2.2.3) and its processes are characterised by finite tomography (definition 2.2.4).*

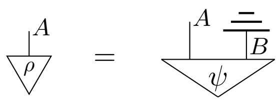
Some key results for this chapter regarding such theories were proved in the previous chapter:

1. causality from proposition 2.3.6,
2. convexity from lemma 4.3.1,
3. linearity again from lemma 4.3.1,

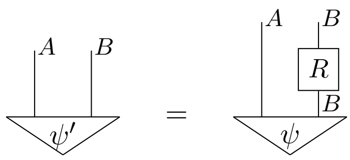
from the above requirements, it can be shown that the set of states, effects, and transformations give rise to real vector spaces, with the effects and transformations acting linearly on the vector space of states [41].

In what follows, we will require our post-quantum theory to satisfy the *purification* postulate, which was first introduced in [41].

Definition 5.2.2 (Purification [41])**.** *For every state on a given system A, there exists a pure bipartite state on some composite system AB, such that the original state arises as a marginalisation of this pure bipartite state:*



Here, ψ is said to purify *ρ. Moreover, any two pure states ψ and ψ' on the same system which purify the same state are connected by a reversible transformation*



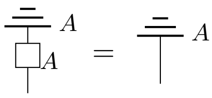
If one considers a pure state to be a state of maximal information, then the purification principle formalises the statement that each state of incomplete information arises in an essentially unique way due to a lack of information about an environment system. In a sense, purification can be thought of as a statement of "information conservation"; any missing information about the state of a given system can always be traced back to lack of information of some environment system. Or, more succinctly: information can only be discarded, not destroyed [44].

The purification principle, in conjunction with another natural principles, implies many quantum information processing [41] and computational primitives [101]. Examples include teleportation, no information without disturbance, no-bit commitment [41], and the existence of reversible controlled transformations. Moreover, purification also leads to a well-defined notion of thermodynamics [45, 47, 48].

5.3 Hyperdecoherence

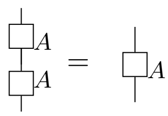
In section 5.1, the quantum to classical transition was modelled by a decoherence map restricting quantum systems to classical ones. We can analogously model a post-quantum to quantum transition with a *hyperdecoherence* map, represented by □, which restricts post-quantum systems —described by a generalised theory, definition 5.2.1 from section 5.2— to quantum ones. We now adapt the three key features of decoherence outlined at the end of section 5.1 to this general setting, ending this section with a formal definition of a post-quantum theory.

As in the quantum to classical transition, we think of this hyperdecoherence map as arising via some deterministic interaction with an environment system, after which the environment is discarded by marginalising with the unique deterministic effect. Hence, as with standard decoherence, hyperdecoherence can be thought of as an irretrievable loss of information to an environment. As deterministic processes are causal, the hyperdecoherence map should be *causal*:



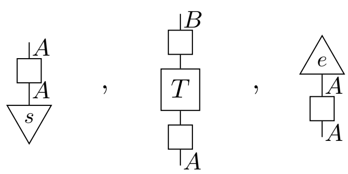
This is the analogue of point 1. from the end of section 5.1.

Moreover, hyperdecohering twice should be the same as hyperdecohering once, as the hyperdecohered system has no more 'post-quantum-coherence' to 'lose'. Hence this map should be *idempotent*:



This is the analogue of point 2. from the end of section 5.1.

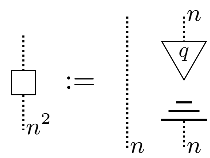
As was the case for classical theory in section 5.1, one can construct the entirety of quantum theory as a sub-theory of the post-quantum theory by appropriately applying $\mathcal{D}$ to states, transformations, and effects from the post-quantum theory. That is, density matrices, completely positive trace non-increasing maps, and POVM elements correspond to



Hence —as the □$_A$ are idempotent— quantum states, transformations, and effects are those left invariant by the hyperdecoherence map. These two conditions are perfectly captured by the leak construction (section 3.2.1) where we demand that the theory given by the construction is quantum theory. However, in section 5.1 there was a third key feature that we have not yet generalised.

As hyperdecoherence arises from an irretrievable loss of information to an environment, if a state resulting from this process is a state of maximal information, then no information can have been lost to the environment. We formalise this by demanding that pure states in the sub-theory are pure in the post-quantum theory. This is the analogue of point 3. from the end of section 5.1.

This requirement will play an important role in our proof, so it is worth discussing in more detail here. Firstly note that we need some assumption in addition to causality and idempotence in order to capture a sensible notion of hyperdecoherence. Indeed, even to adequately capture the standard notion of decoherence, one needs constraints beyond causality and idempotence. To see this, consider the following example. Let n denote a system in classical probability theory. Define systems in a "post-classical theory" by tensoring two n systems together to form $n \otimes n = n^2$, with the decoherence map given by tracing out one of the n systems and preparing a mixed classical state q, in its place. That is, here,



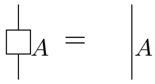
it is easy to see that this decoherence map is trace preserving (i.e. causal), idempotent, and recovers all states of the original n system – albeit tensored with a fixed mixed state. However, this does not properly capture the standard notion of decoherence as the "post-classical theory" is nothing but classical theory itself. Moreover, we can do a similar thing for quantum theory by having a quantum system $\mathcal{H}$ that "hyperdecoheres" from the quantum system $\mathcal{H} \otimes \mathcal{H}$, such that the "post-quantum theory" is nothing but quantum theory itself[3].

Note that these examples are ruled out by our assumption that pure states in the decohered sub-theory are pure in the full theory. One might then ask whether this is the minimal assumption needed to rule out these examples. Indeed, demanding the seemingly weaker constraint that the pre- and post-decohered systems have the same dimension also rules them out. Phrased in operational terms, preserving the dimension corresponds to the hyperdecoherence map preserving the number of pairwise perfectly distinguishable states. This requirement rules out the above

[3] A more complete understanding what it means for an extension of a theory to be non-trivial is the subject of ongoing work.

example. Indeed, if the decohered system has n distinguishable states then the original system has n^2. However, we prove in [103] that —given a strengthened version of purification— one can derive the requirement that pure quantum states are pure post-quantum states from the assumption that hyperdecoherence preserves the number of perfectly distinguishable states. This, in conjunction with the fact that pure classical states are always pure quantum states, leads us to propose the requirement that pure quantum states are pure as a defining feature of hyperdecoherence. See section 5.5 for a further rumination on this point.

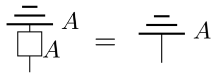
A final requirement of hyperdecoherence is that the original theory is not the same theory as the decohered theory, that is, one of the hyperdecoherence maps must be non-trivial. We say a hyperdecoherence map is *trivial* if it is equal to the identity transformation:



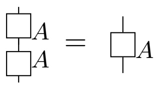
To summarise all of the above, we now formally define a post-quantum theory.

Definition 5.3.1 (Post-quantum theory)**.** *An operational theory (definition 5.2.1) is a post-quantum theory if, for each system type A, there exists a hyperdecoherence map*  *satisfying the following*

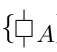
1.  *is causal:*



2.  *is idempotent:*



3. *Pure states in the sub-theory are pure states.*

Moreover, the sub-theory defined by the collection {} *is quantum theory and at least one of the hyperdecoherence maps must be non-trivial.*

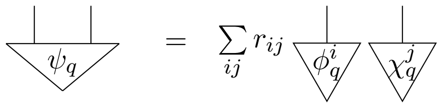
5.4 Result

We can now state our main result.

Main Theorem. *There is no post-quantum theory (definition 5.3.1) satisfying purification (definition 5.2.2).*

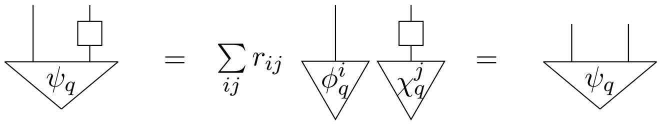
Before we present the proof, we give an intuitive sketch of how it will proceed. We prove that in any post-quantum theory satisfying causality and purification, the hyperdecoherence map must be trivial for all systems. The main idea of the proof is to show that by performing a suitable post-quantum measurement on the quantum Bell state and post-selecting on a suitable post-quantum effect, any post-quantum state can be steered to. As quantum states are left invariant by the hyperdecoherence map (even locally, as we show below), all post-quantum states are left invariant as well—due to the fact that they can be steered to using a quantum state. Hence, for each system, the hyperdecoherence map must be the identity, a contradiction.

Proof. For convenience we denote quantum states with a subscript q. Given a bipartite quantum state ψ_q, it can always be written as



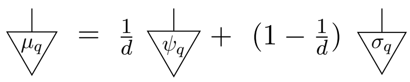
$r_{ij} \in \mathbb{R}$.

Idempotence of the hyperdecoherence map (point 2. of definition (5.3.1)) then gives



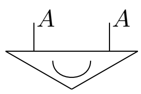
(5.1)

Next, consider the maximally mixed quantum state, $\mu_q := \frac{\mathbb{1}}{d}$, of a d-dimensional system. For any pure quantum state ψ_q, there is a state σ_q such that



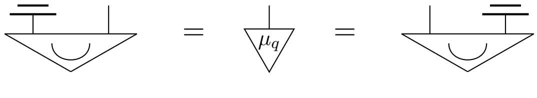
(5.2)

Now, denote the Bell state $\frac{1}{d}\sum_{ij}|ii\rangle\langle jj|$ for a d-dimensional system diagrammatically as:



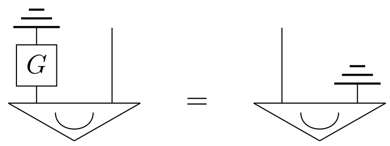
rather than as a bent wire because we do not know that the quantum cup and cap provide a cup and cap for the post-quantum theory.

As the hyperdecoherence map is causal (point 1. of definition (5.3.1)), marginalisation in the post-quantum theory is the same as in quantum theory. Hence, the marginals of the above Bell state are the maximally mixed quantum state

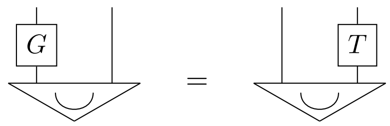


(5.3)

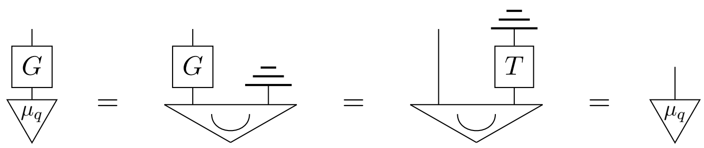
Equation (5.3), in conjunction with the fact that reversible transformations are causal, implies that for any reversible transformation G —including post-quantum transformations— we have



As the marginalised systems are of the same type, the purification principle implies[4] the existence of a reversible transformation T such that



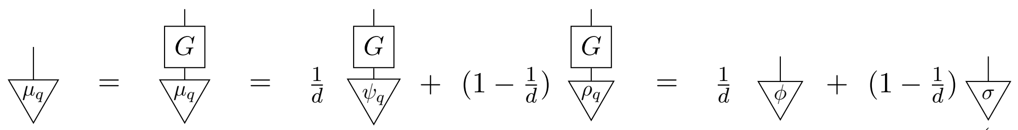
Hence,



(5.4)

the maximally mixed quantum state is invariant under *all* reversible transformations in the post-quantum theory.

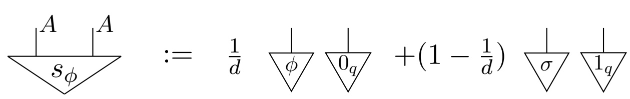
A standard result [41] obtained from purification is *transitivity*: given any two pure states of the same system there exists a reversible transformation between them. This result, in conjunction with equation (5.2), equation (5.4) and the fact that transformations act linearly on states, gives



(5.5)

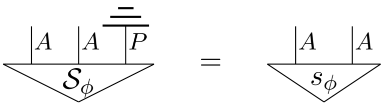
where ϕ is an arbitrary pure state in the theory. Hence, *any* pure state from the post-quantum theory arises in a decomposition of the quantum maximally mixed state.

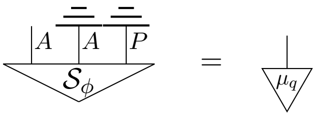
Now, as every (non-trivial) quantum system A has at least two perfectly distinguishable states, $\{|0\rangle\langle 0|, |1\rangle\langle 1|\}$, given the decomposition of equation (5.5), *convexity* implies the following is a state in the post-quantum theory:

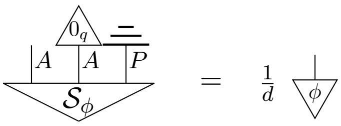


[4] Note that applying a reversible transformation to a pure state results in a pure state [45].

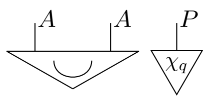
Consider a purification of this state, denoted $\mathcal{S}_\phi$, and note that it has the following properties:

1. 

2. 

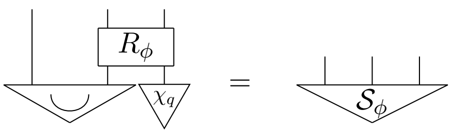
3. 

Where the effect 0_q is the quantum effect $\mathrm{Tr}(|0\rangle\langle 0|\,\_)$ which gives probability 1 for state 0_q and probability 0 for 1_q. As the product of two pure quantum states is a pure quantum state, the definition of hyperdecoherence (point 3. of definition (5.3.1)) implies that the following is another purification of μ_q with the same purifying system $A \otimes P$ as $\mathcal{S}_\phi$

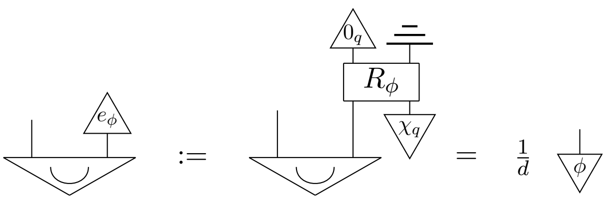


(5.6)

where χ_q is a pure quantum state. The purification principle implies that these two purifications are connected by a reversible transformation R_ϕ:



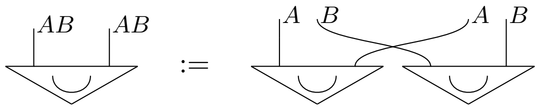
Using point 3. above, it then follows that there is an effect e_ϕ that *steers* the Bell state to ϕ



(5.7)

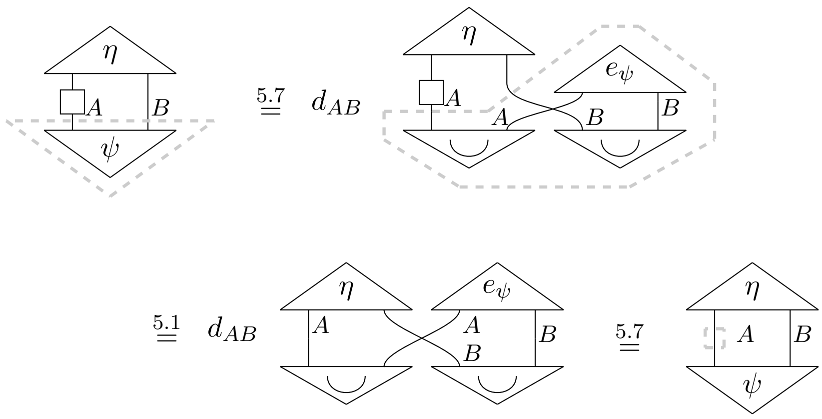
This is true for any pure state ϕ in the theory. That is, despite the fact that we do not know that the cup of quantum theory gives string diagrams for the post-quantum theory, it at least still provides an injective map from pure states to effects.

Using equation (5.1) and equation (5.7), and noting that the Bell state for a composite system is the composite of the Bell states for the single systems

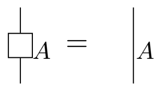


(5.8)

we have, for all pure states ψ and all effects η, that



This result, in conjunction with tomography and convexity, implies that, for all systems A,



□

As we know that there exists a post-classical theory which satisfies causality and purification and decoheres to classical theory, i.e. quantum theory, one might wonder at what stage our proof breaks down when analysing this situation. The main reason is that the maximally correlated state in classical probability theory is mixed and so equation (5.6) is no longer a valid purification. Hence, the reason why quantum theory cannot be extended in the manner proposed here is the existence of pure entangled states. Hence, it is the existence of a pure maximally entangled state in quantum theory that both distinguishes it from other C*-algebraic systems and prevents us from finding a post-quantum theory!

5.5 Discussion

From the famous theorems of Bell [27] and Kochen & Specker [95] to more recent results by Colbeck & Renner [63], and Pusey, Barrett & Rudolph [124], no-go theorems have a long history in the foundations of quantum theory. Most previous no-go theorems have been concerned with ruling out certain classes of hidden variable models from some set of natural assumptions. Hidden variables —or their contemporary incarnation as ontological models [84]— aim to provide quantum theory with an underlying classical description, where non-classical quantum features arise due to the fact that this description is 'hidden' from us.

Unlike these approaches, our result rules out certain classes of operationally-defined physical theories which can supersede quantum theory, yet reduce to it via a suitable process. To the best of our knowledge, our no-go theorem is the first of its kind. This may seem surprising given that it is an obvious question to ask. However, to even begin posing such questions in a rigorous manner requires a consistent way to define operational theories beyond quantum and classical theory. The mathematical underpinnings of such a framework have only recently been developed and investigated in the field of quantum foundations.

As with all no-go theorems, our result is only as strong as the assumptions which underlie it. We now critically examine each of our assumptions, outlining for each one the sense in which it can be considered 'natural', yet also suggesting ways in which a hypothetical post-quantum theory could violate it and hence escape the conclusion of our theorem.

Our first assumption is purification. As noted in section 5.2, the purification principle provides a way of formalising the natural idea that information can only be discarded [44], and any lack of information about the state of a given system arises in an essentially unique way due to a lack of information about some larger environment system. However, proposals for constructing theories in which information can be fundamentally destroyed have been suggested and investigated [117, 13, 139]. Such proposals take their inspiration from the Black Hole Information Loss paradox. Our result can therefore be thought of as providing another manner in which the fundamental status of information conservation can be challenged.

Our second assumption is causality of the classical interface. This principle allows one to uniquely define a notion of "past" and "future" for a given process in a diagram, and is equivalent to the statement that future measurement choices do not affect current experimental outcomes. As such, this principle appears to be fundamental to the scientific method. Despite this, recent work has shown how one can relax this principle to arrive at a notion of "indefinite" causality [120, 119, 43, 77]. In this case, there may be no matter of fact about whether a given process causally precedes another. The indefinite causal order between two processes has even been shown to be a resource which can be exploited to outperform theories satisfying the causality principle in certain information-theoretic tasks [8, 38]. Moreover, it has been suggested that any theory of Quantum Gravity must exhibit indefinite causal order [82, 83]. Hence, as in the previous paragraph, our result provides further motivation for discarding the notion of definite causal order in the search for theories superseding quantum theory.

As purification seems to require a unique way to marginalise multipartite states, one might wonder whether one can define a notion of purification without the causality principle. Indeed, recent work [9] has shown how one can formalise a purification principle in the absence of causality, and the time-symmetric notion of purification of chapter 4 may provide a route to this as well.

Another assumption in our theorem was the manner in which our hyperdecoherence map —the mechanism by which the post-quantum theory reduces to quantum theory— was formalised. It may not be the case that post-quantum physics gives rise to quantum physics via such a mechanism. Indeed, alternate proposals for how some hypothetical post-quantum theory reduces to quantum theory have been proposed [94]. Moreover, there is some evidence from research in quantum gravity that quantum pure states may become mixed at short length scales [113]. This suggests that quantum pure states may not be fundamentally pure in a full theory of quantum gravity. However, we see the necessity of the requirement that quantum pure states are pure in a potential post-quantum theory (point 3. from definition (5.3.1)) in our derivation as a feature rather than a bug. Indeed, it lends evidence to the assertion that to supersede quantum theory one must give up the requirement that states which appear pure within quantum theory are fundamentally pure. Despite this, our understanding of the quantum to classical transition in terms of decoherence suggests hyperdecoherence as the natural mechanism by which this should occur. Moreover, as discussed in the section 5.3 and proved in [103], one can derive that pure quantum states are pure post-quantum states from more primitive notions.

The last assumption underlying our no-go theorem is the generalised framework itself, introduced in section 5.2. While the operational methodology underlying this framework is part and parcel of the scientific method, it may not be the case that the correct way to formalise this methodology is by asserting that pieces of laboratory equipment can be composed together to result in experiments, as described in section 5.2. Indeed, it may be the case that the standard manner in which elements of a theory are composed together needs to be revised in order to go beyond the quantum formalism. Work on a more general compositional framework has already begun [81].

Our result can either be viewed as demonstrating that the fundamental theory of nature is quantum mechanical, or as showing in a rigorous manner that any post-quantum theory must radically depart from a quantum description of the world by abandoning the principle of causality, the principle of purification, or both.

Chapter 6

Summary and future work

This thesis is an exploration of the impact of the process theory framework on quantum foundations. The role of this framework is three-fold. Firstly, it provides physical theories with a convenient and intuitive diagrammatic representation, simplifying calculations and allowing for new insights. Secondly, it allows one to directly explore the impact of physical principles independent of the specifics of the theory or the way it has been formalised. Thirdly, it provides a way to compare different theories and, in particular, to understand the relationships between them via constructions such as the leak construction, biproduct completion and the idea of sub- and super-theories. There were three questions, posed at the beginning of the thesis, that we aimed to understand from this perspective: how are classical and quantum theory related; can we find a more compelling axiomatisation of quantum theory; and what can we learn about any theory that could one day supersede quantum theory? We will now summarise the contribution that this thesis made towards each of these.

The first key result is in understanding emergent classicality from a process theoretic perspective. In particular we show that there is —in the case of quantum theory— an equivalence between two natural process theoretic constructions, each leading to the theory of C*-algebras and completely positive maps. This provides a unified way to describe quantum and classical systems as well as how they interact. Moreover there are other emergent systems, these can be interpreted, via the biproduct completion, as being the result of the branching structure of quantum measurements, or, via the leak construction, as the result of a leaking information to an environment. Indeed, these two perspectives imply that the process theory of C*-algebras is really the 'correct' description of Operational Quantum Theory. Taking this as our starting point we can then precisely pin down the distinction between the quantum and classical systems in terms of their leaking properties, or equivalently, in terms of purity of their maximally correlated states.

Our second main result is that the process theory framework, along with the operational and compositional features discussed in the introduction, brings us very to reconstructing quantum theory. Specifically, all we need to add is the time-symmetric purification postulate to give us a particularly clear and compelling reconstruction of Operational Quantum Theory. A vital step in understanding that Operational Quantum Theory satisfies such a postulate was to introduce a process-theoretic notion of purity. We demonstrate that in certain cases —in particular for the case of classical theory— the standard notions of purity are insufficient and to overcome this limitation we must consider the leaks of a theory. Another key step in the reconstruction was demonstrating how (most of) the structure involved in the generalised probabilistic theory framework can be obtained from the classical interface for a process theory. Of particular interest is that any theory with a classical interface must be causal, demonstrating where the arrow of time arises in our reconstruction.

Finally, we investigated theories that could potential supersede quantum theory. We showed that if any operationally defined theory is able to reduce to quantum theory via a decoherence like mechanism, then it must fundamentally challenge our understanding of the behaviour of information. More precisely, we show that to go beyond quantum theory we must abandon one of the following: the notion of purification; that we can interact with the theory via a classical interface; the process-theory framework; or, the idea that this post-quantum theory is 'hidden' from us by a mechanism analogous to decoherence.

There is however still much work to be done before we can claim a complete understanding of all of these questions, and there are many more questions that this research opens up. Perhaps the most straightforward direction for future work would be to find an improved version of our reconstruction. For example, can we use fewer postulates, or make the existing postulates weaker? Is there a simpler or clearer definition of the sharp-dagger? Can the result be proved more directly without going via properties of the state space and the convex cones formalism, in particular, can we find an entirely diagrammatic proof? More interestingly than these however, is the question of the necessity of the classical interface for the reconstruction. Such an interface is —in one form or another— present in every reconstruction to date. It is an open question as to whether this structure too can be derived from diagrammatic principles rather than being simply postulated. We know that some of the structure of classical systems can be captured diagrammatically as 'spiders', however, this is clearly not sufficient to single out classical theory, can we find further axioms that do? Or, is it even necessary to capture classical theory precisely rather than just its process-theoretic features?

The second avenue of research stems from our no-go theorem, can we either extend this no-go theorem to rule our *all* post-quantum theories, or conversely, can we find some natural theory that gets around it by violating one of the assumptions of the theorem? Of particular interest —given current research in quantum foundations [120, 119, 43, 77, 8, 38, 82, 83]— is the abandonment of causality. However, as we have discussed this stems from the classical interface for a theory, and so, formulating a theory without causality will be challenging and may require a fundamental change to how we approach experimental tests for such a theory. Indeed, we may even have to modify the process theory framework to allow for a more relaxed notion of system types or a more relaxed notion of composition to formulate a post-quantum theory.

A third question to be explored regards the general notion of decoherence that we have introduced in the form of the leak construction and hyperdecoherence. Specifically, what can we say about theories that a) decohere to classical theory and b) cannot hyperdecohere from some other theory. Work has begun in this direction, for example, we recently showed in [126] that if a theory can decohere to classical theory (in a non-trivial way) then it must have entangled states. Can some or all of the structure of quantum theory be obtained from such a perspective?

Fourth, what does our notion of time-symmetric purification tell us about the world? Taken literally, it seems to be saying that the world is fundamentally built of time-symmetric pure processes, and that everything else arises from lack of access to systems to the past or future. This is true of all C*-algebraic systems, so what makes quantum theory special from this perspective is the ability to turn uncertainty about the past into uncertainty about the future. However, these pure processes are not always causal, and so, how (or if) these can be understood operationally is not clear. There have been other recent attempts to formulating quantum theory in a time-symmetric way, it would be interesting to explore whether any connection can be found to these alternative approaches.

Finally, when we introduced the notion of string diagrams, we explained this as a weakening of the constraints imposed on composition for process theories. What happens if we relax the other constraints? Relaxing the assumption that connections are between pairs of systems leads naturally to the definition of 'spiders' for a theory, and that relaxing the assumption that system types must match would necessarily lead to some non-trivial interactions. In fact, it seems plausible that such interactions would allow for a description of the entirety of quantum theory without needing to reference the actual processes at all. This would offer a radically different perspective on Categorical Quantum Mechanics which typically treats the processes as the primitive building blocks.

To conclude, the opening premise of this thesis was that the diagrammatic approach could be the 'correct' language to describe quantum theory. However, whether or not this is the case does not have a universal answer, it depends on what you want to do. If you want specific quantitative results —such as scattering amplitudes in quantum field theory, the secure key rates in quantum key distribution, or the laser frequency needed to drive a transition in a calcium ion— then this diagrammatic approach is unlikely to be providing you with much assistance. On the other hand, if you want to gain a deeper understanding of more qualitative, structural features of a theory, or, if you want to understand the relationship between different theories, then process theories provide a powerful tool. In particular, in this thesis, the process theory framework has allowed for understanding how how different theories relate to one another: we explore the sub-theories of quantum theory and the emergence of the classical world, as well as exploring super-theories and the challenges that await us as we try to go beyond quantum theory.

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