Disjointness graphs of segments

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Abstract

The *disjointness graph* $G = G(\mathcal{S})$ of a set of segments $\mathcal{S}$ in $\mathbb{R}^d$, $d \geq 2$, is a graph whose vertex set is $\mathcal{S}$ and two vertices are connected by an edge if and only if the corresponding segments are disjoint. We prove that the chromatic number of G satisfies $\chi(G) \leq (\omega(G))^4 + (\omega(G))^3$, where $\omega(G)$ denotes the clique number of G. It follows, that $\mathcal{S}$ has $\Omega(n^{1/5})$ pairwise intersecting or pairwise disjoint elements. Stronger bounds are established for lines in space, instead of segments.

We show that computing $\omega(G)$ and $\chi(G)$ for disjointness graphs of lines in space are NP-hard tasks. However, we can design efficient algorithms to compute proper colorings of G in which the number of colors satisfies the above upper bounds. One cannot expect similar results for sets of continuous arcs, instead of segments, even in the plane. We construct families of arcs whose disjointness graphs are triangle-free ($\omega(G) = 2$), but whose chromatic numbers are arbitrarily large.

1 Introduction

Given a set of (geometric) objects, their *intersection graph* is a graph whose vertices correspond to the objects, two vertices being connected by an edge if and only if their intersection is nonempty. Intersection graphs of intervals on a line [H57], more generally, chordal graphs [B61, Di61] and comparability graphs [D50], turned out to be *perfect graphs*, that is, for them and for each of their induced subgraph H, we have $\chi(H) = \omega(H)$, where $\chi(H)$ and $\omega(H)$ denote the chromatic number and the clique number of H, respectively. It was shown [HS58] that the complements of these graphs are also perfect, and based on these results, Berge [B61] conjectured and Lovász [Lo72] proved that the complement of every perfect graph is perfect.

Most geometrically defined intersection graphs are not perfect. However, in many cases they still have nice coloring properties. For example, Asplund and Grünbaum [AsG60] proved that

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every intersection graph G of axis-parallel rectangles in the plane satisfies $\chi(G) = O((\omega(G))^2)$. It is not known if the stronger bound $\chi(G) = O(\omega(G))$ also holds for these graphs. For intersection graphs of chords of a circle, Gyárfás [Gy85] established the bound $\chi(G) = O((\omega(G))^2 4^{\omega(G)})$, which was improved to $O(2^{\omega(G)})$ in [KoK97]. Here we have examples of $\chi(G)$ slightly superlinear in $\omega(G)$ [Ko88]. In some cases, there is no functional dependence between χ and ω. The first such example was found by Burling [Bu65]: there are sets of axis-parallel boxes in $\mathbb{R}^3$, whose intersection graphs are *triangle-free* ($\omega = 2$), but their chromatic numbers are arbitrarily large. Following Gyárfás and Lehel [GyL83], we call a family $\mathcal{G}$ of graphs χ*-bounded* if there exists a function f such that all elements $G \in \mathcal{G}$ satisfy the inequality $\chi(G) \leq f(\omega(G))$. The function f is called a *bounding function* for $\mathcal{G}$. Heuristically, if a family of graphs is χ-bounded, then its members can be regarded "nearly perfect". Consult [GyL85, Gy87, Ko04] for surveys.

At first glance, one might believe that, in analogy to perfect graphs, a family of intersection graphs is χ-bounded if and only if the family of their complements is. Burling's above mentioned constructions show that this is not the case: the family of complements of intersection graphs of axis-parallel boxes in $\mathbb{R}^d$ is χ-bounded with bounding function $f(x) = O(x \log^{d-1} x)$, see [Ka91]. More recently, Pawlik, Kozik, Krawczyk, Lasoń, Micek, Trotter, and Walczak [PKK14] have proved that Burling's triangle-free graphs can be realized as intersection graphs of segments in the plane. Consequently, the family of these graphs is *not* χ-bounded either. On the other hand, the family of their complements is, see Theorem 0.

To simplify the exposition, we call the complement of the intersection graph of a set of objects their *disjointness graph*. That is, in the disjointness graph two vertices are connected by an edge if and only if the corresponding objects are disjoint. Using this terminology, the following is a direct consequence of a result of Larman, Matoušek, Pach, and Törőcsik.

Theorem 0. [LMPT94] *The family of disjointness graphs of segments in the plane is χ-bounded. More precisely, every such graph G satisfies the inequality $\chi(G) \leq (\omega(G))^4$.*

For the proof of Theorem 0, one has to introduce four partial orders on the family of segments, and apply Dilworth's theorem [D50] four times. Although this method does not seem to generalize to higher dimensions, the statement does. We establish the following.

Theorem 1. *The disjointness graph G of any system of segments in $\mathbb{R}^d, d \geq 2$ satisfies the inequality $\chi(G) \leq (\omega(G))^4 + (\omega(G))^3$.*

Moreover, there is a polynomial time algorithm that, given the segments corresponding to the vertices of G, finds a complete subgraph $K \subseteq G$ and a proper coloring of G with at most $|V(K)|^4 + |V(K)|^3$ colors.

If we consider full lines in place of segments, we obtain stronger bounds.

Theorem 2. *(i) Let G be the disjointness graph of a set of lines in $\mathbb{R}^d$, $d \geq 3$. Then we have $\chi(G) \leq (\omega(G))^3$.*

(ii) Let G be the disjointness graph of a set of lines in the projective space $\mathbb{P}^d$, $d \geq 3$. Then we have $\chi(G) \leq (\omega(G))^2$.

In both cases, there are polynomial time algorithms that, given the lines corresponding to the *vertices of G, find complete subgraphs $K \subseteq G$ and proper colorings of G with at most $|V(K)|^3$ and $|V(K)|^2$ colors, respectively.*

Note that the difference between the two scenarios comes from the fact that parallel lines in the Euclidean space are disjoint, but the corresponding lines in the projective space intersect.

Most computational problems for geometric intersection and disjointness graphs are hard. It was shown by Kratochvíl and Nešetřil [KrN90] and by Cabello, Cardinal, and Langerman [CaCL13] that finding the clique number $\omega(G)$ resp. the independence number $\alpha(G)$ of disjointness graphs of segments in the plane are NP-hard. It is also known that computing the chromatic number $\chi(G)$ of disjointness and intersection graphs of segments in the plane is NP-hard [EET86]. Our next theorem shows that some of the analogous problems are also NP-hard for disjointness graphs of lines in space, while others are tractable in this case. In particular, according to Theorem 3(i), in a disjointness graph G of lines, it is NP-hard to determine $\omega(G)$ and $\chi(G)$. In view of this, it is interesting that one can design polynomial time algorithms to find proper colorings and complete subgraphs in G, where the number of colors is bounded in terms of the size of the complete subgraphs, in the way specified in the closing statements of Theorems 1 and 2.

Theorem 3. *(i) Computing the clique number $\omega(G)$ and the chromatic number $\chi(G)$ of disjointness graphs of lines in $\mathbb{R}^3$ or in $\mathbb{P}^3$ are NP-hard problems.*

(ii) Computing the independence number $\alpha(G)$ of disjointness graphs of lines in $\mathbb{R}^3$ or in $\mathbb{P}^3$, and deciding for a fixed k whether $\chi(G) \leq k$, can be done in polynomial time.

The bounding functions in Theorems 0, 1, and 2 are not likely to be optimal. As for Theorem 2 (i), we will prove that there are disjointness graphs G of lines in $\mathbb{R}^3$ for which $\frac{\chi(G)}{\omega(G)}$ are arbitrarily large. Our best constructions for disjointness graphs G' of lines in the projective space satisfy $\chi(G') \geq 2\omega(G') - 1$; see Theorem 2.3.

The proof of Theorem 1 is based on Theorem 0. Any strengthening of Theorem 0 leads to improvements of our results. For example, if $\chi(G) = O((\omega(G))^\gamma)$ holds with any $3 \leq \gamma \leq 4$ for the disjointness graph of every set of segments in the plane, then the proof of Theorem 1 implies the same bound for disjointness graphs of segments in higher dimensions. In fact, it is sufficient to verify this statement in 3 dimensions. For $d \geq 4$, we can find a projection in a generic direction to the 3-dimensional space that does not create additional intersections and then we can apply the 3-dimensional bound. We focus on the case $d = 3$.

It follows immediately from Theorem 0 that the disjointness (and, hence, the intersection) graph of any system of n segments in the plane has a clique or an independent set of size at least $n^{1/5}$. Indeed, denoting by $\alpha(G)$ the maximum number of independent vertices in G, we have

$$\alpha(G) \geq \frac{n}{\chi(G)} \geq \frac{n}{(\omega(G))^4},$$

so that $\alpha(G)(\omega(G))^4 \geq n$. Analogously, Theorem 1 implies that $\max(\alpha(G), \omega(G)) \geq (1 - o(1))n^{1/5}$ holds for disjointness (and intersection) graphs of segments in any dimension $d \geq 2$. For disjointness graphs of n lines in $\mathbb{R}^d$ (respectively, in $\mathbb{P}^d$), we obtain that $\max(\alpha(G), \omega(G))$

is $\Omega(n^{1/4})$ (resp., $\Omega(n^{1/3})$). Using more advanced algebraic techniques, Cardinal, Payne, and Solomon [CPS16] proved the stronger bounds $\Omega(n^{1/3})$ (resp., $\Omega(n^{1/2})$).

If the order of magnitude of the bounding functions in Theorems 0 and 1 are improved, then the improvement carries over to the lower bound on $\max(\alpha(G), \omega(G))$. Despite many efforts [LMPT94, KaPT97, Ky12] to construct intersection graphs of planar segments with small clique and independence numbers, the best known construction, due to Kynčl [Ky12], gives only

$$\max(\alpha(G), \omega(G)) \leq n^{\log 8/\log 169} \approx n^{0.405},$$

where n is the number of vertices. This bound is roughly the square of the best known lower bound.

Our next theorem shows that any improvement of the lower bound on $\max(\alpha(G), \omega(G))$ in the plane, even if it was not achieved by an improvement of the bounding function in Theorem 0, would also carry over to higher dimensions.

Theorem 4. *If the disjointness graph of any set of n segments in the plane has a clique or an independent set of size $\Omega(n^\beta)$ for some fixed $\beta \leq 1/4$, then the same is true for disjointness graphs of segments in $\mathbb{R}^d$ for any $d > 2$.*

A continuous arc in the plane is called a *string*. One may wonder whether Theorem 0 can be extended to disjointness graphs of strings in place of segments. The answer is no, in a very strong sense.

Theorem 5. *There exist triangle-free disjointness graphs of n strings in the plane with arbitrarily large chromatic numbers. Moreover, we can assume that these strings are simple polygonal paths consisting of at most* 4 *segments.*

Very recently, Mütze, Walczak, and Wiechert [MWW17] improved this result. They proved that the statement holds even if the strings are simple polygonal paths of at most 3 segments, moreover, any two intersect at most once.

The following problems remain open.

Problem 6. *(i) Is the family of disjointness graphs of polygonal paths, each consisting of at most two segments, χ-bounded?*

(ii) Is the previous statement true under the additional assumption that any two of the polygonal paths intersect in at most one point?

Problem 7. *Is the family of intersection graphs of lines in $\mathbb{R}^3$ χ-bounded?*

By Theorem 2, the family of *complements* of intersection graphs of lines in $\mathbb{R}^3$ is χ-bounded.

This paper is organized as follows. In the next section, we prove Theorem 2, which is needed for the proof of Theorem 1. Theorem 1 is established in Section 3. The proof of Theorem 4 is presented in Section 4. In Section 5, we construct several examples of disjointness graphs whose chromatic numbers are much larger than their clique numbers. In particular, we prove Theorem 5 and some similar statements. The last section contains the proof of Theorem 3 and remarks on the computational complexity of related problems.

2 Disjointness graphs of lines–Proof of Theorem 2

Claim 2.1. *Let G be the disjointness graph of a set of n lines in $\mathbb{P}^d$. If G has an isolated vertex, then G is perfect.*

Proof. Let $\ell_0 \in V(G)$ be a line representing an isolated vertex of G. Consider the bipartite multigraph H with vertex set $V(H) = A \cup B$, where A consists of all points of ℓ_0 that belong to at least one other line $\ell \in V(G)$, and B is the set of all (2-dimensional) planes passing through ℓ_0 that contain at least one other line $\ell \in V(G)$ different from ℓ_0. We associate with any line $\ell \in V(G)$ different from ℓ_0 an edge e_ℓ of H, connecting the point $p = \ell \cap \ell_0 \in A$ to the plane $\pi \in B$ that contains ℓ. Note that there may be several parallel edges in H. See Figure 1.

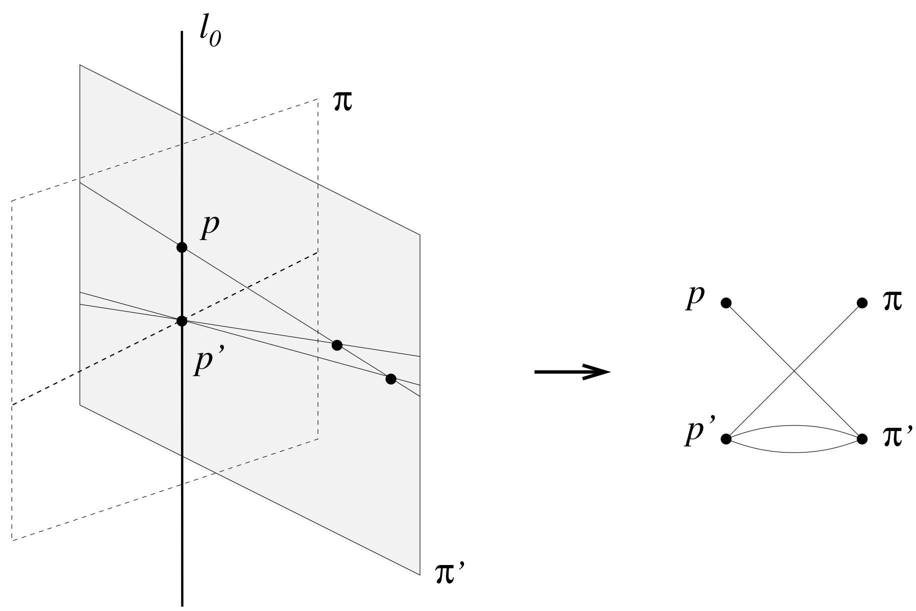

Figure 1: Construction of graph H in the proof of Claim 2.1

Observe that two lines $\ell, \ell' \in V(G) \setminus \{\ell_0\}$ intersect if and only if e_ℓ and $e_{\ell'}$ share an endpoint. This means that G minus the isolated vertex ℓ_0 is isomorphic to the complement of the line graph of H. The line graphs of bipartite multigraphs and their complements are known to be perfect. (For the complements of line graphs, this is the König-Hall theorem; see, e. g., [L93].) The graph G can be obtained by adding the isolated vertex ℓ_0 to a perfect graph, and is, therefore, also perfect. □

Proof of Theorem 2. We start with the proof of part (ii). Let G be a disjointness graph of lines in $\mathbb{P}^d$. Let $C \subseteq G$ be a maximal clique in G. Clearly, $|C| \le \omega(G)$. By the maximality of C, for every $\ell \in V(G) \setminus C$, there exists $c \in C$ that is not adjacent to ℓ in G. Hence, there is a partition of $V(G)$ into disjoint sets $V_c, c \in C$, such that $c \in V_c$ and c is an isolated vertex in the induced subgraph $G[V_c]$ of G. Applying Claim 2.1 separately to each subgraph $G[V_c]$, we obtain

$$\chi(G) \leq \sum_{c\in C} \chi(G[V_c]) = \sum_{c\in C} \omega(G[V_c]) \leq |C|\omega(G) \leq (\omega(G))^2.$$

Now we turn to the proof of part (i) of Theorem 2. Let G be a disjointness graph of lines in $\mathbb{R}^d$. Consider the lines in $V(G)$ as lines in the projective space $\mathbb{P}^d$, and consider the disjointness graph G' of these projective lines. Clearly, G' is a subgraph of G with the lines ℓ, $\ell' \in V(G)$ adjacent in G but not adjacent in G' if and only if ℓ and ℓ' are parallel. Thus, an independent set in G' induces a disjoint union of complete subgraphs in G, where the vertices of each complete subgraph correspond to pairwise parallel lines. If k is the maximal number of pairwise parallel lines in $V(G)$, then $k \leq \omega(G)$ and each independent set in G' can be partitioned into at most k independent sets in G. Applying part (ii), we obtain

$$\chi(G) \leq k\chi(G') \leq \omega(G)(\omega(G'))^2 \leq (\omega(G))^3.$$

Finally, we prove the last claim concerning polynomial time algorithms. In the proof of part (ii), we first took a maximal clique C in G. Such a clique can be efficiently found by a greedy algorithm. The partition of $V(G)$ into subsets $V_c, c \in C$, such that $c \in V_c$ is an isolated vertex in the subgraph $G[V_c]$, can also be done efficiently. It remains to find a clique of maximum size and a proper coloring of each perfect graph $G[V_c]$ with the smallest number of colors. It is well known that for perfect graphs, both of these tasks can be completed in polynomial time. See e.g. Corollary 9.4.8 on page 298 of [GLS88]. Alternatively, notice that in the proof of Claim 2.1 we showed that $G[V_c]$ is, in fact, the complement of the line graph of a bipartite multigraph (plus an isolated vertex). Therefore, finding a maximum size complete subgraph corresponds to finding a maximum size matching in a bipartite graph, while finding an optimal proper coloring of $G[V_c]$ corresponds to finding a minimal size vertex cover in a bipartite graph. This can be accomplished by much simpler and faster algorithms than the general purpose algorithms developed for perfect graphs.

To finish the proof of the algorithmic claim for part (ii), we can simply output as K the set C or one of the largest maximum cliques in $G[V_c]$ over all $c \in C$, whichever is larger. We color each V_c optimally, with pairwise disjoint sets of colors.

For the algorithmic claim about part (i), first color the corresponding arrangement of projective lines, and then refine the coloring by partitioning each color class into at most k smaller classes, where k is the maximum number of parallel lines in the arrangement. It is easy to find the value of k, just partition the lines into groups of parallel lines. Output as K the set we found for the projective lines, or a set of k parallel lines, whichever is larger. □

Theorem 2.3. *(i) There exist disjointness graphs G of families of lines in $\mathbb{R}^3$ for which the ratio $\chi(G)/\omega(G)$ is arbitrarily large.*

(ii) For any k one can find a system of lines in $\mathbb{P}^3$ whose disjointness graph G satisfies $\omega(G) = k$ and $\chi(G) = 2k - 1$.

Proof. First, we prove (i). For some m and d to be determined later, consider the set W_m^d of integer points in the d-dimensional hypercube $[1,m]^d$. That is, $W_m^d = \{1,2,\ldots,m\}^d$. A *combinatorial line* is a sequence of m distinct points of $x^1,\ldots x^m \in W_m^d$ such that for every $1 \le i \le d$, their ith coordinates $(x^j)_i$ are either the same for all $1 \le j \le m$ or we have $(x^j)_i = j$ for all $1 \le j \le m$. Note that the points of any combinatorial line lie on a geometric straight line. Let $\mathcal{L}$ denote the set of these geometric lines.

Let G denote the disjointness graph of $\mathcal{L}$. Since each line in $\mathcal{L}$ passes through m points of W_m^d, and $|W_m^d| = m^d$, we have $\omega(G) \le m^{d-1}$. (It is easy to see that equality holds here, but we do not need this fact for the proof.)

Consider any proper coloring of G. The color classes are families of pairwise crossing lines in $\mathcal{L}$. Observe that any such family has a common point in W_m^d, except some families consisting of 3 lines. Take an optimal proper coloring of G with $\chi(G)$ colors, and split each 3-element color class into two smaller classes. In the resulting coloring, there are at most $2\chi(G)$ color classes, each of which has a point of W_m^d in common. This means that the set of at most $2\chi(G)$ points of W_m^d (the "centers" of the color classes) "hits" every combinatorial line. By the density version of the Hales-Jewett theorem, due to Furstenberg and Katznelson [Bo98, FK91], if d is large enough relative to m, then any set containing fewer than half of the points of W_m^d will miss an entire combinatorial line. Choosing any m and a sufficiently large d depending on m, we conclude that $2\chi(G) \ge m^d/2$ and $\chi(G)/\omega(G) \ge m/4$.

Note that the family $\mathcal{L}$ consists of lines in $\mathbb{R}^d$. To find a similar family in 3-space, simply take the image of $\mathcal{L}$ under a projection to $\mathbb{R}^3$. One can pick a generic projection that does not change the disjointness graph G. This completes the proof of part (i). Note that the same construction does not work for projective lines, as the combinatorial lines in W_m^d fall into $2^d - 1$ parallel classes, so the chromatic number of the corresponding projective disjointness graph is smaller than 2^d.

To establish part (ii), fix a positive integer k, and consider a set S of $2k+1$ points in general position (no four in a plane) in $\mathbb{R}^3 \subseteq \mathbb{P}^3$. Let $\mathcal{L}$ denote the set of $\binom{2k+1}{2}$ lines determined by them. Note that by the general position assumption, two lines in $\mathcal{L}$ intersect if and only if they have a point of S in common. This means that the disjointness graph G of $\mathcal{L}$ is isomorphic to the *Kneser graph* $G^*(2k+1,2)$ formed by all 2-element subsets of a $(2k+1)$-element set. Obviously, $\omega(G^*(n,m)) = \lfloor n/m \rfloor$, so $\omega(G) = k$. By a celebrated result of Lovász [Lo78], $\chi G^*(n,m) = n - 2m + 2$ for all $n \ge 2m-1$. Thus, we have $\chi(G) = 2k-1$, as claimed. □

3 Disjointness graphs of segments–Proof of Theorem 1

If all segments lie in the same plane, then by Theorem 0 we have $\chi(G) \le (\omega(G))^4$. Our next theorem generalizes this result to the case where the segments lie in a bounded number of distinct planes.

Theorem 3.1. *Let G be the disjointness graph of a set of segments in $\mathbb{R}^d, d > 2$, that lie in the* *union of k two-dimensional planes. We have*

$$\chi(G) \leq (k-1)\omega(G) + (\omega(G))^4.$$

Given the segments representing the vertices of G and k planes containing them, there is a polynomial time algorithm to find a complete subgraph $K \subseteq G$ and a proper coloring of G with at most $(k-1)|V(K)| + |V(K)|^4$ colors.

Proof. Let $\pi_1, \pi_2, \ldots, \pi_k$ be the planes containing the segments. Partition the vertex set of G into the classes $V_1, V_2, \ldots, V_k$ by putting a segment s into the class V_i, where i is the *largest* index for which π_i contains s.

For $i = 1, 2, \ldots, k$, we define subsets $W_i, Z_i \subseteq V_i$ with $Z_i \subseteq W_i \subseteq V_i$ by a recursive procedure, as follows. Let $W_1 = V_1$ and let $Z_1 \subseteq W_1$ be a *maximal size* clique in $G[W_1]$.

Assume that the sets $W_1, \ldots, W_i$ and $Z_1, \ldots, Z_i$ have already been defined for some $i < k$. Let W_{i+1} denote the set of all vertices in V_{i+1} that are adjacent to every vertex in $Z_1 \cup Z_2 \cup \ldots \cup Z_i$, and let Z_{i+1} be a maximal size clique in $G[W_{i+1}]$. By definition, $\bigcup_{i=1}^k Z_i$ induces a complete subgraph in G, and we have

$$\sum_{i=1}^{k} |Z_i| \leq \omega(G).$$

Let s be a segment belonging to Z_i, for some $1 \leq i < k$. A point p of s is called a *piercing point* if $p \in \pi_j$ for some $j > i$. Notice that in this case, s "pierces" the plane π_j in a single point, otherwise we would have $s \subset \pi_j$, contradicting our assumption that $s \in V_i$. Letting P denote the set of piercing points of all segments in $\bigcup_{i=1}^k Z_i$, we have

$$|P| \leq \sum_{i=1}^{k} (k-i)|Z_i| \leq (k-1) \sum_{i=1}^{k} |Z_i| \leq (k-1)\omega(G).$$

Let $V_0 = V(G) \setminus \bigcup_{i=1}^k W_i$. We claim that every segment in V_0 contains at least one piercing point. Indeed, if $s \in V_i \setminus W_i$ for some $i \leq k$, then s is not adjacent in G to at least one segment $t \in Z_1 \cup \ldots \cup Z_{i-1}$. Thus, s and t are not disjoint, and their intersection point is a piercing point, at which t pierces the plane π_i.

Assign a color to each piercing point $p \in P$. Coloring every segment in V_0 by the color of one of its piercing points, we get a proper coloring of $G[V_0]$ with $|P|$ colors, so that $\chi(G[V_0]) \leq |P|$.

For every $i \leq k$, all segments of W_i lie in the plane π_i. Therefore, we can apply Theorem 0 to their disjointness graph $G[W_i]$, to conclude that $\chi(G[W_i]) \leq (\omega(G[W_i]))^4$. By definition, Z_i induces a maximum complete subgraph in $G[W_i]$, hence $|Z_i| = \omega(G[W_i])$ and $\chi(G[W_i]) \leq |Z_i|^4$.

Putting together the above estimates, and taking into account that $\bigcup_{i=1}^k Z_i$ induces a complete subgraph in G, we obtain

$$\chi(G) \leq \chi(G[V_0]) + \sum_{i=1}^{k} \chi(G[W_i]) \leq |P| + \sum_{i=1}^{k} |Z_i|^4$$

$$\leq (k-1)\omega(G) + (\sum_{i=1}^{k} |Z_i|)^4 \leq (k-1)\omega(G) + (\omega(G))^4,$$

as required.

We can turn this estimate into a polynomial time algorithm as required, using the fact that the proof of Theorem 0 is constructive. In particular, we use that, given a family of segments in the plane, one can efficiently find a subfamily K of pairwise disjoint segments and a proper coloring of the disjointness graph with at most $|K|^4$ colors. This readily follows from the proof of Theorem 0, based on the four easily computable (semi-algebraic) partial orders on the family of segments, introduced in [LMPT94].

Our algorithm finds the sets V_i, as in the proof. However, finding W_i and a maximum size clique $Z_i \subseteq W_i$ is a challenge. Instead, we use the constructive version of Theorem 0 to find $Z_i \subseteq W_i$ and a proper coloring of $G[W_i]$. The definition of W_i remains unchanged. Next, the algorithm identifies the piercing points.

The algorithm outputs the clique $K = \bigcup Z_i$ and the coloring of G. The latter one is obtained by combining the previously constructed colorings of the subgraphs $G[W_i]$ (using disjoint sets of colors for different subgraphs), and coloring each remaining vertex by a previously unused color, associated with one of the piercing points the corresponding segment passes through. □

Proof of Theorem 1. Consider the set of all lines in the *projective* space $\mathbb{P}^d$ that contain at least one segment belonging to $V(G)$. Let $\bar{G}'$ denote the disjointness graph of these lines. Obviously, we have $\omega(\bar{G}') \leq \omega(G)$. Thus, Theorem 2(ii) implies that

$$\chi(\bar{G}') \leq (\omega(\bar{G}'))^2 \leq (\omega(G))^2.$$

Let C be the set of lines corresponding to the vertices of a maximum complete subgraph in $\bar{G}'$. Fix an optimal proper coloring of $\bar{G}'$. Suppose that we used k "planar" colors (each such color is given to a set of lines that lie in the same plane) and $\chi(\bar{G}') - k$ "pointed" colors (each given to the vertices corresponding to a set of lines passing through a common point).

Consider now G, the disjointness graph of the segments. Let G_0 denote the subgraph of G induced by the set of segments whose supporting lines received one of the k planar colors in the above coloring of $\bar{G}'$. These segments lie in at most k planes. Therefore, applying Theorem 3.1 to G_0, we obtain

$$\chi(G_0) \leq (k-1)\omega(G_0) + (\omega(G_0))^4 \leq (k-1)\omega(G) + (\omega(G))^4.$$

For $i, 1 \leq i \leq \chi(\bar{G}') - k$, let G_i denote the subgraph of G induced by the set of segments whose supporting lines are colored by the ith pointed color. It is easy to see that G_i is the complement of a chordal graph. That is, the complement of G_i contains no induced cycle of length larger than 3. According to a theorem of Hajnal and Surányi [HS58], any graph with this property is perfect, so that

$$\chi(G_i) = \omega(G_i) \leq \omega(G).$$

Putting these bounds together, we obtain that

$$\chi(G) \le \chi(G_0) + \sum_{i=1}^{\chi(\bar{G}')-k} \chi(G_i) \le (k-1)\omega(G) + (\omega(G))^4 + \sum_{i=1}^{\chi(\bar{G}')-k} \omega(G)$$
$$\le ((\omega(\bar{G}'))^2 - 1)\omega(G) + (\omega(G))^4 < (\omega(G))^3 + (\omega(G))^4.$$

To prove the algorithmic claim in Theorem 1, we first apply the algorithm of Theorem 2 to the disjointness graph $\bar{G}'$. We distinguish between planar and pointed color classes and find the subgraphs G_i. We output a coloring of G, where for each $G_i, i > 0$ we use the smallest possible number of colors (G_i is perfect, so its optimal coloring can be found in polynomial time), and we color G_0 by the algorithm described in Theorem 3.1. The subgraphs G_i are colored using pairwise disjoint sets of colors. We output the largest clique K that we can find. This may belong to a subgraph G_i with $i > 0$, or may be found in G_0 or in $\bar{G}'$ by the algorithms given by Theorem 3.1 or Theorem 2, respectively. (In the last case, we need to turn a clique in $\bar{G}'$ into a clique of the same size in G, by picking an arbitrary segment from each of the pairwise disjoint lines.) □

4 Ramsey-type bounds in $\mathbb{R}^2$ vs. $\mathbb{R}^3$–Proof of Theorem 4

As we have pointed out in the Introduction, it is sufficient to establish Theorem 4 in $\mathbb{R}^3$. We rephrase Theorem 4 for this case in the following form.

Theorem 4.1. *Let $f(m)$ be a function with the property that for any disjointness graph G of a system of segments in $\mathbb{R}^2$ with $\max(\alpha(G), \omega(G)) \le m$ we have $|V(G)| \le f(m)$.*

Then for any disjointness graph G of a system of segments in $\mathbb{R}^3$ with $\max(\alpha(G), \omega(G)) \le m$ we have $|V(G)| \le f(m) + m^4$.

Applying Theorem 4.1 with $f(k) = ck^{1/\beta}$, Theorem 4 immediately follows. We prove Theorem 4.1 by adapting the proof of Theorem 3.1.

Proof of Theorem 4.1. Let G be the disjointness graph of a set of segments in $\mathbb{R}^3$ with $\omega(G) \le m$ and $\alpha(G) \le m$.

First, assume that all segments lie in the union of k planes, for some $k \ge 1$. Define the sets of vertices V_i, W_i, and Z_i for every $1 \le i \le k$, as in the proof of Theorem 3.1, and let $V_0 = V(G) \setminus \bigcup_{i=1}^{k} W_i$. Since all elements of W_i lie in the same plane, the subgraph induced by them is a planar segment disjointness graph for every $i \ge 1$. We can clearly represent these graphs by segments in a common plane π such that two segments intersect if and only they come from the same set W_i and there they intersect. In this way, we obtain a system of segments in the plane whose disjointness graph G^* is the *join* of the graphs $G[W_i]$, i.e., G^* is obtained by taking the disjoint union of $G[W_i]$ (for all $i \ge 1$) and adding all edges between W_i and W_j for every pair $i \neq j$. Clearly, we have

$$\omega(G^*) = \sum_{i=1}^{k} \omega(G[W_i]) = \sum_{i=1}^{k} |Z_i| \leq \omega(G) \leq m,$$

and

$$\alpha(G^*) = \max_{i=1}^{k} \alpha(G[W_i]) \leq \alpha(G) \leq m.$$

By our assumption, G^* has at most $f(m)$ vertices, so that $\sum_{i=1}^{k} |W_i| \leq f(m)$. As we have seen in the proof of Theorem 3.1, the total number of piercing points is at most $(k-1)\sum_{i=1}^{k} |Z_i| \leq (k-1)\omega(G) < km$, and each segment in V_0 contains at least one of them. Each piercing point is contained in at most m segments, because these segments induce an independent set in G. Thus, we have $|V_0| < km^2$ and

$$|V(G)| = |V_0| + \bigcup_{i=1}^{k} |W_i| < km^2 + |V(G^*)| \leq km^2 + f(m).$$

Now we turn to the general case, where there is no bound on the number of planes containing the segments. As in the proof of Theorem 1, we consider the disjointness graph $\bar{G}'$ of the supporting lines of the segments in the projective space $\mathbb{P}^3$. Clearly, we have $\omega(\bar{G}') \leq \omega(G) \leq m$, so by Theorem 1 we have $\chi(\bar{G}') \leq m^2$. Following the proof of Theorem 1, take an optimal coloring of $\bar{G}'$, and let G_0 denote the subgraph of G induced by the segments whose supporting lines received one of the planar colors. Letting k denote the number of planar colors, for every $i, 1 \leq i \leq \omega(\bar{G}') - k$, let G_i denote the subgraph of G induced by the set of segments whose supporting lines received the ith pointed color. As in the proof of Theorem 1, every $G_i, i \geq 1$ is perfect and, hence, its number of vertices satisfies

$$|V(G_i)| \leq \chi(G_i)\alpha(G_i) \leq \omega(G_i)\alpha(G_i) \leq \omega(G)\alpha(G) \leq m^2.$$

The segments belonging to $V(G_0)$ lie in at most k planes. In view of the previous paragraph, $|V(G_0)| \leq km^2 + f(m)$ vertices. Combining the above bounds, we obtain

$$|V(G)| = |V(G_0)| + \sum_{i=1}^{\chi(\bar{G}')-k} |V(G_i)| \leq km^2 + f(m) + (\chi(\bar{G}') - k)m^2$$

$$\leq km^2 + f(m) + (m^2 - k)m^2 \leq f(m) + m^4,$$

which completes the proof. $\square$

5 Constructions–Proof of Theorem 5

The aim of this section is to describe various arrangements of geometric objects in 2, 3, and 4 dimensions with triangle-free disjointness graphs, whose chromatic numbers grow logarithmically with the number of objects. (This is much faster than the rate of growth in Theorem 2.3.) Our constructions can be regarded as geometric realizations of a sequence of graphs discovered by Erdős and Hajnal.

Definition 5.1. [EH64]. *Given $m > 1$, let H_m, the m-th* shift graph*, be a graph whose vertex set consists of all ordered pairs (i, j) with $1 \le i < j \le m$, and two pairs (i, j) and (k, l) are connected by an edge if and only if $j = k$ or $l = i$.*

Obviously, H_m is triangle-free for every $m > 1$. It is not hard to show (see, e.g., [L93], Problem 9.26) that $\chi(H_m) = \lceil \log_2 m \rceil$. Therefore, Theorem 5 follows directly from part (vii) of the next theorem.

Theorem 5.2. *For every m, the shift graph H_m can be obtained as a disjointness graph, where each vertex is represented by*
(i) a line minus a point in $\mathbb{R}^2$;
(ii) a two-dimensional plane in $\mathbb{R}^4$;
(iii) the intersection of two general position half-spaces in $\mathbb{R}^3$;
(iv) the union of two segments in $\mathbb{R}^2$;
(v) a triangle in $\mathbb{R}^4$;
(vi) a simplex in $\mathbb{R}^3$;
(vii) a polygonal curve in $\mathbb{R}^2$, consisting of four line segments.

Proof. (i) Let $L_1, \ldots, L_m$ be lines in general position in the plane. For any $1 \le i < j \le m$, let us represent the pair (i, j) by the "pointed line" $p_{ij} = L_i \setminus L_j$.

Fix $1 \le i < j \le m$, $1 \le k < l \le m$, and set $X = p_{ij} \cap p_{kl} = (L_i \cap L_k) \setminus (L_j \cup L_l)$. If $i = k$, then X is an infinite set.

Otherwise, $L_i \cap L_k$ consists of a single point. In this case, X is empty if and only if this point belongs to $L_j \cup L_l$. By the general position assumption, this happens if and only if $j = k$ or $l = i$. Thus, the disjointness graph of the sets p_{ij}, $1 \le i < j \le m$, is isomorphic to the shift graph H_m.

(ii) Let $h_1, \ldots h_m$ be hyperplanes in general position in $\mathbb{R}^4$. For every i, fix another hyperplane h'_i, parallel (but not identical) to h_i. For any $1 \le i < j \le m$, represent the pair (i, j) by the two dimensional plane $p_{ij} = h_i \cap h'_j$.

Given $1 \le i < j \le m$, $1 \le k < l \le m$, the set $X = p_{ij} \cap p_{k,l} = h_i \cap h'_j \cap h_k \cap h'_l$ is the intersection of four hyperplanes. If the four hyperplanes are in general position, then X consists of a single point.

If the hyperplanes are not in general position, then some of the four indices must coincide. If $i = k$ or $j = l$, then two of the hyperplanes coincide and X is a line. In the remaining cases, when $j = k$ or $l = i$, among the four hyperplanes two are parallel, so their intersection X is empty.

(iii) For $i = 1, \ldots, m$, define the half-space h_i as

$$h_i = \{(x, y, z) \in \mathbb{R}^3 \mid ix + i^2 y + i^3 z < 1\}.$$

Note that the bounding planes of these half-spaces are in general position. For any $1 \le i < j \le m$, represent the pair (i, j) by $p_{ij} = h_j \setminus h_i$.

Now let $1 \le i < j \le m$, $1 \le k < l \le m$. If $j = k$ or $l = i$, the sets p_{ij} and p_{kl} are obviously disjoint. If $i = k$ or $j = l$, then $p_{ij} \cap p_{kl}$ is the intersection of at most 3 half-spaces in general position, so it is unbounded and not empty.

It remains to analyze the case when all four indices are distinct. This requires some calculation. We assume without loss of generality that $j < l$. Consider the point $P = (x, y, z) \in \mathbb{R}^3$ with $x = \frac{1}{i} + \frac{1}{j} + \frac{1}{k}$, $y = -\frac{1}{ij} - \frac{1}{jk} - \frac{1}{ki}$ and $z = \frac{1}{ijk}$. This is the intersection point of the bounding planes of h_i, h_j and h_k. Therefore, the polynomial $zu^3 + yu^2 + xu - 1$ vanishes at $u = i, j, k$, and it must be positive at $u = l$, as $l > i, j, k$ and the leading coefficient is positive. This means that P lies in the open half-space h_l. As the bounding planes of h_i, h_j and h_k are in general position, one can find a point P' arbitrarily close to P (the intersection point of these half-planes) with $P' \in h_j \setminus (h_i \cup h_k)$. If we choose P' close enough to P, it will also belong to h_l. Thus, $P' \in p_{ij} \cap p_{kl}$, and so p_{ij} and p_{kl} are not disjoint.

(iv), (v), and (vi) directly follow from (i), (ii) and (iii), respectively, by replacing the unbounded geometric objects representing the vertices with their sufficiently large bounded subsets.

(vii) Let C be an almost vertical, very short curve (arc) in the plane, convex from the right (that is, the set of points to the right of C is convex) lying in a small neighborhood of $(0, 1)$. Let $p_1, p_2, \ldots, p_m$ be a sequence of m points on C such that p_j is above p_i if and only if $j > i$. For every $1 \le i \le m$, let T_i be an equilateral triangle whose base is horizontal, whose upper vertex is p_i, and whose center is on the x-axis. Let q_i and r_i be the lower right and lower left vertices of T_i, respectively. It is easy to see that T_j contains T_i in its interior if $j > i$. Let s_i be a point on $r_i p_i$, very close to p_i.

Let us represent the vertex (i, j) of the shift graph H_m by the polygonal curve $p_{ij} = t_{ij} p_j q_j r_j s_j$, where the point t_{ij} is on the x-axis slightly to the left of the line $p_i p_j$. Note that if C is short enough and close enough to vertical, then t_{ij} can be chosen so that it belongs to the interior of all triangles T_k for $1 \le k \le m$. In particular, the entire polygonal path p_{ij} belongs to T_j.

It depends on our earlier choices of the vertices $p_{i'}$, how close we have to choose s_i to p_i. Analogously, it depends on our earlier choices of $p_{i'}$ and $s_{i'}$, how close we have to choose t_{ij} to the line. Instead of describing an explicit construction, we simply claim that with proper choices of these points, we obtain a disjointness representation of the shift graph.

To see this, let $1 \le i < j \le m$, $1 \le k < l \le m$. If $j = l$, then three of the four line segments in p_{ij} and p_{kl} are the same, so they intersect. Otherwise, assume without loss of generality that $j < l$. As noted above, p_{ij} belongs to the triangle T_j, which, in turn, lies in the interior of T_l. Three segments of p_{kl} lie on the edges of T_l, so if p_{ij} and p_{kl} meet, the fourth segment, $t_{kl} p_l$, must meet p_{ij}. This segment enters the triangle T_j, so it meets one of its edges. Namely, for

$j > k$ it follows from the convexity of the curve C that the segment $t_{kl}p_l$ intersects the edge p_jq_j and, hence, also p_{ij}. Analogously, if $j < k$, then $t_{kl}p_l$ intersects the interior of the edge r_jp_j. This is true even if t_{kl} were chosen *on* the line p_kp_l, so choosing s_j close enough to p_j, one can make sure that $t_{kl}p_l$ intersects r_js_j and, hence, also p_{ij}. On the other hand, if $j = k$, we choose t_{kl} so that $t_{kl}p_l$ is just slightly to the left of $p_j = p_k$, so it enters T_j through the interior of the segment s_jp_j that is *not* contained in p_{ij}. To see that in this case p_{ij} and p_{kl} are disjoint, it is enough to check that $t_{kl}p_l$ and $t_{ij}p_j$ are disjoint. This is true, because p_j is on the right of $t_{kl}p_l$ and (from the convexity of C) the slope of the segments is such that p_j is the closest point of the segment $t_{ij}p_j$ to $t_{kl}p_l$. □

6 Complexity issues–Proof of Theorem 3

The aim of this section is to outline the proof of Theorem 3 and to establish some related complexity results. For simplicity, we only consider systems of lines in the *projective* space $\mathbb{P}^3$. It is easy to see that by removing a generic hyperplane (not containing any of the intersection points), we can turn a system of projective lines into a system of lines into $\mathbb{R}^3$ without changing the corresponding disjointness graph.

It is more convenient to speak about intersection graphs rather than their complement in formulating the next theorem.

Theorem 6.1 *(i) If G is a graph with maximum degree at most* 3*, then G is an intersection graph of lines in $\mathbb{P}^3$.*

(ii) For an arbitrary graph G the line graph *of G is an intersection graph of lines in $\mathbb{P}^3$.*

Proof. (i) Suppose first that G is triangle-free. Let $V(G) = \{v_1, \dots, v_k\}$. Let vertex v_1 be represented by an arbitrary line L_1. Suppose, recursively, that the line L_j representing vertex j has already been defined for every $j < i$. We will maintain the "general position" property that no doubly ruled surface contains more than 3 pairwise disjoint lines. We must choose L_i representing v_i such that

(a) it intersects the lines representing the neighbors v_j of v_i with $j < i$,

(b) it does not intersect the lines representing the non-neighbors v_j with $j < i$, and

(c) we maintain our general position conditions.

These are simple algebraic conditions. The vertex v_i has at most 3 neighbors among v_j for $j < i$, and they must be represented by pairwise disjoint lines. Thus, the Zariski-closed conditions from (a) determine an irreducible variety of lines, so unless they force the violation of a specific other (Zariski-open) condition from (b) or (c), all of those conditions can be satisfied with a generic line through the lines representing the neighbors. In case v_i has three neighbors v_j with $j < i$, the corresponding condition forces L_i to be in one of the two families of lines on a doubly ruled surface Σ. This further forces L_i to intersect *all* lines of the other family on Σ, but due to the general position condition, none of the vertices of G is represented by lines there, except the three neighbors of v_i. We would violate the general position condition with the new line L_i if the family we choose it from already had three members representing vertices. However, this would mean that the degrees of the neighbors of v_i would be at least 4, a contradiction. In case v_i has fewer than 3 neighbors, the requirement of L_i intersecting the corresponding lines does not force L_i to intersect any further lines or to lie on any doubly ruled surface.

We prove the general case by induction on $|V(G)|$. Suppose that $a, b, c \in V(G)$ form a triangle in G and that the subgraph of G induced by $V(G)\setminus\{a, b, c\}$ can be represented as the intersection graph of distinct lines in $\mathbb{P}^3$. Note that each of a, b and c has at most a single neighbor in the rest of the graph. We extend the representation of the subgraph by adding three lines L_a, L_b and L_c, representing the vertices of the triangle. We choose these lines in a generic way so that they pass through a common point p, and L_a intersects the line representing the neighbor of a (in case it exists), and similarly for L_b and L_c. It is clear that we have enough degrees of freedom (at least six) to avoid creating any further intersection. For instance, it suffices to choose p outside all lines in the construction and all planes determined by intersecting pairs of lines.

(ii) Assign distinct points of $\mathbb{P}^3$ to the vertices of G so that no four points lie in a plane. Represent each edge $xx' \in E(G)$ by the line connecting the points assigned to x and x'. As no four points are coplanar, two lines representing a pair of edges will cross if and only if the edges share an endpoint. Therefore, the intersection graph of these lines is isomorphic to the edge graph of G. □

The following theorem implies Theorem 3, as the disjointness graph $H = \bar{G}$ is the complement of the intersection graph G, and we have $\omega(G) = \alpha(H)$, $\alpha(G) = \omega(H)$, $\chi(G) = \theta(H)$, and $\theta(G) = \chi(H)$. Here $\theta(H)$ denotes the *clique covering number* of H, that is, the smallest number of complete subgraphs of H whose vertex sets cover $V(H)$.

Theorem 6.2. *Let H be an intersection graph of n lines in the Euclidean space $\mathbb{R}^3$ or in the projective space $\mathbb{P}^3$.*

(i) Computing $\alpha(H)$, the independence number of H, is NP-hard.

(ii) Computing $\theta(H)$, the clique covering number of H, is NP-hard.

(iii) Deciding whether $\chi(H) \leq 3$, that is, whether H is 3-colorable, is NP-complete.

(iv) Computing $\omega(H)$, the clique number of H, is in P.

(v) Deciding whether $\theta(H) \leq k$ for a fixed k is in P.

(vi) All the above statements remain true if H is not given as an abstract graph, but with its intersection representation with lines.

Proof. We only deal with the case where the lines are in $\mathbb{P}^3$. The reduction of the Euclidean case to this case is easy.

(i) The problem of determining the independence number of 3-regular graphs is NP-hard; see [AK00]. By Theorem 6.1(i), all 3-regular graphs are intersection graphs of lines in $\mathbb{P}^3$.

(ii) The *vertex cover number* of a graph H is the smallest number of vertices with the property that every edge of H is incident to at least one of them. Note that the vertex cover number of H is $|V(H)| - \alpha(H)$. In [Po74], it was shown that the problem of determining the *vertex cover number* is NP-hard even for triangle-free graphs. We can reduce this problem to the problem of determining the clique covering number of an intersection graph of lines. For this, note that each complete subgraph of the line graph H' of H corresponds to a star of H and thus $\theta(H')$ is the vertex cover number of H. The reduction is complete, as H' is the intersection graph of lines in $\mathbb{P}^3$, by Theorem 6.1(ii).

(iii) Deciding whether the *chromatic index* (chromatic number of the line graph) of a 3-regular graph is 3 is NP-complete, see [Ho81]. Using that the line graph of any graph is an intersection graph of lines in $\mathbb{P}^3$ (Theorem 6.1(ii)), the statement follows.

(iv) A maximal complete subgraph corresponds to a set of lines passing through the same point p or lying in the same plane Π. Any such point p or plane Π is determined by two lines, and in both cases we can verify for each remaining line whether it belongs to the corresponding complete subgraph (whether it passes through p or belongs to Π, respectively). This gives an $O(n^3)$-time algorithm, but we suspect that the running time can be much improved.

(v) As we have seen in part (iv), there are polynomially many maximal complete subgraphs in H. We can check all k-tuples of them, and decide whether they cover all vertices in H.

(vi) For this, we need to consider the constructions of lines in the representations described in the proof of Theorem 6.1, and show that they can be built in polynomial time. This is obvious in part (ii) of the theorem. For part (i), the situation is somewhat more complex. To find many possible representations of the next vertex intersecting the lines it should, is an algebraically simple task. In polynomial time, we can find one of them that is generic in the sense needed for the construction. However, if the coordinates of each line would be twice as long as those of the preceding line (a condition that is hard to rule out *a priori*), then the whole construction takes more than polynomial time.

A simple way to avoid this problem is the following. First, color the vertices of the triangle-free graph G of maximal degree at most 3 by at most 4 colors, by a simple greedy algorithm. Find the lines representing the vertices in the following order: first for the first color class, next for second color class, etc. The coordinates of each line will be just slightly more complex than the coordinates of the lines representing vertices in *earlier color classes*. Therefore, the construction can be performed in polynomial time. A similar argument works also for graphs G with triangles: First we find a maximal subset of pairwise vertex-disjoint triangles in G. Let G_0 be the graph obtained from G by removing these triangles. Then we construct an auxiliary graph G' with these triangles as vertices by connecting two of them with an edge if there is an edge in G between the triangles. The graph G' has maximum degree at most 3, so it can be greedily 4-colored. If we construct G by adding back the triangles to G_0, in the order determined by their colors, then the procedure will end in polynomial time. □

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