DURFEE'S CONJECTURE ON THE SIGNATURE OF SMOOTHINGS OF SURFACE SINGULARITIES

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with an appendix by
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Abstract. In 1978 Durfee conjectured various inequalities between the signature σ and the geometric genus p_g of a normal surface singularity. Since then a few counter examples have been found and positive results established in some special cases.

We prove a 'strong' Durfee–type inequality for any smoothing of a Gorenstein singularity, provided that the intersection form the resolution is unimodular, and the conjectured 'weak' inequality for all hypersurface singularities and for sufficiently large multiplicity strict complete intersections. The proofs establish general inequalities valid for any normal surface singularity.

1. Introduction

Durfee's conjectures. Let $(X,0)$ be a complex analytic normal surface singularity and $\tilde{X}\to X$ a resolution. The *geometric genus* p_g is defined as $h^1(\mathcal{O}_{\tilde{X}})$. For any one–parameter smoothing with generic (Milnor) fiber F, the rank of the second homology $H_2(F,\mathbb{Z})$ is the *Milnor number* of the smoothing μ. Furthermore, $H_2(F,\mathbb{Z})$ has a natural intersection form with Sylvester invariants (μ_+,μ_0,μ_-). Then $\mu=\mu_++\mu_0+\mu_-$ and $\sigma:=\mu_+-\mu_-$ is called the *signature* of the smoothing. The Milnor number and the signature usually depend on the choice of the smoothing. For more details see the monographs [1, 2, 18, 22] or [16, 19, 32]. Formulas for various classes of singularities can be found in [8, 9, 10, 11, 13, 14, 15, 20].

These local invariants should be viewed as analogs of the most important global invariants: Todd genus, Euler number and signature.

Durfee proved that $2p_g=\mu_0+\mu_+$ [5] and μ_0 equals the first Betti number $b_1(L_X)$ of the link L_X of $(X,0)$.

Examples show that for a surface singularity μ_- is usually large compared to the other Sylvester invariants. Equivalently, p_g is essentially smaller than μ and σ tends to be rather negative. These observations led to the formulation of Durfee's Conjectures [5].

Strong inequality: If $(X,0)$ is an isolated complete intersection surface singularity (ICIS) then $6p_g\le\mu$.
Weak inequality: If $(X,0)$ is a normal surface singularity, then for any smoothing $4p_g\le\mu+\mu_0$. Equivalently, $\sigma\le 0$.
Semicontinuity of σ: If $\{(X_t,0)\}_{t\in(\mathbb{C},0)}$ is a flat family of isolated surface singularities then $\sigma(X_{t=0})\le\sigma(X_{t\neq 0})$.

Other invariants are provided by the combinatorics of a resolution $\pi:\tilde{X}\to X$. Let s denote the number of irreducible π-exceptional curves and K the canonical class of $\tilde{X}$. Then K^2+s is independent of the resolution and, for Gorenstein singularities,

$$(1)\qquad \mu=12p_g+K^2+s-\mu_0 \quad\text{and}\quad -\sigma=8p_g+K^2+s;$$

see [5, 16, 29, 32]. Therefore, an inequality of type $\mu+\mu_0\ge C\cdot p_g$ (for some constant C) transforms into $(12-C)p_g+K^2+s\ge 0$, or $-\sigma\ge(C-4)p_g$.

The resolution defines the *minimal cycle* Z_{min} (also called the *Artin* or *fundamental* cycle) and the *maximal cycle* Z_{max}. The former is the smallest integral effective cycle $\sum e_i E_i$ such that $(E_j, \sum e_i E_i) \le 0$ for every π-exceptional curve $E_j \subset \tilde{X}$ and the latter is the divisor corresponding to the ideal sheaf $\pi^{-1}\mathfrak{m}_{X,0} \cdot \mathcal{O}_{\tilde{X}}$. It is clear that $Z_{min} \le Z_{max}$.

Other invariants of $(X,0)$ are the *multiplicity,* denoted by ν, and the *embedding dimension,* denoted by e.

Known results 2. A counterexample to the *weak inequality* was given by Wahl [32, page 240]; it is a minimally elliptic normal surface singularity (not ICIS) with $\nu = 12$, $\mu = 3$, $\mu_0 = 0$, $p_g = 1$ and $\sigma = 1$. Nevertheless, both the strong and the weak inequalities hold in most examples and the intrinsic structure responsible for the positivity/negativity of the signature of a given germ has not been understood.

A counterexample to the *semicontinuity* of the signature was found in [12]; this excludes degeneration arguments in possible proofs of the inequalities.

The articles [13, 14] show that the *strong inequality* also fails for some non–hypersurface ICIS, and without other restrictions the best that we can expect is the weak inequality.

For hypersurfaces we have the following ‘positive’ results:

$8p_g < \mu$ for $(X,0)$ of multiplicity 2, Tomari [30],
$6p_g \le \mu - 2$ for $(X,0)$ of multiplicity 3, Ashikaga [3],
$6p_g \le \mu - \nu + 1$ for quasi-homogeneous singularities, Xu–Yau [33],
$6p_g \le \mu$ for suspension singularities $\{g(x,y) + z^k = 0\}$, Némethi [24, 25],
$6p_g \le \mu$ for absolutely isolated singularities, Melle–Hernández [21].
For a short proof of $\sigma \le 0$ in the suspension case see [26].

In this note we estimate the expression $8p_g + K^2 + s$ using properties of the dual graph of the minimal resolution. For smoothable Gorenstein singularities we obtain the following.

Theorem 3. *Let $(X,0)$ be a normal Gorenstein surface singularity with embedding dimension e and geometric genus p_g. Let σ denote the signature of a smoothing. Then*

(1) *If the resolution intersection form is unimodular then $-\sigma \ge 2^{4-e}(p_g + 1)$.*
(2) *If $(X,0)$ is a (non smooth) hypersurface singularity then $-\sigma \ge 1 + \mu_0$.*

We prove several inequalities that hold without the Gorenstein assumption. At each step we ‘loose something’. Analyzing these steps should lead to better estimates in many cases. Our aim is not to over-exploit these technicalities, but to show conceptually the general principles behind the inequalities.

It seems that $-\sigma \ge 0$ for all ‘sufficiently complicated’ complete intersections, but we can prove this only for *strict complete intersection* singularities where a local ring $(\mathcal{O}_{X,0}, \mathfrak{m}_{X,0})$ is called a strict complete intersection iff the corresponding graded ring $\mathrm{Gr}_{\mathfrak{m}_{X,0}}(\mathcal{O}_{X,0})$ is a complete intersection; see [4].

Proposition 4. *Fix e and consider the set of strict ICIS of embedding dimension e. Then $-\sigma$ tends to infinity whenever the multiplicity ν tends to infinity.*

Example 5. [13, 14] Assume that $(X,0)$ is a homogeneous ICIS of codimension $r = e - 2$ and multidegree $(d, \dots, d)$. Then $\nu = d^r$ and

$$\frac{p_g}{\nu} = \frac{r(d-1)(d-2)}{6} + \frac{r(r-1)(d-1)^2}{8}; \quad \frac{\mu + 1 - \nu}{\nu} = r(d^2 - 3d + 2) + \frac{r(r-1)(d-1)^2}{2}.$$

(a) If $r = 1$ then $6p_g = \mu + 1 - \nu$.

(b) If $r \ge 2$ is fixed then $\frac{\mu}{p_g}$ asymptotically tends to $C_{2,r} := \frac{4(r+1)}{r+1/3}$, although $C_{2,r} \cdot p_g \le \mu + 1$ does not hold in general. (The constant 4 is the best bound valid for any d.)

(c) For any r the inequality $4p_g \le \mu + 1 - \nu$ is valid. In fact, for any fixed d

$$4 \cdot \frac{(d-1)(r-1) + 2(d-2)}{(d-1)(r-1) + \frac{4}{3}(d-2)} \cdot p_g \le \mu + 1 - \nu.$$

For $d = 2$ the coefficient of p_g is 4, this coefficient is increasing if d is increasing.

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2. Minimal Euler characteristic of a resolution

Let $(X, 0)$ be a normal surface singularity with minimal resolution $\tilde{X} \to X$. We write $L = H_2(\tilde{X}, \mathbb{Z})$, $(\cdot, \cdot)$ denotes the intersection form on L and L' is the dual lattice $\mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{Z})$ with natural inclusions $L \subset L' \subset L \otimes \mathbb{Q}$.

Let $Z_K \in L'$ be the anticanonical cycle, that is, $(Z_K, E_i) = -(K, E_i)$ for every exceptional curve E_i. By the minimality of the resolution $(Z_K, l) \le 0$ for any effective rational cycle l and $Z_K \ge 0$. A singularity is called *numerically Gorenstein* if $Z_K \in L$.

Set $\chi(l') = -(l', l' - Z_K)/2$ for any $l' \in L \otimes \mathbb{Q}$. By Riemann–Roch and the adjunction formula, $\chi(l) = \chi(\mathcal{O}_l)$ for any effective cycle $l \in L$. We set

$$\min \chi := \min_{l \in L} \chi(l).$$

It is a topological invariant of $(X, 0)$, strongly related to arithmetical properties of the lattice L. Usually it is hard to compute explicitly. In the literature $1 - \min \chi = p_a$ is called the *arithmetic genus* of $(X, 0)$ [31].

(The expression $\min \chi$ is also the normalization term of the Seiberg–Witten invariant of the link expressed in terms of the lattice cohomology [27]. The comparison of $\min \chi$ with the d–invariant of the link provided by the Heegaard–Floer theory and the involved topological inequalities lead the authors to the ideas of the present note.)

The quantity $\min \chi$ satisfies two obvious inequalities. Since $h^0(\mathcal{O}_l) - h^1(\mathcal{O}_l) \ge 1 - p_g$ we get $\min \chi \ge 1 - p_g$. Also, since the real quadratic function $\chi(x) = -(x, x - Z_K)/2$ has its minimum at $Z_K/2$, and $\chi(Z_K/2) = K^2/8$, we get that $\min \chi \ge K^2/8$.

We wish to understand how sharp these inequalities are. The first inequality $\min \chi \ge 1 - p_g$ will be improved to $\min \chi \ge -Cp_g$ for a certain constant $0 < C < 1$. This will be applied in the form $p_g + \chi(l) \ge (1 - C)p_g$ for any l.

On the other hand, we wish to bound the difference $\min \chi - K^2/8$ from above. The strategy is the following. Assume that for some rational cycle ξ one has $Z_K - \xi = 2l \in 2L$. Then $\chi(l) = (K^2 - \xi^2)/8$, hence $\chi(l) - K^2/8$ is minimal exactly when $-\xi^2/8$ is minimal among the rational cycles ξ satisfying $Z_K - \xi \in 2L$. Thus the existence of a cycle ξ with $\xi^2 + s \ge 0$ implies that $(K^2 + s)/8 \ge \min \chi$, which combined with the first inequality gives $p_g + (K^2 + s)/8 \ge (1 - C)p_g$.

Lemma 6. *Let $(X, 0)$ be a numerically Gorenstein singularity. Then $\min \chi$ is achieved by a cycle $l \in L$ satisfying $Z_K/2 \le l \le Z_K$.*

Proof. Assume that $\chi(l) = \min \chi$ and write $l = a - b$, where $a, b \in L$ are effective and have no common components. Then $\chi(a + b) - \chi(a - b) = (b, Z_K - 2a) \le 0$, thus $\chi(a + b) \le \chi(a - b)$. Thus we may assume the l is effective. Similarly, write $l = Z_K - a + b$. Then $\chi(Z_K - a + b) - \chi(Z_K - a - b) = (b, 2a - Z_K) \ge 0$. These two inequalities applied repeatedly show that the minimum is achieved for some $l \in L$ with $0 \le l \le Z_K$.

Take such a cycle and write it as $l = Z_K/2 + a - b$, $a, b \in \frac{1}{2}L$, effective and without common components. Then $\chi(Z_K/2 + a + b) - \chi(l) = -2(a, b) \le 0$. $\square$

If $(X, 0)$ is a Du Val singularity then $Z_K = 0$ hence $\min \chi(l)$ is realized by the empty cycle $l = 0$. This tends to mess up our formulas and we exclude them in the sequel. If $(X, 0)$ is numerically Gorenstein but not Du Val then the support of Z_K, and hence the support of $l \geq Z_K/2$, is the whole exceptional set of the resolution.

Proposition 7. *Set $\epsilon = 1$ if $(X, 0)$ is Gorenstein, and $\epsilon = 0$ otherwise. Then for any numerically Gorenstein, non-Du Val surface singularity $p_g + \min \chi \geq 2^{\epsilon - e}(p_g + 1)$.*

Proof. Fix $l \in L$ such that $Z_K/2 \leq l \leq Z_K$ and $\min \chi = \chi(l)$. In the non-Du Val case $Z_K > 0$, hence $l > 0$ too and

$$p_g + \chi(l) = p_g - h^1(\mathcal{O}_l) + h^0(\mathcal{O}_l) \geq h^0(\mathcal{O}_l).$$

Note that for any effective $m \in L$ we have

$$h^0(\mathcal{O}_m) \geq \dim\big(H^0(\mathcal{O}_{\tilde X})/H^0(\mathcal{O}_{\tilde X}(-m))\big).$$

The inequality is usually strict but if $m = Z_K$ then Grauert–Riemenschneider vanishing implies that

$$h^0(\mathcal{O}_{Z_K}) = \dim\big(H^0(\mathcal{O}_{\tilde X})/H^0(\mathcal{O}_{\tilde X}(-Z_K))\big) = p_g.$$

Note that $H^0(\mathcal{O}_{\tilde X})$ equals the local ring R of $(X, 0)$ and each $H^0(\mathcal{O}_{\tilde X}(-m))$ can be identified with an ideal sheaf $I(m) \subset R$. This correspondence is sub-multiplicative, that is, $I(m_1) \cdot I(m_2) \subset I(m_1 + m_2)$. Thus, for every m, Lemma 25 shows that

$$\dim\big(H^0(\mathcal{O}_{\tilde X})/H^0(\mathcal{O}_{\tilde X}(-m))\big) \geq 2^{-e}\big(1 + \dim\big(H^0(\mathcal{O}_{\tilde X})/H^0(\mathcal{O}_{\tilde X}(-2m))\big)\big).$$

Putting these together gives that

$$\begin{array}{rcl} p_g + \chi(l) & \geq & \dim\big(H^0(\mathcal{O}_{\tilde X})/H^0(\mathcal{O}_{\tilde X}(-l))\big) \\ & \geq & \frac{1}{2^e}\big(1 + \dim\big(H^0(\mathcal{O}_{\tilde X})/H^0(\mathcal{O}_{\tilde X}(-2l))\big)\big) \\ & \geq & \frac{1}{2^e}\big(1 + \dim\big(H^0(\mathcal{O}_{\tilde X})/H^0(\mathcal{O}_{\tilde X}(-Z_K))\big)\big) \\ & = & \frac{1}{2^e}(p_g + 1). \end{array}$$

Let $0 \leq m \leq Z_K$ be a cycle and set $\bar m = Z_K - m$. In the Gorenstein case duality gives that

$$\begin{array}{rcl} h^1(\mathcal{O}_m) = h^0(\mathcal{O}_m(-\bar m)) & = & \dim\big(H^0(\mathcal{O}_{\tilde X}(-\bar m))/H^0(\mathcal{O}_{\tilde X}(-Z_K))\big) \\ & = & p_g - \dim\big(H^0(\mathcal{O}_{\tilde X})/H^0(\mathcal{O}_{\tilde X}(-\bar m))\big), \end{array}$$

hence, using Lemma 25 in the 3rd line we get that

$$\begin{array}{rcl} p_g + \chi(m) & = & p_g - h^1(\mathcal{O}_m) + h^0(\mathcal{O}_m) \\ & \geq & \dim\big(H^0(\mathcal{O}_{\tilde X})/H^0(\mathcal{O}_{\tilde X}(-\bar m))\big) + \dim\big(H^0(\mathcal{O}_{\tilde X})/H^0(\mathcal{O}_{\tilde X}(-m))\big) \\ & \geq & \frac{1}{2^{e-1}}\big(1 + \dim\big(H^0(\mathcal{O}_{\tilde X})/H^0(\mathcal{O}_{\tilde X}(-Z_K))\big)\big) \\ & = & \frac{1}{2^{e-1}}(p_g + 1). \end{array} \tag{8}$$

For $m = l$ this gives the claimed inequality. □

3. Inequalities in the unimodular case.

Assume that the intersection form of L is unimodular, that is $L = L'$. Note that this holds iff the first integral homology of the link of $(X, 0)$ is torsion free since this torsion group is isomorphic to L'/L by [23].

Theorem 9. *Let $(X, 0)$ be a normal surface singularity of embedding dimension e. Let $\tilde X \to X$ be the minimal resolution with canonical class K and s exceptional curves. Assume that the resolution intersection form is unimodular. Then*

(1) $(K^2 + s)/8 \geq \min \chi$ *and*

(2) $p_g + (K^2+s)/8 \geq 2^{\epsilon-e}\,(p_g+1)$, *equivalently,* $(K^2+s)/8 \geq -(1-2^{\epsilon-e})\,p_g + 2^{\epsilon-e}$, *where* ϵ *is as in Proposition 7.*

Proof. By a result of Elkies [7], there is a $\xi \in L$ such that $\xi^2 + s \geq 0$ and $(m, m-\xi)$ is even for every $m \in L$. (That is, ξ is a *characteristic element* of small norm.) If E is an irreducible exceptional curve then $(E, E-Z_K) = 2g(E)-2$ is even, thus $(m, m-Z_K)$ is even for every $m \in L$. Therefore $(m, Z_K-\xi)$ is even for every $m \in L$ and $l := \frac{1}{2}(Z_K-\xi) \in L$. (We used unimodularity here and it is also needed in [7].)

Then $(K^2+s)/8 = \chi(l) + (\xi^2+s)/8 \geq \chi(l)$ and we can apply Proposition 7. □

If, in addition, $(X,0)$ is Gorenstein, then $\epsilon = 1$ thus (2) of Theorem 9 and the second formula of (1) give that

$$-\sigma = 8p_g + K^2 + s \geq 2^{4-e}\,(p_g+1). \tag{10}$$

This completes the proof of part (1) of Theorem 3. □

The above theorem shows that the torsionfreeness of the first homology of the link has more substantial effect on the negativity of the signature than the embedded properties, like being a hypersurface or an ICIS.

Example 11. Assume that $(X,0)$ is a hypersurface singularity with $L = L'$. Then $-\sigma \geq 2p_g + 2$, or equivalently, $\mu + \mu_0 \geq 6p_g + 2$. In particular, if the link of a hypersurface singularity is an integral homology sphere (hence $\mu_0 = 0$ too), then it satisfies the strong Durfee inequality $6p_g \leq \mu - 2$ with the optimal asymptotic constant 6.

4. The general case

In this section we assume that $(X,0)$ is numerically Gorenstein but not Du Val. Set $x := 2\{Z_K/2\} \in L$ and $\bar{x} := E - x$, where E is the reduced exceptional curve. Then $m := (Z_K - x)/2 = \lfloor Z_K/2 \rfloor \in L$. We write Σ for $8p_g + K^2 + s$. (Thus, in the Gorenstein case, $\sigma = -\Sigma$.)

Since $8\chi(m) = K^2 - x^2$, by Proposition 7

$$\Sigma = 8\big(p_g + \chi(m)\big) + x^2 + s \geq 2^{\epsilon+3-e}(p_g+1) + x^2 + s. \tag{12}$$

Similarly,

$$\Sigma = 8(p_g + \chi(m+E)) + (E+\bar{x})^2 + s \geq 2^{\epsilon+3-e}(p_g+1) + (E+\bar{x})^2 + s. \tag{13}$$

Since $x = E - \bar{x}$, adding the equations (12) and (13) gives that

$$\Sigma \geq 2^{\epsilon+3-e}(p_g+1) + E^2 + \bar{x}^2 + s. \tag{14}$$

For each cycle $y = x,\ \bar{x}$ and E write the relation $y^2 = -2\chi(y) + (y, Z_K)$ and add the equations (12) and (14). We get that

$$\Sigma \geq 2^{\epsilon+3-e}(p_g+1) + s - \chi(x) - \chi(\bar{x}) - \chi(E) + (E, Z_K). \tag{15}$$

Since $x,\ \bar{x},\ E$ are reduced, $\chi(x) + \chi(\bar{x}) + \chi(E) \leq s + 1 - b_1(L_X)$ (since $b_1(L_X) = b_1(E)$). Hence (15) can be rewritten as

Proposition 16. $\Sigma \geq 2^{\epsilon+3-e}(p_g+1) - 1 + b_1(L_X) + (E, Z_K)$ *where* (E, Z_K) *also equals* $E^2 + 2\chi(E)$. *Furthermore,* $-1 + b_1(L_X) + (E, Z_K) = E^2 + \chi(\Gamma)$ *where* $\chi(\Gamma)$ *is the Euler characteristic of the topological realization of the resolution graph* Γ. □

Although the term (E, Z_K) is negative, in many cases (e.g. hypersurfaces, ICIS) it is much smaller than p_g. We do not have a good general estimate, but the following argument gives a bound that implies the negativity of the signature in several cases.

In order to simplify the notation let us denote the constant $2^{\epsilon+3-e}-1+b_1(L_X)$ by A. Let $Z=Z_{max}\in L$ be the maximal cycle. Hence $Z\geq E$, which implies that $(E,Z_K)\geq(Z,Z_K)$. For any $t\geq e-\epsilon-3$ write $(2^{t+1}Z,Z_K)$ as $(2^{t+1}Z)^2+2\chi(2^{t+1}Z)$, hence we obtain that

$$\Sigma\geq\big(\tfrac{1}{2^{e-\epsilon-3}}-\tfrac{1}{2^t}\big)p_g+\tfrac{1}{2^t}\big(p_g+\chi(2^{t+1}Z)\big)+2^{t+1}Z^2+A. \tag{17}$$

Then using $Z^2\geq-\nu$ (cf. [31]) and Proposition 7 we get the following.

Lemma 18. *For $t\geq e-\epsilon-3$ one has*

$$\Sigma\geq\big(\tfrac{1}{2^{e-\epsilon-3}}-\tfrac{1}{2^t}+\tfrac{1}{2^{t+e-\epsilon}}\big)p_g-2^{t+1}\nu+A+\tfrac{1}{2^{t+e-\epsilon}}.\quad\square$$

With different choices of t the coefficient of p_g can be arranged to be as close to $1/2^{e-\epsilon-3}$ as we wish, but the price is a more negative coefficient for ν. This expression shows that for an arbitrary normal surface singularity we should expect an inequality of the form

$$\Sigma\geq C_1p_g-C_2\nu+C_3\quad\text{for some constants}\quad C_1,C_2>0\text{ and }C_3>-1$$

that depend on the embedding dimension e. If ν dominates p_g—as in the example of Wahl—then Σ can be negative. However, if p_g dominates the multiplicity, then Σ becomes positive, as in the next examples.

The case of strict complete intersections. By Theorem (2.17) of Bennett [4], every strict ICIS is a normally flat deformation of an isolated homogeneous complete intersection singularity. (Under such deformation p_g is semicontinuous and ν is constant.)

In order to prove that $-\sigma=\Sigma$ is positive for large ν and fixed $e=r+2$, by Lemma 18 it is enough to show that p_g/ν tends to infinity with ν for homogeneous complete intersections. In that case, if $d_1,\dots,d_r$ ($d_i\geq2$) are the degrees of the defining equations, then

$$\frac{p_g}{\nu}=\sum_i\frac{(d_i-1)(d_i-2)}{6}+\sum_{i<j}\frac{(d_i-1)(d_j-1)}{4} \tag{19}$$

and $\nu=\prod_i d_i$, cf. [13, 14].

Note that (19) does not imply the negativity of the signature for every strict ICIS, but it gives a much stronger result asymptotically. This suggests that the positivity of Σ (or, the negativity of the signature in the presence of Gorenstein smoothing) is guided by the ratio p_g/ν. This seem to be a general phenomenon, not specifically related to embedded properties.

The case of hypersurfaces. Assume that $e=3$, hence $\epsilon=1$ too. Our goal is to prove the negativity of the signature without any multiplicity restriction. The inequality (17) with $t=-1$ becomes

$$\Sigma\geq2(p_g+\chi(Z))+1+b_1(L_X)-\nu. \tag{20}$$

Using (8) for $m=Z$ shows that $p_g+\chi(Z)\geq\frac14(p_g+1)$, thus

$$-\sigma\geq\tfrac12(p_g+1)+1+\mu_0-\nu. \tag{21}$$

By semicontinuity of the geometric genus $p_g\geq\binom{\nu}{3}$, since the geometric genus of a degree ν homogeneous singularity is $\binom{\nu}{3}$. Hence we get that

$$-\sigma\geq\tfrac12\tbinom{\nu}{3}-\nu+\tfrac32+\mu_0. \tag{22}$$

In the right hand side $\frac12\binom{\nu}{3}\geq\nu$ for $\nu\geq5$, hence $-\sigma\geq\frac32+\mu_0$. If $\nu=2$ or 3 then $-\sigma\geq1+\mu_0$ follows from the results of Tomari and Ashikaga mentioned in Paragraph 2. Finally, if $\nu=4$ then $p_g\geq\binom43=4$ hence $p_g+\chi(Z)\geq\frac14(p_g+1)\geq\frac54$. Since $p_g+\chi(Z)$ is an integer, it has to be ≥2 thus (20) becomes

$$-\sigma\geq2\cdot2+1+\mu_0-4=1+\mu_0. \tag{23}$$

This completes the proof of part (2) of Theorem 3. □

It is possible to analyze this case further and prove stronger lower bounds for $-\sigma$. For instance, for an isolated hypersurface singularity with $\nu \geq 4$ one can show that

$$-\sigma \geq \tfrac{2}{3}\big(p_g - \tbinom{\nu}{3}\big) + 2\tbinom{\nu-1}{3} - \nu + 3 + \mu_0.$$

5. Speculations regarding generalizations of Elkies's result

The result of Elkies—valid for unimodular definite lattices—lies behind the 'strong' inequalities of Theorem 9. It is somewhat surprising that comparable inequalities can be obtained by the alternative methods of Section 4.

In this section we analyze different possibilities to extend [7] to the non-unimodular case, and the effect of such extensions on Durfee-type inequalities.

Owens and Strle prove that there exists a characteristic element $\xi \in L'$ such that $\xi^2 + s \geq 0$. Thus there exists $l' \in L'$ such that $Z_K - \xi = 2l' \in 2L'$. However, this is not really helpful to us if $l' \notin L$. We need to approximate l' with an integral cycle $l \in L$ and the final output is not better than the results of Section 4.

Therefore, we need a generalization of the Elkies theorem that guarantees the existence of some ξ with ξ^2 not very negative, and $Z_K - \xi \in 2L$. Examples show that in general we cannot expect $\xi^2 + s \geq 0$. (For instance, this would contradict the existence of singularities with positive signature.)

The results of [7] are valid for any abstract lattice, and in this general context we do not have any guess about the right form of a weaker inequality. However, lattices coming from singularities have distinguished bases and a positive cone of effective divisors. Having these in mind, and also the type of inequalities we already obtained, we can speculate on how to weaken the Elkies inequality using a combinatorial object of the lattice related to the multiplicity of the singularity. Computation of several examples supports the following conjecture.

Conjecture 24. *Let $Z_{min} \in L$ be the minimal cycle of the singularity lattice L. Then there exists a cycle $\xi \in L'$, with $Z_K - \xi \in 2L$ such that*

$$\xi^2 + s \geq Z_{min}^2.$$

As in the proof of Theorem 9, the conjecture would imply that, for any normal surface singularity

$$-\sigma \geq \tfrac{1}{2^{e-\epsilon-3}} \cdot (p_g + 1) - \nu.$$

For hypersurfaces this becomes $-\sigma \geq 2(p_g + 1) - \nu$. Keeping in mind that for hypersurfaces $p_g \geq \binom{\nu}{3}$ (that is, $\nu \leq C p_g^{1/3}$), this inequality is a good replacement for the expected strong inequality. These methods would imply that $-\sigma \geq 0$ for every large multiplicity ICIS but they fall short in general. (However, the constants are better than those in (18)). We believe that small multiplicity ICIS should be studied by techniques specific to them.

Nevertheless, we hope that the above conjecture has interesting arithmetical and geometrical meaning and that it is related to the topological d–invariant of the link as well.

6. Appendix by Tommaso de Fernex: Colength of a product of ideals

Let R be a local ring with maximal ideal $\mathfrak{m}$, essentially of finite type over a field k. Let e be the embedded dimension of R. For any $\mathfrak{m}$-primary ideal $\mathfrak{a}$, denote by $\ell(R/\mathfrak{a})$ the length of $R/\mathfrak{a}$.

Lemma 25. *For any finite collection of $\mathfrak{m}$-primary ideals $\mathfrak{a}_1, \dots, \mathfrak{a}_d \subset R$, we have*

$$d^{e-1} \textstyle\sum_{i=1}^{d} \ell(R/\mathfrak{a}_i) \geq \ell\big(R/(\mathfrak{a}_1 \cdots \mathfrak{a}_d)\big),$$

and the inequality is strict if $d \geq 2$ and $e \geq 2$.

Proof. By Cohen's structure theorem, there is a surjection $k[[x_1, \dots, x_e]] \to \widehat{R}$, where $\widehat{R}$ is the $\mathfrak{m}$-adic completion of R. After taking the inverse image of the ideals $\mathfrak{a}_i\widehat{R}$ to $k[[x_1, \dots, x_e]]$ and restricting to $k[x_1, \dots, x_e]$, we reduce to prove the lemma when $R = k[x_1, \dots, x_e]$ and $\mathfrak{m} = (x_1, \dots, x_e)$. If we fix a monomial order which gives a flat degeneration to monomial ideals, and denote by $\mathrm{in}(\mathfrak{a})$ the initial ideal of an ideal $\mathfrak{a} \subset R$, then $\ell(R/\mathfrak{a}) = \ell(R/\mathrm{in}(\mathfrak{a}))$ and $\prod_{i=1}^d \mathrm{in}(\mathfrak{a}_i) \subset \mathrm{in}(\prod_{i=1}^d \mathfrak{a}_i)$. We can therefore assume that each $\mathfrak{a}_i$ is monomial.

Let $\mathfrak{a} = \prod_{i=1}^d \mathfrak{a}_i$. For $\mathbf{u} = (u_1, \dots, u_e) \in \mathbb{Z}^e_{\geq 0}$, we denote $\mathbf{x}^\mathbf{u} = \prod_{j=1}^e x_j^{u_j}$. Let

$$Q_i = \bigcup_{\mathbf{x}^\mathbf{u} \in \mathfrak{a}_i} \left(\mathbf{u} + \mathbb{R}^e_{\geq 0}\right) \quad \text{and} \quad Q = \bigcup_{\mathbf{x}^\mathbf{u} \in \mathfrak{a}} \left(\mathbf{u} + \mathbb{R}^e_{\geq 0}\right).$$

Notice that $\ell(R/\mathfrak{a}_i) = \mathrm{Vol}\left(\mathbb{R}^e_{\geq 0} \smallsetminus Q_i\right)$ and $\ell(R/\mathfrak{a}) = \mathrm{Vol}\left(\mathbb{R}^e_{\geq 0} \smallsetminus Q\right)$, where the volumes are computed with respect to the Euclidean metric. We consider the radial sum

$$Q' = \mathop{\bigstar}_{i=1}^{d} Q_i := \bigcup_W \sum_{i=1}^d (Q_i \cap W)$$

introduced in [6]: the union runs over all rays $W \subset \mathbb{R}^e_{\geq 0}$, and the sum appearing in the right-hand side is the usual sum of subsets of a vector space.

For every $\mathbf{v} \in Q'$, we can find $\mathbf{v}_i \in Q_i$ such that $\mathbf{v} = \sum_{i=1}^d \mathbf{v}_i$. For each i, we have $\mathbf{v}_i \in \left(\mathbf{u}_i + \mathbb{R}^e_{\geq 0}\right)$ for some $\mathbf{u}_i \in \mathbb{Z}^e_{\geq 0}$ such that $\mathbf{x}^{\mathbf{u}_i} \in \mathfrak{a}_i$. Then, setting $\mathbf{u} = \sum_{i=1}^d \mathbf{u}_i$, we have $\mathbf{x}^\mathbf{u} \in \mathfrak{a}$ and $\mathbf{v} \in \left(\mathbf{u} + \mathbb{R}^e_{\geq 0}\right)$, and therefore $\mathbf{v} \in Q$. This means that $Q' \subset Q$, and hence

$$\mathrm{Vol}\left(\mathbb{R}^e_{\geq 0} \smallsetminus Q'\right) \geq \mathrm{Vol}\left(\mathbb{R}^e_{\geq 0} \smallsetminus Q\right). \tag{26}$$

Then, to prove the inequality stated in the lemma, it suffices to show that

$$d^{e-1}\left(\sum_{i=1}^d \mathrm{Vol}\left(\mathbb{R}^e_{\geq 0} \smallsetminus Q_i\right)\right) \geq \mathrm{Vol}\left(\mathbb{R}^e_{\geq 0} \smallsetminus Q'\right). \tag{27}$$

To this end, we fix spherical coordinates $(\theta, \rho) \in S \times \mathbb{R}_{\geq 0}$ where S is the intersection of the unit sphere with $\mathbb{R}^e_{\geq 0}$. For any $\theta \in S$, we define $r_i(\theta) = \inf\{\rho \mid (\theta, \rho) \in Q_i\}$ and $r(\theta) = \inf\{\rho \mid (\theta, \rho) \in Q'\}$. By the definition of Q', we have $r(\theta) = \sum_{i=1}^d r_i(\theta)$. We have

$$\mathrm{Vol}(\mathbb{R}^e_{\geq 0} \smallsetminus Q_i) = \int_S \int_0^{r_i(\theta)} \rho^{e-1}\, d\rho\, \omega(\theta) = \int_S \frac{r_i(\theta)^e}{e}\, \omega(\theta)$$

and

$$\mathrm{Vol}(\mathbb{R}^e_{\geq 0} \smallsetminus Q') = \int_S \int_0^{r(\theta)} \rho^{e-1}\, d\rho\, \omega(\theta) = \int_S \frac{r(\theta)^e}{e}\, \omega(\theta)$$

for some volume form ω on S. Then the desired inequality follows from

$$d^{e-1} \sum_{i=1}^d r_i(\theta)^e \geq r(\theta)^e, \tag{28}$$

which follows from Hölder's inequality.

To conclude, we show that the inequality is strict if $d \geq 2$ and $e \geq 2$. First observe that (27) is a strict inequality unless (28) is an equality for almost all $\theta \in S$, which can only happen if $\mathfrak{a}_i = \mathfrak{a}_1$ for every i. Suppose this is the case, so that $\mathfrak{a} = \mathfrak{a}_1^d$. Notice that in this case Q' is a polyhedron. Let a, b be the smallest integers such that $x_1^a \in \mathfrak{a}_1$ and $x_1^{a'} x_2^b \in \mathfrak{a}_1$ for some $a' < a$. Then $x_1^{(d-1)a+a'} x_2^b \in \mathfrak{a}$, and hence the vector $\mathbf{v} = ((d-1)a + a', b, 0, \dots, 0)$ belongs to Q. Note, on the contrary, that $\mathbf{v}$ is not in Q'. Hence $Q' \subsetneq Q$, and since these sets are polyhedra, it follows that (26) is a strict inequality. Therefore the inequality stated in the lemma, which follows as a combination of (26) and (27), is strict. □

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