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Control by time delayed feedback near a Hopf bifurcation point

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Abstract. In this paper we study the stabilization of rotating waves using time delayed feedback control. It is our aim to put some recent results in a broader context by discussing two different methods to determine the stability of the target periodic orbit in the controlled system: 1) by directly studying the Floquet multipliers and 2) by use of the Hopf bifurcation theorem. We also propose an extension of the Pyragas control scheme for which the controlled system becomes a functional differential equation of neutral type. Using the observation that we are able to determine the direction of bifurcation by a relatively simple calculation of the root tendency, we find stability conditions for the periodic orbit as a solution of the neutral type equation.

Keywords: Pyragas control, time delayed feedback control, Hopf bifurcation, neutral equations.

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Stabilization of motion is a subject of interest in applications, where one often wishes the observed motion to be stable. Pyragas control [14], a form of time-delayed feedback control, provides a method to stabilize unstable periodic solutions of ordinary differential equations which has been successfully implemented in experimental set-ups [6, 12]. It can also be used to stabilize rotating waves in lasers [4] and in coupled networks [1]. To be able to apply Pyragas control effectively, one is interested for which strength of the control term stability can be achieved. Furthermore, in physical set-ups it is also relevant to have knowledge of the overall dynamics of the controlled system. Since by applying Pyragas control we turn a finite dimensional system into an infinite dimensional system, one expects the dynamics of the system to change significantly. Therefore, the controlled system is an interesting object of study in itself [9].

A number of variations to the Pyragas control scheme have been proposed in the literature. For example, in [15] the control term contains an infinite number of delay terms in which each delay is chosen to be a multiple of the period of the target periodic orbit; and in [11] the control matrix is chosen to be non-autonomous.

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In this article we continue an analysis started in [9] and apply Pyragas control to the differential equation

$$\dot{z}(t) = (\lambda + i)z(t) + (1 + i\gamma) |z(t)|^2 z(t) \tag{0.1}$$

where $\lambda, \gamma \in \mathbb{R}$ are parameters and $z : \mathbb{R} \to \mathbb{C}$.

Solutions of the form $A(x, t) = z(t)e^{i\alpha x}$ of the Ginzburg–Landau equation

$$\frac{\partial A}{\partial t}(x, t) = (\lambda + i)\frac{\partial^2}{\partial x^2}A(x, t) + (1 + i\gamma) |A(x, t)|^2 A(x, t), \quad x \in \mathbb{R} \text{ and } t \geq 0$$

reduce, after rescaling, to solutions of (0.1) [17]. Equation (0.1) can be used to model a range of physical phenomena, and arises as a model for Stuart–Landau oscillators [10, 16] and laser dynamics [4].

A useful property of (0.1) is that we can explicitly find a periodic solution of which we can analytically determine its stability. Indeed, for $\lambda < 0$, system (0.1) has a periodic solution given by

$$z(t) = \sqrt{-\lambda}e^{i(1-\gamma\lambda)t} \tag{0.2}$$

with period $T = 2\pi/(1 - \gamma\lambda)$. For $\gamma\lambda < 1$, (0.2) is unstable as a solution of (0.1) (see Section 1).

As in [9], we write for the controlled system

$$\dot{z}(t) = (\lambda + i)z(t) + (1 + i\gamma) |z(t)|^2 z(t) - Ke^{i\beta} [z(t) - z(t - \tau)] \tag{0.3}$$

with $K \in \mathbb{R}, \tau \geq 0$ and $\beta \in [0, \pi]$. The controlled system is designed such that for $\tau = T = 2\pi/(1 - \gamma\lambda)$, the function (0.2) is still a solution of (0.3).

In [5], the periodic solution (0.2) of (0.1) was used as a counterexample to the claim that periodic orbits with an odd number of Floquet multipliers outside the unit circle cannot be stabilized using Pyragas control. In [9], the bifurcation diagram of the controlled system (0.3) was studied in more detail, and it was shown that the stability of (0.2) as a solution of (0.3) can be determined using the Hopf bifurcation theorem. In fact, it was shown that the periodic solution (0.2) of the system (0.3) emmanates from a Hopf bifurcation. By using the direction of the Hopf bifurcation (i.e. whether the Hopf bifurcation is sub- or supercritical), one is then able, for λ near the bifurcation point and given γ, to find conditions on the parameters K, β that ensure that the periodic orbit (0.2) is stable as a solution of (0.3).

The paper is organized as follows. In Sections 1–4, we place the results from [9] in a broader context using the theory developed for delay equations in [2] and, in particular discuss and compare different methods to determine the stability of (0.2) as a solution of (0.3). We start by exploring the dynamics of the uncontrolled system (0.1) in Section 1. In Section 2 we give necessary conditions for (0.2) to be stable as a solution of (0.3) by direct investigation of the Floquet multipliers. As a different approach to determine the stability of (0.2) as a solution of (0.3), we use – inspired by [9] – the Hopf bifurcation theorem. In Section 3 we approach the bifurcation point over a different curve in the parameter plane than was done in [9]. This enables us to give stability conditions for a wider range of parameter values. We choose the curve through parameter plane in such a way that we a priori know for which points on the curve a periodic solution exists. A relatively simple calculation of the root tendency of the roots of the characteristic equation then directly yields the direction of the bifurcation. In Section 4, we give a direct proof of the result from [9] using the explicit closed-form formula's to determine the direction of the Hopf bifurcation developed in [2].

In Section 5 we propose a variation to the Pyragas control scheme for which the controlled system becomes a functional differential equation of neutral type. We apply the proposed control scheme to the system (0.1) and use the methods developed in Section 3 to determine the stability of the target periodic orbit.

1 Dynamics of the uncontrolled system

We start with some definitions and notations used throughout the article.

Definition 1.1. Let $r > 0, \mathcal{C} = C([-r,0], \mathbb{R}^n)$ equipped with the norm $\|\phi\|_\infty = \sup_{\theta\in[-r,0]} |\phi(\theta)|$. Let $F : \mathcal{C} \to \mathbb{R}^n$. Let us study the retarded functional differential equation

$$\dot{x}(t) = F(x_t), \quad t \geq 0 \tag{1.1}$$

where $x_t(\theta) = x(t+\theta)$ for $\theta \in [-r,0]$. Denote by $T(t)$ the semiflow associated to (1.1). Let x_0 be an equilibrium of (1.1). Then we say that x_0 is *stable* if it is asymptotically stable, i.e. the following two conditions are satisfied: 1) For every $\epsilon > 0$ there exists a $\delta > 0$ such that if $\|\phi - x_0\|_\infty < \delta$ for $\phi \in \mathcal{C}$, then $\|T(t)\phi - x_0\|_\infty < \epsilon$ for all $t \geq 0$. 2) There exists a $b > 0$ such that if $\|\phi - x_0\| < b$ for $\phi \in \mathcal{C}$, then $\lim_{t\to\infty} \|T(t)\phi - x_0\|_\infty = 0$. We say that x_0 is *unstable* if it is not asymptotically stable.

Note that we do not require exponential stability. However, for retarded functional differential equations, when we determine that a fixed point is stable by establishing that all the associated eigenvalues are in the left half of the complex plane, exponential stability automatically follows.

To study the uncontrolled system (0.1), we can take the real and imaginary parts and view (0.1) as a system on $\mathbb{R}^2$ given by

$$\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix} = \begin{pmatrix} \lambda & -1 \\ 1 & \lambda \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} + (x^2(t) + y^2(t)) \begin{pmatrix} 1 & -\gamma \\ \gamma & 1 \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}. \tag{1.2}$$

Note that $(x, y) = (0, 0)$ is an equilibrium of this system, and the linearization of (1.2) can be used to determine its stability.

Lemma 1.2. *If $\lambda < 0$, the equilibrium $(x, y) = (0, 0)$ of (1.2) is stable. If $\lambda > 0$, the equilibrium $(x, y) = (0, 0)$ of (1.2) is unstable.*

Proof. Linearizing the system (1.2) around the zero solution gives:

$$\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix} = \begin{pmatrix} \lambda & -1 \\ 1 & \lambda \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}. \tag{1.3}$$

The eigenvalues of the matrix in the RHS of (1.3) are given by $\mu_\pm = \lambda \pm i$. This shows that the equilibrium point $(x, y) = (0, 0)$ is stable for $\lambda < 0$ and unstable for $\lambda > 0$. □

We recall that a *Hopf bifurcation* of an equilibrium occurs if we have exactly one pair of non zero roots at the imaginary axis, and that this pair of roots crosses the axis with non zero speed as we vary the parameters. Indeed, in the case of (1.2) we see that for $\lambda = 0$, the eigenvalues $\mu_\pm$ cross the imaginary axis at non zero speed, since $\frac{d}{d\lambda}\mathrm{Re}(\mu_\pm(\lambda)) = 1 \neq 0$. Thus, we find that for $\lambda = 0$ a Hopf bifurcation of the origin of system (0.1) takes place. The Hopf bifurcation theorem now implies that for parameter values λ near the bifurcation point $\lambda = 0$, an unique periodic solution of (1.2) exists.

It turns out that we can explicitly compute this periodic solution of (1.2). By substituting $z(t) = r(t)e^{i\phi(t)}$ into (0.1) with $r(t), \phi(t) \in \mathbb{R}$, we find that for $\lambda < 0$ a periodic solution of (0.1) is given by (0.2). Using that we know for which parameter values λ a periodic orbit exists, we can easily determine whether the Hopf bifurcation is sub- or supercritical. This is summarized for retarded functional differential equations in the following theorem.

Theorem 1.3. *Consider the delay equation*

$$\dot{x}(t) = F(x_t, \lambda) \tag{1.4}$$

where $r > 0, \lambda \in \mathbb{R}$, $F : \mathcal{C}([-r, 0], \mathbb{R}^n) \times \mathbb{R} \to \mathbb{R}^n$ satisfies $F(0, \lambda) = 0$ for all $\lambda \in \mathbb{R}$ and x_t is defined as $x_t(\theta) = x(t + \theta)$ for $\theta \in [-r, 0]$. Let us assume that for $\lambda = \lambda_0$ a Hopf bifurcation of the origin of system (1.4) *takes place. Let us write $\Delta(\mu, \lambda)$ for the characteristic equation of the linearization of* (1.4)*. Denote by $\mu_0 = \mu_0(\lambda)$ the root of the characteristic equation $\Delta(\mu_0(\lambda), \lambda) = 0$ that satisfies $\mu_0(\lambda_0) = i\omega_0$ for some $\omega_0 \in \mathbb{R}\backslash\{0\}$. Furthermore, let us assume that for $\lambda < \lambda_0$, a periodic solution $\overline{x}_\lambda$ of the system* (1.4) *exists. Then we find that the Hopf bifurcation is subcritical if $\mathrm{Re}(\mu_0(\lambda)) < 0$ for $\lambda < \lambda_0$ in a neighbourhood of λ_0; the Hopf bifurcation is supercritical if $\mathrm{Re}(\mu_0(\lambda)) > 0$ for $\lambda < \lambda_0$ in a neighbourhood of λ_0.*

Proof. Since by assumption for $\lambda = \lambda_0$ a Hopf bifurcation of the origin of system (1.4) takes place, we find by the Hopf bifurcation theorem (see for example [2] for the Hopf bifurcation theorem for retarded functional differential equations) that an unique periodic solution of (1.4) exists for parameters λ near the bifurcation point $\lambda = \lambda_0$. Since $\overline{x}_\lambda$ is a periodic solution of (1.4) for $\lambda < \lambda_0$, we conclude that this periodic solution arises from the Hopf bifurcation at $\lambda = \lambda_0$.

If now $\mathrm{Re}(\mu_0(\lambda)) < 0$ for $\lambda < \lambda_0$ in a neighbourhood of λ_0, we find that the periodic solution arising from the Hopf bifurcation exists for parameter values λ for which $\mu_0(\lambda)$ is in the left half of the complex plane. This implies that the Hopf bifurcation is subcritical. Similarly, if $\mathrm{Re}(\mu_0(\lambda)) > 0$ for $\lambda < \lambda_0$ in a neighbourhood of λ_0, we find that the periodic solution arising from the Hopf bifurcation exists for parameters λ for which $\mu_0(\lambda)$ is in the right half of the complex plane. This implies that the Hopf bifurcation is supercritical. □

Since in the case of system (0.1) a periodic solution exists for $\lambda < 0$, combining Lemma 1.2 with Lemma 1.3 yields the following corollary.

Corollary 1.4. *The Hopf bifurcation at $\lambda = 0$ of system* (0.1) *is subcritical and the periodic solution* (0.2) *of* (0.1) *is unstable for parameters $\lambda < 0$ near the bifurcation point $\lambda = 0$.*

We see that the Hopf bifurcation theorem gives us information on the stability of the periodic solution (0.2) of (0.1) for parameters in $\lambda < 0$ in a neighbourhood of the bifurcation point $\lambda = 0$.

For general parameters $\lambda < 0$, the stability of the periodic orbit (0.2) of (0.1) is determined by its Floquet multipliers.

Lemma 1.5. *Let $\lambda < 0$. Then the periodic solution* (0.2) *of* (0.1) *is stable if $\gamma\lambda > 1$ and unstable if $\gamma\lambda < 1$.*

Proof. In order to compute the Floquet multipliers, we first compute the linear variational equation. As it turns out that the linear variational equation is autonomous, the computation of the Floquet multipliers is then relatively straightforward.

As in [9], we write small deviations around the periodic solution (0.2) as

$$z(t) = R_p e^{i\omega_p t}[1 + r(t) + i\phi(t)] \tag{1.5}$$

with $r(t), \phi(t) \in \mathbb{R}$ and where $R_p = \sqrt{-\lambda}$ denote the modulus and $\tau_p = 1 - \gamma\lambda$ the argument of the complex function (0.2). For (1.5) to be a solution of (0.1), we should have that

$$\begin{aligned} i\omega_p R_p e^{i\omega_p t}\left(1 + r(t) + i\phi(t)\right)\right) + R_p e^{i\omega_p t}\left(\dot{r}(t) + i\dot{\phi}(t)\right) \\ = (\lambda + i)R_p e^{i\omega_p t}\left(1 + r(t) + i\phi(t)\right) \\ + (1 + i\gamma)R_p^3 e^{i\omega_p t}\left|1 + r(t) + i\phi(t)\right|^2\left(1 + r(t) + i\phi(t)\right). \end{aligned} \tag{1.6}$$

Up to first order, this expression reduces to

$$\begin{aligned} i\omega_p R_p e^{i\omega_p t}(1 + r(t) + i\phi(t)) + R_p e^{i\omega_p t}\left(\dot{r}(t) + i\dot{\phi}(t)\right) \\ = (\lambda + i)R_p e^{i\omega_p t}(1 + r(t) + i\phi(t)) + (1 + i\gamma)R_p^3 e^{i\omega_p t}(1 + 3r(t) + i\phi(t)). \end{aligned} \tag{1.7}$$

Using that (0.2) is a solution of (0.1), we arrive at

$$i\omega_p R_p e^{i\omega_p t} = (\lambda + i)R_p e^{i\omega_p t} + (1 + i\gamma)R_p^3 e^{i\omega_p t}.$$

Cancelling out factors $R_p e^{i\omega_p t}$ on both sides of (1.7), we have

$$i\omega_p(r(t) + i\phi(t)) + \dot{r}(t) + i\dot{\phi}(t) = (\lambda + i)(r(t) + i\phi(t)) + (1 + i\gamma)R_p^2(3r(t) + i\phi(t)).$$

Using that $R_p^2 = -\lambda$ and $\omega_p = 1 - \gamma\lambda$, leads to the linear variational equation

$$\dot{r}(t) + i\dot{\phi} = -2\lambda r(t) - 2i\gamma\lambda r(t). \tag{1.8}$$

Taking real and imaginary parts, the linear system on $\mathbb{R}^2$ is given by

$$\begin{pmatrix} \dot{r}(t) \\ \dot{\phi}(t) \end{pmatrix} = \begin{pmatrix} -2\lambda & 0 \\ -2\gamma\lambda & 0 \end{pmatrix} \begin{pmatrix} r(t) \\ \phi(t) \end{pmatrix}. \tag{1.9}$$

Put

$$A = \begin{pmatrix} -2\lambda & 0 \\ -2\gamma\lambda & 0 \end{pmatrix}.$$

The Floquet multipliers of (1.9) are given by

$$\nu_i = e^{\mu_i T}, \qquad i = 1, 2,$$

where μ_1, μ_2 are the eigenvalues of A and $T = \frac{2\pi}{1-\gamma\lambda}$ the minimal period of the periodic solution (0.2). The eigenvalues of A are given by $\mu_1 = 0, \mu_2 = -2\lambda$; therefore $\nu_1 = 1$ (the trivial Floquet multiplier) and

$$\nu_2 = e^{-2\lambda\frac{2\pi}{1-\gamma\lambda}}.$$

Since the periodic orbit exists for $\lambda < 0$, we conclude that the periodic orbit (0.2) of (0.1) is stable if $\gamma\lambda > 1$ and unstable if $\gamma\lambda < 1$. □

See Figure 1.1 for the bifurcation diagram of system (0.1).

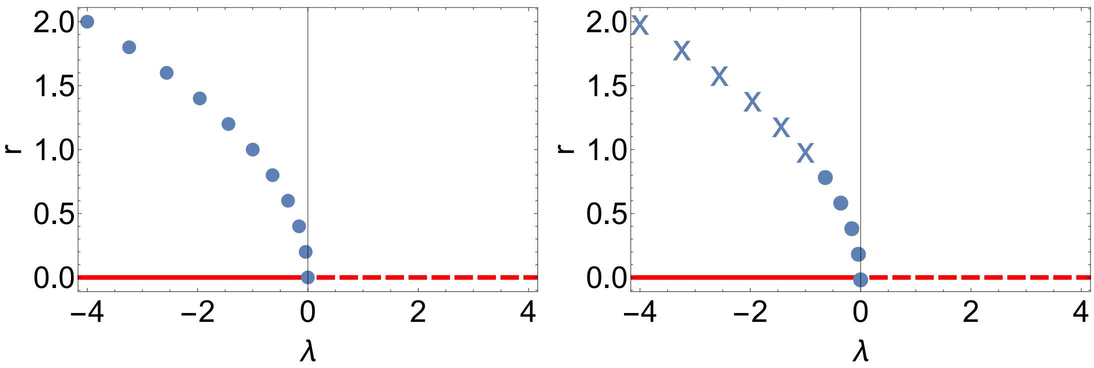

Figure 1.1: Bifurcation diagram of system (0.1) for $\gamma = 10$ (left) and $\gamma = -10$ (right). The solid line indicates a stable equilibrium, the dashed line an unstable equilibrium, the dotted line an unstable periodic orbit and the crosses a stable periodic orbit. Furthermore, r denotes the modulus of the periodic orbit.

2 Floquet multipliers in the controlled system

In Section 1, we used Floquet theory to determine the stability of the periodic solution (0.2) as a solution of the ODE (0.1). As we have seen in Lemma 1.5, the linear variational equation becomes autonomous in this case, and the computation of the Floquet multipliers reduces to the calculation of eigenvalues of a 2×2-matrix.

In this section we use Floquet theory to gain information on the stability of (0.2) as a solution of the delay equation (0.3). We again find that the linear variational equation is autonomous, but the computation of the Floquet multipliers is more involved, because the characteristic matrix function now becomes transcendental. We will first present a necessary condition for (0.2) to be stable as a solution of (0.3), and then, in Sections 3 and 4, we use the Hopf bifurcation theorem to show that for $\lambda < 0$ small, this condition is also sufficient.

Lemma 2.1. *Let us consider the system* (0.3) *with* $\gamma\lambda < 1$. *A necessary condition for* (0.2) *to be stable as a solution of* (0.3) *with* $\tau = \frac{2\pi}{1-\gamma\lambda}$, *is that*

$$1 + \tau K(\cos\beta + \gamma \sin\beta) < 0.$$

Proof. We start by determining the linear variational equation of (0.3) around the periodic solution (0.2) by writing small deviations around the solution (0.2) as in (1.5).

We note that we go from system (0.1) to system (0.3) by adding the linear term

$$Ke^{i\beta}\left[z(t) - z(t-\tau)\right].$$

Using that we already determined the linearization of system (1.6) around the periodic solution (0.2) in the proof of Lemma 1.5, we find that the linearization of system (0.3) around the solution (0.2) satisfies

$$\dot{r}(t) + i\dot{\phi}(t) = -2\lambda r(t) - 2i\gamma\lambda\phi(t) - Ke^{i\beta}\left[r(t) + i\phi(t) - r(t-\tau) - i\phi(t-\tau)\right]$$

where $\tau = \frac{2\pi}{1-\gamma\lambda}$ is the period of the solution (0.2). Taking real and imaginary parts, we see that the linear variational equation of system (0.3) around the solution (0.2) is given by

$$\begin{pmatrix} \dot{r}(t) \\ \dot{\phi}(t) \end{pmatrix} = \begin{pmatrix} -2\lambda & 0 \\ -2\lambda\gamma & 0 \end{pmatrix} \begin{pmatrix} r(t) \\ \phi(t) \end{pmatrix} - K \begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix} \begin{pmatrix} r(t) - r(t-\tau) \\ \phi(t) - \phi(t-\tau) \end{pmatrix}. \tag{2.1}$$

Note that the linear variational equation is autonomous. Therefore, the Floquet exponents are given by the roots of the characteristic equation corresponding to (2.1). The characteristic function reads

$$\begin{aligned}\det \Delta(\mu) = &\left(\mu + 2\lambda + K\cos\beta(1 - e^{-\mu\tau}\right)\left(\mu + K\cos\beta(1 - e^{-\mu\tau})\right) \\ &+ \left(2\lambda\gamma + K\sin\beta(1 - e^{-\mu\tau})\right) K\sin\beta(1 - e^{-\mu\tau}).\end{aligned} \tag{2.2}$$

Observe that we have indeed a trivial Floquet multiplier, as predicted by Floquet theory, since $\det\Delta(0) = 0$ for all values of $\lambda, \gamma, K, \beta$.

Let us now consider the stability of (0.2) as a solution of (0.3) in the parameter plane $H = \{(\lambda, K) \mid \lambda < 0, K \in \mathbb{R}\}$ and fix a point $(\lambda_0, K_0) \in H$. For $K = 0$; system (0.3) reduces to (0.1) and Lemma 1.5 gives that for $(\lambda, K) = (\lambda_0, 0)$ we have exactly one Floquet exponent in the right half of the complex plane.

If a Floquet exponent moves from the right to the left half of the complex plane or vice versa, it should cross the imaginary axis [2]. If the Floquet exponent crosses the imaginary axis at the point $i\omega$ with $\omega \neq 0$, then the number of Floquet exponents in the right half of the complex plane changes by two, since if $\Delta(i\omega) = 0$, then also $\Delta(-i\omega) = 0$.

Now let us move from $(\lambda_0, 0)$ to the point (λ_0, K_0) and suppose that we do not cross a point (λ_0, K') such that for $\lambda = \lambda_0, K = K'$, $\mu = 0$ is a non trivial solution of (2.2), then the previous remarks imply that on the way from $(\lambda_0, 0)$ to (λ_0, K_0) the number of unstable Floquet exponents can only change by an even number; since for $(\lambda_0, 0)$ the number of unstable Floquet exponents is one, this gives that for (λ_0, K_0) the number of unstable Floquet exponents is odd. Since the number of unstable Floquet exponents is always non-negative, we see that it is at least one. Therefore, the periodic solution (0.2) of (0.3) is unstable for $(\lambda, K) = (\lambda_0, K_0)$. Thus, we find that a necessary condition for (0.2) to be stable as a solution of (0.3) for $(\lambda, K) = (\lambda_0, K_0)$ is that on the way from $(\lambda_0, 0)$ to (λ_0, K_0) we cross a point such that $\mu = 0$ is a non trivial solution of (2.2).

It holds that $\mu = 0$ is a non trivial root of $\det\Delta(\mu) = 0$ if and only if $(\det\Delta(\mu))/\mu = 0$. Using (2.2) gives that

$$\begin{aligned}\frac{\det\Delta(\mu)}{\mu} = &\ \mu + 2K\cos\beta(1 - e^{-\mu\tau}) + 2\lambda + 2\lambda K\cos\beta\frac{1 - e^{-\mu\tau}}{\mu} \\ &+ K^2\frac{(1 - e^{-\mu\tau})^2}{\mu} + 2\lambda\gamma K\sin\beta\frac{1 - e^{-\mu\tau}}{\mu}.\end{aligned}$$

Combining this with

$$\frac{1 - e^{-\mu\tau}}{\mu} = \tau_p + \mathcal{O}(\mu)$$

gives that $\mu = 0$ is a non trivial root of $\det\Delta(\mu) = 0$ if and only if

$$2\lambda(1 + \tau K(\cos\beta + \gamma\sin\beta)) = 0.$$

For $\lambda < 0$, we now find that $\mu = 0$ is a non trivial root of $\det\Delta(\mu) = 0$ if and only if $1 + 2\tau K(\cos\beta + \gamma\sin\beta) = 0$.

We note that the equation $1 + 2\tau K(\cos\beta + \gamma\sin\beta) = 0$ defines a curve ℓ in the parameter plane H. Let (λ_0, K_0) be as above; since for $K = 0$ we have that $1 + 2\pi K(\cos\beta + \gamma\sin\beta) = 1 > 0$, we cross the curve ℓ on the way from $(\lambda_0, 0)$ to (λ_0, K_0) if and only if $1 + \tau K(\cos\beta + \gamma\sin\beta) < 0$ for $(\lambda, K) = (\lambda_0, K_0)$. This proves the lemma. □

We note that in the above proof, we use the fact that the period orbit of the uncontrolled system (0.1) has an odd number of unstable Floquet multipliers. It was commonly believed that such periodic orbits cannot be stabilized using the Pyragas control scheme. In [5], the system (0.1) was used as a counterexample to this claim. However, for general systems there can still be obstructions to successful stabilization using Pyragas control: in [8] analytical conditions are given under which a periodic orbit in an autonomous system cannot be stabilized via the Pyragas control scheme.

3 Hopf bifurcation and stability conditions

In the previous section, we used Floquet theory to determine necessary conditions for the periodic orbit (0.2) of (0.3) to be stable. In this section, we use – inspired by [5] and [9] – the Hopf bifurcation theorem to find sufficient conditions for the periodic orbit (0.2) to be stable as a solution of (0.3) for parameter values near the bifurcation point. In particular, we find conditions for which the periodic solution (0.2) of (0.3) arises from a Hopf bifurcation. Using that a Hopf bifurcation is either subcritical (an unstable periodic orbit arises for parameter values where the fixed point is stable) or supercritical (a stable periodic orbit arises for parameter values where the fixed point is unstable), we then determine for which parameter values (0.2) is (un)stable as a solution of (0.3).

We note that in the Hopf bifurcation theorem (see Theorem 3.3 below), the parameters are varied along a curve in parameter space. In order to apply the Hopf bifurcation theorem to system (0.3), we should therefore choose a one-dimensional curve through the parameter space to approach the bifurcation point. There are, of course, different ways to do this and different curves of approach will give us different information on the behaviour of the controlled system.

Following [9], we introduce the following definition:

Definition 3.1. We define the *Pyragas curve* as the curve in (λ, τ)-parameter space given by the graph of $\tau(\lambda) = \frac{2\pi}{1-\gamma\lambda}$ with λ in the domain $(-\infty, 0)\backslash\{\frac{1}{\gamma}\}$.

By construction, we know that for parameter values on the Pyragas curve, the system (0.3) has a periodic orbit (see Figure 3.1). In the uncontrolled system (0.1) the family of periodic orbits (0.2) arises from a Hopf bifurcation at $\lambda = 0$ (see Section 1). We therefore expect that that in the controlled system (0.3) the family of periodic orbits (0.2) arises from a Hopf bifurcation at $(\lambda, \tau) = (0, 2\pi)$. To prove this, we would now like to use the Pyragas curve as curve of approach for the Hopf bifurcation theorem: this enables us to use Lemma 1.3 to determine the stability of the periodic orbit (0.2). In order to use the Pyragas curve as curve of approach for the Hopf bifurcation theorem, we have to extend the curve to the other side of the Hopf bifurcation point $(\lambda, \tau) = (0, 2\pi)$; see Figure 3.1. This motivates the following definition:

Definition 3.2. We define the *extended Pyragas curve* as the curve in (λ, τ)-parameter space given by the graph of $\tau(\lambda) = \frac{2\pi}{1-\gamma\lambda}$ with λ in the domain $(-\infty, \frac{1}{\gamma})$ if $\gamma > 0$ and λ in the domain $(-\frac{1}{\gamma}, \infty)$ if $\gamma < 0$.

In this section, we approach the point $(\lambda, \tau) = (0, 2\pi)$ over the extended Pyragas curve. We show that, under certain conditions on parameter values, we find a Hopf bifurcation of the origin for $(\lambda, \tau) = (0, 2\pi)$. Uniqueness of the periodic orbit arising from the Hopf bifurcation

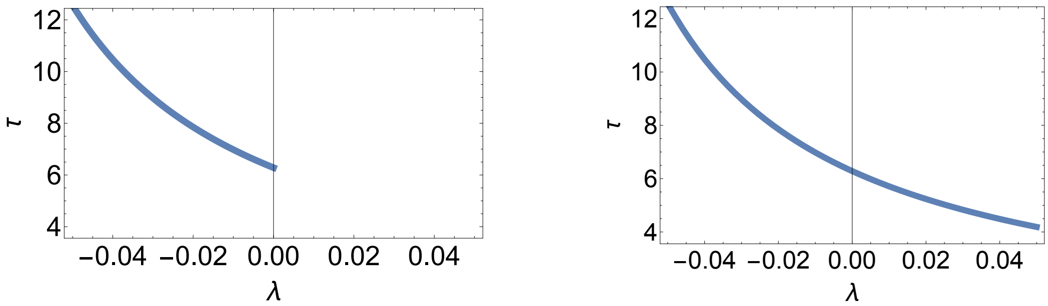

Figure 3.1: The Pyragas curve (left) and the extended Pyragas curve (right) for $\gamma = -10$.

now directly guarantees that the periodic orbit (0.2) of (0.3) arises from a Hopf bifurcation for parameter values near the bifurcation point.

We first state Theorem X.2.7 and Theorem X.3.9 from [2] on the Hopf bifurcation for differential delay equations.

Theorem 3.3 (Occurrence of a Hopf bifurcation). *Let us consider the differential delay equation*

$$\begin{cases} \dot{x}(t) = A(\lambda)x(t) + B(\lambda)x(t-\tau) + g(x_t, \lambda), & \text{for } t \geq 0 \\ x(t) = \phi(t), & \text{for } -\tau \leq t \leq 0 \end{cases} \tag{3.1}$$

where λ is a scalar parameter, $A(\lambda)$ and $B(\lambda)$ are $n \times n$-matrices, $\lambda \mapsto A(\lambda)$, $\lambda \mapsto B(\lambda)$ are smooth maps, $g : C([-\tau, 0], \mathbb{R}^n) \times \mathbb{R} \to \mathbb{R}^n$ is at least C^2, $g(0, \lambda) = D_1 g(0, \lambda) = 0$ for all λ and $\phi \in C([-\tau, 0], \mathbb{R}^n)$. Denote the characteristic function of (3.1) by $\Delta(\mu, \lambda)$. Assume that there exists an $\omega_0 \in \mathbb{R} \backslash \{0\}$ and a $\lambda_0 \in \mathbb{R}$ such that $\Delta(i\omega_0, \lambda_0) = 0$. Let $p, q \in \mathbb{C}^n$ satisfy

$$\Delta(i\omega_0, \lambda_0)p = 0, \qquad \Delta(i\omega_0, \lambda_0)^T q = 0, \qquad qD_1\Delta(i\omega_0, \lambda_0)p = 1. \tag{3.2}$$

If $\mathrm{Re}(q \cdot D_2\Delta(i\omega_0, \lambda_0)p) < 0$, $i\omega_0$ is a simple root of $\Delta(z, \lambda_0)$ and no other roots of $\Delta(z, \lambda_0)$ than $\pm i\omega_0$ belong to $i\omega_0\mathbb{Z}$, a Hopf bifurcation of the origin of (3.1) occurs.

We remark that the condition that $\mathrm{Re}(q \cdot D_2\Delta(i\omega_0, \lambda_0)p) < 0$ ensures that the eigenvalue on the imaginary axis that exists for $\lambda = \lambda_0$, moves to the right half of the complex plane if we vary λ.

Theorem 3.4 (Direction of the Hopf bifurcation). *Let us study the system (3.1) with A, B, g, p, q, λ_0 and ω_0 as in Theorem 3.3. Define $\psi(\theta) = e^{i\omega_0\theta}p$ for $\theta \in [-\tau, 0]$. If we introduce*

$$m = \frac{\mathrm{Re}(c)}{\mathrm{Re}(q \cdot D_2\Delta(i\omega_0, \lambda_0)p)} \tag{3.3}$$

with

$$\begin{aligned} c = {} & \frac{1}{2} q \cdot D_1^3 g(0, \lambda_0)(\psi, \psi, \overline{\psi}) + q \cdot D_1^2 g(0, \lambda_0)(e^{0\cdot}\Delta(0, \lambda_0)^{-1} D_1^2 g(0, \lambda_0)(\psi, \overline{\psi}), \psi) \\ & + \frac{1}{2} q \cdot D_1^2 g(0, \lambda_0)(e^{2i\omega_0\cdot}\Delta(2i\omega_0, \lambda_0)^{-1} D_1^2 g(0, \lambda_0)(\psi, \psi), \overline{\psi}), \end{aligned} \tag{3.4}$$

then for $m < 0$, the Hopf bifurcation is subcritical; for $m > 0$, the Hopf bifurcation is supercritical.

In order to apply Theorem 3.3 and 3.4 to system (0.3), we first note that system (0.3) is equivalent to the following system on $\mathbb{R}^2$:

$$\begin{aligned}\begin{pmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{pmatrix} &= \begin{pmatrix}\lambda - K\cos\beta & -1 + K\sin\beta\\ 1 - K\sin\beta & \lambda - K\cos\beta\end{pmatrix}\begin{pmatrix}x_1(t)\\ x_2(t)\end{pmatrix}\\ &\quad + \begin{pmatrix}x_1(t), & x_2(t)\end{pmatrix}\begin{pmatrix}x_1(t)\\ x_2(t)\end{pmatrix}\begin{pmatrix}1 & -\gamma\\ \gamma & 1\end{pmatrix}\begin{pmatrix}x_1(t)\\ x_2(t)\end{pmatrix}\\ &\quad + K\begin{pmatrix}\cos\beta & -\sin\beta\\ \sin\beta & \cos\beta\end{pmatrix}\begin{pmatrix}x_1(t-\tau)\\ x_2(t-\tau)\end{pmatrix}.\end{aligned} \tag{3.5}$$

The characteristic matrix of the linearization around zero is given by

$$\Delta(\mu,\lambda,\tau) = \mu I - \begin{pmatrix}\lambda - K\cos\beta & -1 + K\sin\beta\\ 1 - K\sin\beta & \lambda - K\cos\beta\end{pmatrix} - Ke^{-\mu\tau}\begin{pmatrix}\cos\beta & -\sin\beta\\ \sin\beta & \cos\beta\end{pmatrix}. \tag{3.6}$$

The non-linear term in (3.5), is given by the function $g : \mathcal{C}([-\tau,0],\mathbb{R}^2)\times\mathbb{R}\to\mathbb{R}^2$ given by

$$g(x_t,\lambda) = \langle x_t(0), x_t(0)\rangle\, Cx_t(0) \quad \text{with} \quad C = \begin{pmatrix}1 & -\gamma\\ \gamma & 1\end{pmatrix}. \tag{3.7}$$

An application of Theorem 3.3 yields the following result.

Theorem 3.5. *Consider the system* (0.3). *Assume*

$$1 + 2\pi K e^{i\beta} \neq 0. \tag{3.8}$$

If

$$1 + 2\pi K\left[\cos\beta + \gamma\sin\beta\right] > 0, \tag{3.9}$$

then we find a Hopf bifurcation at $(\lambda,\tau) = (0,2\pi)$ *if we approach the point* $(\lambda,\tau) = (0,2\pi)$ *over the extended Pyragas curve from the left.*

If

$$1 + 2\pi K\left[\cos\beta + \gamma\sin\beta\right] < 0, \tag{3.10}$$

then we find a Hopf bifurcation at $(\lambda,\tau) = (0,2\pi)$ *if we approach the point* $(\lambda,\tau) = (0,2\pi)$ *over the extended Pyragas curve from the right.*

Proof. We note that for $(\lambda,\tau) = (0,2\pi)$, $\mu = i$ is a root of the characteristic equation $\det\Delta(z) = 0$, where $\Delta(z)$ is given by (3.6). Using this fact in combination with the definition of p, q as in Theorem 3.3, we find that

$$p = \begin{pmatrix}1\\ -i\end{pmatrix} \quad \text{and} \quad q = \alpha\begin{pmatrix}1\\ i\end{pmatrix}. \tag{3.11}$$

The normalization factor $\alpha\in\mathbb{C}$ in (3.11) should be chosen such that for $\lambda = 0$,

$$q\cdot D_1\Delta(i\omega,\lambda)p = 1 \tag{3.12}$$

(see (3.2)). Using (3.6), we note that

$$D_1\Delta(i,0) = I + K\tau e^{-i2\pi}\begin{pmatrix}\cos\beta & -\sin\beta\\ \sin\beta & \cos\beta\end{pmatrix} = I + K\tau\begin{pmatrix}\cos\beta & -\sin\beta\\ \sin\beta & \cos\beta\end{pmatrix}.$$

Thus we find for $\lambda = 0$ that

$$
\begin{aligned}
q \cdot D_1\Delta(i\omega,\lambda)p &= \alpha\left((1,i)\begin{pmatrix}1\\-i\end{pmatrix} + K\tau(1,i)\begin{pmatrix}\cos\beta & -\sin\beta\\ \sin\beta & \cos\beta\end{pmatrix}\begin{pmatrix}1\\-i\end{pmatrix}\right)\\
&= 2\alpha\left(1+K\tau e^{i\beta}\right).
\end{aligned}
$$

Condition (3.12) therefore yields

$$
\alpha = \frac{1}{2(1+K\tau e^{i\beta})}. \tag{3.13}
$$

If we approach the point $(\lambda,\tau) = (0,2\pi)$ over the extended Pyragas curve from the left, we can parametrize the path by

$$
(\lambda(\theta),\tau(\theta)) = \left(\theta, \frac{2\pi}{1-\gamma\theta}\right), \qquad \theta \in \mathbb{R}\backslash\left\{\frac{1}{\gamma}\right\}. \tag{3.14}
$$

Using (3.6), we find that, for parameter values on this curve, the characteristic matrix is given by

$$
\Delta(\mu,\theta) = \mu I - \begin{pmatrix}\theta - K\cos\beta & -1+K\sin\beta\\ 1-K\sin\beta & \theta - K\cos\beta\end{pmatrix} - Ke^{-\mu\tau(\theta)}\begin{pmatrix}\cos\beta & -\sin\beta\\ \sin\beta & \cos\beta\end{pmatrix}.
$$

We are interested in the Hopf bifurcation at $(\lambda,\tau) = (0,2\pi)$. We note that the path parametrized by (3.14) reaches this point for $\theta = 0$. We find that

$$
\begin{aligned}
D_2\Delta(i,0) = &-\left.\frac{d}{d\theta}\right|_{\theta=0}\begin{pmatrix}\theta - K\cos\beta & -1+K\sin\beta\\ 1-K\sin\beta & \theta - K\cos\beta\end{pmatrix}\\
&- Ke^{-i\tau(0)}\begin{pmatrix}\cos\beta & -\sin\beta\\ \sin\beta & \cos\beta\end{pmatrix}\left(-i\left.\frac{d\tau}{d\theta}\right|_{\theta=0}\right)\\
= &-I + 2\pi i K\gamma\begin{pmatrix}\cos\beta & -\sin\beta\\ \sin\beta & \cos\beta\end{pmatrix}.
\end{aligned}
$$

We note that

$$
\begin{aligned}
q\begin{pmatrix}\cos\beta & -\sin\beta\\ \sin\beta & \cos\beta\end{pmatrix}p &= \alpha(1,i)\begin{pmatrix}\cos\beta & -\sin\beta\\ \sin\beta & \cos\beta\end{pmatrix}\begin{pmatrix}1\\-i\end{pmatrix}\\
&= \alpha(1,i)\begin{pmatrix}\cos\beta + i\sin\beta\\ \sin\beta - i\cos\beta\end{pmatrix}\\
&= 2\alpha(\cos\beta + i\sin\beta) = 2\alpha e^{i\beta}.
\end{aligned}
$$

Since α is given by (3.13), we find that

$$
\begin{aligned}
q \cdot D_2\Delta(i,0)p &= -2\alpha + 4\pi i K\gamma\alpha e^{i\beta}\\
&= \frac{-1+2\pi i\gamma K e^{i\beta}}{1+K\tau e^{i\beta}},
\end{aligned}
$$

which gives

$$
\mathrm{Re}(q \cdot D_2\Delta(i,0)p) = -\frac{1+2\pi K(\cos\beta + \gamma\sin\beta)}{|1+K2\pi e^{i\beta}|^2}.
$$

We conclude that if (3.9) holds, we have that $\mathrm{Re}(q \cdot D_2\Delta(i,0)p) < 0$. Condition (3.8) ensures that $\mu = i$ has multiplicity one as a root of $\Delta(\mu, 0)$ and one easily verifies that $\mu = i$ is the only root of $\Delta(\mu, 0)$ of the form $i\mathbb{Z}$. Therefore if (3.8) – (3.9) hold, we obtain a Hopf bifurcation if we approach the point $(\lambda, \tau) = (0, 2\pi)$ over the extended Pyragas curve from left.

Similarly, if we approach the point $(\lambda, \tau) = (0, 2\pi)$ over the extended Pyragas curve from the right, we parametrize the path by (3.14) by replacing $\theta \mapsto -\theta$. Denote by $\tilde{\Delta}$ the characteristic matrix of system (0.3) for parameter values (λ, τ) on this path. A similar analysis then shows that

$$\mathrm{Re}(q \cdot D_2\tilde{\Delta}(i,0)p) = \frac{1 + 2\pi K(\cos\beta + \gamma \sin\beta)}{\left|1 + K2\pi e^{i\beta}\right|^2}.$$

Thus, $\mathrm{Re}(q \cdot D_2\tilde{\Delta}(i,0)p) < 0$ if (3.10) is satisfied. Therefore, if (3.10) and (3.8) hold, we find a Hopf bifurcation at $(\lambda, \tau) = (0, 2\pi)$ if we approach this point over the extended Pyragas curve from the right. □

Now that we have derived conditions for a Hopf bifurcation in the origin to occur, we determine the direction of the bifurcation using Theorem 3.4. As outlined before, the direction of the Hopf bifurcation will give us conditions for (0.2) to be (un)stable as a solution of (0.3).

Theorem 3.6. *If we approach the Hopf bifurcation point $(\lambda, \tau) = (0, 2\pi)$ over the extended Pyragas curve from the left, the value of m as defined in Theorem 3.4 is given by*

$$m = -4.$$

If we approach the Hopf bifurcation point $(\lambda, \tau) = (0, 2\pi)$ over the extended Pyragas curve from the right, the value of m as defined in Theorem 3.4 is given by

$$m = 4.$$

Proof. Computing the derivative of (3.7) gives (see [18] for more details):

$$D_1^2 g(0, \lambda) = 0 \quad \text{for all } \lambda \in \mathbb{R} \tag{3.15}$$

$$D_1^3 g(\phi, \lambda)(f_1, f_2, f_3) = \sum_{\sigma \in S_3} \left\langle f_{\sigma(1)}(0), f_{\sigma(2)}(0) \right\rangle C f_{\sigma(3)}(0) \tag{3.16}$$

for all $\phi, f_1, f_2, f_3 \in \mathcal{C}\left([-\tau, 0], \mathbb{R}^2\right)$. Here, S_3 denotes the permutation group of three objects. Using this, we find that

$$\begin{aligned}
c &= \frac{1}{2} q \cdot D_1^3 g(0, \lambda)(\psi, \psi, \overline{\psi}) + 0 + 0 \\
&= \frac{1}{2} q \cdot \left(2\left\langle \psi(0), \psi(0) \right\rangle C\overline{\psi(0)} + 2\left\langle \overline{\psi(0)}, \psi(0) \right\rangle C\psi(0) + 2\left\langle \psi(0), \overline{\psi(0)} \right\rangle C\psi(0)\right) \\
&= q \cdot \left(\langle p, p \rangle C\overline{p} + \langle p, \overline{p} \rangle Cp + \langle \overline{p}, p \rangle Cp\right) \\
&= \frac{4(1 + i\gamma)}{1 + K2\pi e^{i\beta}}.
\end{aligned}$$

Taking real parts yields

$$\mathrm{Re}\, c = \frac{4\left(1 + K2\pi\left(\cos\beta + \gamma \sin\beta\right)\right)}{\left|1 + K2\pi e^{i\beta}\right|^2}.$$

Let us now approach the point $(\lambda, \tau) = (0, 2\pi)$ over the extended Pyragas curve from the left. We find as in the proof of Lemma 3.5 that

$$\mathrm{Re}(q \cdot D_2\Delta(i, 0)p) = -\frac{1 + 2\pi K(\cos\beta + \gamma \sin\beta)}{\left|1 + K2\pi e^{i\beta}\right|^2}.$$

It follows that $m = -4$.

Similarly, if we approach the point $(\lambda, \tau) = (0, 2\pi)$ over the extended Pyragas curve from the right and denote by $\tilde{\Delta}$ the corresponding characteristic matrix, we find as in the proof of Lemma 3.5 that

$$\mathrm{Re}(q \cdot D_2\tilde{\Delta}(i, 0)p) = \frac{1 + 2\pi K(\cos\beta + \gamma \sin\beta)}{\left|1 + K2\pi e^{i\beta}\right|^2}.$$

Combining this with the value of $\mathrm{Re}\, c$, we find that $m = 4$. □

We are now able to determine for which parameter values (0.2) is (un)stable as a solution of (0.3).

Corollary 3.7. *Let $1 + 2\pi K e^{i\beta} \neq 0$. If*

$$1 + 2\pi K\left[\cos\beta + \gamma \sin\beta\right] > 0, \tag{3.17}$$

then for small λ, (0.2) is an unstable periodic solution of (0.3). Furthermore, if for $\lambda = 0, \tau = 2\pi$ no roots of the characteristic equation $\det\Delta(\mu) = 0$ with $\Delta(\mu)$ as in (3.6) are in the right half of the complex plane and

$$1 + 2\pi K\left[\cos\beta + \gamma \sin\beta\right] < 0, \tag{3.18}$$

then for small λ, (0.2) is a stable periodic solution of (0.3).

Proof. If (3.17) is satisfied, then Lemma 3.5 shows that we find a Hopf bifurcation at the point $(\lambda, \tau) = (0, 2\pi)$ if we approach this point over the extended Pyragas curve from the left. Combining Lemma 3.6 with Theorem 3.4, we find that this Hopf bifurcation is subcritical. Thus, there exists an unstable periodic solution for parameter values (λ, τ) on the (extended) Pyragas curve to the left of the point $(0, 2\pi)$. By the Hopf bifurcation theorem, the periodic solution for these parameter values is unique. By definition of the Pyragas curve, (0.2) is a periodic solution of (0.3) for (λ, τ) near $(0, 2\pi)$, i.e. this is the periodic solution generated by the Hopf bifurcation. We conclude that for (λ, τ) on the Pyragas curve near $(0, 2\pi)$, (0.2) is an unstable periodic solution of (0.3).

If (3.18) is satisfied, we have by Lemma 3.5 that we find a Hopf bifurcation at the point $(\lambda, \tau) = (0, 2\pi)$ if we approach this point over the extended Pyragas curve from the right. Combining Lemma 3.6 with Theorem 3.4, we find that this Hopf bifurcation is supercritical if no roots of $\det\Delta(\mu) = 0$ with $\Delta(\mu)$ as in (3.6) are in the right half of the complex plane.

Therefore, we find an unique, stable periodic solution of (0.3) for (λ, τ) on the Pyragas curve near $(0, 2\pi)$. Since (0.2) is a periodic solution of (0.3) for (λ, τ) on the Pyragas curve, we conclude that for (λ, τ) on the Pyragas curve near $(0, 2\pi)$, this solution is in fact stable if for $\lambda = 0, \tau = 2\pi$ no roots of the characteristic equation are in the right half of the complex plane. □

Recall that in Section 1 we determined the direction of Hopf bifurcation when we vary λ. A similar approach can be followed for the controlled system (0.3) to give an alternative proof of Corollary 3.7 using Lemma 1.3.

Proof (of Corollary 3.7). The characteristic function corresponding to the linearization of (0.3) around $z = 0$ is given by

$$\Delta(\mu) = \mu - (\lambda + i) + Ke^{i\beta}\left[1 - e^{-\mu\tau}\right]. \tag{3.19}$$

We recall from the proof of Lemma 3.5 that for $\lambda = 0$, $\mu = i$ is a root of (3.19) and that there are no other roots on the imaginary axis. Furthermore, if $1 + 2\pi Ke^{i\beta} \neq 0$, then $\mu = i$ has multiplicity one as a solution of $\Delta(\mu) = 0$. Therefore, if $\mu = i$ crosses the imaginary axis with non zero speed as we cross the point $(\lambda, \tau) = (0, 2\pi)$ over the Pyragas curve, a Hopf bifurcation of the origin occurs for $\lambda = 0$.

Parametrize the Pyragas curve as in (3.14) and, for small θ, $\mu = \mu(\theta)$ for the root satisfying $\Delta(\mu(\theta)) = 0$ for $\lambda = \lambda(\theta)$, and $\tau = \tau(\theta)$ as in (3.14) with $\mu(0) = i$. Differentiation of (3.19) gives that

$$0 = \left.\frac{d\mu}{d\theta}\right|_{\theta=0} - 1 + Ke^{i\beta}\left(\left.\frac{d\mu}{d\theta}\right|_{\theta=0} 2\pi + 2\pi\gamma i\right),$$

which we can rewrite as

$$\left.\frac{d\mu}{d\theta}\right|_{\theta=0}\left(1 + 2\pi Ke^{i\beta}\right) = 1 - 2\pi\gamma iKe^{i\beta},$$

which gives

$$\begin{aligned}\left.\frac{d\mu}{d\theta}\right|_{\theta=0} &= \frac{1}{\left|1 + 2\pi Ke^{i\beta}\right|^2}\left(1 - 2\pi\gamma iKe^{i\beta}\right)\left(1 + 2\pi Ke^{-i\beta}\right) \\ &= \frac{1}{\left|1 + 2\pi Ke^{i\beta}\right|^2}\left(1 + 2\pi Ke^{-i\beta} - 2\pi\gamma iKe^{i\beta} - 4\pi^2\gamma K^2 i\right).\end{aligned}$$

Taking real parts yields

$$\left.\frac{d\,\mathrm{Re}\,\mu}{d\theta}\right|_{\theta=0} = \frac{1}{\left|1 + 2\pi Ke^{i\beta}\right|^2}\left(1 + 2\pi K\cos\beta + 2\pi\gamma K\sin\beta\right).$$

In particular, if $1 + 2\pi K(\cos\beta + \gamma\sin\beta) \neq 0$, then the root $\mu = i$ that exists for $(\lambda, \tau) = (0, 2\pi)$ crosses the imaginary axis with non zero speed as we cross the point $(\lambda, \tau) = (0, 2\pi)$ over the Pyragas curve. This shows that there is a Hopf bifurcation at the origin. An application of Lemma 1.3 now yields the result. □

We remark that this alternative proof of Corollary 3.7 exploits the fact that the extended Pyragas curve is defined in such a way that we a priori know for which points on the curve a periodic solution of the system (0.3) exists. We will us this observation again in Section 5 when we introduce a variation of Pyragas control scheme to system (0.1).

4 Hopf bifurcation and dynamics of the controlled system

In the previous section, we approached the Hopf bifurcation point $(\lambda, \tau) = (0, 2\pi)$ over the extended Pyragas curve. As remarked before, there are of course many different ways to approach this bifurcation point. In this section, we approach the bifurcation point parallel to the λ-axis, as was done in [9]. This again enables us to determine stability conditions for (0.2) as a solution of (0.3) and gives us more insight in the dynamics of the controlled system.

Using Theorem 3.3, we can determine conditions for a Hopf bifurcation of system (0.3) to occur if we vary λ and leave all the other parameters fixed.

Lemma 4.1. *Let us consider the system (0.3) where we leave all parameters but λ fixed. Let $(\lambda, \tau) \neq (0,0)$ be such that*

$$\lambda = K\left[\cos\beta - \cos(\beta - \phi)\right], \tag{4.1}$$

$$\tau = \frac{\phi}{1 - K\left[\sin\beta - \sin(\beta - \phi)\right]} \tag{4.2}$$

for some $\phi \in \mathbb{R}\backslash\{0\}$. Furthermore, assume that

$$1 + K\tau e^{i(\beta-\phi)} \neq 0, \tag{4.3}$$

$$1 + K\tau\cos(\beta - \phi) > 0. \tag{4.4}$$

Then a Hopf bifurcation of the origin of system (0.3) occurs.

Proof. We note that we can cast system (3.5) in the form of Theorem 3.3 by setting

$$A = \begin{pmatrix} \lambda - K\cos\beta & -1 + K\sin\beta \\ 1 - K\sin\beta & \lambda - K\cos\beta \end{pmatrix}, \qquad B = K\begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix} \tag{4.5}$$

and g as in (3.7). We see that $\lambda \mapsto A(\lambda), \lambda \mapsto B(\lambda)$ is smooth and g is C^2 with $g(0,\lambda) = D_1 g(0,\lambda) = 0$ for all λ.

We note that the characteristic equation of the linearized equation of (0.3) is given by

$$\Delta(\mu, \lambda) = \mu - \lambda - i + Ke^{i\beta}\left[1 - e^{-\mu\tau}\right].$$

Writing $\mu = i\omega, \omega \neq 0$ and taking real and imaginary parts of the equation $\Delta(\mu) = 0$, we find that

$$0 = \lambda - K\left[\cos\beta - \cos(\beta - \omega\tau)\right],$$
$$\omega = 1 - K\left[\sin\beta - \sin(\beta - \omega\tau)\right].$$

Introducing the notation $\phi = \omega\tau$ and rewriting yields (4.1) and (4.2). Thus, for (λ, τ) satisfying (4.1) and (4.2) we find a non zero root of the characteristic equation on the imaginary axis.

We note that

$$D_1\Delta(i\omega, \lambda) = 1 - Ke^{i\beta}e^{-i\omega\tau}(-\tau) = 1 + K\tau e^{i(\beta-\phi)}.$$

Thus, if (4.3) is satisfied, we find that $\mu = i\omega$ is a simple zero of the characteristic equation. By construction, $\mu = i\omega$ is the only zero of the characteristic equation of the form $\mu = i\omega n$, $n \in \mathbb{Z}$.

A similar computation as in the proof of Lemma 3.5 yields

$$p = \begin{pmatrix} 1 \\ -i \end{pmatrix} \quad \text{and} \quad q = \frac{1}{2(1 + K\tau e^{i(\beta-\phi)})}\begin{pmatrix} 1 \\ i \end{pmatrix}.$$

We have that

$$\begin{aligned} \operatorname{Re}\left(q \cdot D_2\Delta(i\omega, \lambda)p\right) &= \operatorname{Re}\left(-\frac{1}{1 + K\tau e^{i(\beta-\phi)}}\right) \\ &= -\frac{1 + K\tau\cos(\beta - \phi)}{\left|1 + K\tau e^{i(\beta-\phi)}\right|^2}. \end{aligned}$$

Thus, $\operatorname{Re}\left(q \cdot D_2\Delta(i\omega, \lambda)p\right) < 0$ if and only if (4.4) is satisfied. Using Theorem 3.3, we conclude that if the conditions (4.1) - (4.4) are satisfied a Hopf bifurcation of the origin occurs. □

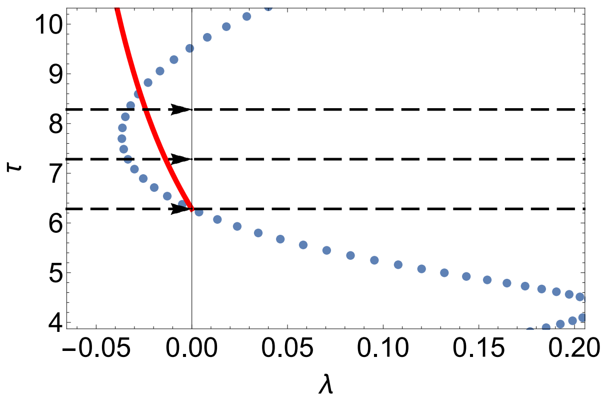

Figure 4.1: Approaching the Hopf bifurcation points parallel to the λ-axis for $\gamma = -10, \beta = \frac{\pi}{4}$ and $K = 0.035$. The solid line indicates the Pyragas curve, the dotted line the Hopf bifurcation curve and the dashed lines indicate the curves of approach.

As in [9], we define the *Hopf bifurcation curve* as the curve in (λ, τ)-parameter space parametrized by (4.1)–(4.2) for $\phi \in \mathbb{R}$. We note that the Pyragas curve (see Definition 3.1) ends on the Hopf bifurcation point at $(\lambda, \tau) = (0, 2\pi)$ (see Figure 4.1). We can now try to choose the parameters in such a way that the periodic solution (0.2) of (0.3) emanates from a supercritical Hopf bifurcation; then (0.2) is a stable solution of (0.3) for parameter values near the bifurcation point.

In [9], the direction of the Hopf bifurcation was determined using a normal form reduction. Here, we rederive this result directly as an application of Theorem 3.4.

Theorem 4.2. *Let (λ, τ) be a point on the Hopf bifurcation curve and let $\phi \in \mathbb{R} \backslash \{0\}$ satisfy (4.1)–(4.2). If λ varies while all other parameters remain fixed, then the value of m as defined in (3.3) is given by*

$$m = -\frac{4(1 + K\tau\,(\cos(\beta - \phi) + \gamma \sin(\beta - \phi))}{1 + K\tau \cos(\beta - \phi)}. \tag{4.6}$$

Proof. We recall that if (λ, τ) lies on the Hopf bifurcation curve, then there exists an $\omega \in \mathbb{R}$ satisfying $\phi = \omega\tau$ such that $\Delta(i\omega, \lambda) = 0$. Now let p, q be as in the proof of Lemma 4.1, then we have that p, q satisfy (3.2). Using (3.6), we obtain

$$D_2\Delta(i\omega, \lambda, \tau) = -I,$$

which gives

$$\begin{aligned} q \cdot D_2\Delta(i\omega, \lambda)p &= -q \cdot p \\ &= -\frac{1}{1 + K\tau e^{i(\beta - \phi)}}. \end{aligned}$$

Taking the real part yields

$$\mathrm{Re}(q \cdot D_2\Delta(i\omega, \lambda)p) = -\frac{1 + K\tau \cos(\beta - \phi)}{\left|1 + K\tau e^{i(\beta - \phi)}\right|^2}.$$

Using (3.15)–(3.16), we can now explicitly compute c:

$$
\begin{aligned}
c &= \frac{1}{2} q \cdot D_1^3 g(0,\lambda)(\psi,\psi,\overline{\psi}) + 0 + 0 \\
&= \frac{1}{2} q \cdot \left(2\left\langle \psi(0), \psi(0) \right\rangle C\overline{\psi(0)} + 2\left\langle \overline{\psi(0)}, \psi(0) \right\rangle C\psi(0) + 2\left\langle \psi(0), \overline{\psi(0)} \right\rangle C\psi(0) \right) \\
&= q \cdot \left(\langle p, p \rangle C\overline{p} + \langle p, \overline{p} \rangle Cp + \langle \overline{p}, p \rangle Cp \right) \\
&= \frac{4(1+i\gamma)}{1 + K\tau e^{i(\beta-\phi)}}.
\end{aligned}
$$

Thus we find

$$
\operatorname{Re} c = \frac{4(1 + K\tau\left(\cos(\beta-\phi) + \gamma\sin(\beta-\phi)\right)}{\left|1 + K\tau e^{i(\beta-\phi)}\right|^2}.
$$

Using the definition of m as in Theorem 3.4, we arrive at equation (4.6). This completes the proof. □

We are able to determine the direction of the Hopf bifurcation for parameter values (λ, τ) for which a Hopf bifurcation of the origin of system (0.3) occurs; cf. eq. (8) in [9].

Corollary 4.3. *Let (λ, τ) be such that a Hopf bifurcation of the origin of system (0.3) occurs, i.e., let the conditions of Theorem 4.1 be satisfied for some $\phi \in \mathbb{R}\backslash\{0\}$. If*

$$
1 + K\tau\left[\cos(\beta-\phi) + \gamma\sin(\beta-\phi)\right] > 0, \tag{4.7}
$$

then the Hopf bifurcation at (λ, τ) is subcritical. If

$$
1 + K\tau\left[\cos(\beta-\phi) + \gamma\sin(\beta-\phi)\right] < 0, \tag{4.8}
$$

the Hopf bifurcation at (λ, τ) is supercritical.

Proof. If the conditions of Theorem 4.1 are satisfied, then (4.4) holds and

$$
1 + K\tau\cos(\beta-\phi) > 0.
$$

Combining this inequality with Theorem 4.2, we find that $m < 0$ if (4.7) holds. Using Theorem 3.4 this shows that the Hopf bifurcation is subcritical. Similarly, if (4.8) holds, then $m > 0$ and again by Theorem 3.4 the Hopf bifurcation is supercritical. □

We can determine the orientation of the Pyragas curve with respect to the Hopf bifurcation curve at the point $(\lambda, \tau) = (0, 2\pi)$ by computing the slopes of the curves at $(\lambda, \tau) = (0, 2\pi)$. Combining this with the direction of the Hopf bifurcation curve, we are able to give conditions for (0.2) to be (un)stable as a solution of (0.3). If the Hopf bifurcation at $(\lambda, \tau) = (0, 2\pi)$ is subcritical and the Pyragas curve is locally to the left of the Hopf bifurcation curve, we expect the periodic solution (0.2), that exists for parameter values on the Pyragas curve, to arise from the Hopf bifurcation and therefore be unstable. By an analogous argument, we find that the solution (0.2) of (0.3) is stable if the Hopf bifurcation at $(\lambda, \tau) = (0, 2\pi)$ is supercritical and the Pyragas curve is locally to the right of the Hopf bifurcation curve. Following [9], this leads to the following corollary.

Corollary 4.4. *Let the parameters K, β, γ be such that a Hopf bifurcation of system (0.3) occurs for $(\lambda, \tau) = (0, 2\pi)$, i.e. let*

$$1 + 2\pi K e^{i\beta} \neq 0, \tag{4.9}$$

$$1 + 2\pi K \cos\beta > 0. \tag{4.10}$$

If $1 + 2\pi K[\cos\beta + \gamma \sin\beta] < 0$, the Pyragas curve is locally to the right of the Hopf bifurcation curve and no roots of the characteristic equation $\det \Delta(\mu) = 0$ with $\Delta(\mu)$ as in (3.6) are in the right half of the complex plane, then the periodic solution (0.2) of (0.3) is stable for small λ. If $1 + 2\pi K[\cos\beta + \gamma \sin\beta] > 0$ and the Pyragas curve is locally to the left of the Hopf bifurcation curve, then the periodic solution (0.2) of (0.3) is unstable for small λ.

As we have seen in Sections 3–4, applying the Hopf bifurcation theorem with respect to different curves yields different results. Comparing Corollary 4.4 with Corollary 3.7, we see that Corollary 3.7 gives us weaker conditions for (0.2) to be (un)stable as a solution of (0.3) for small λ. In particular, we can drop the condition (4.10) and we no longer have to take the orientation of the Pyragas curve with respect to the Hopf bifurcation curve into account. Using Corollary 3.7, we are therefore able to determine upon the (in)stability of the periodic solution (0.2) of (0.3) for a wider range of parameter values than if we use Corollary 4.3.

The approach we have used in Section 4 gives more insight in the dynamics of the controlled system (0.3). If $1 + K\tau[\cos\beta + \gamma\sin\beta] > 0$, then (4.7) holds for ϕ in a small neighbourhood of 2π. Applying Corollary 4.3, we find that for parameter values (λ, τ) in a neighbourhood of $(\lambda, \tau) = (0, 2\pi)$ to the left of the Hopf bifurcation curve, a periodic orbit exists. Similarly, if $1 + K\tau[\cos\beta + \gamma\sin\beta] < 0$, a periodic orbit exists for all parameter values (λ, τ) in a neighbourhood of $(\lambda, \tau) = (0, 2\pi)$ to the right of the Hopf bifurcation curve. We conclude that by applying Pyragas control, a new set of periodic orbits is created, see also [13].

5 A variation in control term

In previous sections, we discussed three different methods to determine the stability of periodic orbit (0.2) of system (0.3). In this section, we return to the general problem of Pyragas control. Let us study the system

$$\dot{x}(t) = f(x(t)), \qquad x(0) = x_0 \tag{5.1}$$

with $f : \mathbb{R}^n \to \mathbb{R}^n$. Let us assume that an unstable periodic solution $u(t)$ of this system exists; denote its period by T. In the Pyragas control scheme, we add a term to the system (5.1) in such a way that the periodic solution $u(t)$ is a also a solution of the controlled system. Usually, we write for the controlled system

$$\dot{x}(t) = f(x(t)) + K[x(t) - x(t-T)] \tag{5.2}$$

with K a $n \times n$-matrix. There are, however, variations to this scheme possible. We remark that $u(t)$ is also a periodic solution of the system

$$\dot{x}(t) = f(x(t)) + K_1[x(t) - x(t-T)] + K_2[\dot{x}(t) - \dot{x}(t-T)] \tag{5.3}$$

where K_1, K_2 are $n \times n$-matrices. We can investigate for which values of K_1, K_2 the solution $u(t)$ of (5.3) is stable, and how these values of K_1, K_2 compare to the values of K for which $u(t)$ is stable as a solution to (5.2).

Applying the type of control given in (5.3) to (0.1) yields the system

$$\begin{aligned}\dot{z}(t) = (\lambda + i)z(t) + (1 + i\gamma)\,|z(t)|^2\, z(t) - K_1 e^{i\beta_1}\,[z(t) - z(t-\tau)] \\ - K_2 e^{i\beta_2}\,[\dot{z}(t) - \dot{z}(t-\tau)]\,,\end{aligned} \tag{5.4}$$

which can be rewritten as

$$\begin{aligned}\dot{z}(t) - \frac{K_2 e^{i\beta_2}}{1 + K_2 e^{i\beta_2}} \dot{z}(t-\tau) = \frac{1}{1 + K_2 e^{i\beta_2}} \Big((\lambda + i) z(t) + (1 + i\gamma)\,|z(t)|^2\, z(t) \Big) \\ - \frac{K_1 e^{i\beta_1}}{1 + K_2 e^{i\beta_2}}\,[z(t) - z(t-\tau)]\,.\end{aligned} \tag{5.5}$$

We note that (5.5) is a neutral functional differential equation. Neutral functional differential equations have very different properties from retarded functional differential equations. For example, for retarded functional differential equations the solution operator $T(t)$ is compact for $t \geq r$ (where r denotes the delay of the system), but for neutral functional differential equations this property does in general not hold. Also, if we fix $\alpha, \beta \in \mathbb{R}$, then for neutral functional differential equations we can have an infinite number of roots of the characteristic equation in a strip $\{z \in \mathbb{C} \mid \alpha \leq \operatorname{Re} z \leq \beta\}$. This cannot occur for retarded functional differential equations. Since we can have an infinite number of eigenvalues in a strip $\{z \in \mathbb{C} \mid \alpha \leq \operatorname{Re} z \leq \beta\}$, it can also occur that all the eigenvalues are in the left half of the complex plane, but the eigenvalues get arbitrary close to the imaginary axis. In this case, it is possible that all eigenvalues are in the left half of the complex plane, but the fixed point of the equation is not stable. However, if we have a so called spectral gap, i.e. there exists a $\gamma < 0$ such that all the eigenvalues are in the set $\{z \in \mathbb{C} \mid \operatorname{Re} z < \gamma\}$, then stability of the fixed point is guaranteed. In the case of a spectral gap, we can use the same methods as in the retarded case to find a Hopf bifurcation theorem for neutral equations.

Lemma 5.1. *Let $K_1, K_2, \beta_1, \beta_2$ be such that for $\lambda = 0$, there exists a $\gamma < 0$ such that all roots, except the root $\mu = i$, of (5.7) are in the set $\{z \in \mathbb{C} \mid \operatorname{Re} z < \gamma\}$. If*

$$1 + 2\pi K_1 \left(\cos(\beta_1) + \gamma \sin(\beta_1)\right) - 2\pi K_2 \left(\sin(\beta_2) - \gamma \cos(\beta_2)\right) > 0,$$

then the periodic solution (0.2) of (5.4) that exists for $\lambda < 0$ is unstable for small $\lambda < 0$. If

$$1 + 2\pi K_1 \left(\cos(\beta_1) + \gamma \sin(\beta_1)\right) - 2\pi K_2 \left(\sin(\beta_2) - \gamma \cos(\beta_2)\right) < 0, \tag{5.6}$$

the periodic solution (0.2) of (5.4) that exists for $\lambda < 0$ is stable for small $\lambda < 0$.

Proof. We note that the characteristic equation corresponding to the linearization of (5.4) around $z = 0$ is given by

$$\Delta(\mu) = \mu - (\lambda + i) + K_1 e^{i\beta_1}(1 - e^{-\mu\tau}) + K_2 e^{i\beta_2} \mu (1 - e^{-\mu\tau}). \tag{5.7}$$

We have that $\Delta(i) = 0$ for $\lambda = 0$ and $\tau = 2\pi$. We determine whether the root $\mu = i$ moves in our out of the right half of the complex plane if approach the point $(\lambda, \tau) = (0, 2\pi)$ over the extended Pyragas curve from the left.

Parametrize the extended Pyragas curve as in (3.14). For θ near 0, write $\mu = \mu(\theta)$ satisfying $\Delta(\mu(\theta)) = 0$ for $\lambda = \lambda(\theta)$ and $\tau = \tau(\theta)$ with $\mu(0) = i$. Then differentation of (5.7) with respect to θ yields

$$\begin{aligned} 0 &= \left.\frac{d\mu}{d\theta}\right|_{\theta=0} - 1 + K_1 e^{i\beta_1} e^{-\mu(0)\tau(0)} \left(\mu(\theta) \left.\frac{d\tau}{d\theta}\right|_{\theta=0} + \left.\frac{d\mu}{d\theta}\right|_{\theta=0} \tau(0) \right) \\ &\quad + K_2 e^{i\beta_2} \left.\frac{d\mu}{d\theta}\right|_{\theta=0} (1 - e^{-\mu(0)\tau(0)}) + K_2 e^{i\beta_2} \mu(0) \left(\mu(0) \left.\frac{d\tau}{d\theta}\right|_{\theta=0} + \left.\frac{d\mu}{d\theta}\right|_{\theta=0} \tau(0) \right) \\ &= \left.\frac{d\mu}{d\theta}\right|_{\theta=0} - 1 + K_1 e^{i\beta_1} \left(2\pi i\gamma + 2\pi \left.\frac{d\mu}{d\theta}\right|_{\theta=0} \right) + K_2 e^{i\beta_2} i \left(2\pi i\gamma + 2\pi \left.\frac{d\mu}{d\theta}\right|_{\theta=0} \right), \end{aligned}$$

which can be rewritten as

$$\left.\frac{d\mu}{d\theta}\right|_{\theta=0} \left(1 + 2\pi K_1 e^{i\beta_1} + 2\pi i K_2 e^{i\beta_2} \right) = 1 - 2\pi\gamma i K_1 e^{i\beta_1} + 2\pi\gamma K_2 e^{i\beta_2}.$$

With $a = 1 + 2\pi K_1 e^{i\beta_1} + 2\pi i K_2 e^{i\beta_2}$ this gives

$$\begin{aligned} \left.\frac{d\mu}{d\theta}\right|_{\theta=0} &= \frac{1}{|a|^2} \left(1 - 2\pi\gamma i K_1 e^{i\beta_1} + 2\pi\gamma K_2 e^{i\beta_2} \right) \left(1 + 2\pi K_1 e^{-i\beta_1} - 2\pi i K_2 e^{-i\beta_2} \right) \\ &= \frac{1}{|a|^2} \Big(1 + 2\pi K_1 e^{-i\beta_1} - 2\pi K_2 i e^{-i\beta_2} - 2\pi\gamma i K_1 e^{i\beta_1} - 4\pi^2 \gamma i K_1^2 \\ &\qquad - K_1 K_2 4\pi^2 \gamma e^{i(\beta_1 - \beta_2)} + 2\pi\gamma K_2 e^{i\beta_2} + 4\pi^2 \gamma K_1 K_2 e^{i(\beta_2 - \beta_1)} - 4\pi^2 \gamma i K_2^2 \Big). \end{aligned}$$

After taking the real part we arrive at

$$\begin{aligned} \operatorname{Re} \left.\frac{d\mu}{d\theta}\right|_{\theta=0} &= 1 + 2\pi K_1 \cos\beta_1 - 2\pi K_2 \sin\beta_2 + 2\pi\gamma K_1 \sin\beta_1 + 2\pi\gamma K_2 \cos\beta_2 \\ &= 1 + 2\pi K_1 (\cos\beta_1 + \gamma \sin\beta_1) - 2\pi K_2 (\sin\beta_2 - \gamma\cos\beta_2). \end{aligned}$$

If $1 + 2\pi K_1 (\cos\beta_1 + \gamma\sin\beta_1) - 2\pi K_2(\sin\beta_2 - \gamma\cos\beta_2) \neq 0$ and for $\lambda = 0$ all the roots of (5.7) except $\mu = i$ are in the left half of the complex plane, then the conditions of the Hopf bifurcation theorem for neutral functional differential equations are satisfied. An application of Lemma 1.3 now yields the result. □

Assume that $\beta_1 = \beta_2$ and let us study the case $\gamma = -10$ and $\beta_1 = \frac{\pi}{4}$. In order to apply Lemma 5.1 we are interested in values of K_1, K_2 such that there exists a $\gamma < 0$ such that all roots, expect the root $\mu = i$, of (5.7) are in the set $\{z \in \mathbb{C} \mid \operatorname{Re} z < \gamma\}$. We note that if

$$\left| \frac{K_2 e^{i\beta_2}}{1 + K_2 e^{i\beta_2}} \right| < 1 \tag{5.8}$$

(i.e. we have a stable D-operator), and all the roots of the characteristic equation are in the left half of the complex plane, then this condition is automatically satisfied. Now let us choose K_1 close to zero; using DDE-Biftool [3], we find that for $K_2 = 0$ and some (fixed) K_1 small, the characteristic equation (5.7) has no roots in the right half of the complex plane. One can prove that a root of (5.7) must cross the imaginary axis to move from the left to the right half of the complex plane. Using this, one can draw a stability chart to show that for points inside the region whose boundary is parametrized by

$$\begin{aligned} K_1 &= \frac{1}{2\sin(\omega\pi)} (1-\omega)\cos(\omega\pi - \beta), \\ K_2 &= \frac{1}{2\omega\sin(\omega\pi)} (1-\omega)\sin(\omega\pi - \beta) \end{aligned} \tag{5.9}$$

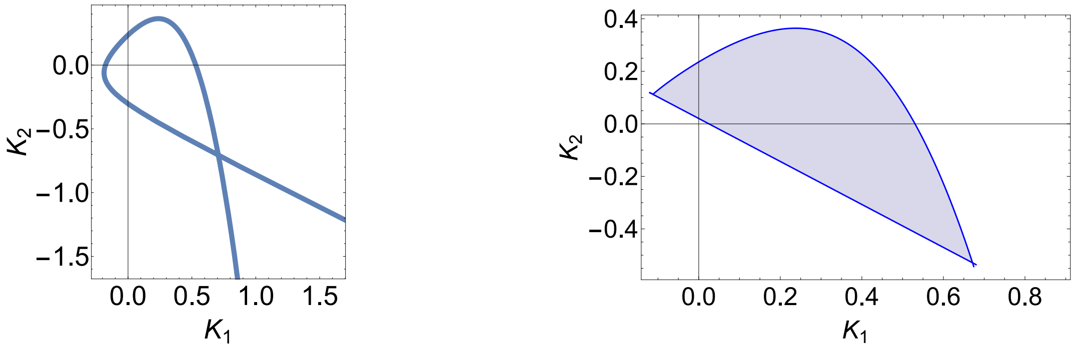

Figure 5.1: Let $\gamma = -10$ and $\beta_1 = \beta_2 = \frac{\pi}{4}$. The left figure shows the curve parametrized by (5.9) for $\omega \in (0,2)$. The right figure indicates the region in the (K_1, K_2)–plane where we have no roots of (5.7) in the right half of the complex plane, and the conditions (5.8), (5.6) are satisfied.

with $\omega \in (0,2)$ no roots of (5.7) are in the right half of the complex plane (see Figure 5.1).

If now K_1, K_2 are such that no roots of (5.7) are in the right half of the complex plane and the condition (5.8) is satisfied, we have a spectral gap. If then also (5.6) is satisfied, we can apply Lemma 5.1 to find that the periodic solution (0.2) of (5.4) is stable for small $\lambda < 0$ (see Figure 5.1).

Next consider the case $\beta_1 \neq \beta_2$ and choose $\gamma = -10$, $\beta_1 = -\frac{\pi}{4}$, $\beta_2 = \frac{3}{4}\pi$. One can show that for points on the curve parametrized by

$$
\begin{aligned}
K_1 &= \frac{1}{2\sin(\omega\pi)}(1-\omega)\cos\left(\omega\pi + \frac{\pi}{4}\right), \\
K_2 &= \frac{1}{2\omega\sin(\omega\pi)}(\omega-1)\sin\left(\omega\pi + \frac{\pi}{4}\right)
\end{aligned}
\tag{5.10}
$$

the equation (5.7) has a pair of roots on the imaginary axis. Using this, one can show that in the region in the (K_1, K_2)-plane indicated in Figure 5.2 the periodic solution (0.2) of (5.4) is stable for small $\lambda < 0$.

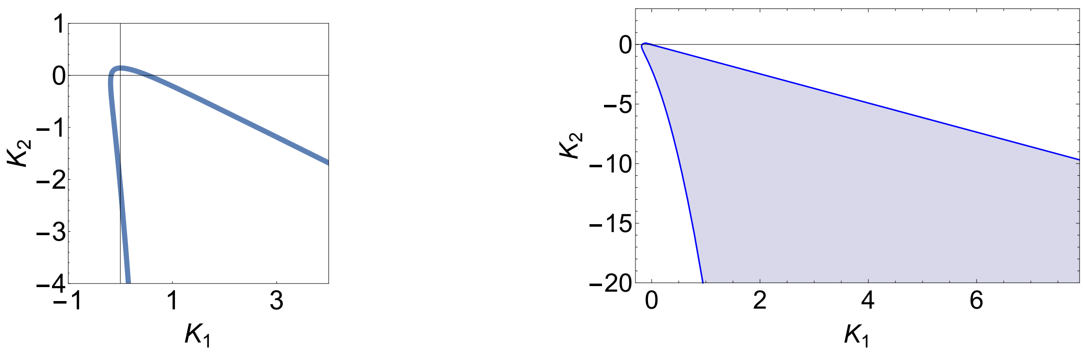

Figure 5.2: Let $\gamma = -10, \beta_1 = -\frac{\pi}{4}$ and $\beta_2 = \frac{3\pi}{4}$. The left figure shows the curve parametrized by (5.10) for $\omega \in (0,2)$. The shaded area in the right figure indicates the region in the (K_1, K_2)-plane where we have no roots of (5.7) in the right half of the complex plane, and the conditions (5.8), (5.6) are satisfied.

Now that we have determined stability conditions for (0.2) to be stable as a solution of (5.4), a number of questions arise naturally. For the specific example discussed here, one is interested how the range of values of λ for which the periodic orbit (0.2) is (un)stable as a solution of (5.4) compares to the range of values of λ for which (0.2) is (un)stable as a solution of (0.3). Furthermore, if (0.2) is stable as a solution of both (5.4) and (0.3), it is also interesting to study how the basin of attraction in both situations compare. More generally, one would like to apply the control scheme (5.3) to various systems or consider different control schemes including a 'neutral term'. We hope to return to these questions in the future.

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