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Asymptotic behaviour for a thermoelastic problem of a microbeam with thermoelasticity of type III

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Abstract. In this paper we study the asymptotic behavior of a equation modeling a microbeam moving transversally, coupled with an equation describing a heat pulse on it. Such pulse is given by a type III of the Green–Naghdi model, providing a more realistic model of heat flow from a physics point of view. We use semigroups theory to prove existence and uniqueness of solutions of our model, and multiplicative techniques to prove exponentially stable of its associated semigroup.

Keywords: C_0-semigroup, functional analysis, transverse vibration, coupled system, exponential stability.

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1 Introduction

We begin by recalling Green and Naghdi [6, 7] seminal work from about two decades ago, where they introduced new thermoelastic theories by a novel approach based on entropy equality instead of usual entropy inequality. They derived three theories under different assumptions, they are currently known as thermoelasticity type I, Type II and Type III respectively. These theories constitute a refined sequence of models addressing progressively certain anomalies such as infinite speed heat propagation induced by heat conduction classical theory under Type I model and so on.

On other hand, Abouelregal and Zenkour [1] propose a model given by the first equation from the system given below, based on Euler–Bernoulli beam's model. We will assume that such beam is moving along the x axis with constant velocity κ, and it is subject to a heat pulse governed by the so-called Green and Naghdi Theory (type III), resulting in a system given by:

$$u_{tt} + (p(x)\, u_{xx})_{xx} + 2\, q(x)\, u_t + 2\, \delta\, u_{xt} - \kappa^2\, u_{xx} + \eta\, \theta_{txx} = 0, \tag{1.1}$$

$$\theta_{tt} + \theta_t - \kappa\, \theta_{xx} - \eta\, u_{xxt} - \xi\, \theta_{xxt} = 0, \tag{1.2}$$

where $u = u(x, t)$ is a real valued function, representing the transverse displacement on the axis x which is fixed at both ends, $\theta = \theta(x, t)$ is the difference of temperature between the

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actual state and a reference temperature, and η is the coupling constant. We will assume throughout this paper $q(x)$ and $p(x)$ are positive definite functions, where $q(x) \in L^{\infty}(0, L)$, $p(x) \in H^{2}(0, L)$ and there are constants α_1, α_2, β_1 and β_2 such that

$$0 < \alpha_1 \leq p(x) \leq \alpha_2, \quad \forall\, x \in [0, L], \tag{1.3}$$
$$0 < \beta_1 \leq q(x) \leq \beta_2, \quad \forall\, x \in [0, L] \tag{1.4}$$

under the following boundary conditions

$$u(0, t) = u(L, t) = 0, \quad u_x(0, t) = u_x(L, t) = 0, \quad \theta(0, t) = \theta(L, t) = 0, \tag{1.5}$$

with initial values

$$u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad \theta(x, 0) = \theta_0(x), \quad x \in [0, L]. \tag{1.6}$$

We will establish stabilization results of the system (1.1)–(1.6) showing the energy is exponentially stable. Similar results are well known for stabilization for various flexible structures showing the energy decay exponentially, for instance see [3,8] and references therein.
For (1.1)–(1.6) we have the following estimate of the energy.

Lemma 1.1. *For every solution of the system (1.1)–(1.6) the total energy $\mathcal{E} : \mathbb{R}^+ \to \mathbb{R}^+$ is given at time t by*

$$\mathcal{E}(t) = \frac{1}{2}\int_0^L \left(u_t^2 + p(x)\, u_{xx}^2 + \kappa^2 u_x^2 + \theta_t^2 + \kappa\, \theta_x^2 \right) dx \tag{1.7}$$

and satisfies

$$\frac{d}{dt}\mathcal{E}(t) = -\, 2\int_0^L q(x)\, u_t^2\, dx - \int_0^L \theta_t^2\, dx - \xi \int_0^L \theta_{xt}^2\, dx. \tag{1.8}$$

Proof. We multiply (1.1) by u_t and integrating with respect to x over $[0, L]$, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_0^L u_t^2\, dx + \int_0^L p(x)\, u_{xx}\, u_{xxt}\, dx + 2\int_0^L q(x)\, u_t^2\, dx \\ &\quad + \delta \int_0^L \frac{1}{2}\frac{d}{dx}\left(u_t^2\right)\, dx + \kappa^2 \int_0^L u_x\, u_{xt}\, dx + \eta \int_0^L \theta_{txx}\, u_t\, dx = 0. \end{aligned}$$

then

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_0^L u_t^2\, dx + \frac{1}{2}\frac{d}{dt}\int_0^L p(x)\, u_{xx}^2\, dx + 2\int_0^L q(x)\, u_t^2\, dx \\ &\quad + \delta \int_0^L \frac{1}{2}\frac{d}{dx}\left(u_t^2\right)\, dx + \frac{1}{2}\kappa^2 \frac{d}{dt}\int_0^L u_x^2\, dx + \eta \int_0^L \theta_{txx}\, u_t\, dx = 0. \end{aligned}$$

Using the boundary condition (1.5), we have

$$\frac{1}{2}\frac{d}{dt}\int_0^L \left(u_t^2 + p(x)\, u_{xx}^2 + \kappa^2 u_x^2 \right) dx + 2\int_0^L q(x)\, u_t^2\, dx + \eta \int_0^L \theta_{txx}\, u_t\, dx = 0. \tag{1.9}$$

On the other hand, multiplying (1.2) by θ_t and integrating with respect to x over $[0, L]$, using the boundary conditions (1.5) and performing straightforward calculations, we have

$$\frac{1}{2}\frac{d}{dt}\int_0^L \left(\theta_t^2 + \kappa\, \theta_x^2 \right)\, dx - \eta \int_0^L \theta_{txx}\, u_t\, dx + \xi \int_0^L \theta_{xt}^2\, dx + \int_0^L \theta_t^2\, dx = 0. \tag{1.10}$$

adding (1.9) and (1.10), the proof of lemma is complete. □

The goal in this paper is to prove the following theorem.

Theorem 1.2. *Let u, θ be solutions of the system (1.1)–(1.6) . Then there exist positive constants K and γ such that*

$$\mathcal{E}(t) \le K\,\mathcal{E}(0)\, e^{-\gamma\, t}, \qquad \forall\, t \ge 0.$$

This paper is organized as follows: Section 2: we will develop the necessary tools to prove our main result; and Section 3 and 4: we show well-posedness and the exponential stability of the system (1.1)–(1.6), and final section with conclusions and remarks.

2 Setting of the semigroup

Before proving our main result, we will obtain the phase space and the domain of the operator associated to the system (1.1)–(1.6).

We will use the following standard $L^2(0, L)$ space, the scalar product and norm are denoted by

$$\langle u, v\rangle_{L^2(0,L)} = \int_0^L u\,\overline{v}\, dx, \qquad \|u\|^2_{L^2(0,L)} = \int_0^L u^2\, dx.$$

To prove the theorem 1.2 , we need the following two inequalities.

I. The Poincaré inequality

$$\|u\|^2_{L^2(0,L)} \le C_P\, \|u_x\|^2_{L^2(0,L)}, \qquad \forall\, u \in H^1_0(0, L).$$

where C_P is the Poincaré constant.

II. Young-type inequality

$$\int_0^L \phi\,\psi\, dx \le \int_0^L |\phi\,\psi|\, dx \le \frac{1}{2}\left(\varepsilon \int_0^L \phi^2\, dx + \frac{1}{\varepsilon}\int_0^L \psi^2\, dx\right), \quad \forall\, \varepsilon > 0, \quad \forall\, \phi, \psi \in L^2(0, L).$$

Taking $u_t = v$ and $\phi = \theta_t$ the initial value problem (1.1)–(1.6) can be reduced to the following abstract initial value problem for a first-order evolution equation

$$\frac{d}{dt}U(t) = \mathcal{A}\,U(t), \qquad U(0) = U_0, \qquad \forall\, t > 0, \tag{2.1}$$

where $U(t) = (u, v, \theta, \phi)^T$ and $U_0(t) = (u_0, v_0, \theta_0, \phi_0)^T$, where the linear operator $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset \mathcal{H} \to \mathcal{H}$ is given by

$$\mathcal{A}\begin{pmatrix} u \\ v \\ \theta \\ q \end{pmatrix} = \begin{pmatrix} v \\ -\,(p(x)\,u_{xx})_{xx} - 2\,q(x)\,v - 2\,\delta\,v_x + \kappa^2\,u_{xx} - \eta\,\phi_{xx} \\ \phi \\ -\phi + \kappa\,\theta_{xx} + \eta\,v_{xx} + \xi\,\phi_{xx} \end{pmatrix}. \tag{2.2}$$

We introduce the phase space $\mathcal{H} = H^2_0(0,L) \times L^2(0,L) \times H^1_0(0,L) \times L^2(0,L)$ endowed with the inner product given by

$$\begin{aligned}\langle (u, v, \theta, \phi), (u_1, v_1, \theta_1, \phi_1)\rangle_{\mathcal{H}} = &\int_0^L v\,\overline{v}_1\, dx + \int_0^L p(x)\,u_{xx}\,\overline{u}_{1xx}\, dx + \kappa^2 \int_0^L u_x\,\overline{u}_{1x}\, dx \\ &+ \kappa \int_0^L \theta_x\,\overline{\theta}_{1x}\, dx + \int_0^L \phi\,\overline{\phi}_1\, dx,\end{aligned}$$

where $U=(u,\,v,\,\theta,\,\phi)$, $\widetilde{U}=(u_1,\,v_1,\,\theta_1,\,\phi_1)$ and the norm

$$\begin{aligned}
&\|(u,\,v,\,\theta,\,\phi)\|^2_{\mathcal{H}}\\
&\quad=\int_0^L u_t^2\,dx+\int_0^L p(x)\,u_{xx}^2\,dx+\kappa^2\int_0^L u_x^2\,dx+\kappa\int_0^L \theta_x^2\,dx+\int_0^L \phi^2\,dx\\
&\quad=\|v\|^2_{L^2(0,L)}+\left\|\sqrt{p(x)}\,u_{xx}\right\|^2_{L^2(0,L)}+\kappa^2\,\|u_x\|^2_{L^2(0,L)}+\kappa\,\|\theta_x\|^2_{L^2(0,L)}+\|\phi\|^2_{L^2(0,L)}.
\end{aligned}$$

Instead of dealing with (1.1)–(1.6) we will consider (2.1) in the Hilbert space $\mathcal{H}$, with domain $\mathcal{D}(\mathcal{A})$ of the operator $\mathcal{A}$ given by

$$\mathcal{D}(\mathcal{A})=\Big\{(u,\,v,\,\theta,\,\phi)\in\mathcal{H}:v\in H^2_0(0\,,L),\;\theta\in H^1_0(0\,,L),\;-\,p(x)\,u_{xx}+\kappa^2\,u-\eta\,\phi\in H^2(0\,,L)\Big\}\,.$$

Firstly, we show that the operator $\mathcal{A}$ generates a C_0-semigroup of contractions on the space $\mathcal{H}$.

3 Well posedness

Proposition 3.1. *The operator $\mathcal{A}$ generates a C_0-semigroup $\mathcal{S}_{\mathcal{A}}(t)$ of contractions on the space $\mathcal{H}$.*

Proof. We will show that $\mathcal{A}$ is a dissipative operator and 0 belongs to the resolvent set of $\mathcal{A}$, denoted by $\varrho(\mathcal{A})$. Then our conclusion will follow using the well know Lumer–Phillips theorem [10].

$$\begin{aligned}
\langle\mathcal{A}U\,,U\rangle_{\mathcal{H}}=&\int_0^L\big[(-\,p(x)\,u_{xx})_{xx}\,\overline{v}-2\,q(x)\,v\,\overline{v}-2\,\delta\,v_x\,\overline{v}+\kappa^2\,u_{xx}\,\overline{v}-\eta\,\phi_{xx}\,\overline{v}\\
&\quad+\;\kappa^2\,v_x\,\overline{u}_x+p(x)\,v_{xx}\,\overline{u}_{xx}+\kappa\,\theta_{xx}\,\overline{\phi}-\phi\,\overline{\phi}+\eta\,v_{xx}\,\overline{\phi}+\xi\,\phi_{xx}\,\overline{\phi}+\kappa\,\phi_x\,\overline{\theta}_x\big]\,dx\\
=&\int_0^L\big[-\,p(x)\,u_{xx}\,\overline{v}_{xx}+p(x)\,v_{xx}\,\overline{u}_{xx}-2\,\delta\,v_x\,\overline{v}-\kappa^2\,u_x\,\overline{v}_x+\kappa^2\,v_x\,\overline{u}_x\\
&\quad-\;\kappa\,\theta_x\,\overline{\phi}_x+\kappa\,\phi_x\,\overline{\theta}_x-\phi\,\overline{\phi}-\eta\,\phi\,\overline{v}_{xx}+\eta\,v_{xx}\,\overline{\phi}-2\,q(x)\,v^2-\xi\,\phi_x^2\big]\,dx\\
=&\,2\,i\ \mathrm{Im}\int_0^L p(x)\,v_{xx}\,\overline{u}_{xx}\,dx-2\,\delta\int_0^L v_x\,\overline{v}\,dx+2\,i\,\kappa^2\ \mathrm{Im}\int_0^L v_x\,\overline{u}_x\,dx\\
&+\;2\,i\,\kappa\ \mathrm{Im}\int_0^L \theta_x\,\overline{\phi}_x\,dx+2\,i\,\eta\,Im\int_0^L v_{xx}\,\overline{\phi}\,dx-2\int_0^L q(x)\,v^2\,dx\\
&-\;\xi\int_0^L\phi_x^2\,dx-\int_0^L\phi^2\,dx.
\end{aligned}\tag{3.1}$$

Taking real parts in (3.1), we obtain

$$\mathrm{Re}\,\langle\mathcal{A}U\,,U\rangle_{\mathcal{H}}=-\;2\int_0^L q(x)\,v^2\,dx-\int_0^L\phi_x^2\,dx-2\delta\ \mathrm{Re}\int_0^L v_x\,\overline{v}\,dx-\int_0^L\phi^2\,dx.\tag{3.2}$$

On the other hand

$$\frac{d}{dx}\left(|v|^2\right)=\frac{1}{2}\ \mathrm{Re}(v_x\,\overline{v}).$$

Then

$$2\ \mathrm{Re}\int_0^L v_x\,\overline{v}\,dx=\frac{1}{2}\int_0^L\frac{d}{dx}\left(|v|^2\right)dx=0.$$

Replacing into (3.2) , we obtain

$$\mathrm{Re}\,\langle \mathcal{A}U\,,U\rangle_{\mathcal{H}} = -\,2\int_0^L q(x)\,v^2\,dx - \int_0^L \phi_x^2\,dx - \int_0^L \phi^2\,dx. \tag{3.3}$$

Hence $\mathcal{A}$ is a dissipative operator.

On the other hand, we have that $0 \in \varrho(\mathcal{A})$. In fact, given $F = (f_1,\, f_2,\, f_3,\, f_4) \in \mathcal{H}$, we must show that there exists a unique $U = (u,\, v,\, \theta,\, \phi)$ in $\mathcal{D}\,(\mathcal{A})$ such that $\mathcal{A}U = F$. Indeed,

$$v = f_1 \in H_0^2(0,\,L), \tag{3.4}$$

$$-\,(p(x)\,u_{xx})_{xx} - 2\,q(x)\,v - 2\,\delta\,v_x + \kappa^2\,u_{xx} - \eta\,\phi_{xx} = f_2 \in L^2(0,\,L), \tag{3.5}$$

$$\phi = f_3 \in H_0^1(0,\,L), \tag{3.6}$$

$$-\phi + \kappa\theta_{xx} + \eta v_{xx} + \xi\phi_{xx} = \psi\,f_4 \in L^2(0,\,L). \tag{3.7}$$

Replacing (3.4) into (3.5) we have

$$\left[-\,p(x)\,u_{xx} + \kappa^2\,u - \eta\,\phi\right]_{xx} - 2\,q(x)\,f_1 - 2\,\delta\,f_{1x} = f_2 \in L^2(0,\,L), \tag{3.8}$$

then

$$\left[-\,p(x)\,u_{xx} + \kappa^2\,u - \eta\,\phi\right]_{xx} = f_2 + 2\,q(x)\,f_1 + 2\,\delta\,f_{1x} \in L^2(0,\,L).$$

Hence

$$\left[-\,p(x)\,u_{xx} + \kappa^2\,u - \eta\,\phi\right]_{xx} \in L^2(0,\,L). \tag{3.9}$$

It is well know there is an unique

$$-\,p(x)\,u_{xx} + \kappa^2\,u - \eta\,\phi \in H^2(0,\,L)$$

satisfying (3.9) and

$$\left\|[-\,p(x)\,u_{xx} + \kappa^2\,u - \eta\,\phi]_{xx}\right\|_{L^2(0,L)} \leq \|f_2 + 2\,q(x)\,f_1 + 2\,\delta\,f_{1x}\| \leq C\,\|F\|$$

for a positive constant C.

Moreover, substituting (3.4) and (3.6) into (3.7) we have

$$\kappa\,\theta_{xx} = f_4 + f_3 - \eta\,f_{1xx} - \xi\,f_{3xx} \in H^{-1}(0,\,L),$$

then $\theta \in H_0^1(0,\,L)$.

It is easy to show that $\|U\|_{\mathcal{H}} \leq C\,\|F\|_{\mathcal{H}}$ for a positive constant C. Therefore we conclude that $0 \in \varrho(\mathcal{A})$. □

From Proposition 3.1 we can state the following result [10].

Theorem 3.2. *Assume that $U_0 \in \mathcal{D}(\mathcal{A})$, then there exists a unique solution $U(t) = (u,\, v,\, \theta,\, \phi)$ of (1.1)–(1.5) with boundary conditions (1.6) satisfying*

$$(u,\, v,\, \theta,\, \phi) \in C\left([0,\,\infty)\,:\,\mathcal{D}(\mathcal{A})\right) \cap C^1\left([0,\,\infty)\,:\,\mathcal{H}\right),$$

or equivalently, the abstract Cauchy problem (2.1) *satisfies*

$$u \in C\left([0,\infty):\ H_0^2(0,L)\right)\cap C^1\left([0,\infty[:\ L^2(0,L)\right),$$
$$\theta \in C\left([0,\infty[:\ H_0^1(0,L)\right)\cap C^1\left([0,\infty[:\ L^2(0,L)\right).$$

However, if $U_0 \in \mathcal{D}(\mathcal{A})$*, then*

$$u \in C^2\left([0,\infty[:\ H_0^2(\Omega)\right)\cap C^2\left([0,\infty[:\ L^2(0,L)\right),$$
$$\theta \in C^1\left([0,\infty[:\ H_0^1(0,L)\right),$$
$$-\,p(x)\,u_{xx}+\kappa^2 u-\eta\,\phi \in C\left([0,\infty[:\ H^2(0,L)\right).$$

4 Asymptotic behaviour

In this section, we will show that the energy decays uniformly with time. This is given by means of an exponential energy decay estimate, i.e. the solution of the system (1.1)–(1.6) converges uniformly to zero as the time t tends to infinity. The idea is to use the multipliers techniques, presented by the following lemmas.

Lemma 4.1. *For every solution* $u,\ \theta$ *of the system* (1.1)–(1.6)*, the time derivative of the functional* $\mathcal{F}_1(t)$*, defined by*

$$\mathcal{F}_1(t) := 2\int_0^L u\,u_t\,dx + 2\int_0^L q(x)\,u^2\,dx, \tag{4.1}$$

satisfies

$$\frac{d}{dt}\mathcal{F}_1(t) = -\,2\int_0^L p(x)\,u_{xx}^2\,dx + 4\,\delta\int_0^L u_x\,u_t\,dx - 2\,\kappa^2\int_0^L u_x^2\,dx - \ 2\,\eta\int_0^L \theta_t\,u_{xx}\,dx + 2\int_0^L u_t^2\,dx.$$

Moreover the functional $\mathcal{F}_1(t)$ *given by* (4.1) *satisfies the inequality*

$$-\,\mu_0\,\mathcal{E}(t) \le \mathcal{F}_1(t) \le (\mu_0+\mu_1)\ \mathcal{E}(t), \qquad \forall\ t\ge 0, \tag{4.2}$$

where

$$\mu_0 = \max\left\{1\,,\frac{C_P}{\kappa^2}\right\}, \qquad \mu_1 = \frac{2\,\beta_2\,C_P}{\kappa^2}.$$

Proof. Differentiating (4.1) in t-variable, using (1.1) and integrating by parts we have

$$\begin{aligned}
\frac{d}{dt}\mathcal{F}_1(t) &= 2\int_0^L u\,u_{tt}\,dx + 2\int_0^L u_t^2\,dx + 2\frac{d}{dt}\int_0^L q(x)\,u^2\,dx\\
&= 2\int_0^L u\left[-\,(p(x)\,u_{xx})_{xx} - 2\,q(x)\,v - 2\,\delta\,v_x + \kappa^2\,u_{xx} - \eta\,\theta_{txx}\right]\,dx\\
&\quad +\ 2\int_0^L u_t^2 dx + 2\,\frac{d}{dt}\int_0^L q(x)\,u^2\,dx\\
&= -\ 2\int_0^L p(x)\,u_{xx}^2\,dx - 2\,\frac{d}{dt}\int_0^L q(x)\,u^2\,dx + 4\,\delta\int_0^L u_x\,u_t\,dx - 2\,\kappa^2\int_0^L u_x^2\,dx\\
&\quad -\ 2\,\eta\int_0^L \theta_t\,u_{xx}\,dx + 2\int_0^L u_t^2\,dx + 2\,\frac{d}{dt}\int_0^L p(x)\,u^2\,dx\\
&= -\ 2\int_0^L p(x)\,u_{xx}^2 + 4\,\delta\int_0^L u_x\,u_t\,dx - 2\,\kappa^2\int_0^L u_x^2\,dx - \ 2\,\eta\int_0^L \theta_t\,u_{xx}\,dx + 2\int_0^L u_t^2\,dx.
\end{aligned}$$

On the other hand, using the Young and Poincaré inequalities in (4.1), we have

$$\begin{aligned}\left|2\int_0^L u\,u_t\,dx\right| &\le C_P\int_0^L u_x^2\,dx+\int_0^L u_t^2\,dx\\ &\le \frac{C_P}{\kappa^2}\int_0^L \kappa^2\,u_x^2\,dx+\int_0^L u_t^2\,dx\\ &\le \max\left\{\frac{C_P}{\kappa^2},1\right\}\left(\int_0^L \kappa^2\,u_x^2\,dx+\int_0^L u_t^2\,dx\right)\\ &= \mu_0\,\mathcal{E}(t).\end{aligned}$$

For the second part of the functional (4.1), we have

$$2\int_0^L q(x)\,u^2\,dx \le 2\,\beta_2\,C_P\int_0^L u_x^2\,dx \le \frac{2\,\beta_2\,C_P}{\kappa^2}\int_0^L \kappa^2 u_x^2\,dx \le \mu_1\,\mathcal{E}(t).$$

Then

$$-\,\mu_0\,\mathcal{E}(t)\le \mathcal{F}_1(t)\le(\mu_0+\mu_1)\,\mathcal{E}(t),\qquad \forall\, t\ge 0.$$

Hence the lemma follows. □

Lemma 4.2. *For every solution u, θ of the system (1.1)–(1.6), the time derivative of the functional $\mathcal{F}_2(t)$ defined by*

$$\mathcal{F}_2(t):=2\int_0^L u\,\theta_t\,dx+\int_0^L \theta_x^2\,dx+\eta\int_0^L u_x^2\,dx, \tag{4.3}$$

satisfies

$$\frac{d}{dt}\mathcal{F}_2(t)=-\,2\int_0^L u\,\theta_t\,dx-2\,\kappa\int_0^L u_x\,\theta_x\,dx+2\,\xi\int_0^L u_{xx}\,\theta_t\,dx+\,2\int_0^L u_t\,\theta_t\,dx+2\int_0^L \theta_x\,\theta_{xt}\,dx.$$

Moreover,

$$|\mathcal{F}_2(t)|\le \mu_3\,\mathcal{E}(t),$$

where

$$\mu_3=\max\left\{1,\ C_P,\ \frac{1}{\kappa},\ \frac{\eta}{\kappa^2}\right\}.$$

Proof. Differentiating (4.3) in t-variable, using (1.2) and integrating by parts we have

$$\begin{aligned}\frac{d}{dt}\mathcal{F}_2(t) &= 2\int_0^L u\,\theta_{tt}\,dx+2\int_0^L u_t\,\theta_t\,dx+2\int_0^L \theta_x\,\theta_{xt}\,dx+2\,\eta\int_0^L u_x\,u_{xt}\,dx\\ &= 2\int_0^L u\,[-\,\phi+\kappa\,\theta_{xx}+\eta\,v_{xx}+\xi\,\phi_{xx}]\,dx+2\int_0^L u_t\,\theta_t\,dx\\ &\quad +2\int_0^L \theta_x\,\theta_{xt}\,dx+\,2\,\eta\int_0^L u_x\,u_{xt}\,dx\\ &= -\,2\int_0^L u\,\theta_t\,dx-2\,\kappa\int_0^L u_x\,\theta_x\,dx-2\,\eta\int_0^L u_x\,v_x\,dx+2\,\xi\int_0^L \theta_t\,u_{xx}\,dx\\ &\quad +\,2\int_0^L u_t\,\theta_t\,dx+2\int_0^L \theta_x\,\theta_{xt}\,dx+2\,\eta\int_0^L u_x\,v_x\,dx\\ &= -\,2\int_0^L u\,\theta_t\,dx-2\,\kappa\int_0^L u_x\,\theta_x\,dx+2\,\xi\int_0^L u_{xx}\,\theta_t\,dx\\ &\quad +2\int_0^L u_t\,\theta_t\,dx+\,2\int_0^L \theta_x\,\theta_{xt}\,dx.\end{aligned}$$

On the other hand using the Poincaré and Young inequality into (4.3), we obtain

$$\begin{aligned}|\mathcal{F}_2(t)| &\leq C_P \int_0^L u_x^2\,dx + \int_0^L \theta_t^2\,dx + \frac{1}{\kappa}\int_0^L \kappa\,\theta_x^2\,dx + \frac{\eta}{\kappa^2}\int_0^L \kappa^2\,u_x^2\,dx \\ &\leq \mu_3\,\mathcal{E}(t).\end{aligned}$$

Hence the lemma is follows. □

Lemma 4.3. *The time derivative of the functional $\mathcal{G}(t)$ defined by*

$$\mathcal{G}(t) := \mathcal{F}_1(t) + \mathcal{F}_2(t)$$

satisfies

$$\frac{d}{dt}\mathcal{G}(t) = -\,2\,\mathcal{E}(t) + \mathcal{R},$$

where

$$\begin{aligned}\mathcal{R} = &-\int_0^L p(x)\,u_{xx}^2\,dx + 4\,\delta\int_0^L u_x\,u_t\,dx - \kappa^2\int_0^L u_x^2\,dx - 2\,\eta\int_0^L \theta_t\,u_{xx}\,dx \\ &+\;3\int_0^L u_t^2\,dx + \int_0^L \theta_t^2\,dx + \kappa\int_0^L \theta_x^2\,dx - 2\int_0^L u\,\theta_t\,dx - 2\,\kappa\int_0^L u_x\,\theta_x\,dx \\ &+\;2\,\xi\int_0^L u_{xx}\,\theta_t\,dx + 2\int_0^L u_t\,\theta_t\,dx + 2\int_0^L \theta_x\,\theta_{xt}\,dx.\end{aligned} \tag{4.4}$$

Moreover the remainder $\mathcal{R}$ satisfies the following inequality

$$\begin{aligned}\mathcal{R} \leq\; &\left[-1 + \frac{\eta\,\varepsilon_2}{\alpha_1} + \frac{\xi\,\varepsilon_5}{\alpha_1}\right]\int_0^L p(x)\,u_{xx}^2\,dx + \left[\frac{2\,\delta}{\varepsilon_1} + 3 + \varepsilon_6\right]\int_0^L u_t^2\,dx \\ &+\left[2\,\delta\,\varepsilon_1 - \kappa^2 + C_P\,\varepsilon_3 + \frac{\kappa}{\varepsilon_4}\right]\int_0^L u_x^2\,dx + \left[\frac{\eta}{\varepsilon_2} + 1 + \frac{1}{\varepsilon_3} + \frac{\xi}{\varepsilon_5} + \frac{1}{\varepsilon_6}\right]\int_0^L \theta_t^2\,dx \\ &+ [C_\kappa + \kappa\,\varepsilon_4 + \varepsilon_7]\int_0^L \theta_x^2\,dx + \frac{1}{\varepsilon_7}\int_0^L \theta_{xt}^2\,dx\end{aligned} \tag{4.5}$$

for all $\varepsilon_i > 0$, $i = 1, \ldots, 7$, and where $C_\kappa \in \mathbb{R}^+$, $C_\kappa > \kappa$.

Proof. Differentiating $\mathcal{G}(t)$ in t-variable and adding terms we have

$$\begin{aligned}\frac{d}{dt}\mathcal{G}(t) &= \frac{d}{dt}\mathcal{F}_1(t) + \frac{d}{dt}\mathcal{F}_1(t) \\ &= -\;2\int_0^L p(x)\,u_{xx}^2\,dx + 4\,\delta\int_0^L u_x\,u_t\,dx - 2\,\kappa^2\int_0^L u_x^2\,dx - 2\,\eta\int_0^L \theta_t\,u_{xx}\,dx \\ &\quad+\;2\int_0^L u_t^2\,dx - 2\int_0^L u\,\theta_t\,dx - 2\,\kappa\int_0^L u_x\,\theta_x\,dx + 2\,\xi\int_0^L u_{xx}\,\theta_t\,dx \\ &\quad+\;2\int_0^L u_t\,\theta_t\,dx + 2\int_0^L \theta_x\,\theta_{xt}\,dx + 3\int_0^L u_t^2\,dx - 3\int_0^L u_t^2\,dx \\ &\quad+\int_0^L \theta_t^2\,dx - \int_0^L \theta_t^2\,dx + \kappa\int_0^L \theta_x^2\,dx - \kappa\int_0^L \theta_x^2\,dx \\ &= -\;2\,\mathcal{E}(t) + \mathcal{R}.\end{aligned} \tag{4.6}$$

On the other hand from (1.3), we have

$$\int_0^L u_{xx}^2\,dx = \int_0^L \frac{1}{p(x)}\,p(x)\,u_{xx}^2\,dx \leq \frac{1}{\alpha_1}\int_0^L p(x)\,u_{xx}^2\,dx. \tag{4.7}$$

Using (4.7) and the inequalities of Young and Poincaré in (4.4), we obtain

$$\begin{aligned}\mathcal{R} \leq & -\int_0^L p(x)\, u_{xx}^2\, dx + 2\,\delta\,\varepsilon_1 \int_0^L u_x^2\, dx + \frac{2\,\delta}{\varepsilon_1} \int_0^L u_t^2\, dx - \kappa^2 \int_0^L u_x^2\, dx \\ & + \frac{\eta\,\varepsilon_2}{\alpha_1} \int_0^L p(x)\, u_{xx}^2\, dx + \frac{\eta}{\varepsilon_2} \int_0^L \theta_t^2\, dx + 3 \int_0^L u_t^2\, dx + \int_0^L \theta_t^2\, dx + C_\kappa \int_0^L \theta_x^2\, dx \\ & + C_P\,\varepsilon_3 \int_0^L u_x^2\, dx + \frac{1}{\varepsilon_3} \int_0^L \theta_t^2\, dx + \frac{\kappa}{\varepsilon_4} \int_0^L u_x^2\, dx + \kappa\,\varepsilon_4 \int_0^L \theta_x^2\, dx \\ & + \frac{\xi\,\varepsilon_5}{\alpha_1} \int_0^L p(x)\, u_{xx}^2\, dx + \frac{\xi}{\varepsilon_5} \int_0^L \theta_t^2\, dx + \varepsilon_6 \int_0^L u_t^2\, dx + \frac{1}{\varepsilon_6} \int_0^L \theta_t^2\, dx \\ & + \varepsilon_7 \int_0^L \theta_x^2\, dx + \frac{1}{\varepsilon_7} \int_0^L \theta_{xt}^2\, dx. \end{aligned} \tag{4.8}$$

Adding terms into (4.8) we have

$$\begin{aligned}\mathcal{R} \leq & \left[-1 + \frac{\eta\,\varepsilon_2}{\alpha_1} + \frac{\xi\,\varepsilon_5}{\alpha_1} \right] \int_0^L p(x)\, u_{xx}^2\, dx + \left[\frac{2\,\delta}{\varepsilon_1} + 3 + \varepsilon_6 \right] \int_0^L u_t^2\, dx \\ & + \left[2\,\delta\,\varepsilon_1 - \kappa^2 + C_P\,\varepsilon_3 + \frac{\kappa}{\varepsilon_4} \right] \int_0^L u_x^2\, dx + \left[\frac{\eta}{\varepsilon_2} + 1 + \frac{1}{\varepsilon_3} + \frac{\xi}{\varepsilon_5} + \frac{1}{\varepsilon_6} \right] \int_0^L \theta_t^2\, dx \\ & + [C\kappa + \kappa\,\varepsilon_4 + \varepsilon_7] \int_0^L \theta_x^2\, dx + \frac{1}{\varepsilon_7} \int_0^L \theta_{xt}^2\, dx. \end{aligned}$$

Therefore the lemma is proved. □

Since Lemmas 4.1 and 4.2 yields for $\mathcal{G}(t)$ the following estimate

$$-\,\mu_4\,\mathcal{E}(t) \leq \mathcal{G}(t) \leq (\mu_1 + \mu_4)\,\mathcal{E}(t), \qquad \forall\, t \geq 0, \tag{4.9}$$

where

$$\mu_4 := \mu_0 + \mu_3.$$

Now, we proceed following closely Gorain [5] and Komornik [8] approaches by introducing an energy $\mathcal{V}(t)$ a Lyapunov functional defined by $\mathcal{V}(t) := \mathcal{E}(t) + \delta_1\,\mathcal{G}(t)$ where $\delta_1 > 0$ is a small enough to be chosen later. The Lemmas 4.1 and 4.2 yields the following $\mathcal{V}(t)$ estimates:

$$(1 - 2\,\delta_1\,\mu_4)\,\mathcal{E}(t) \leq \mathcal{V}(t) \leq [1 + (\mu_1 + \mu_4)\,\delta_1]\;\mathcal{E}(t), \qquad \forall\, t \geq 0, \tag{4.10}$$

where we choose $\delta_1 < \frac{1}{2\,\mu_4}$ so that $\mathcal{V}(t) \geq 0$ for $t \geq 0$.

We state our main result as follows.

Theorem 4.4. *Let u, θ be solutions of the system (1.1)–(1.6) . Then there exist positive constants γ and K such that*

$$\mathcal{E}(t) \leq K\,\mathcal{E}(0)\, e^{-\gamma\, t}, \qquad \forall\, t \geq 0.$$

Proof. From Lemma 1.1 we have

$$\mathcal{E}(t) = \frac{1}{2} \int_0^L \left(u_t^2 + p(x)\, u_{xx}^2 + \kappa^2\, u_x^2 + \theta_t^2 + \kappa\,\theta_x^2 \right) dx, \tag{4.11}$$

and

$$\frac{d}{dt}\mathcal{E}(t) = -\,2 \int_0^L q(x)\, u_t^2\, dx - \int_0^L \theta_t^2\, dx - \xi \int_0^L \theta_{xt}^2\, dx. \tag{4.12}$$

Let us now define an energy like Lyapunov functional

$$\mathcal{V}(t) := \mathcal{E}(t) + \delta_1 \mathcal{G}(t)$$

taking time derivative, using (4.12) and (4.6) we obtain

$$\begin{aligned}\frac{d}{dt}\mathcal{V}(t) = \frac{d}{dt}\mathcal{E}(t) + \delta_1 \frac{d}{dt}\mathcal{G}(t) &= -\,2\int_0^L q(x)\,u_t^2\,dx - \int_0^L \theta_t^2\,dx \\ &-\,\xi\int_0^L \theta_{xt}^2\,dx - 2\,\delta_1\,\mathcal{E}(t) + \delta_1\,\mathcal{R}.\end{aligned} \tag{4.13}$$

Then using (4.11) and (4.5) into (4.13), we have

$$\begin{aligned}\frac{d}{dt}\mathcal{V}(t) \leq &-\,2\int_0^L q(x)\,u_t^2\,dx - \int_0^L \theta_t^2\,dx - \xi\int_0^L \theta_{xt}^2\,dx \\ &-\,\delta_1\int_0^L u_t^2\,dx - \delta_1\int_0^L p(x)\,u_{xx}^2\,dx - \kappa^2\delta_1\int_0^L u_x^2\,dx - \delta_1\int_0^L \theta_t^2\,dx \\ &-\,\kappa\,\delta_1\int_0^L \theta_x^2\,dx + \delta_1\left[-\,1 + \frac{\eta\,\varepsilon_2}{\alpha_1} + \frac{\xi\,\varepsilon_5}{\alpha_1}\right]\int_0^L p(x)\,u_{xx}^2\,dx \\ &+\,\delta_1\left[\frac{2\,\delta}{\varepsilon_1} + 3 + \varepsilon_6\right]\int_0^L u_t^2\,dx + \delta_1\left[2\,\delta\,\varepsilon_1 - \kappa^2 + C_P\,\varepsilon_3 + \frac{\kappa}{\varepsilon_4}\right]\int_0^L u_x^2\,dx \\ &+\,\delta_1\left[\frac{\eta}{\varepsilon_2} + 1 + \frac{1}{\varepsilon_3} + \frac{\xi}{\varepsilon_5} + \frac{1}{\varepsilon_6}\right]\int_0^L \theta_t^2\,dx \\ &+\,\delta_1\left[C_\kappa + \kappa\,\varepsilon_4 + \epsilon_7\right]\int_0^L \theta_x^2\,dx + \frac{\delta_1}{\varepsilon_7}\int_0^L \theta_{xt}^2\,dx.\end{aligned}$$

How $q(x)$ satisfies $0 < \beta_1 \leq q(x) \leq \beta_2$, $\forall\, x \in [0,\, L]$, we have

$$\begin{aligned}\frac{d}{dt}\mathcal{V}(t) \leq &-\,2\,\beta_1\int_0^L u_t^2\,dx - \int_0^L \theta_t^2\,dx - \xi\int_0^L \theta_{xt}^2\,dx \\ &-\,\delta_1\int_0^L u_t^2\,dx - \delta_1\int_0^L p(x)\,u_{xx}^2\,dx - \kappa^2\,\delta_1\int_0^L u_x^2\,dx - \delta_1\int_0^L \theta_t^2\,dx \\ &-\,\kappa\,\delta_1\int_0^L \theta_x^2\,dx + \delta_1\left[-\,1 + \frac{\eta\,\varepsilon_2}{\alpha_1} + \frac{\xi\,\varepsilon_5}{\alpha_1}\right]\int_0^L p(x)\,u_{xx}^2\,dx \\ &+\,\delta_1\left[\frac{2\,\delta}{\varepsilon_1} + 3 + \varepsilon_6\right]\int_0^L u_t^2\,dx + \delta_1\left[2\,\delta\,\varepsilon_1 - \kappa^2 + C_P\,\varepsilon_3 + \frac{\kappa}{\varepsilon_4}\right]\int_0^L u_x^2\,dx \\ &+\,\delta_1\left[\frac{\eta}{\varepsilon_2} + 1 + \frac{1}{\varepsilon_3} + \frac{\xi}{\varepsilon_5} + \frac{1}{\varepsilon_6}\right]\int_0^L \theta_t^2\,dx \\ &+\,\delta_1\left[C_\kappa + \kappa\,\varepsilon_4 + \varepsilon_7\right]\int_0^L \theta_x^2\,dx + \frac{\delta_1}{\varepsilon_7}\int_0^L \theta_{xt}^2\,dx.\end{aligned} \tag{4.14}$$

Adding terms into (4.14), we obtain

$$
\begin{aligned}
\frac{d}{dt}\mathcal{V}(t) \leq & -\left[2\,\beta_1 - \delta_1\left(\frac{2\,\delta}{\varepsilon_1} + 2 + \varepsilon_6\right)\right]\int_0^L u_t^2\,dx \\
& -\left[2\,\delta_1 - \frac{\eta\,\varepsilon_2\,\delta_1}{\alpha_1} - \frac{\xi\,\varepsilon_5\,\delta_1}{\alpha_1}\right]\int_0^L p(x)\,u_{xx}^2\,dx \\
& -\left[2\,\kappa^2\,\delta_1 - 2\,\delta\,\varepsilon_1\,\delta_1 - C_P\,\varepsilon_3\,\delta_1 - \frac{\kappa\,\delta_1}{\varepsilon_4}\right]\int_0^L u_x^2\,dx \\
& -\left[1 - \delta_1\left(\frac{\eta}{\varepsilon_2} + \frac{1}{\varepsilon_3} + \frac{\xi}{\varepsilon_5} + \frac{1}{\varepsilon_6}\right)\right]\int_0^L \theta_t^2\,dx \\
& -\left[\kappa\,\delta_1 - C_\kappa\,\delta_1 - \kappa\,\varepsilon_4\,\delta_1 - \varepsilon_7\,\delta_1\right]\int_0^L \theta_x^2\,dx \\
& -\left[\xi - \frac{\delta_1}{\varepsilon_7}\right]\int_0^L \theta_{xt}^2\,dx.
\end{aligned}
$$

We define the following positive constants

$$
\begin{aligned}
\mathcal{C}_1 &= 2\,\beta_1 - \delta_1\left(\frac{2\,\delta}{\varepsilon_1} + 2 + \varepsilon_6\right), \\
\mathcal{C}_2 &= 2\,\delta_1 - \frac{\eta\,\varepsilon_2\,\delta_1}{\alpha_1} - \frac{\xi\,\varepsilon_5\,\delta_1}{\alpha_1}, \\
\mathcal{C}_3 &= 2\,\kappa^2\,\delta_1 - 2\,\delta\,\varepsilon_1\,\delta_1 - C_P\,\varepsilon_3\,\delta_1 - \frac{\kappa\,\delta_1}{\varepsilon_4}, \\
\mathcal{C}_4 &= 1 - \delta_1\left(\frac{\eta}{\varepsilon_2} + \frac{1}{\varepsilon_3} + \frac{\xi}{\varepsilon_5} + \frac{1}{\varepsilon_6}\right), \\
\mathcal{C}_5 &= \kappa\,\delta_1 - C_\kappa\,\delta_1 - \kappa\,\varepsilon_4\,\delta_1 - \varepsilon_7\,\delta_1, \\
\mathcal{C}_6 &= \xi - \frac{\delta_1}{\varepsilon_7},
\end{aligned}
$$

where $\varepsilon_i > 0$, $\forall\, i = 1, \ldots, 7$. Hence

$$\mathcal{C}_1,\ \mathcal{C}_4,\ \mathcal{C}_6$$

are strictly positive if and only if

$$\delta_1 < \min\left\{\frac{2\,\beta_1}{2 + \varepsilon_6 + \frac{2\,\delta}{\varepsilon_1}},\ \frac{1}{\frac{\eta}{\varepsilon_2} + \frac{1}{\varepsilon_3} + \frac{\xi}{\varepsilon_5} + \frac{1}{\varepsilon_6}},\ \xi\,\varepsilon_7\right\}.$$

We will provide extra conditions on ε_i so that $\mathcal{C}_2$, $\mathcal{C}_3$, $\mathcal{C}_5$ are positive constants. We will have

$$\mathcal{C}_2 = 2\,\delta_1 - \frac{\eta\,\varepsilon_2\,\delta_1}{\alpha_1} - \frac{\xi\,\varepsilon_5\,\delta_1}{\alpha_1} > 0$$

if and only if

$$0 < \varepsilon_2 < \frac{2\,\alpha_1 - \xi\,\varepsilon_1}{\eta}.$$

In an analogous way for $\mathcal{C}_3$

$$\mathcal{C}_3 = 2\,\kappa^2\,\delta_1 - 2\,\delta\,\varepsilon_1\,\delta_1 - C_P\,\varepsilon_3\,\delta_1 - \frac{\kappa\,\delta_1}{\varepsilon_4} > 0$$

if and only if

$$0 < \varepsilon_1 < \frac{2\,\kappa^2 - C_P\,\varepsilon_3 - \frac{\kappa\,\delta_1}{\varepsilon_4}}{2\,\delta},$$

and also for $\mathcal{C}_5$, we have

$$\mathcal{C}_5 = \kappa\,\delta_1 - C_\kappa\,\delta_1 - \kappa\,\varepsilon_4\,\delta_1 - \varepsilon_7\,\delta_1 > 0$$

if and only if

$$0 < \varepsilon_7 < \kappa - C_\kappa - \kappa\,\varepsilon_4.$$

Since the above calculations, we obtain

$$\begin{aligned}\frac{d}{dt}\mathcal{V}(t) \le &- \mathcal{C}_1\int_0^L u_t^2\,dx - \mathcal{C}_2\int_0^L p(x)\,u_{xx}^2\,dx - \mathcal{C}_3\int_0^L u_x^2\,dx - \mathcal{C}_4\int_0^L \theta_t^2\,dx\\ &- \mathcal{C}_5\int_0^L \theta_x^2\,dx - \mathcal{C}_6\int_0^L \theta_{xt}^2\,dx.\end{aligned}$$

then

$$\begin{aligned}\frac{d}{dt}\mathcal{V}(t) \le &- \mathcal{C}_1\int_0^L u_t^2\,dx - \mathcal{C}_2\int_0^L p(x)\,u_{xx}^2\,dx - \mathcal{C}_3\int_0^L u_x^2\,dx - \mathcal{C}_4\int_0^L \theta_t^2\,dx\\ &- \mathcal{C}_5\int_0^L \theta_x^2\,dx.\end{aligned} \tag{4.15}$$

Since $\delta_1 > 0$ is small enough, we assume that

$$0 < \delta_1 < \delta_2 := \min\left\{\frac{2\,\beta_1}{2+\varepsilon_6+\frac{2\,\delta}{\varepsilon_1}},\ \frac{1}{\frac{\eta}{\varepsilon_2}+\frac{1}{\varepsilon_3}+\frac{\xi}{\varepsilon_5}+\frac{1}{\varepsilon_6}},\ \xi\,\varepsilon_7,\ \frac{1}{2\,\mu_4}\right\}.$$

From (4.15) we get the differential inequality

$$\frac{d}{dt}\mathcal{V}(t) \le -\,\delta_1\,\mathcal{E}(t). \tag{4.16}$$

Using (4.10), we have

$$\frac{d}{dt}\mathcal{V}(t) \le \frac{-\,\delta_1\mathcal{V}(t)}{1+(\mu_1+\mu_4)\,\delta_1}, \tag{4.17}$$

then

$$\frac{d}{dt}\mathcal{V}(t) \le -\,\gamma\,\mathcal{V}(t), \tag{4.18}$$

where

$$\gamma := \frac{\delta_1}{1+(\mu_1+\mu_4)\,\delta_1}.$$

Multiplying (4.18) by $e^{\lambda\,t}$ and integrating over $[0,\,t]$ for any $t \ge 0$, we get

$$\mathcal{V}(t) \le e^{-\gamma\,t}\,\mathcal{V}(0). \tag{4.19}$$

Applying (4.10) to (4.19), we obtain

$$\mathcal{E}(t) \le \frac{1+(\mu_1+\mu_4)\,\delta_1}{(1-2\,\mu_4\,\delta_1)}\,\mathcal{E}(0)\,e^{-\gamma\,t}.$$

Hence Theorem 4.4 is proved. □

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