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Infinitely many solutions for Schrödinger–Kirchhoff-type equations involving indefinite potential

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Abstract. In this paper, we study the multiplicity of solutions for the following Schrödinger–Kirchhoff-type equation

$$\begin{cases} -\left(a+b\int_{\mathbb{R}^N}|\nabla u|^2dx\right)\triangle u+V(x)u=f(x,u)+g(x,u), & x\in\mathbb{R}^N,\\ u\in H^1(\mathbb{R}^N), \end{cases}$$

where $N\geq 3$, a, $b>0$ are constants and the potential V may be unbounded from below. Under some mild conditions on the nonlinearities f and g, we obtain the existence of infinitely many solutions for this problem. Recent results from the literature are generalized and significantly improved.

Keywords: Schrödinger–Kirchhoff-type equation, symmetric mountain pass lemma, variational method.

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1 Introduction and main results

In this paper, we consider the following Schrödinger–Kirchhoff-type equation

$$\begin{cases} -\left(a+b\int_{\mathbb{R}^N}|\nabla u|^2dx\right)\triangle u+V(x)u=f(x,u)+g(x,u), & x\in\mathbb{R}^N,\\ u\in H^1(\mathbb{R}^N), \end{cases}\tag{1.1}$$

where $N\geq 3$ and a, $b>0$ are constants. If in (1.1), we set $V(x)\equiv 0$ and replace $\mathbb{R}^N$ by a smooth bounded domain Ω, then (1.1) reduces to the following Dirichlet problem

$$\begin{cases} -\left(a+b\int_{\Omega}|\nabla u|^2dx\right)\triangle u=f(x,u), & x\in\Omega,\\ u=0, & x\in\partial\Omega. \end{cases}\tag{1.2}$$

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Problem (1.2) is related to the stationary analogue of the Kirchhoff equation

$$u_{tt} - \left(a + b\int_{\Omega} |\nabla u|^2 dx\right) \triangle u = f(x,u),$$

which was presented by Kirchhoff in 1883 [8] as a generalization of the classical D'Alembert's wave equation for free vibrations of elastic strings. Kirchhoff's model takes into account the changes in length of the string produced by transverse vibrations. In [11], Lions first introduced an abstract functional analysis framework to this model. After that, problems like type (1.2) have been studied by many authors, see [3, 4, 15, 16, 20, 25, 27] and the references therein.

More recently, with the aid of variational methods, the existence and multiplicity of various solutions for equations of type (1.1) have also been extensively investigated in the literature, see, for instance, [1,2,5,6,9,10,12,21–24,26,29] and the references therein. Here we emphasize that almost in all these mentioned papers the conditions imposed on the potential V always imply that V is bounded from below, which is crucial for the corresponding results.

In the present paper, different from the references mentioned above, we are going to study the existence of infinitely many solutions for (1.1) in the case where the potential V may be unbounded from below. Specifically, we first assume that V satisfies

(S_1) $V \in L^q_{\text{loc}}(\mathbb{R}^N)$ and $V^- := \min\{V, 0\} \in L^\infty(\mathbb{R}^N) + L^q(\mathbb{R}^N)$ for some $q \in [2,\infty) \cap (\frac{N}{2},\infty)$.

This type of assumptions on the potential V has already been introduced in [13] to study Schrödinger equations (see also [28]), which ensures that the Schrödinger operator $\mathcal{S} := -a\Delta + V$, defined as a form sum, is self-adjoint and semibounded on $L^2(\mathbb{R}^N)$ (see Theorem A.2.7 in [19]). We denote by $\sigma(\mathcal{S}) \subset \mathbb{R}$ the spectrum, $\sigma_{ess}(\mathcal{S})$ the essential spectrum and $\sigma_{pp}(\mathcal{S})$ the pure point spectrum of $\mathcal{S}$ respectively.

Consider the nondecreasing sequence of min-max values defined by

$$\lambda_k = \inf_{U\in\mathcal{U}_k} \sup_{u\in U\setminus\{0\}} \frac{\int_{\mathbb{R}^N} \left(a|\nabla u|^2 + V(x)u^2\right) dx}{\int_{\mathbb{R}^N} u^2 dx}, \qquad \forall k \in \mathbb{N},$$

where $\mathcal{U}_k$ is the family of all k-dimensional subspaces of $C_0^\infty(\mathbb{R}^N)$. It is known that $\lambda_\infty := \lim_{k\to\infty} \lambda_k = \inf \sigma_{ess}(\mathcal{S})$. Moreover, $\lambda_k \in \sigma_{pp}(\mathcal{S})$ whenever $\lambda_k < \lambda_\infty$ (cf. [17, 18] for details). Then we make the further assumption on V.

(S_2) $\lambda_\infty > 0$.

For the nonlinearities, we present the following assumptions.

(S_3) The function $f \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$ is odd in u, and there exist constants $\nu \in (1,2)$ and $\mu \in (2^*/(2^*-\nu), 2/(2-\nu)]$ and a nonnegative function $\xi \in L^\mu(\mathbb{R}^N)$ such that

$$|f(x,u)| \leq \xi(x)|u|^{\nu-1}, \qquad \forall (x,u) \in \mathbb{R}^N \times \mathbb{R},$$

where $2^* := 2N/(N-2)$ is the critical exponent.

(S_4) There exist an $x_0 \in \mathbb{R}^N$ and a constant $r_0 > 0$ such that

$$\liminf_{u\to 0} \left(\inf_{x\in B_{r_0}(x_0)} u^{-2}F(x,u) \right) > -\infty,$$

and

$$\limsup_{u\to 0}\left(\inf_{x\in B_{r_0}(x_0)} u^{-2}F(x,u)\right)=+\infty$$

where $B_{r_0}(x_0)$ is the ball in $\mathbb{R}^N$ centered at x_0 with radius r_0 and

$$F(x,u):=\int_0^u f(x,t)dt.$$

(S_5) $g\in C(\mathbb{R}^N\times\mathbb{R},\mathbb{R})$ is odd in u, and there exists $d\in(0,\lambda_\infty)$ such that

$$|g(x,u)|\le d|u|,\qquad \forall\,(x,u)\in\mathbb{R}^N\times\mathbb{R}.$$

Our main result reads as follows.

Theorem 1.1. *Suppose that* (S_1)–(S_5) *are satisfied. Then* (1.1) *possesses a sequence of nontrivial solutions* $\{u_k\}_{k\in\mathbb{N}}\subset H^1(\mathbb{R}^N)$ *with* $u_k\to 0$ *in* $H^1(\mathbb{R}^N)$ *as* $k\to\infty$.

Remark 1.2. In Theorem 1.1, the potential V satisfying (S_1) and (S_2) may not be coercive or bounded from below. Moreover, the nonlinear term f satisfying (S_3) and (S_4) may be partially oscillatory near the origin. This is in sharp contrast with the aforementioned references. To the best of our knowledge, there is little literature concerning infinitely many solutions for (1.1) in this situation. In fact, it is easy to see that conditions (S_1) and (S_2) are rather weaker than the usual one in the existing literature that the potential $V\in C(\mathbb{R}^N)$ with $\lim_{|x|\to\infty}V(x)=+\infty$ or $\inf_{x\in\mathbb{R}^N}V(x)>0$.

Remark 1.3. Theorem 1.1 also essentially improves some related results in the existing literature. Compared to Theorem 6 in [26], our conditions (S_1) and (S_2) on the potential V are weaker than (V_1) there, and our conditions (S_3) and (S_4) on the nonlinear term f are much weaker than (f_5) there if we just take $g=0$ in (1.1). In fact, there are many functions V and f which satisfy our conditions (S_1)–(S_4) but do not satisfy the condition (V_1) and (f_5) in [26]. For instance, let

$$V(x)=V_0(x)+\overline{V},$$

where $V_0\in L^q(\mathbb{R}^N)$ for some $q\ge 2$ is a given non-positive function and unbounded from below. Then it is evident that V satisfies (S_1) and (S_2) if the positive constant $\overline{V}$ is chosen to be large enough. Moreover, V is also unbounded from below. In addition, let

$$F(x,u)=\begin{cases} e^{-|x|^2}|u|^{\alpha}\sin^2(|u|^{-\epsilon}), & \forall\,x\in\mathbb{R}^N,\ 0<|u|<\pi^{-1/\epsilon},\\ 0, & \forall\,x\in\mathbb{R}^N,\ u=0 \text{ or } |u|\ge\pi^{-1/\epsilon}\end{cases}$$

be the primitive function of f with respect to u, where $\epsilon>0$ is small enough and $\alpha\in(1+\epsilon,2)$. Then it is easy to check that f satisfies conditions (S_3) and (S_4) with $\nu=\alpha-\epsilon$ and $\xi(x)=(\alpha+\epsilon)e^{-|x|^2}$.

2 Notations and preliminaries

Throughout this paper, we always use the following notations:

- $H^1(\mathbb{R}^N)$ is the usual Sobolev space equipped with the standard norm
$$\|u\|_{H^1}^2 = \int_{\mathbb{R}^N} \left(|\nabla u|^2 + u^2\right) dx,$$
and $H^{-1}(\mathbb{R}^N)$ is the dual space of $H^1(\mathbb{R}^N)$.
- $D^{1,2}(\mathbb{R}^N)$ is the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm
$$\|u\|_{D^{1,2}}^2 = \int_{\mathbb{R}^N} |\nabla u|^2 dx.$$
- $L^p(\Omega), 1 \leq p \leq \infty, \Omega \subseteq \mathbb{R}^N$, denotes a Lebesgue space, and the norm in $L^p(\Omega)$ is denoted by $\|u\|_{p,\Omega}$ when Ω is a proper subset of $\mathbb{R}^N$, by $\|u\|_p$ when $\Omega = \mathbb{R}^N$.
- For any $R > 0$, B_R denotes the ball in $\mathbb{R}^N$ centered at 0 with radius R.
- $\to$ (resp. $\rightharpoonup$) denotes the strong (resp. weak) convergence.

In what follows it will always be assumed that (S_1) and (S_2) are satisfied. As pointed out in [13], the form domain of the Schrödinger operator $\mathcal{S}$ is
$$E := \left\{ u \in H^1(\mathbb{R}^N) \mid \int_{\mathbb{R}^N} \left(a|\nabla u|^2 + V(x)u^2\right) dx < \infty \right\},$$
which becomes a Hilbert space if it is equipped with the inner product
$$(u,v)_0 := \int_{\mathbb{R}^N} \left(a\nabla u \cdot \nabla v + V(x)uv + l_0 uv\right) dx, \qquad \forall\, u,v \in E,$$
where $l_0 > -\inf \sigma(\mathcal{S}) = -\lambda_1$ is a fixed positive constant. We denote by $\|\cdot\|_0$ the associated norm.

Lemma 2.1. *E is continuously embedded into $H^1(\mathbb{R}^N)$, that is,*
$$\|u\|_{H^1} \leq c_0 \|u\|_0, \qquad \forall\, u \in E$$
for some $c_0 > 0$.

Proof. Arguing indirectly, we assume that there exists a sequence $\{u_n\}_{n\in\mathbb{N}} \subset E$ such that
$$\|u_n\|_{H^1}^2 = \int_{\mathbb{R}^N} \left(|\nabla u_n|^2 + u_n^2\right) dx \equiv 1, \quad \forall\, n \in \mathbb{N} \tag{2.1}$$
and
$$\|u_n\|_0^2 = \int_{\mathbb{R}^N} \left(a|\nabla u_n|^2 + V(x)u_n^2 + l_0 u_n^2\right) dx \to 0 \quad \text{as } n \to \infty. \tag{2.2}$$
Since $l_0 > -\inf \sigma(S)$, then it holds that
$$\int_{\mathbb{R}^N} \left(a|\nabla u_n|^2 + V(x)u_n^2 + l_0 u_n^2\right) dx \geq c_1 \int_{\mathbb{R}^N} u_n^2 dx \tag{2.3}$$
for some $c_1 > 0$. By (2.2) and (2.3), we get
$$\|u_n\|_2 = \left(\int_{\mathbb{R}^N} u_n^2 dx\right)^{1/2} \to 0 \quad \text{as } n \to \infty. \tag{2.4}$$

Let

$$V^- = V_1^- + V_2^-$$

with $V_1^- \in L^\infty(\mathbb{R}^N)$ and $V_2^- \in L^q(\mathbb{R}^N)$, where V^- and q are given in (S_1). Combining (2.1), (2.4), Hölder's inequality and the Gagliardo–Nirenberg inequality, we have

$$\begin{aligned}
\left|\int_{\mathbb{R}^N} V^- u_n^2 dx\right| &= \left|\int_{\mathbb{R}^N} V_1^- u_n^2 dx + \int_{\mathbb{R}^N} V_2^- u_n^2 dx\right| \\
&\le \int_{\mathbb{R}^N} |V_1^- u_n^2| dx + \int_{\mathbb{R}^N} |V_2^- u_n^2| dx \\
&\le \|V_1^-\|_\infty \|u_n\|_2^2 + \|V_2^-\|_q \|u_n\|_{2q/(q-1)}^2 \\
&\le \|V_1^-\|_\infty \|u_n\|_2^2 + c_2 \|V_2^-\|_q \|\nabla u_n\|_2^{N/q} \|u_n\|_2^{(2q-N)/q} \to 0 \quad \text{as } n \to \infty,
\end{aligned}$$

where $c_2 > 0$ is a constant depending on q. This together with (2.2) and (2.4) yields

$$\int_{\mathbb{R}^N} \left(|\nabla u_n|^2 + u_n^2\right) dx \to 0 \quad \text{as } n \to \infty,$$

which contradicts (2.1). The proof is completed. □

For later use, we introduce the new inner product in E as follows. Choose $\bar{d} \in (d, \lambda_\infty)$ such that $\bar{d} \neq \lambda_k$ for all $k \in \mathbb{N}$, where d is the constant given in (S_5). Denote by λ_{k_0} the first eigenvalue of the Schrödinger operator $\mathcal{S}$ greater than $\bar{d}$. Let E^- be the subspace of E spanned by the eigenfunctions with corresponding eigenvalues less than $\bar{d}$. Note the fact that $\lambda_\infty = \lim_{k\to\infty} \lambda_k$ and $\lambda_k \in \sigma_{pp}(\mathcal{S})$ whenever $\lambda_k < \lambda_\infty$. Then it is evident that E^- is a finite dimensional subspace of E. If there is no eigenvalue of the Schrödinger operator $\mathcal{S}$ greater than $\bar{d}$, then we set $\lambda_{k_0} = \lambda_\infty$ and E^- is empty in this case. Let E^+ be the orthogonal complement of E^- in E with respect to the inner product $(\cdot,\cdot)_0$. Then E possesses the orthogonal decomposition $E = E^- \oplus E^+$. By definition, it holds that

$$\int_{\mathbb{R}^N} \left(a|\nabla u|^2 + V(x)u^2\right) dx \ge \lambda_{k_0} \int_{\mathbb{R}^N} u^2 dx, \qquad \forall\, u \in E^+. \tag{2.5}$$

Now we can define the new inner product $(\cdot,\cdot)$ and the induced norm $\|\cdot\|$ in E by

$$\begin{aligned}
(u,v) = &\int_{\mathbb{R}^N} \left(a\nabla u^+ \cdot \nabla v^+ + V(x)u^+v^+ - \bar{d}u^+v^+\right) dx \\
&- \int_{\mathbb{R}^N} \left(a\nabla u^- \cdot \nabla v^- + V(x)u^-v^- - \bar{d}u^-v^-\right) dx,
\end{aligned} \tag{2.6}$$

$$\|u\| = \sqrt{(u,u)} \tag{2.7}$$

for all $u = u^- + u^+$, $v = v^- + v^+ \in E$ with $u^\pm, v^\pm \in E^\pm$. Note the fact that E^- and E^+ are also orthogonal with respect to the usual inner product in $L^2(\mathbb{R}^N)$. Then it is evident that E possesses the same orthogonal decomposition $E = E^- \oplus E^+$ with respect to the new inner product $(\cdot,\cdot)$. Moreover, we have

$$\int_{\mathbb{R}^N} \left(a|\nabla u|^2 + V(x)u^2 - \bar{d}u^2\right) dx = \|u^+\|^2 - \|u^-\|^2 \tag{2.8}$$

for all $u = u^- + u^+ \in E$ with $u^\pm \in E^\pm$.

Lemma 2.2. *The norms $\|\cdot\|$ and $\|\cdot\|_0$ are equivalent in E.*

Proof. It suffices to show that $\|\cdot\|$ and $\|\cdot\|_0$ are equivalent in E^+ since E^- is finite dimensional. On the one hand, by (2.8), there holds

$$\begin{aligned} \|u\|^2 &= \int_{\mathbb{R}^N} \left(a|\nabla u|^2 + V(x)u^2 - \bar{d}u^2\right) dx \\ &\le \int_{\mathbb{R}^N} \left(a|\nabla u|^2 + V(x)u^2 + l_0 u^2\right) dx = \|u\|_0^2, \qquad \forall\, u \in E^+. \end{aligned} \tag{2.9}$$

On the other hand, invoking (2.5) and (2.8), we get

$$\begin{aligned} \|u\|^2 &= \int_{\mathbb{R}^N} \left(a|\nabla u|^2 + V(x)u^2 - \bar{d}u^2\right) dx \\ &\ge \frac{\lambda_{k_0} - \bar{d}}{\lambda_{k_0}} \int_{\mathbb{R}^N} \left(a|\nabla u|^2 + V(x)u^2\right) dx \\ &\ge \frac{\lambda_{k_0} - \bar{d}}{2\lambda_{k_0}} \int_{\mathbb{R}^N} \left(a|\nabla u|^2 + V(x)u^2\right) dx + \frac{\lambda_{k_0} - \bar{d}}{2l_0} \int_{\mathbb{R}^N} l_0 u^2 dx \\ &\ge c_3 \int_{\mathbb{R}^N} \left(a|\nabla u|^2 + V(x)u^2 + l_0 u^2\right) dx \\ &= c_3 \|u\|_0^2, \qquad \forall\, u \in E^+, \end{aligned} \tag{2.10}$$

where $c_3 = \min\{(\lambda_{k_0} - \bar{d})/2\lambda_{k_0}, (\lambda_{k_0} - \bar{d})/2l_0\} > 0$ by the choice of $\bar{d}$ and λ_{k_0}. Combining (2.9) and (2.10), we know that $\|\cdot\|$ and $\|\cdot\|_0$ are equivalent in E^+. The proof is completed. □

Hereafter, we always use the inner product $(\cdot,\cdot)$ and the induced norm $\|\cdot\|$ in E. Moreover, we write E^* for the dual space of E, and $\langle\cdot,\cdot\rangle : E^* \times E \to \mathbb{R}$ for the dual pairing. From Lemma 2.1 and Lemma 2.2, we immediately know that E is continuously embedded into $H^1(\mathbb{R}^N)$. Furthermore, using the Sobolev embedding theorem, we also get the following lemma.

Lemma 2.3. *E is continuously embedded into $D^{1,2}(\mathbb{R}^N)$ and $L^p(\mathbb{R}^N)$ for all $p \in [2, 2^*]$, and hence there exist constants c_4, $\tau_p > 0$ such that*

$$\|u\|_{D^{1,2}} \le c_4 \|u\|, \qquad \forall\, u \in E \tag{2.11}$$

and

$$\|u\|_p \le \tau_p \|u\|, \qquad \forall\, u \in E \text{ and } p \in [2, 2^*]. \tag{2.12}$$

Moreover, for any bounded domain $\Omega \subset \mathbb{R}^N$, E is compactly embedded into $L^p(\Omega)$ for all $p \in [1, 2^*)$.

3 Variational setting and proof of the main result

In this section, we will first introduce the variational setting for (1.1). To this end, we define functionals $\Psi_i (i = 1, 2, 3)$ and Φ on E by

$$\begin{aligned} \Psi_1(u) &= \frac{b}{4}\left(\int_{\mathbb{R}^N} |\nabla u|^2 dx\right)^2, \\ \Psi_2(u) &= \int_{\mathbb{R}^N} F(x,u) dx, \\ \Psi_3(u) &= \int_{\mathbb{R}^N} \left(\frac{\bar{d}}{2}u^2 - G(x,u)\right) dx \end{aligned}$$

and

$$
\begin{aligned}
\Phi(u) &= \frac{1}{2}\int_{\mathbb{R}^N}\left(a|\nabla u|^2+V(x)u^2\right)dx+\frac{b}{4}\left(\int_{\mathbb{R}^N}|\nabla u|^2dx\right)^2-\int_{\mathbb{R}^N}\left(F(x,u)+G(x,u)\right)dx\\
&= \frac{1}{2}\int_{\mathbb{R}^N}\left(a|\nabla u|^2+V(x)u^2-\bar{d}u^2\right)dx+\frac{b}{4}\left(\int_{\mathbb{R}^N}|\nabla u|^2dx\right)^2-\int_{\mathbb{R}^N}F(x,u)dx\\
&\quad+\int_{\mathbb{R}^N}\left(\frac{\bar{d}}{2}u^2-G(x,u)\right)dx\\
&= \frac{1}{2}\|u^+\|^2-\frac{1}{2}\|u^-\|^2+\Psi_1(u)-\Psi_2(u)+\Psi_3(u)
\end{aligned}
\tag{3.1}
$$

for all $u=u^-+u^+\in E$ with $u^\pm\in E^\pm$. Here $\bar{d}$ is the constant in (2.8) and $G(x,u):=\int_0^u g(x,t)dt$ is the primitive function of $g(x,u)$ with respect to u.

Proposition 3.1. *Assume that* (S_1)–(S_3) *and* (S_5) *are satisfied. Then* $\Psi_i\in C^1(E,\mathbb{R})$ *for* $i=1,2,3$ *with* $\Psi_i':E\to E^*$ *being completely continuous for* $i=2,3$*, and hence* $\Phi\in C^1(E,\mathbb{R})$*. Moreover,*

$$
\langle\Psi_1'(u),v\rangle=b\left(\int_{\mathbb{R}^N}|\nabla u|^2dx\right)\int_{\mathbb{R}^N}\nabla u\cdot\nabla v dx, \tag{3.2}
$$

$$
\langle\Psi_2'(u),v\rangle=\int_{\mathbb{R}^N}f(x,u)vdx, \tag{3.3}
$$

$$
\langle\Psi_3'(u),v\rangle=\int_{\mathbb{R}^N}(\bar{d}u-g(x,u))vdx, \tag{3.4}
$$

$$
\begin{aligned}
\langle\Phi'(u),v\rangle &= (u^+,v^+)-(u^-,v^-)+\langle\Psi_1'(u),v\rangle-\langle\Psi_2'(u),v\rangle+\langle\Psi_3'(u),v\rangle\\
&= \left(a+b\int_{\mathbb{R}^N}|\nabla u|^2dx\right)\int_{\mathbb{R}^N}\nabla u\cdot\nabla v dx+\int_{\mathbb{R}^N}V(x)uvdx\\
&\quad-\int_{\mathbb{R}^N}(f(x,u)+g(x,u))vdx
\end{aligned}
\tag{3.5}
$$

for all $u=u^-+u^+$, $v=v^-+v^+\in E$ *with* $u^\pm$, $v^\pm\in E^\pm$*. In addition, if* $u\in E\subseteq H^1(\mathbb{R}^N)$ *is a critical point of* Φ *on* E*, then it is a solution of* (1.1)*.*

Proof. First, we show that $\Psi_1\in C^1(E,\mathbb{R})$ and (3.2) holds. Define a functional Ψ_0 on $D^{1,2}(\mathbb{R}^N)$ by

$$
\Psi_0(u)=\frac{b}{4}\left(\int_{\mathbb{R}^N}|\nabla u|^2dx\right)^2.
$$

Evidently, $\Psi_0\in C^1(D^{1,2}(\mathbb{R}^N),\mathbb{R})$ and

$$
\langle\Psi_0'(u),v\rangle=b\left(\int_{\mathbb{R}^N}|\nabla u|^2dx\right)\int_{\mathbb{R}^N}\nabla u\cdot\nabla v dx,\qquad\forall\,u,\,v\in D^{1,2}(\mathbb{R}^N). \tag{3.6}
$$

Let $\iota:E\to D^{1,2}(\mathbb{R}^N)$ be the continuous embedding in Lemma 2.3. Since $\Psi_1=\Psi_0\circ\iota$, we immediately know by (3.6) that $\Psi_1\in C^1(E,\mathbb{R})$ and (3.2) holds.

Next, we verify (3.3) by definition and prove that $\Psi_2\in C^1(E,\mathbb{R})$ with $\Psi_2':E\to E^*$ being completely continuous. By (S_3), there holds

$$
|F(x,u)|\le\nu^{-1}\xi(x)|u|^\nu,\qquad\forall\,(x,u)\in\mathbb{R}^N\times\mathbb{R}. \tag{3.7}
$$

For notational simplicity, we set

$$
\mu^*:=\frac{\mu\nu}{\mu-1}. \tag{3.8}
$$

Since $\nu \in (1,2)$ and $\mu \in (2^*/(2^*-\nu), 2/(2-\nu)]$ in (S$_3$), we get $\mu^* \in [2, 2^*)$. Then for any $u \in E$, by (2.12), (3.7) and Hölder's inequality, we have

$$\begin{aligned}\int_{\mathbb{R}^N} |F(x,u)|dx &\leq \int_{\mathbb{R}^N} \nu^{-1}\xi(x)|u|^\nu dx \\ &\leq \nu^{-1}\|\xi\|_\mu \|u\|_{\mu^*}^\nu \\ &\leq \nu^{-1}\tau_{\mu^*}^\nu \|\xi\|_\mu \|u\|^\nu < \infty,\end{aligned} \tag{3.9}$$

where τ_{μ^*} is the constant given in (2.12). Thus Ψ_2 is well defined. For any given $u \in E$, define an associated linear operator $\mathcal{J}(u) : E \to \mathbb{R}$ as follows:

$$\langle \mathcal{J}(u), v\rangle = \int_{\mathbb{R}^N} f(x,u)v dx, \qquad \forall\, v \in E. \tag{3.10}$$

By (S$_3$), (2.12) and Hölder's inequality, there holds

$$\begin{aligned}|\langle \mathcal{J}(u), v\rangle| &\leq \int_{\mathbb{R}^N} \xi(x)|u|^{\nu-1}|v|dx \\ &\leq \|\xi\|_\mu \|u\|_{\mu^*}^{\nu-1}\|v\|_{\mu^*} \\ &\leq \tau_{\mu^*}^\nu \|\xi\|_\mu \|u\|^{\nu-1}\|v\|, \qquad \forall\, v \in E,\end{aligned} \tag{3.11}$$

which shows that $\mathcal{J}(u)$ is well defined and bounded. On the other hand, it follows from (S$_3$) that

$$|f(x, u+\eta v)v| \leq 2^{\nu-1}\xi(x)(|u|^{\nu-1}|v| + |v|^\nu), \qquad \forall\, x \in \mathbb{R}^N,\ \eta \in [0,1] \text{ and } u, v \in \mathbb{R}. \tag{3.12}$$

Then for any $u, v \in E$, combining (3.9)-(3.12), the mean value theorem and Lebesgue's dominated convergence theorem, we have

$$\begin{aligned}\lim_{t\to 0}\frac{\Psi_2(u+tv) - \Psi_2(u)}{t} &= \lim_{t\to 0}\int_{\mathbb{R}^N} f(x, u+\theta(x)tv)v dx \\ &= \int_{\mathbb{R}^N} f(x,u)v dx \\ &= \langle \mathcal{J}(u), v\rangle,\end{aligned} \tag{3.13}$$

where $\theta(x) \in [0,1]$ depends on u, v, t. This shows that Ψ_2 is Gâteaux differentiable on E and the Gâteaux derivative of Ψ_2 at u is $\mathcal{J}(u)$.

In order to prove that $\Psi_2 \in C^1(E, \mathbb{R})$ and $\Psi_2' : E \to E^*$ is completely continuous, it suffices to prove that $\mathcal{J} : E \to E^*$ is completely continuous. To this end, we claim that if $u_n \rightharpoonup u$ in E, then for any $R > 0$,

$$\int_{B_R} |f(x,u_n) - f(x,u)|^{p_0} dx \to 0 \quad \text{as } n \to \infty, \tag{3.14}$$

where $p_0 := \max\{2^*/(2^*-1), \mu/(\mu(\nu-1)+1)\}$ with μ and ν given in (S$_3$). Arguing indirectly, by Lemma 2.3, we assume that there exist constants $R_0, \varepsilon_0 > 0$ and a subsequence $\{u_{n_k}\}_{k\in\mathbb{N}}$ such that

$$u_{n_k} \to u \text{ in } L^{p_0^*}(B_{R_0}) \text{ and } u_{n_k} \to u \text{ a.e. in } B_{R_0} \quad \text{as } k \to \infty \tag{3.15}$$

but

$$\int_{B_{R_0}} |f(x,u_{n_k}) - f(x,u)|^{p_0} dx \geq \varepsilon_0, \qquad \forall\, k \in \mathbb{N}, \tag{3.16}$$

where $p_0^* := p_0\mu(\nu-1)/(\mu-p_0) \in [1, 2^*)$ by (S_3) and the choice of p_0 above. Due to (3.15), passing to a subsequence if necessary, we can further assume that

$$\sum_{k=1}^{\infty} \|u_{n_k} - u\|_{p_0^*, B_{R_0}} < +\infty.$$

Let $w(x) = \sum_{k=1}^{\infty} |u_{n_k}(x) - u(x)|$ for all $x \in B_{R_0}$, then $w \in L^{p_0^*}(B_{R_0})$. By virtue of (S_3) and Hölder's inequality, we get

$$\begin{aligned}
&|f(x,u_{n_k}) - f(x,u)|^{p_0} \\
&\quad \le (|f(x,u_{n_k})| + |f(x,u)|)^{p_0} \\
&\quad \le \xi(x)^{p_0} \left(|u_{n_k}|^{\nu-1} + |u|^{\nu-1}\right)^{p_0} \\
&\quad \le 2^{p_0}\xi(x)^{p_0} \left(|u_{n_k}|^{p_0(\nu-1)} + |u|^{p_0(\nu-1)}\right) \\
&\quad \le 2^{p_0\nu+1}\xi(x)^{p_0} \left(|u_{n_k} - u|^{p_0(\nu-1)} + |u|^{p_0(\nu-1)}\right) \\
&\quad \le 2^{p_0\nu+1}\xi(x)^{p_0} \left(|w|^{p_0(\nu-1)} + |u|^{p_0(\nu-1)}\right), \qquad \forall\, k \in \mathbb{N} \text{ and } x \in B_{R_0}
\end{aligned} \tag{3.17}$$

and

$$\int_{B_{R_0}} \xi(x)^{p_0} \left(|w|^{p_0(\nu-1)} + |u|^{p_0(\nu-1)}\right) dx \le \|\xi\|_{\mu}^{p_0} \left(\|w\|_{p_0^*, B_{R_0}}^{p_0(\nu-1)} + \|u\|_{p_0^*, B_{R_0}}^{p_0(\nu-1)}\right) < +\infty. \tag{3.18}$$

Combining (3.15), (3.17), (3.18) and Lebesgue's dominated convergence theorem, we have

$$\lim_{k\to\infty} \int_{B_{R_0}} |f(x,u_{n_k}) - f(x,u)|^{p_0} dx = 0,$$

which contradicts (3.16). Thus the claim is true.

Now let $u_n \rightharpoonup u$ in E as $n \to \infty$, then $\{u_n\}$ is bounded in E and hence there exists a constant $D_0 > 0$ such that

$$\|u_n\|^{\nu} + \|u_n\|\|u\|^{\nu-1} \le D_0, \qquad \forall\, n \in \mathbb{N}. \tag{3.19}$$

For any $\epsilon > 0$, by (S_3), there exists $R_\epsilon > 0$ such that

$$\left(\int_{\mathbb{R}^N \setminus B_{R_\epsilon}} \xi(x)^{\mu} dx\right)^{1/\mu} < \frac{\epsilon}{2D_0\tau_{\mu^*}^{\nu}}. \tag{3.20}$$

Combining (S_3), (3.19), (3.20) and Hölder's inequality, we have

$$\begin{aligned}
\int_{\mathbb{R}^N \setminus B_{R_\epsilon}} |f(x,u_n) - f(x,u)||v| dx &\le \int_{\mathbb{R}^N \setminus B_{R_\epsilon}} (|f(x,u_n)| + |f(x,u)|)\, |v| dx \\
&\le \int_{\mathbb{R}^N \setminus B_{R_\epsilon}} \xi(x) \left(|u_n|^{\nu-1} + |u|^{\nu-1}\right) |v| dx \\
&\le \left(\int_{\mathbb{R}^N \setminus B_{R_\epsilon}} \xi(x)^{\mu} dx\right)^{1/\mu} \left(\|u_n\|_{\mu^*}^{\nu-1} + \|u\|_{\mu^*}^{\nu-1}\right) \|v\|_{\mu^*} \\
&\le \tau_{\mu^*}^{\nu} \left(\int_{\mathbb{R}^N \setminus B_{R_\epsilon}} \xi(x)^{\mu} dx\right)^{1/\mu} \left(\|u_n\|^{\nu-1} + \|u\|^{\nu-1}\right) \\
&< \frac{\epsilon}{2}, \qquad \forall\, n \in \mathbb{N} \text{ and } \|v\| = 1.
\end{aligned} \tag{3.21}$$

For the R_ϵ given in (3.20), by Hölder's inequality and (3.14), there exists $N_\epsilon \in \mathbb{N}$ such that

$$\begin{aligned}\int_{B_{R_\epsilon}} |f(x,u_n)-f(x,u)||v|dx &\le \left(\int_{B_{R_\epsilon}} |f(x,u_n)-f(x,u)|^{p_0}dx\right)^{1/p_0} \|v\|_{\bar{p}_0, B_{R_\epsilon}} \\ &\le \left(\int_{B_{R_\epsilon}} |f(x,u_n)-f(x,u)|^{p_0}dx\right)^{1/p_0} \|v\|_{\bar{p}_0} \\ &\le \tau_{\bar{p}_0}\left(\int_{B_{R_\epsilon}} |f(x,u_n)-f(x,u)|^{p_0}dx\right)^{1/p_0} \\ &< \frac{\epsilon}{2}, \qquad \forall\, n \ge N_\epsilon \text{ and } \|v\| = 1,\end{aligned} \tag{3.22}$$

where $\bar{p}_0 := p_0/(p_0-1) \in (1, 2^*]$ and p_0 is the constant given in (3.14), and $\tau_{\bar{p}_0}$ is the constant given in (2.12). Combining (3.21) and (3.22), we have

$$\begin{aligned}\|\mathcal{J}(u_n)-\mathcal{J}(u)\|_{E^*} &= \sup_{\|v\|=1} |\langle \mathcal{J}(u_n)-\mathcal{J}(u), v\rangle| \\ &= \sup_{\|v\|=1} \left|\int_{\mathbb{R}^N} (f(x,u_n)-f(x,u))\, v dx\right| \\ &\le \sup_{\|v\|=1} \int_{B_{R_\epsilon}} |f(x,u_n)-f(x,u)||v|dx \\ &\quad + \sup_{\|v\|=1} \int_{\mathbb{R}^N \setminus B_{R_\epsilon}} |f(x,u_n)-f(x,u)||v|dx \\ &\le \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon, \qquad \forall\, n \ge N_\epsilon.\end{aligned}$$

This shows that $\mathcal{J} : E \to E^*$ is completely continuous.

Then, taking (S_5) into account and using similar arguments to those above, one can also prove that $\Psi_3 \in C^1(E, \mathbb{R})$ with $\Psi_3' : E \to E^*$ being completely continuous and (3.4) holds. For simplicity, we omit the proof here.

Finally, combining (2.6) and (3.1)–(3.4), we immediately know that $\Phi \in C^1(E, \mathbb{R})$ and (3.5) holds. In addition, it is known that any critical point $u \in E \subseteq H^1(\mathbb{R}^N)$ of the functional Φ is a solution of (1.1). The proof is completed. □

We will use the following variant symmetric mountain pass lemma due to [7] (see also [14]) to prove that (1.1) possesses a sequence of weak solutions. Before stating this theorem, we first recall the notion of genus.

Let E be a Banach space and A a subset of E. A is said to be symmetric if $u \in A$ implies $-u \in A$. Denote by Γ the family of all closed symmetric subset of E which does not contain 0. For any $A \in \Gamma$, define the genus $\gamma(A)$ of A by the smallest integer k such that there exists an odd continuous mapping from A to $\mathbb{R}^k \setminus \{0\}$. If there does not exist such a k, define $\gamma(A) = \infty$. Moreover, set $\gamma(\emptyset) = 0$. For each $k \in \mathbb{N}$, let $\Gamma_k = \{A \in \Gamma \mid \gamma(A) \ge k\}$.

Theorem 3.2 ([7, Theorem 1]). *Let E be an infinite dimensional Banach space and $\Phi \in C^1(E, \mathbb{R})$ an even functional with $\Phi(0) = 0$. Suppose that Φ satisfies*

(Φ_1) *Φ is bounded from below and satisfies* (PS) *condition.*

(Φ_2) *For each $k \in \mathbb{N}$, there exists an $A_k \in \Gamma_k$ such that $\sup_{u \in A_k} \Phi(u) < 0$.*

Then either (i) or (ii) below holds.

(i) *There exists a critical point sequence* $\{u_k\}$ *such that* $\Phi(u_k) < 0$ *and* $\lim_{k\to\infty} u_k = 0$.

(ii) *There exist two critical point sequences* $\{u_k\}$ *and* $\{v_k\}$ *such that* $\Phi(u_k) = 0$, $u_k \neq 0$, $\lim_{k\to\infty} u_k = 0$, $\Phi(v_k) < 0$, $\lim_{k\to\infty} \Phi(v_k) = 0$, *and* $\{v_k\}$ *converges to a non-zero limit.*

In order to apply Theorem 3.2, we will show in the following lemmas that the functional Φ defined in (3.1) satisfies conditions (Φ_1) and (Φ_2) in Theorem 3.2. The proof of these lemmas is partially motivated by [21] and [7].

Lemma 3.3. *Let* (S_1)–(S_3) *and* (S_5) *be satisfied. Then* Φ *is coercive and bounded from below.*

Proof. We first prove that Φ is coercive. Arguing indirectly, we assume that for some sequence $\{u_n\}_{n\in\mathbb{N}} \subset E$ with $\|u_n\| \to \infty$, there is a constant $M > 0$ such that $\Phi(u_n) \leq M$ for all $n \in \mathbb{N}$. Let $u_n = u_n^- + u_n^+$ with $u_n^\pm \in E^\pm$. If we set $v_n = u_n/\|u_n\|$ for all $n \in \mathbb{N}$, then $\|v_n\| \equiv 1$, and $v_n = v_n^- + v_n^+$ with $v_n^\pm = u_n^\pm/\|u_n\| \in E^\pm$. Note that E^- is finite dimensional. Thus, passing to a subsequence if necessary, we can assume by Lemma 2.3 that

$$v_n \rightharpoonup v,\ v_n^- \to v^-,\ v_n^+ \rightharpoonup v^+ \text{ and } v_n \to v \text{ a.e. in } \mathbb{R}^N \quad \text{as } n \to \infty \tag{3.23}$$

for some $v = v^- + v^+ \in E$ with $v^\pm \in E^\pm$. By (S_5), there hold

$$0 \leq \frac{\bar{d} - d}{2} u^2 \leq \frac{\bar{d}}{2} u^2 - G(x,u) \leq \frac{\bar{d} + d}{2} u^2, \qquad \forall\, (x,u) \in \mathbb{R}^N \times \mathbb{R} \tag{3.24}$$

and

$$\bar{d} u^2 - g(x,u)u \geq (\bar{d} - d)u^2 \geq 0, \quad \forall\, (x,u) \in \mathbb{R}^N \times \mathbb{R} \tag{3.25}$$

since $\bar{d}$ is chosen to be greater than d in Section 2. Combining (3.1), (3.9) and (3.24), we have

$$\begin{aligned} M \geq \Phi(u_n) &\geq \frac{1}{2}\|u_n^+\|^2 - \frac{1}{2}\|u_n^-\|^2 - \int_{\mathbb{R}^N} |F(x,u_n)|dx \\ &\geq \frac{1}{2}\|u_n^+\|^2 - \frac{1}{2}\|u_n^-\|^2 - \nu^{-1}\tau_{\mu^*}^{\nu}\|\xi\|_\mu \|u_n\|^\nu, \qquad \forall\, n \in \mathbb{N}. \end{aligned} \tag{3.26}$$

Multiplying both sides of (3.26) by $\|u_n\|^{-2}$, we get

$$\|v_n^+\|^2 \leq \|v_n^-\|^2 + o(1) \quad \text{as } n \to \infty \tag{3.27}$$

since $\nu < 2$ in (S_3) and $\|u_n\| \to \infty$.

If $v = 0$, then $v_n^- \to 0$ and hence $v_n^+ \to 0$ by (3.27). This implies $v_n \to 0$, which leads to a contradiction since $\|v_n\| \equiv 1$. Therefore, $v \neq 0$. Note that $v_n \rightharpoonup v$ in $D^{1,2}(\mathbb{R}^N)$ since E is continuously embedded into $D^{1,2}(\mathbb{R}^N)$. Then it follows from the weak lower semi-continuity of the norm $\|\cdot\|_{D^{1,2}}$ in $D^{1,2}(\mathbb{R}^N)$ that

$$\begin{aligned} \liminf_{n\to\infty} \left[\frac{b}{4\|u_n\|^4} \left(\int_{\mathbb{R}^N} |\nabla u_n|^2 dx \right)^2 \right] &= \liminf_{n\to\infty} \left[\frac{b}{4} \left(\int_{\mathbb{R}^N} |\nabla v_n|^2 dx \right)^2 \right] \\ &= \liminf_{n\to\infty} \frac{b}{4} \|v_n\|_{D^{1,2}}^4 \\ &= \frac{b}{4} \left(\liminf_{n\to\infty} \|v_n\|_{D^{1,2}} \right)^4 \\ &\geq \frac{b}{4} \|v\|_{D^{1,2}}^4 > 0. \end{aligned} \tag{3.28}$$

Combining (3.1), (3.9) and (3.24), we have

$$M \geq \Phi(u_n) \geq \frac{1}{2}\|u_n^+\|^2 - \frac{1}{2}\|u_n^-\|^2 + \frac{b}{4}\left(\int_{\mathbb{R}^N} |\nabla u_n|^2 dx\right)^2 - \int_{\mathbb{R}^N} |F(x,u_n)| dx$$
$$\geq \frac{1}{2}\|u_n^+\|^2 - \frac{1}{2}\|u_n^-\|^2 + \frac{b}{4}\left(\int_{\mathbb{R}^N} |\nabla u_n|^2 dx\right)^2 - \nu^{-1}\tau_{\mu^*}^{\nu}\|\xi\|_{\mu}\|u_n\|^{\nu},$$

or equivalently,

$$\frac{b}{4}\left(\int_{\mathbb{R}^N} |\nabla u_n|^2 dx\right)^2 \leq \frac{1}{2}\|u_n^-\|^2 - \frac{1}{2}\|u_n^+\|^2 + \nu^{-1}\tau_{\mu^*}^{\nu}\|\xi\|_{\mu}\|u_n\|^{\nu} + M, \tag{3.29}$$

where μ^* and τ_{μ^*} are the constants given in (3.8) and (2.12) respectively. Multiplying both sides of (3.29) by $\|u_n\|^{-4}$ and letting $n \to \infty$, we get

$$\liminf_{n\to\infty}\left[\frac{b}{4\|u_n\|^4}\left(\int_{\mathbb{R}^N} |\nabla u_n|^2 dx\right)^2\right] \leq 0,$$

which contradicts (3.28). Therefore, Φ is coercive.

Next, we show that Φ is bounded from below. Combining (2.11), (2.12), (3.1), (3.9) and (3.24), we have

$$\begin{aligned}|\Phi(u)| &\leq \frac{1}{2}\|u\|^2 + \frac{b}{4}\left(\int_{\mathbb{R}^N} |\nabla u|^2 dx\right)^2 + \int_{\mathbb{R}^N} |F(x,u)| dx + \int_{\mathbb{R}^N}\left(\frac{\bar{d}}{2}u^2 - G(x,u)\right) dx \\ &\leq \frac{1}{2}\|u\|^2 + \frac{bc_4^4}{4}\|u\|^4 + \nu^{-1}\tau_{\mu^*}^{\nu}\|\xi\|_{\mu}\|u\|^{\nu} + \frac{\bar{d}+d}{2}\tau_2^2\|u\|^2,\end{aligned} \tag{3.30}$$

where c_4 and τ_2 are the constants given in (2.11) and (2.12) respectively. This implies that Φ maps bounded sets in E into bounded sets in $\mathbb{R}$. Then it follows from the coercivity that Φ is bounded from below. The proof is completed. □

Lemma 3.4. *Assume that* (S_1)–(S_3) *and* (S_5) *are satisfied. Then Φ satisfies* (PS) *condition.*

Proof. Let $\{u_n\}_{n\in\mathbb{N}} \subset E$ be a (PS)-sequence, i.e.,

$$|\Phi(u_n)| \leq D_1 \quad \text{and} \quad \Phi'(u_n) \to 0 \quad \text{as } n \to \infty \tag{3.31}$$

for some $D_1 > 0$. Note first that Φ is coercive by Lemma 3.3. This together with (3.31) implies that $\{u_n\}_{n\in\mathbb{N}}$ is bounded in E. Thus there exists a subsequence $\{u_{n_k}\}_{k\in\mathbb{N}}$ such that

$$u_{n_k} \rightharpoonup u_0 \quad \text{as } k \to \infty \tag{3.32}$$

for some $u_0 \in E$. Let

$$u_{n_k} = u_{n_k}^- + u_{n_k}^+ \quad \text{and} \quad u_0 = u_0^- + u_0^+$$

with $u_{n_k}^{\pm}, u_0^{\pm} \in E^{\pm}$. Since E^- is finite dimensional, we get

$$u_{n_k}^- \to u_0^- \quad \text{and} \quad u_{n_k}^+ \rightharpoonup u_0^+ \quad \text{as } k \to \infty. \tag{3.33}$$

By (3.5), it holds

$$
\begin{aligned}
&\langle \Phi'(u_{n_k}) - \Phi'(u_0), u_{n_k} - u_0 \rangle \\
&\quad = \|u_{n_k}^+ - u_0^+\|^2 - \|u_{n_k}^- - u_0^-\|^2 - \langle \Psi_2'(u_{n_k}) - \Psi_2'(u_0), u_{n_k} - u_0 \rangle \\
&\qquad + \langle \Psi_3'(u_{n_k}) - \Psi_3'(u_0), u_{n_k} - u_0 \rangle + b \int_{\mathbb{R}^N} |\nabla u_{n_k}|^2 dx \int_{\mathbb{R}^N} \nabla u_{n_k} \cdot \nabla (u_{n_k} - u_0) dx \\
&\qquad - b \int_{\mathbb{R}^N} |\nabla u_0|^2 dx \int_{\mathbb{R}^N} \nabla u_0 \cdot \nabla (u_{n_k} - u_0) dx \\
&\quad = \|u_{n_k}^+ - u_0^+\|^2 - \|u_{n_k}^- - u_0^-\|^2 - \langle \Psi_2'(u_{n_k}) - \Psi_2'(u_0), u_{n_k} - u_0 \rangle \\
&\qquad + \langle \Psi_3'(u_{n_k}) - \Psi_3'(u_0), u_{n_k} - u_0 \rangle + b \int_{\mathbb{R}^N} |\nabla u_{n_k}|^2 dx \int_{\mathbb{R}^N} |\nabla (u_{n_k} - u_0)|^2 dx \\
&\qquad + b \left(\int_{\mathbb{R}^N} |\nabla u_{n_k}|^2 dx - \int_{\mathbb{R}^N} |\nabla u_0|^2 dx \right) \int_{\mathbb{R}^N} \nabla u_0 \cdot \nabla (u_{n_k} - u_0) dx, \qquad \forall k \in \mathbb{N}.
\end{aligned}
\tag{3.34}
$$

By virtue of (3.32) and Lemma 2.3, we have $u_{n_k} \rightharpoonup u_0$ in $D^{1,2}(\mathbb{R}^N)$. Then it follows that

$$
\left(\int_{\mathbb{R}^N} |\nabla u_{n_k}|^2 dx - \int_{\mathbb{R}^N} |\nabla u_0|^2 dx \right) \int_{\mathbb{R}^N} \nabla u_0 \cdot \nabla (u_{n_k} - u_0) dx \to 0 \quad \text{as } k \to \infty. \tag{3.35}
$$

Due to (3.31) and (3.32), there holds

$$
\langle \Phi'(u_{n_k}) - \Phi'(u_0), u_{n_k} - u_0 \rangle \to 0 \quad \text{as } k \to \infty. \tag{3.36}
$$

Moreover, from (3.32) and Proposition 3.1, we know that

$$
\langle \Psi_i'(u_{n_k}) - \Psi_i'(u_0), u_{n_k} - u_0 \rangle \to 0 \quad \text{as } k \to \infty \tag{3.37}
$$

for $i = 2, 3$. Combining (3.33)-(3.37), we obtain

$$
\begin{aligned}
\|u_{n_k}^+ - u_0^+\|^2 &\le \|u_{n_k}^- - u_0^-\|^2 + \langle \Phi'(u_{n_k}) - \Phi'(u_0), u_{n_k} - u_0 \rangle \\
&\qquad + \langle \Psi_2'(u_{n_k}) - \Psi_2'(u_0), u_{n_k} - u_0 \rangle - \langle \Psi_3'(u_{n_k}) - \Psi_3'(u_0), u_{n_k} - u_0 \rangle \\
&\qquad - b \left(\int_{\mathbb{R}^N} |\nabla u_{n_k}|^2 dx - \int_{\mathbb{R}^N} |\nabla u_0|^2 dx \right) \int_{\mathbb{R}^N} \nabla u_0 \cdot \nabla (u_{n_k} - u_0) dx \\
&= o(1) \quad \text{as } k \to \infty,
\end{aligned}
$$

which implies that $u_{n_k}^+ \to u_0^+$ in E. This together with (3.33) shows that $u_{n_k} \to u_0$ in E. Therefore, Φ satisfies (PS) condition. The proof is completed. □

Lemma 3.5. *Let* (S$_1$)–(S$_5$) *be satisfied. Then for each $k \in \mathbb{N}$, there exists an $A_k \subseteq E$ with genus $\gamma(A_k) = k$ such that $\sup_{u \in A_k} \Phi(u) < 0$.*

Proof. We follow the idea of the geometric construction introduced in [7]. By coordinate translation, we can assume $x_0 = 0$ in (S$_4$). Let $\mathcal{C}$ denote the cube

$$
\mathcal{C} := \{x = (x_1, x_2, \dots, x_N) \mid -r_0/2 \le x_i \le r_0/2,\ i = 1, 2, \dots, N\},
$$

where r_0 is the positive constant given in (S$_4$). Evidently, $\mathcal{C} \subseteq B_{r_0}$. By (S$_4$), there exist constants δ, $\varrho > 0$ and two sequences of positive numbers $\delta_n \to 0$, $M_n \to \infty$ as $n \to \infty$ such that

$$
F(x, u) \ge -\varrho u^2, \qquad \forall\, x \in \mathcal{C} \text{ and } |u| \le \delta \tag{3.38}
$$

and

$$F(x, \pm\delta_n)/\delta_n^2 \geq M_n, \qquad \forall\, x \in \mathcal{C} \text{ and } n \in \mathbb{N}. \tag{3.39}$$

For any fixed $k \in \mathbb{N}$, let $m \in \mathbb{N}$ be the smallest positive integer satisfying $m^N \geq k$. We divide the cube $\mathcal{C}$ equally into m^N small cubes by planes parallel to each face of $\mathcal{C}$ and denote them by $\mathcal{C}_i$ with $1 \leq i \leq m^N$. Then the edge of each $\mathcal{C}_i$ has the length of $l := r_0/m$. For each $1 \leq i \leq k$, we make a cube $\mathcal{D}_i$ in $\mathcal{C}_i$ such that $\mathcal{D}_i$ has the same center as that of $\mathcal{C}_i$, the faces of $\mathcal{D}_i$ and $\mathcal{C}_i$ are parallel and the edge of $\mathcal{D}_i$ has the length of $l/2$.

Choose a function $\psi \in C_0^\infty(\mathbb{R}, \mathbb{R})$ such that $\psi(t) \equiv 1$ for $t \in [-l/4, l/4]$, $\psi(t) \equiv 0$ for $t \in \mathbb{R} \setminus [-l/2, l/2]$, and $0 \leq \psi(t) \leq 1$ for all $t \in \mathbb{R}$. Define

$$\varphi(x) := \psi(x_1)\psi(x_2)\cdots\psi(x_N), \qquad \forall x = (x_1, x_2, \ldots, x_N) \in \mathbb{R}^N.$$

For each $1 \leq i \leq k$, let $y_i \in \mathbb{R}^N$ be the center of both $\mathcal{C}_i$ and $\mathcal{D}_i$, and define

$$\varphi_i(x) = \varphi(x - y_i), \qquad \forall\, x \in \mathbb{R}^N.$$

Then it is easy to see that

$$\operatorname{supp}\varphi_i \subseteq \mathcal{C}_i, \tag{3.40}$$

and

$$\varphi_i(x) = 1,\ \forall\, x \in \mathcal{D}_i, \quad 0 \leq \varphi_i(x) \leq 1,\ \forall\, x \in \mathbb{R}^N \tag{3.41}$$

for all $1 \leq i \leq k$. Set

$$\mathcal{V}_k := \left\{ (s_1, s_2, \ldots, s_k) \in \mathbb{R}^k \mid \max_{1\leq i\leq k} |s_i| = 1 \right\}$$

and

$$\mathcal{W}_k := \left\{ \sum_{i=1}^k s_i\varphi_i \mid (s_1, s_2, \ldots, s_k) \in \mathcal{V}_k \right\}.$$

Evidently, $\mathcal{V}_k$ is homeomorphic to the unit sphere in $\mathbb{R}^k$ by an odd mapping. Thus $\gamma(\mathcal{V}_k) = k$. If we define the mapping $\mathcal{H} : \mathcal{V}_k \to \mathcal{W}_k$ by

$$\mathcal{H}(s_1, s_2, \ldots, s_k) = \sum_{i=1}^k s_i\varphi_i, \qquad \forall\, (s_1, s_2, \ldots, s_k) \in \mathcal{V}_k,$$

then $\mathcal{H}$ is odd and homeomorphic. Therefore $\gamma(\mathcal{W}_k) = \gamma(\mathcal{V}_k) = k$. Moreover, it is evident that $\mathcal{W}_k$ is compact and hence there is a constant $C_k > 0$ such that

$$\|u\| \leq C_k, \quad \forall\, u \in \mathcal{W}_k. \tag{3.42}$$

For each $\delta_n \in (0, \delta)$ given in (3.39) and any $u = \sum_{i=1}^k s_i\varphi_i \in \mathcal{W}_k$, combining (2.11), (2.12), (3.1), (3.24), (3.40) and (3.41) , we have

$$\begin{aligned}
\Phi(\delta_n u) &= \frac{1}{2}\|\delta_n u^+\|^2 - \frac{1}{2}\|\delta_n u^-\|^2 + \frac{b}{4}\left(\int_{\mathbb{R}^N} |\nabla(\delta_n u)|^2 dx\right)^2 \\
&\quad - \int_{\mathbb{R}^N} F(x, \delta_n \sum_{i=1}^k s_i\varphi_i) dx + \int_{\mathbb{R}^N} \left(\frac{\bar{d}}{2}(\delta_n u)^2 - G(x, \delta_n u)\right) dx \\
&\leq \frac{\delta_n^2}{2}(\|u^+\|^2 - \|u^-\|^2) + \frac{b\delta_n^4}{4}\left(\int_{\mathbb{R}^N} |\nabla u|^2 dx\right)^2 \\
&\quad - \sum_{i=1}^k \int_{\mathcal{C}_i} F(x, \delta_n s_i\varphi_i) dx + \frac{\bar{d} + d}{2}\delta_n^2 \int_{\mathbb{R}^N} u^2 dx \\
&\leq \frac{\delta_n^2}{2}\|u\|^2 + \frac{bc_4^4\delta_n^4}{4}\|u\|^4 + \frac{\bar{d} + d}{2}\tau_2^2\delta_n^2\|u\|^2 - \sum_{i=1}^k \int_{\mathcal{C}_i} F(x, \delta_n s_i\varphi_i) dx,
\end{aligned} \tag{3.43}$$

where c_4 and τ_2 are the constants given in (2.11) and (2.12) respectively. By the definition of $\mathcal{V}_k$, there exists some integer $1 \leq i_u \leq k$ such that $|s_{i_u}| = 1$. Then it follows that

$$\begin{aligned}\sum_{i=1}^{k}\int_{\mathcal{C}_i} F(x,\delta_n s_i\varphi_i)dx = \int_{\mathcal{D}_{i_u}} F(x,\delta_n s_{i_u}\varphi_{i_u})dx + \int_{\mathcal{C}_{i_u}\setminus\mathcal{D}_{i_u}} F(x,\delta_n s_{i_u}\varphi_{i_u})dx \\ + \sum_{i\neq i_u}\int_{\mathcal{C}_i} F(x,\delta_n s_i\varphi_i)dx.\end{aligned} \tag{3.44}$$

By (3.38) and (3.41), there holds

$$\int_{\mathcal{C}_{i_u}\setminus\mathcal{D}_{i_u}} F(x,\delta_n s_{i_u}\varphi_{i_u})dx + \sum_{i\neq i_u}\int_{\mathcal{C}_i} F(x,\delta_n s_i\varphi_i)dx \geq -\varrho r_0^N\delta_n^2, \tag{3.45}$$

Here we use the fact that the volume of cube $\mathcal{C}$ in $\mathbb{R}^N$ is r_0^N. Combining (3.39) and (3.42)–(3.45), we have

$$\begin{aligned}\Phi(\delta_n u) &\leq \frac{C_k^2\delta_n^2}{2} + \frac{bc_4^4C_k^4\delta_n^4}{4} + \frac{(\bar{d}+d)\tau_2^2C_k^2\delta_n^2}{2} + \varrho r_0^N\delta_n^2 - \int_{\mathcal{D}_{i_u}} F(x,\delta_n s_{i_u}\varphi_{i_u})dx \\ &\leq \delta_n^2\left(\frac{C_k^2}{2} + \frac{b\delta_n^2c_4^4C_k^4}{4} + \frac{(\bar{d}+d)\tau_2^2C_k^2}{2} + \varrho r_0^N - \frac{l^N M_n}{2^N}\right).\end{aligned} \tag{3.46}$$

where M_n is the constant given in (3.39). Here we use the fact that $|\delta_n s_{i_u}\varphi_{i_u}(x)| \equiv \delta_n$ for all $x \in \mathcal{D}_{i_u}$ and the volume of cube $\mathcal{D}_{i_u}$ in $\mathbb{R}^N$ is $(l/2)^N$. Since $M_n \to \infty$ as $n \to \infty$, we can choose $n_0 \in \mathbb{N}$ large enough such that the right-hand side of (3.46) is negative. Define

$$A_k := \{\delta_{n_0}u \mid u \in \mathcal{W}_k\}. \tag{3.47}$$

Then we have

$$\gamma(A_k) = \gamma(\mathcal{W}_k) = k \quad \text{and} \quad \sup_{u\in A_k}\Phi(u) < 0.$$

The proof is completed. □

Now we are in a position to give the proof of our main result.

Proof of Theorem 1.1. Evidently, the functional Φ defined in (3.1) is an even functional with $\Phi(0) = 0$. Besides, Proposition 3.1 and Lemmas 3.3–3.5 show that $\Phi \in C^1(E,\mathbb{R})$ and satisfies conditions (Φ_1) and (Φ_2) in Theorem 3.2. Thus, by Theorem 3.2, we get a sequence of nontrivial critical points $\{u_k\}_{k\in\mathbb{N}}$ of Φ satisfying $\Phi(u_k) \leq 0$ for all $k \in \mathbb{N}$ and $u_k \to 0$ in E as $k \to \infty$. Taking into account Proposition 3.1 again and the fact that E is continuously embedding into $H^1(\mathbb{R}^N)$, we know that $\{u_k\}_{k\in\mathbb{N}}$ is a sequence of nontrivial solutions of (1.1) with $u_k \to 0$ in $H^1(\mathbb{R}^N)$ as $k \to \infty$. This ends the proof. □

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