# Multiple nontrivial solutions for a nonhomogeneous Schrödinger-Poisson system in $\mathbb{R}^{3}$ 

Sofiane Khoutir ${ }^{\boxtimes}$ and Haibo Chen<br>School of Mathematics and Statistics, Central South University, Changsha, Hunan 410083, PR China

Received 4 January 2017, appeared 2 May 2017
Communicated by Dimitri Mugnai


#### Abstract

In this paper, we study the following Schrödinger-Poisson system $$
\begin{cases}-\Delta u+V(x) u+\phi u=f(x, u)+g(x), & x \in \mathbb{R}^{3}, \\ -\Delta \phi=u^{2}, & x \in \mathbb{R}^{3} .\end{cases}
$$

Under appropriate assumptions on $V, f$ and $g$, using the Mountain Pass Theorem and the Ekeland's variational principle, we establish two existence theorems to ensure that the above system has at least two different solutions. Recent results from the literature are extended and improved.


Keywords: Schrödinger-Poisson system, Mountain Pass Theorem, Ekeland's variational principle, multiple solutions.
2010 Mathematics Subject Classification: 35J20, 35J60.

## 1 Introduction and main results

In this paper, we consider the following nonlinear Schrödinger-Poisson system

$$
\begin{cases}-\Delta u+V(x) u+\phi u=f(x, u)+g(x), & x \in \mathbb{R}^{3},  \tag{1.1}\\ -\Delta \phi=u^{2}, & x \in \mathbb{R}^{3},\end{cases}
$$

where $V \in C\left(\mathbb{R}^{3}, \mathbb{R}\right), f \in C\left(\mathbb{R}^{3} \times \mathbb{R}, \mathbb{R}\right)$ and the conditions on $g$ will be given later.
System (1.1) is also called Schrödinger-Maxwell system, arises in an interesting physical context. In fact, according to a classical model, the interaction of a charge particle with an electromagnetic field can be described by coupling the nonlinear Schrödinger's and Poisson's equations. For more information on the physical relevance of the Schrödinger-Poisson system, we refer the readers to the papers $[3,23]$ and the references therein.

If $g(x)=0$, system (1.1) becomes the well known Schrödinger-Poisson system, which has been extensively investigated in the last years by the aid of the modern variational methods and critical point theory. Moreover, since the pioneering work of Benci and Fortunato [5],

[^0]there is huge literature on the studies of the existence and behavior of solutions of the system (1.1) with $g(x)=0$, see for example $[1,2,7,9,10,14,16-18,21,24-26,30-32,34]$ and the references therein.

Compared to the homogeneous case (i.e., $g(x)=0$ ), there are few papers concerning the case where $g(x) \neq 0$, see for example [ $8,12,15,22,28,35$ ]. Particularly, in [8] the authors obtained the existence of two nontrivial solutions for system (1.1) by using Ekeland's variational principle and the Mountain Pass Theorem when $g \in L^{2}\left(\mathbb{R}^{3}\right), g \not \equiv 0$, and $f$ and $V$ satisfy the following assumptions, respectively:
$\left(V_{0}\right) V(x) \in C\left(\mathbb{R}^{3}, \mathbb{R}\right)$ satisfies $\inf _{x \in \mathbb{R}^{3}} V(x) \geq V_{0}>0$, where $V_{0}$ is a constant. Moreover, for every $M>0$, meas $\left\{x \in \mathbb{R}^{3}: V(x) \leq M\right\}<\infty$, where (and in the sequel) meas(•) denotes the Lebesgue measure in $\mathbb{R}^{3}$.
$\left(f_{1}\right) f \in C\left(\mathbb{R}^{3} \times \mathbb{R}\right)$, and there exist constants $a>0$ and $p \in(2,6)$ such that

$$
|f(x, u)| \leq a\left(1+|u|^{p-1}\right), \quad \forall(x, u) \in \mathbb{R}^{3} \times \mathbb{R},
$$

where $6=2^{*}=\frac{2 N}{N-2}$ is the critical Sobolev exponent;
( $f_{2}$ ) $\lim _{u \rightarrow 0} \frac{f(x, u)}{u}=0$ uniformly for $x \in \mathbb{R}^{3}$;
$\left(f_{3}\right)$ there exists $\mu>4$ such that

$$
\begin{equation*}
\mu F(x, u) \leq f(x, u) u, \quad \forall(x, u) \in \mathbb{R}^{3} \times \mathbb{R}, \tag{1.2}
\end{equation*}
$$

where (and in the sequel) $F(x, t)=\int_{0}^{t} f(x, s) d s$;
$\left(f_{4}\right)$

$$
\inf _{x \in \mathbb{R}^{3},|u|=1} F(x, u)>0 .
$$

Specifically, the authors established the following theorem in [8].
Theorem 1.1 ([8]). Suppose that $g \in L^{2}\left(\mathbb{R}^{3}\right), g \not \equiv 0$. Let $\left(V_{0}\right)$ and $\left(f_{1}\right)-\left(f_{4}\right)$ hold, then there exists a constant $m_{0}>0$ such that problem (1.1) admits at least two different solutions when $\|g\|_{L^{2}} \leq m_{0}$.

It is worth pointing out that the combination of $\left(f_{3}\right)-\left(f_{4}\right)$ implies that the rang of $p$ in condition $\left(f_{1}\right)$ should be $4<p<6$. In fact, for any $x \in \mathbb{R}^{3}, u \in \mathbb{R}$, define

$$
h(t)=F\left(x, t^{-1} u\right) t^{\mu}, \quad \forall t \in[1,+\infty) .
$$

Then, for $|u| \geq 1$ and $t \in[1,|u|]$, it follows from (1.2) that

$$
h^{\prime}(t)=\left[\mu F\left(x, t^{-1} u\right)-f\left(x, t^{-1} u\right) t^{-1} u\right] t^{\mu-1} \leq 0
$$

Therefore, $h(1) \geq h(|u|)$. Hence, $\left(f_{4}\right)$ implies that

$$
\begin{equation*}
F(x, u) \geq F\left(x, \frac{u}{|u|}\right)|u|^{\mu} \geq c|u|^{\mu}, \quad \forall x \in \mathbb{R}^{3} \text { and }|u| \geq 1, \tag{1.3}
\end{equation*}
$$

where, $c=\inf _{x \in \mathbb{R}^{3},|u|=1} F(x, u)>0$. If $p \leq 4$, by $\left(f_{1}\right)$ we have

$$
|F(x, u)| \leq \int_{0}^{1}|f(x, t u) u| d t \leq a \int_{0}^{1}\left(1+|t u|^{p-1}\right)|u| d t \leq a\left(|t|+|t|^{p}\right), \quad \forall(x, u) \in \mathbb{R}^{3} \times \mathbb{R}
$$

which implies that

$$
\limsup _{t \rightarrow+\infty} \frac{F(x, t)}{t^{4}} \leq a \quad \text { uniformly in } x \in \mathbb{R}^{3} .
$$

This contradicts (1.3). Thus, $4<p<6$.
Inspired by the above facts, in the present paper we shall consider the nonhomogeneous Schrödinger-Poisson system, and we are interested in looking for multiple solutions for the problem (1.1). Under much more relaxed assumptions on the nonlinearity $f$ and the potential function $V$, using some special proof techniques especially the verification of the boundedness of Palais-Smale sequence, new results on the existence of multiple nontrivial solutions for the system (1.1) are obtained, which extend and sharply improve some recent results in the literature. In order to state the main results of this paper, we make the following assumptions.
( $V$ ) $V \in C\left(\mathbb{R}^{3}, \mathbb{R}\right)$ satisfies $\inf _{x \in \mathbb{R}^{3}} V(x) \geq V_{0}>0$, where $V_{0}$ is a constant. Moreover, there exists $r_{0}>0$ such that

$$
\lim _{|y| \rightarrow \infty} \operatorname{meas}\left\{x \in \mathbb{R}^{3}:|x-y| \leq r_{0}, V(x) \leq M\right\}=0, \quad \forall M>0 .
$$

$\left(H_{1}\right) f \in C\left(\mathbb{R}^{3} \times \mathbb{R}, \mathbb{R}\right)$, and there exist constants $c_{1}, c_{2}>0$ and $p \in(4,6)$ such that

$$
|f(x, t)| \leq c_{1}|t|+c_{2}|t|^{p-1}, \quad \forall(x, t) \in \mathbb{R}^{3} \times \mathbb{R} .
$$

$\left(H_{2}\right) \lim _{t \rightarrow 0} \frac{f(x, t)}{t}<\mu^{*}$ uniformly for $x \in \mathbb{R}^{3}$ where

$$
\mu^{*}=\inf \left\{\int_{\mathbb{R}^{3}}\left(|\nabla u|^{2}+V(x) u^{2}\right) d x: u \in H^{1}\left(\mathbb{R}^{3}\right), \int_{\mathbb{R}^{3}} u^{2} d x=1\right\} .
$$

$\left(H_{3}\right) \lim _{t \rightarrow \infty} \frac{F(x, t)}{t^{4}}=\infty$ uniformly in $x \in \mathbb{R}^{3}$.
$\left(H_{4}\right)$ There exist $c_{3}>0$ and $L>0$ such that

$$
4 F(x, t) \leq f(x, t) t+c_{3} t^{2}, \quad \text { for a.e. } x \in \mathbb{R}^{3} \quad \text { and } \quad \forall|t| \geq L .
$$

$\left(H_{4}^{\prime}\right)$ There exists $L>0$ such that

$$
4 F(x, t) \leq f(x, t) t, \quad \text { for a.e. } x \in \mathbb{R}^{3} \quad \text { and } \quad \forall|t| \geq L
$$

$\left(H_{5}\right) g \in L^{p^{\prime}}\left(\mathbb{R}^{3}\right), g \not \equiv 0$, where $\frac{1}{p^{\prime}}+\frac{1}{p}=1, p$ is defined by $\left(H_{1}\right)$.
Now, we are ready to state the main results of this paper as follows.
Theorem 1.2. Assume that $(V)$ and $\left(H_{1}\right)-\left(H_{5}\right)$ hold. Then, there exists $m_{0}>0$ such that for any $g \in L^{p^{\prime}}\left(\mathbb{R}^{3}\right)$ with $\|g\|_{p^{\prime}} \leq m_{0}$, the system (1.1) possesses at least two different nontrivial solutions, one is negative energy solution, and the other is positive energy solution.

The other aim of this paper is to study the existence of at least two different nontrivial solutions for problem (1.1) involving a concave-convex nonlinearity. We also consider the effect of the parameter $\lambda$ and the perturbation term $g$ on the existence of solutions.

Theorem 1.3. Let $g \in L^{2}\left(\mathbb{R}^{3}\right), g \not \equiv 0$. Assume that $(V)$ and
$\left(H_{6}\right) f(x, u)=\lambda h_{1}(x)|u|^{\sigma-2} u+h_{2}(x)|u|^{p-2} u$ with $1<\sigma<2,4<p<6$ for all $(x, u) \in \mathbb{R}^{3} \times \mathbb{R}$, in which $h_{1} \in L^{\sigma_{0}}\left(\mathbb{R}^{3}\right) \cap L^{\infty}\left(\mathbb{R}^{3}\right)$ and $h_{2} \in L^{\infty}\left(\mathbb{R}^{3}\right)$ with $\sigma_{0}=2 /(2-\sigma)$. Moreover, there exists a nonempty bounded domain $\Omega \subset \mathbb{R}^{3}$ such that $h_{2}>0$ in $\Omega$.

Then there exist $\lambda_{0}, m_{0}>0$ such that for all $\lambda \in\left(0, \lambda_{0}\right)$, the system (1.1) possesses at least two different nontrivial solutions whenever $\|g\|_{2} \leq m_{0}$, one is negative energy solution, and the other is positive energy solution.

Obviously, the condition $\left(H_{4}^{\prime}\right)$ implies the condition $\left(H_{4}\right)$, so we have the following corollary.

Corollary 1.4. If we replace $\left(H_{4}\right)$ with $\left(H_{4}^{\prime}\right)$ in Theorem 1.2, then the conclusion of Theorem 1.2 remains valid.

Remark 1.5. Since the problem (1.1) is defined in the whole space $\mathbb{R}^{3}$, the main difficulty of this problem is the lack of compactness for Sobolev embedding theorem. To overcome this difficulty, the condition $(V)$, which was firstly introduced by Bartsch et al. [4], is always assumed to preserve the compactness of the embedding of the working space. Furthermore, condition $(V)$ is weaker than condition $\left(V_{0}\right)$, and there are functions $V(x)$ satisfying $(V)$ but not satisfying $\left(V_{0}\right)$, see for example Remark 2 in [33].
Remark 1.6.
(1) Theorem 1.2 sharply improves Theorem 1.1. If fact, from Remark 3. in [33], we know that the condition $\left(H_{1}\right)$ is much weaker than the combination of $\left(f_{1}\right)$ and $\left(f_{2}\right)$, and conditions $\left(H_{3}\right)-\left(H_{4}\right)$ are much weaker than $\left(f_{3}\right)-\left(f_{4}\right)$.
(2) The condition $\left(H_{2}\right)$ which gives the behaviour of $f(x, u) / u$ for $u$ near to the origin, is very essential for obtain the positive energy solution in Theorem 1.2. Moreover, it seems to be nearly optimal for obtain a such existence result.
(3) As a function $f$ satisfying the assumptions $\left(H_{1}\right)-\left(H_{4}\right)$, one can take

$$
f(x, u)= \begin{cases}u^{3}(4 \ln |u|+1), & |u| \geq 1 \\ -(2 v-1) u^{2}+2 v u, & |u| \leq 1\end{cases}
$$

where $0<v<\frac{\mu^{*}}{2}$ ( $\mu^{*}$ is given by $\left(H_{2}\right)$ ). A straightforward computation deduces that

$$
F(x, u)= \begin{cases}u^{4} \ln |u|+\frac{v+1}{3}, & |u| \geq 1 \\ -\frac{2 v-1}{3} u^{3}+v u^{2}, & |u| \leq 1\end{cases}
$$

and

$$
f(x, u) u-4 F(x, u)=u^{4}-\frac{4}{3}(v+1), \quad \forall x \in \mathbb{R}^{3}, \quad|u| \geq 1
$$

Hence, it is easy to check that $f$ satisfies the assumptions $\left(H_{1}\right)-\left(H_{4}\right)$. However, it does not satisfy the assumptions of Theorem 1.1. In fact, we have $\lim _{t \rightarrow 0} \frac{f(x, t)}{t}=2 v>0$ uniformly for $x \in \mathbb{R}^{3}$, which implies that $f$ does not satisfy the condition $\left(f_{2}\right)$. Moreover, for any $\mu>4$, we have

$$
f(x, u) u-\mu F(x, u)=-(\mu-4) u^{4} \ln |u|+u^{4}-\frac{\mu}{3}(v+1) \rightarrow-\infty, \quad \text { as }|u| \rightarrow \infty
$$

which shows that the condition $\left(f_{3}\right)$ is not satisfying for our choice.

Remark 1.7. The assumptions of Theorems 1.2 and 1.3 can be used to deal with the existence of nontrivial solutions for the following nonhomogeneous Kirchhoff-type equations

$$
\begin{cases}-\left(a+b \int_{\mathbb{R}^{3}}|\nabla u|^{2} d x\right) \Delta u+V(x) u=f(x, u)+g(x), & \text { in } \mathbb{R}^{3}, \\ u(x) \rightarrow 0 & \text { as }|x| \rightarrow \infty,\end{cases}
$$

where $a>0, b \geq 0$ are constants. So, the conclusions of Theorems 1.2 and 1.3 still hold for the above problem.

The paper is organized as follows. In Section 2, we present some preliminary results. Section 3 is devoted to the proof of Theorems 1.2 and 1.3.

## 2 Preliminaries

In the following, we will introduce the variational setting for Problem (1.1). In the sequel, we denote by $\|\cdot\|_{p}$ the usual norm of the space $L^{p}\left(\mathbb{R}^{3}\right), c_{i}, C_{i}$ or $C$ stand for different positive constants.

As usual, for $1 \leq p<+\infty$, we let

$$
\|u\|_{p}:=\left(\int_{\mathbb{R}^{3}}|u|^{p} d x\right)^{\frac{1}{p}}, \quad u \in L^{p}\left(\mathbb{R}^{3}\right)
$$

and

$$
\|u\|_{\infty}:=\underset{x \in \mathbb{R}^{3}}{\operatorname{ess} \sup }|u(x)|, \quad u \in L^{\infty}\left(\mathbb{R}^{3}\right) .
$$

Let

$$
H^{1}\left(\mathbb{R}^{3}\right)=\left\{u \in L^{2}\left(\mathbb{R}^{3}\right): \nabla u \in L^{2}\left(\mathbb{R}^{3}\right)\right\},
$$

with the inner product and norm

$$
\langle u, v\rangle_{H^{1}}=\int_{\mathbb{R}^{3}}(\nabla u \nabla v+u v) d x, \quad\|u\|_{H^{1}}=\langle u, u\rangle_{H^{1}}^{\frac{1}{2}} .
$$

Define our working space

$$
E=\left\{u \in H^{1}\left(\mathbb{R}^{3}\right): \int_{\mathbb{R}^{3}} V(x)|u|^{2} d x<+\infty\right\} .
$$

Then $E$ is a Hilbert space equipped with the inner product and norm

$$
\langle u, v\rangle=\int_{\mathbb{R}^{3}}(\nabla u \nabla v+V(x) u v) d x, \quad\|u\|=\langle u, u\rangle^{\frac{1}{2}} .
$$

Let $\mathcal{D}^{1,2}\left(\mathbb{R}^{3}\right)$ be the completion of $C_{0}^{\infty}\left(\mathbb{R}^{3}\right)$ with respect to the norm

$$
\|u\|_{\mathcal{D}^{1,2}}^{2}=\int_{\mathbb{R}^{3}}|\nabla u|^{2} d x .
$$

Then, the embedding $\mathcal{D}^{1,2}\left(\mathbb{R}^{3}\right) \hookrightarrow L^{6}\left(\mathbb{R}^{3}\right)$ is continuous (see for instance [29]). Since the embedding $H^{1}\left(\mathbb{R}^{3}\right) \hookrightarrow L^{s}\left(\mathbb{R}^{3}\right)(2 \leq s \leq 6)$ is continuous, then the embedding $E \hookrightarrow L^{s}\left(\mathbb{R}^{3}\right)$ $(2 \leq s \leq 6)$ is continuous under the condition $(V)$, that is, there exist $\eta_{s}>0$ such that

$$
\begin{equation*}
\|u\|_{s} \leq \eta_{s}\|u\|, \quad \forall u \in E, \quad s \in[2,6] . \tag{2.1}
\end{equation*}
$$

Moreover, we have the following compactness results from [4, Lemma 3.1.].

Lemma 2.1 ([4]). Under the assumption $(V)$, the embedding $E \hookrightarrow L^{s}\left(\mathbb{R}^{N}\right)$ is compact for $s \in[2,6)$.
Recall that $\mu \in \mathbb{R}$ is called an eigenvalue of the operator $-\Delta+V(x)$ provided there exists a nontrivial weak solution $u_{0}$ of the equation:

$$
-\Delta u+V(x) u=\mu u, \quad x \in \mathbb{R}^{3}
$$

i.e., for any $\varphi \in E$,

$$
\int_{\mathbb{R}^{3}}\left(\nabla u_{0} \nabla \varphi+V(x) u_{0} \varphi\right) d x=\mu \int_{\mathbb{R}^{3}} u_{0} \varphi d x
$$

Lemma 2.2. Assume that $(V)$ holds. Then $\mu^{*}$ is an eigenvalue of the operator $-\Delta+V(x)$ and there exists a corresponding eigenfunction $\varphi_{1}$ with $\varphi_{1}>0$ for all $x \in \mathbb{R}^{3}$.

Proof. The proof of this lemma is almost the same to the one of Lemma 2.3 in [13]. So we omit it here.

For every $u \in H^{1}\left(\mathbb{R}^{3}\right)$, by the Lax-Milgram theorem, we know that there exists a unique $\phi_{u} \in \mathcal{D}^{1,2}\left(\mathbb{R}^{3}\right)$ such that

$$
\begin{equation*}
-\Delta \phi_{u}=u^{2}, \quad \text { in } \mathbb{R}^{3} \tag{2.2}
\end{equation*}
$$

Furthermore, $\phi_{u}$ has the following integral expression

$$
\begin{equation*}
\phi_{u}(x)=\frac{1}{4 \pi} \int_{\mathbb{R}^{3}} \frac{u^{2}(y)}{|x-y|} d y \geq 0 \tag{2.3}
\end{equation*}
$$

From (2.1), for any $u \in E$, using the Hölder inequality we obtain

$$
\begin{equation*}
\left\|\phi_{u}\right\|_{\mathcal{D}^{1,2}}^{2}=\int_{\mathbb{R}^{3}} \phi_{u} u^{2} d x \leq\left\|\phi_{u}\right\|_{6}\|u\|_{12 / 5}^{2} \leq C\left\|\phi_{u}\right\|_{\mathcal{D}^{1,2}}\|u\|_{12 / 5}^{2} \tag{2.4}
\end{equation*}
$$

Therefore

$$
\begin{equation*}
\left\|\phi_{u}\right\|_{\mathcal{D}^{1,2}} \leq C\|u\|_{12 / 5}^{2} \tag{2.5}
\end{equation*}
$$

By (2.4), (2.5) and the Sobolev inequality, we obtain

$$
\begin{equation*}
\frac{1}{4 \pi} \iint_{\mathbb{R}^{3} \times \mathbb{R}^{3}} \frac{u^{2}(x) u^{2}(y)}{|x-y|} d y d x=\int_{\mathbb{R}^{3}} \phi_{u} u^{2} d x \leq C_{1}\|u\|^{4} \tag{2.6}
\end{equation*}
$$

Moreover, $\phi_{u}$ has the following properties (for a proof, see $[6,21]$ ).
Lemma 2.3. For $u \in E$ we have
(i) $\phi_{t u}=t^{2} \phi_{u}$, for all $t \geq 0$;
(ii) If $u_{n} \rightharpoonup u$ in $E$, then $\phi_{u_{n}} \rightharpoonup \phi_{u}$ in $\mathcal{D}^{1,2}\left(\mathbb{R}^{3}\right)$ and

$$
\lim _{n \rightarrow \infty} \int_{\mathbb{R}^{3}} \phi_{u_{n}} u_{n}^{2} d x=\int_{\mathbb{R}^{3}} \phi_{u} u^{2} d x
$$

Now, we define the energy functional $J: E \rightarrow \mathbb{R}$ associated with problem (1.1) by

$$
\begin{equation*}
J(u)=\frac{1}{2} \int_{\mathbb{R}^{3}}\left(|\nabla u|^{2}+V(x)|u|^{2}\right) d x+\frac{1}{4} \int_{\mathbb{R}^{3}} \phi_{u} u^{2} d x-\int_{\mathbb{R}^{3}} F(x, u) d x-\int_{\mathbb{R}^{3}} g(x) u d x . \tag{2.7}
\end{equation*}
$$

Therefore, combining (2.5), (2.6), $\left(H_{1}\right)-\left(H_{2}\right)$ and Lemma 2.1, $J$ is well defined and $J \in C^{1}(E, \mathbb{R})$ with

$$
\begin{align*}
\left\langle J^{\prime}(u), v\right\rangle= & \int_{\mathbb{R}^{3}}(\nabla u \nabla v+V(x) u v) d x+\int_{\mathbb{R}^{3}} \phi_{u} u v d x  \tag{2.8}\\
& -\int_{\mathbb{R}^{3}} f(x, u) v d x-\int_{\mathbb{R}^{3}} g(x) v d x, \quad \forall v \in E .
\end{align*}
$$

Moreover, if $u \in E$ is a critical point of $J$, then the pair $\left(u, \phi_{u}\right)$ is a solution of system (1.1).
Recall that a sequence $\left\{u_{n}\right\} \subset E$ is said to be a Palais-Smale sequence at the level $c \in \mathbb{R}$ $\left((P S)_{c}\right.$-sequence for short) if $J\left(u_{n}\right) \rightarrow c$ and $J^{\prime}\left(u_{n}\right) \rightarrow 0, J$ is said to satisfy the Palais-Smale condition at the level $c\left((P S)_{c}\right.$-condition for short) if any $(P S)_{c}$-sequence has a convergent subsequence.

In order to prove the existence of positive energy solution for problem (1.1), we shall use the following Mountain Pass Theorem (cf. [20,29]).

Proposition $2.4([20,29])$. Let $E$ be a Banach space, $J \in C^{1}(E, \mathbb{R})$ satisfies the $(P S)$-condition for any $c>0, J(0)=0$, and
(i) there exist $\rho, \alpha>0$ such that $\left.J\right|_{\partial B_{\rho}} \geq \alpha$;
(ii) there exists $e \in E \backslash B_{\rho}$ such that $J(e) \leq 0$.

Then J has at least a critical value $c \geq \alpha$.
On the other hand, the following Ekeland's variational principle is the main tool to obtain the negative energy solution for problem (1.1)

Proposition 2.5 ([19, Theorem 4.1]). Let $M$ be a complete metric space with metric $d$ and let $J$ : $M \mapsto(-\infty,+\infty]$ be a lower semicontinuous function, bounded from below and not identical to $+\infty$. Let $\varepsilon>0$ be given and $u \in M$ be such that

$$
J(u) \leq \inf _{M} J+\varepsilon .
$$

Then, there exists $v \in M$ such that

$$
J(v) \leq J(u), \quad d(u, v) \leq 1,
$$

and for each $w \in M$, one has

$$
J(v) \leq J(w)+\varepsilon d(v, w)
$$

We also need the following auxiliary result, see [27].
Lemma 2.6. Assume that $p_{1}, p_{2}>1, r, q \geq 1$ and $\Omega \subseteq \mathbb{R}^{N}$. Let $f(x, t)$ be a Carathéodory function on $\Omega \times \mathbb{R}$ satisfying

$$
|f(x, t)| \leq a_{1}|t|^{\left(p_{1}-1\right) / r}+a_{2}|t|^{\left(p_{2}-1\right) / r}, \quad \forall(x, t) \in \Omega \times \mathbb{R},
$$

where, $a_{1}, a_{2} \geq 0$. If $u_{n} \rightarrow u_{0}$ in $L^{p_{1}}(\Omega) \cap L^{p_{2}}(\Omega)$, and $u_{n} \rightarrow u_{0}$ a.e. $x \in \Omega$, then for any $v \in L^{p_{1} q}(\Omega) \cap L^{p_{2} q}(\Omega)$,

$$
\lim _{n \rightarrow \infty} \int_{\Omega}\left|f\left(x, u_{n}\right)-f\left(x, u_{0}\right)\right|^{r}|v|^{q} d x \rightarrow 0
$$

## 3 Proof of main results

In this section we shall prove Theorems 1.2 and 1.3. We first prove some lemmas, which are crucial to prove our main results.

Lemma 3.1. Assume that the assumptions $(V),\left(H_{1}\right),\left(H_{2}\right)$ and $\left(H_{5}\right)$ hold. Then, there exist $\rho, \alpha$ and $m_{0}>0$ such that $J(u) \geq \alpha$ whenever $\|u\|=\rho$ and $\|g\|_{p^{\prime}}<m_{0}$.
Proof. By $\left(H_{1}\right)$ and $\left(H_{2}\right)$, there exist $\varepsilon_{0}>0$ and $c_{4}>0$ such that

$$
\begin{equation*}
F(x, u) \leq \frac{\mu^{*}-\varepsilon_{0}}{2}|u|^{2}+c_{4}|u|^{p}, \quad \forall(x, u) \in \mathbb{R}^{3} \times \mathbb{R} \tag{3.1}
\end{equation*}
$$

Combining (2.1), (2.3), (2.7) and (3.1), we have

$$
\begin{align*}
J(u) & =\frac{1}{2}\|u\|^{2}+\frac{1}{4} \int_{\mathbb{R}^{3}} \phi_{u} u^{2} d x-\int_{\mathbb{R}^{3}} F(x, u) d x-\int_{\mathbb{R}^{3}} g(x) u d x \\
& \geq \frac{1}{2}\|u\|^{2}-\int_{\mathbb{R}^{3}} F(x, u) d x-\int_{\mathbb{R}^{3}} g(x) u d x \\
& \geq \frac{1}{2}\|u\|^{2}-\frac{\mu^{*}-\varepsilon_{0}}{2} \int_{\mathbb{R}^{3}}|u|^{2}-c_{4} \int_{\mathbb{R}^{3}}|u|^{p} d x-\|g\|_{p^{\prime}}\|u\|_{p}  \tag{3.2}\\
& \geq \frac{\varepsilon_{0}}{2 \mu^{*}}\|u\|^{2}-c_{4} \eta_{p}^{p}\|u\|^{p}-\eta_{p}\|g\|_{p^{\prime}}\|u\| .
\end{align*}
$$

Taking

$$
\rho=\left[\frac{\varepsilon_{0}}{4 \mu^{*}\left(c_{4} \eta_{p}^{p}+\eta_{p}\right)}\right]^{\frac{1}{p-2}},
$$

$m_{0}=\rho^{p-1}$ in (3.2), we then get

$$
J(u) \geq \frac{\varepsilon_{0}}{4 \mu^{*}} \rho^{2}=\alpha>0, \quad \forall\|u\|=\rho
$$

The proof is completed.
Lemma 3.2. Assume that the assumptions $(V),\left(H_{1}\right),\left(H_{3}\right)$ and $\left(H_{5}\right)$ hold. Then there exists $e \in E$ with $\|e\|>\rho$ such that $J(e) \leq 0$, where $\rho$ is given in Lemma 3.1.

Proof. By $\left(H_{1}\right)$ and $\left(H_{3}\right)$ we have, for any $M>0$, there exists $C_{M}>0$ such that

$$
\begin{equation*}
F(x, u) \geq M|u|^{4}-C_{M}|u|^{2}, \quad \forall(x, u) \in \mathbb{R}^{3} \times \mathbb{R} . \tag{3.3}
\end{equation*}
$$

Consequently, it follows from (2.6), (2.7) and (3.3) that

$$
\begin{align*}
J\left(t \varphi_{1}\right)= & \frac{t^{2}}{2}\left\|\varphi_{1}\right\|^{2}+\frac{1}{4} \int_{\mathbb{R}^{3}} \phi_{t \varphi_{1}}\left(t \varphi_{1}\right)^{2} d x-\int_{\mathbb{R}^{3}} F\left(x, t \varphi_{1}\right) d x-t \int_{\mathbb{R}^{3}} g(x) \varphi_{1}(x) d x \\
\leq & \frac{t^{2}}{2}\left\|\varphi_{1}\right\|^{2}+\frac{t^{4}}{4} C_{1}\left\|\varphi_{1}\right\|^{4}-t^{4} M \int_{\mathbb{R}^{3}}\left|\varphi_{1}\right|^{4} d x  \tag{3.4}\\
& +t^{2} C_{M} \int_{\mathbb{R}^{3}}\left|\varphi_{1}\right|^{2} d x-t \int_{\mathbb{R}^{3}} g(x) \varphi_{1}(x) d x \\
\leq & \frac{t^{2}}{2}\left(1+2 C_{M}\right)\left\|\varphi_{1}\right\|^{2}-\frac{t^{4}}{4}\left(4 M\left\|\varphi_{1}\right\|_{4}^{4}-C_{1}\left\|\varphi_{1}\right\|^{4}\right)-t \int_{\mathbb{R}^{3}} g(x) \varphi_{1}(x) d x .
\end{align*}
$$

Therefore, choosing $M>0$ such that $4 M\left\|\varphi_{1}\right\|_{4}^{4}-C_{1}\left\|\varphi_{1}\right\|^{4}>0$, then, it follows from (3.4) that $J\left(t \varphi_{1}\right) \rightarrow-\infty$ as $t \rightarrow+\infty$. Hence, there exists $t_{1}>0$ so large that $\left\|t_{1} \varphi_{1}\right\|>\rho$ and $J\left(t_{1} \varphi_{1}\right)<0$. Thus, the lemma is proved by taking $e=t_{1} \varphi_{1}$.

Lemma 3.3. Assume that $(V),\left(H_{1}\right)-\left(H_{5}\right)$ hold. Then $J$ satisfies the $(P S)$-condition on $E$.
Proof. Let $\left\{u_{n}\right\} \subset E$ be such that

$$
\begin{equation*}
J\left(u_{n}\right) \rightarrow c \quad \text { and } \quad J^{\prime}\left(u_{n}\right) \rightarrow 0 \tag{3.5}
\end{equation*}
$$

We first show that $\left\{u_{n}\right\}$ is bounded in $E$. Otherwise, set $v_{n}=\frac{u_{n}}{\left\|u_{n}\right\|}$, then $\left\|v_{n}\right\|=1$ and $\left\|v_{n}\right\|_{p} \leq \eta_{p}\left\|v_{n}\right\|=\eta_{p}$ (see 2.1). It follows from $\left(H_{1}\right)$ that

$$
\begin{align*}
|F(x, u)| & =|F(x, u)-F(x, 0)| \\
& =\left|\int_{0}^{1} f(x, t u) u d t\right| \\
& \leq \int_{0}^{1}\left(c_{1}|u|^{2} t+c_{2}|u|^{p} t^{p-1}\right) d t  \tag{3.6}\\
& =\frac{c_{1}}{2}|u|^{2}+\frac{c_{2}}{p}|u|^{p}, \quad \forall(x, u) \in \mathbb{R}^{3} \times \mathbb{R}
\end{align*}
$$

Let $\mathcal{F}\left(x, u_{n}\right)=f\left(x, u_{n}\right) u_{n}-4 F\left(x, u_{n}\right)$. Therefore, for $x \in \mathbb{R}^{3}$ and $|u(x)|<L$, by (3.6), we have

$$
\begin{aligned}
|f(x, u) u-4 F(x, u)| & \leq|f(x, u) u|+4|F(x, u)| \\
& \leq\left(c_{1}|u|^{2}+c_{2}|u|^{p}\right)+\left(2 c_{1}|u|^{2}+\frac{4 c_{2}}{p}|u|^{p}\right) \\
& \leq\left(3 c_{1}+\frac{4+p}{p} c_{2} L^{p-2}\right)|u|^{2} \\
& =c_{7}|u|^{2}
\end{aligned}
$$

where $L>0$ is given by $\left(H_{4}\right)$. Combining the above inequality with $\left(H_{4}\right)$, we conclude that there exists $c_{8}>0$ such that

$$
\begin{equation*}
f(x, u) u-4 F(x, u) \geq-c_{8}|u|^{2}, \quad \forall(x, u) \in \mathbb{R}^{3} \times \mathbb{R} \tag{3.7}
\end{equation*}
$$

By $\left(\mathrm{H}_{5}\right),(2.7),(2.8),(3.5),(3.7)$ and the Hölder inequality, without loss of generality, we may assume that for all $n \in \mathbb{N}$, we have

$$
\begin{aligned}
1+c+\left\|u_{n}\right\| & \geq J\left(u_{n}\right)-\frac{1}{4}\left\langle J^{\prime}\left(u_{n}\right), u_{n}\right\rangle \\
& =\frac{1}{4}\left\|u_{n}\right\|^{2}+\frac{1}{4} \int_{\mathbb{R}^{3}} \mathcal{F}\left(x, u_{n}\right) d x-\frac{3}{4} \int_{\mathbb{R}^{3}} g(x) u_{n} d x \\
& \geq \frac{1}{4}\left\|u_{n}\right\|^{2}-\frac{c_{8}}{4} \int_{\mathbb{R}^{3}}\left|u_{n}\right|^{2} d x-\frac{3}{4}\|g\|_{p^{\prime}}\left\|u_{n}\right\|_{p} \\
& \geq \frac{1}{4}\left\|u_{n}\right\|^{2}-\frac{c_{8}}{4}\left\|u_{n}\right\|_{2}^{2}-\frac{3}{4} \eta_{p}\|g\|_{p^{\prime}}\left\|u_{n}\right\|
\end{aligned}
$$

which implies that

$$
\frac{\left\|u_{n}\right\|_{2}^{2}}{\left\|u_{n}\right\|^{2}} \geq \frac{1}{c_{8}}-\frac{1}{c_{8}}\left[\frac{4(c+1)}{\left\|u_{n}\right\|^{2}}+\frac{3 \eta_{p}\|g\|_{p^{\prime}}}{\left\|u_{n}\right\|}\right]
$$

Therefore, for sufficiently large $n$ such that $\frac{4(c+1)}{\left\|u_{n}\right\|^{2}}+\frac{3 \eta_{p}\|g\|_{p^{\prime}}}{\left\|u_{n}\right\|} \leq \frac{1}{2}$, we then get

$$
\frac{\left\|u_{n}\right\|_{2}^{2}}{\left\|u_{n}\right\|^{2}} \geq \frac{1}{2 c_{8}}>0
$$

Consequently, we conclude that

$$
\begin{equation*}
\left\|v_{n}\right\|_{2}>0 . \tag{3.8}
\end{equation*}
$$

Let $\Omega_{n}=\left\{x \in \mathbb{R}^{3}:\left|u_{n}(x)\right| \leq L\right\}$ and $A_{n}=\left\{x \in \mathbb{R}^{3}: v_{n}(x) \neq 0\right\}$, then meas $\left(A_{n}\right)>0$. Moreover, since $\left\|u_{n}\right\| \rightarrow \infty$ as $n \rightarrow \infty$, we obtain

$$
\left|u_{n}(x)\right| \rightarrow \infty \quad \text { as } n \rightarrow \infty \quad \text { for } x \in A_{n} .
$$

Hence, $A_{n} \subseteq \mathbb{R}^{3} \backslash \Omega_{n}$ for $n \in \mathbb{N}$ large enough. It follows from $\left(H_{5}\right)$ and the Hölder inequality that for any $\beta \in(1,6)$, one has

$$
\begin{equation*}
\left|\int_{\mathbb{R}^{3}} \frac{g(x) u_{n}}{\left\|u_{n}\right\|^{\beta}} d x\right| \leq \frac{\|g\|_{p^{\prime}}\left\|u_{n}\right\|_{p}}{\left\|u_{n}\right\|^{\beta}} \leq \eta_{p} \frac{\|g\|_{p^{\prime}}}{\left\|u_{n}\right\|^{\beta-1}} \rightarrow 0 \tag{3.9}
\end{equation*}
$$

since $\left\|u_{n}\right\| \rightarrow \infty$ as $n \rightarrow \infty$. By ( $H_{1}$ ), ( $H_{3}$ ), (2.1), (2.6), (3.5), (3.6), (3.8), (3.9) and Fatou's lemma, we have

$$
\begin{align*}
0 & =\lim _{n \rightarrow \infty} \frac{J\left(u_{n}\right)}{\left\|u_{n}\right\|^{4}} \\
& =\lim _{n \rightarrow \infty}\left[\frac{1}{2\left\|u_{n}\right\|^{2}}+\frac{1}{4\left\|u_{n}\right\|^{4}} \int_{\mathbb{R}^{3}} \phi_{u_{n}} u_{n}^{2} d x-\int_{\mathbb{R}^{3}} \frac{F\left(x, u_{n}\right)}{\left\|u_{n}\right\|^{4}} d x-\int_{\mathbb{R}^{3}} \frac{g(x) u_{n}}{\left\|u_{n}\right\|^{4}} d x\right] \\
& \leq C_{1}-\lim _{n \rightarrow \infty}\left[\int_{\Omega_{n}} \frac{F\left(x, u_{n}\right)}{u_{n}^{4}} v_{n}^{4} d x+\int_{\mathbb{R}^{3} \backslash \Omega_{n}} \frac{F\left(x, u_{n}\right)}{u_{n}^{4}} v_{n}^{4} d x\right] \\
& \leq C_{1}-\lim _{n \rightarrow \infty}\left[\frac{1}{\left\|u_{n}\right\|^{2}}\left(\frac{c_{1}}{2}+\frac{c_{2}}{p} L^{p-2}\right) \eta_{2}^{2}+\int_{\mathbb{R}^{3} \backslash \Omega_{n}} \frac{F\left(x, u_{n}\right)}{u_{n}^{4}} v_{n}^{4} d x\right]  \tag{3.10}\\
& \leq C_{1}-\liminf _{n \rightarrow \infty} \int_{\mathbb{R}^{3} \backslash \Omega_{n}} \frac{F\left(x, u_{n}\right)}{u_{n}^{4}} v_{n}^{4} d x \\
& \leq C_{1}-\int_{A_{n}} \liminf _{n \rightarrow \infty} \frac{F\left(x, u_{n}\right)}{u_{n}^{4}} v_{n}^{4} d x \\
& =C_{1}-\int_{\mathbb{R}^{3}} \liminf _{n \rightarrow \infty} \frac{F\left(x, u_{n}\right)}{u_{n}^{4}}\left[\chi_{A_{n}}(x)\right] v_{n}^{4} d x \\
& \rightarrow-\infty, \quad \text { as } n \rightarrow \infty .
\end{align*}
$$

This is an obvious contradiction. Hence $\left\{u_{n}\right\} \subset E$ is bounded. So, up to a subsequence we may assume that $u_{n} \rightharpoonup u_{0}$ weakly in $E$. By Lemma $2.1, u_{n} \rightarrow u_{0}$ strongly in $L^{s}\left(\mathbb{R}^{3}\right)$ for $2 \leq s<6$ and $u_{n}(x) \rightarrow u_{0}(x)$ a.e. on $\mathbb{R}^{3}$. It follows from (2.7) and (2.8) that

$$
\begin{align*}
\left\|u_{n}-u_{0}\right\|^{2}= & \left\langle J^{\prime}\left(u_{n}\right)-J^{\prime}\left(u_{0}\right), u_{n}-u_{0}\right\rangle+\int_{\mathbb{R}^{3}}\left[f\left(x, u_{n}\right)-f\left(x, u_{0}\right)\right]\left(u_{n}-u_{0}\right) d x  \tag{3.11}\\
& -\int_{\mathbb{R}^{3}}\left(\phi_{u_{n}} u_{n}-\phi_{u_{0}} u_{0}\right)\left(u_{n}-u_{0}\right) d x .
\end{align*}
$$

Obviously, $\left\langle J^{\prime}\left(u_{n}\right)-J^{\prime}\left(u_{0}\right), u_{n}-u_{0}\right\rangle \rightarrow 0$ as $n \rightarrow \infty$. Let us take $r=q=1$ in Lemma 2.6 and combine with $u_{n} \rightarrow u_{0}$ strongly in $L^{s}\left(\mathbb{R}^{3}\right)$ for $2 \leq s<6$, to get

$$
\begin{equation*}
\int_{\mathbb{R}^{3}}\left[f\left(x, u_{n}\right)-f\left(x, u_{0}\right)\right]\left(u_{n}-u_{0}\right) d x \rightarrow 0 . \tag{3.12}
\end{equation*}
$$

Furthermore, from Lemma 2.3 (ii), we have that $\int_{\mathbb{R}^{3}}\left(\phi_{u_{n}} u_{n}-\phi_{u_{0}} u_{0}\right)\left(u_{n}-u_{0}\right) d x \rightarrow 0$. Consequently, $u_{n} \rightarrow u_{0}$ in $E$. This completes the proof.

Proof of Theorem 1.2. The proof is divided in two steps, the first one for the negative energy solution, the second one for the positive energy solution.
Step 1. By using Ekeland's variational principle, we first show that there exists a function $u_{0} \in E$ such that $J^{\prime}\left(u_{0}\right)=0$ and $J\left(u_{0}\right)<0$. By (3.3) fixing $M>0$ a constant $C_{M}>0$ exists such that

$$
F(x, u) \geq M|u|^{4}-C_{M}|u|^{2}, \quad \forall(x, u) \in \mathbb{R}^{3} \times \mathbb{R} .
$$

Since $g \in L^{p^{\prime}}\left(\mathbb{R}^{3}\right)$ and $g \not \equiv 0$, we may choose a function $v \in E$ such that

$$
\int_{\mathbb{R}^{3}} g(x) v(x) d x>0 .
$$

Therefore,

$$
\begin{aligned}
J(t v) & =\frac{t^{2}}{2}\|v\|^{2}+\frac{t^{4}}{4} \int_{\mathbb{R}^{3}} \phi_{v} v^{2} d x-\int_{\mathbb{R}^{3}} F(x, t v) d x-t \int_{\mathbb{R}^{3}} g(x) v(x) d x \\
& \leq \frac{t^{2}}{2}\|v\|^{2}+C_{1} \frac{t^{4}}{4}\|v\|^{4}-M t^{4}\|v\|_{4}^{4}+C_{M} t^{2}\|v\|_{2}^{2}-t \int_{\mathbb{R}^{3}} g(x) v(x) d x<0,
\end{aligned}
$$

for $t>0$ small enough, which implies that

$$
\inf \left\{J(u): u \in \bar{B}_{\rho}\right\}<0,
$$

where $\rho>0$ is given by Lemma 3.1, and $\bar{B}_{\rho}=\{u \in E:\|u\| \leq \rho\}$. On the other hand, by (3.2), one has

$$
\begin{aligned}
J(u) & \geq \frac{\varepsilon_{0}}{2 \mu^{*}}\|u\|^{2}-c_{4} \eta_{p}^{p}\|u\|^{p}-\eta_{p}\|g\|_{p^{\prime}}\|u\| \\
& \geq-c_{4} \eta_{p}^{p}\|u\|^{p}-\eta_{p}\|g\|_{p^{\prime}}\|u\|,
\end{aligned}
$$

which implies that $J$ is bounded below in $\bar{B}_{\rho}$. Thus, we obtain

$$
-\infty<c_{0}=\inf \left\{J(u): u \in \bar{B}_{\rho}\right\}<0
$$

By Ekeland's variational principle, there exists a sequence $\left\{u_{n}\right\} \subset \bar{B}_{\rho}$ such that

$$
c_{0} \leq J\left(u_{n}\right) \leq c_{0}+\frac{1}{n},
$$

and

$$
J\left(u_{n}\right) \leq J(w)+\frac{1}{n}\left\|u_{n}-w\right\|, \quad \forall w \in \bar{B}_{\rho} .
$$

Then, following the idea of [11] (see pp. 534-535), we can show that $\left\{u_{n}\right\}$ is a bounded PalaisSmale sequence of $J$. Therefore, by Lemma 3.3, $\left\{u_{n}\right\}$ has a strongly convergent subsequence, still denoted by $\left\{u_{n}\right\}$ and $u_{n} \rightarrow u_{0} \in \bar{B}_{\rho}$ as $n \rightarrow \infty$. Hence, we conclude that there exists $u_{0} \in E$ such that $J\left(u_{0}\right)=\inf _{u \in \bar{B}_{\rho}} J(u)=c_{0}<0$ and $J^{\prime}\left(u_{0}\right)=0$, this completes the Step 1 .
Step 2. Now, we show that there exists a function $\bar{u}_{0} \in E$ such that $J\left(\bar{u}_{0}\right)=\bar{c}_{0}>0$ and $J^{\prime}\left(\bar{u}_{0}\right)=0$ by means of the Mountain Pass Theorem. Obviously, $J \in C^{1}(E, \mathbb{R})$ and $J(0)=0$. By Lemmas 3.1 and 3.2, the functional $J$ satisfies the geometric property of the mountain pass theorem whenever $\|g\|_{p^{\prime}} \leq m_{0}$. Lemma 3.3 implies that $J$ satisfies the (PS)-condition. Therefore, applying Proposition 2.4, we deduce that there exists $\bar{u}_{0} \in E$ such that $J\left(\bar{u}_{0}\right)=\bar{c}_{0} \geq$ $\alpha>0$ and $J^{\prime}\left(\bar{u}_{0}\right)=0$, we complete the Step 2.

Therefore, by the above two steps the proof of Theorem 1.2 is completed.

Next, we will give the proof of Theorem 1.3. Under the assumption $\left(H_{6}\right)$, we can easily find that the energy functional associated to problem (1.1)

$$
\begin{equation*}
J(u)=\frac{1}{2}\|u\|^{2}+\frac{1}{4} \int_{\mathbb{R}^{3}} \phi_{u} u^{2} d x-\frac{\lambda}{\sigma} \int_{\mathbb{R}^{3}} h_{1}(x)|u|^{\sigma} d x-\frac{1}{p} \int_{\mathbb{R}^{3}} h_{2}(x)|u|^{p} d x-\int_{\mathbb{R}^{3}} g(x) u d x, \tag{3.13}
\end{equation*}
$$

is of class $C^{1}$ on $E$ and for any $v \in E$, we have

$$
\begin{align*}
\left\langle J^{\prime}(u), v\right\rangle= & \int_{\mathbb{R}^{3}}(\nabla u \nabla v+V(x) u v) d x+\int_{\mathbb{R}^{3}} \phi_{u} u v d x-\lambda \int_{\mathbb{R}^{3}} h_{1}(x)|u|^{\sigma-2} u v d x  \tag{3.14}\\
& -\int_{\mathbb{R}^{3}} h_{2}(x)|u|^{p-2} u v d x-\int_{\mathbb{R}^{3}} g(x) v d x .
\end{align*}
$$

Lemma 3.4. Suppose that the assumptions $(V)$ and $\left(H_{6}\right)$ are satisfied. Then, there exist $\rho, \alpha$ and $m_{0}>0$ such that $J(u) \geq \alpha$ whenever $\|u\|=\rho$ and $\|g\|_{2}<m_{0}$.

Proof. By the Hölder inequality, we have

$$
\int_{\mathbb{R}^{3}}\left|h_{1}(x)\left\|\left.u\right|^{\sigma} d x \leq\right\| h_{1}\left\|_{\sigma_{0}}\right\| u\left\|_{2}^{\sigma} \leq V_{0}^{-\frac{\sigma}{2}}\right\| h_{1}\left\|_{\sigma_{0}}\right\| u \|^{\sigma}\right.
$$

where $\sigma_{0}=2 /(2-\sigma)$. On the other hand, by (2.1), we have

$$
\int_{\mathbb{R}^{3}}\left|h_{2}(x)\left\|\left.u\right|^{p} d x \leq\right\| h_{2}\left\|_{\infty}\right\| u\left\|_{p}^{p} \leq \eta_{p}^{p}\right\| h_{2}\left\|_{\infty}\right\| u \|^{p} .\right.
$$

Similarly, we have by Young's inequality,

$$
\int_{\mathbb{R}^{3}}\left|g(x)\|u \mid d x \leq\| g\left\|_{2}\right\| u\left\|_{2} \leq V_{0}^{-\frac{1}{2}}\right\| g\left\|_{2}\right\| u\left\|\leq \frac{1}{4}\right\| u\left\|^{2}+\frac{1}{V_{0}}\right\| g \|_{2}^{2}\right.
$$

Therefore, it follows from (2.3) and (3.13) that

$$
\begin{equation*}
J(u) \geq \frac{1}{4}\|u\|^{2}-\lambda \beta_{1}\|u\|^{\sigma}-\beta_{2}\|u\|^{p}-V_{0}^{-1}\|g\|_{2}^{2} \tag{3.15}
\end{equation*}
$$

where, $\beta_{1}=\frac{1}{\sigma} V_{0}^{-\frac{\sigma}{2}}\left\|h_{1}\right\|_{\sigma_{0}}, \beta_{2}=\frac{1}{p} \eta_{p}^{p}\left\|h_{2}\right\|_{\infty}$. Let

$$
\xi(t)=\lambda \beta_{1} t^{\sigma-2}+\beta_{2} t^{p-2}, \quad t>0 .
$$

We claim $\xi\left(t_{0}\right)<\frac{1}{4}$ for some $t_{0}>0$. Note that $\xi(t) \rightarrow+\infty$ as $t \rightarrow 0^{+}$or $t \rightarrow+\infty$. Then, $\xi(t)$ has a minimum at $t_{0}>0$. In order to find $t_{0}$, note

$$
\xi^{\prime}\left(t_{0}\right)=\lambda \beta_{1}(\sigma-2) t_{0}^{\sigma-3}+\beta_{2}(p-2) t_{0}^{p-3}=0 \quad \text { and } \quad t_{0}=\lambda^{1 /(p-\sigma)}\left(\frac{\beta_{1}(2-\sigma)}{\beta_{2}(p-2)}\right)^{1 /(p-\sigma)}>0
$$

Thus, $\xi\left(t_{0}\right)=\lambda^{(p-2) /(p-\sigma)}\left(\beta_{1} \beta_{0}^{(\sigma-2) /(p-\sigma)}+\beta_{2} \beta_{0}^{(p-2) /(p-\sigma)}\right)$ with $\beta_{0}=\beta_{1}(2-\sigma) / \beta_{2}(p-2)$. This shows that there exists $\lambda_{0}>0$ such that for all $\lambda \in\left(0, \lambda_{0}\right), \xi\left(t_{0}\right)<\frac{1}{4}$. Hence, (3.15) implies that there exists $m_{0}, \alpha>0$ such that $J(u) \geq \alpha$ whenever $\|u\|=t_{0}=\rho$ and $\|g\|_{2}<m_{0}$.

Lemma 3.5. Suppose that the assumptions $(V)$ and $\left(H_{6}\right)$ are satisfied. Then there exists $e \in E$ with $\|e\|>\rho$ such that $J(e) \leq 0$, where $\rho$ is given in Lemma 3.4.

Proof. Choose $\varphi_{2} \in C_{0}^{\infty}(\Omega), \varphi_{2} \geq 0, \varphi_{2} \not \equiv 0$. By $\left(H_{6}\right)$, we know that $h_{2}>0$ in $\Omega$, then

$$
\begin{aligned}
J\left(t \varphi_{2}\right)= & \frac{t^{2}}{2}\left\|\varphi_{2}\right\|^{2}+\frac{t^{4}}{4} \int_{\mathbb{R}^{3}} \phi_{\varphi_{2}} \varphi_{2}^{2} d x \\
& -\frac{\lambda t^{\sigma}}{\sigma} \int_{\Omega} h_{1}(x)\left|\varphi_{2}\right|^{\sigma} d x-\frac{t^{p}}{p} \int_{\Omega} h_{2}(x)\left|\varphi_{2}\right|^{p} d x-t \int_{\Omega} g(x) \varphi_{2} d x \\
\rightarrow & -\infty
\end{aligned}
$$

as $t \rightarrow+\infty$ with $1<\sigma<2$ and $p>4$. Thus, there exists $t_{2}>0$ large enough, such that $J\left(t_{2} \varphi_{2}\right)<0$. Thus, we complete the proof by taking $e=t_{2} \varphi_{2}$.

Lemma 3.6. Assume that $(V)$ and $\left(H_{6}\right)$ hold. Then $J$ satisfies the (PS)-condition on E.
Proof. Let $\left\{u_{n}\right\} \subset E$ satisfying (3.5). We claim that $\left\{u_{n}\right\}$ is bounded in $E$. For $n$ large enough, it follows from (2.3), (3.5), (3.13) and (3.14) that

$$
\begin{aligned}
1+c+\left\|u_{n}\right\| \geq & J\left(u_{n}\right)-\frac{1}{p}\left\langle J^{\prime}\left(u_{n}\right), u_{n}\right\rangle \\
= & \left(\frac{1}{2}-\frac{1}{p}\right)\left\|u_{n}\right\|^{2}+\left(\frac{1}{4}-\frac{1}{p}\right) \int_{\mathbb{R}^{3}} \phi_{u_{n}} u_{n}^{2} d x-\lambda\left(\frac{1}{\sigma}-\frac{1}{p}\right) \int_{\mathbb{R}^{3}} h_{1}(x)\left|u_{n}\right|^{\sigma} d x \\
& -\left(1-\frac{1}{p}\right) \int_{\mathbb{R}^{3}} g(x) u_{n} d x \\
\geq & \left(\frac{1}{2}-\frac{1}{p}\right)\left\|u_{n}\right\|^{2}-\lambda\left(\frac{1}{\sigma}-\frac{1}{p}\right) V_{0}^{-\frac{\sigma}{2}}\left\|h_{1}\right\|_{\sigma_{0}}\left\|u_{n}\right\|^{\sigma}-\left(1-\frac{1}{p}\right) V_{0}^{-\frac{1}{2}}\|g\|_{2}\left\|u_{n}\right\| .
\end{aligned}
$$

Because $1<\sigma<2$ and $p>4$, we deduce that $\left\{u_{n}\right\}$ is bounded in $E$. Therefore, there exists $u \in E$ such that, up to a subsequence, we have $u_{n} \rightharpoonup u$ weakly in $E, u_{n} \rightarrow u$ strongly in $L^{s}\left(\mathbb{R}^{3}\right)$ for $2 \leq s<6$ and $u_{n}(x) \rightarrow u(x)$ a.e. on $\mathbb{R}^{3}$. Similar to the proof of Lemma 3.3 (see (3.11)), in order to prove that $u_{n} \rightarrow u$ strongly in $E$, it sufficient to show that

$$
\int_{\mathbb{R}^{3}} f\left(x, u_{n}\right)\left(u_{n}-u\right) d x=\int_{\mathbb{R}^{3}}\left(\lambda h_{1}(x)\left|u_{n}\right|^{\sigma-2} u_{n}+h_{2}(x)\left|u_{n}\right|^{p-2} u_{n}\right)\left(u_{n}-u\right) d x \rightarrow 0 .
$$

Since $u_{n} \rightarrow u$ strongly in $L^{s}\left(\mathbb{R}^{3}\right)$ for $2 \leq s<6$, the Hölder inequality implies that

$$
\int_{\mathbb{R}^{3}}\left|h_{1}\right|\left|u_{n}\right|^{\sigma-1}\left|u_{n}-u\right| d x \leq\left\|h_{1}\right\|_{\sigma_{0}}\left\|u_{n}\right\|_{2}^{\sigma-1}\left\|u_{n}-u\right\|_{2} \rightarrow 0
$$

and

$$
\int_{\mathbb{R}^{3}}\left|h_{2}\right|\left|u_{n}\right|^{p-1}\left|u_{n}-u\right| d x \leq\left\|h_{2}\right\|_{\infty}\left\|u_{n}\right\|_{p}^{p-1}\left\|u_{n}-u\right\|_{p} \rightarrow 0 .
$$

Therefore, $J$ satisfies the (PS)-condition.
Proof of Theorem 1.3. Similar to the proof of Theorem 1.2, we also divide the proof into two steps.

Step 1. As the proof of Step 1 in Theorem 1.2, we first prove the existence of negative energy solution via Ekeland's variational principle (cf. Proposition 2.5). Since $g \in L^{2}\left(\mathbb{R}^{3}\right)$ and $g \not \equiv 0$, we can choose a function $v \in E$ such that

$$
\int_{\mathbb{R}^{3}} g(x) v(x) d x>0 .
$$

It follows from (3.13) that

$$
\begin{aligned}
J(t v)= & \frac{t^{2}}{2}\|v\|^{2}+\frac{t^{4}}{4} \int_{\mathbb{R}^{3}} \phi_{v} v^{2} d x-\frac{\lambda t^{\sigma}}{\sigma} \int_{\mathbb{R}^{3}} h_{1}(x)|v|^{\sigma} d x \\
& -\frac{t^{p}}{p} \int_{\mathbb{R}^{3}} h_{2}(x)|v|^{p} d x-t \int_{\mathbb{R}^{3}} g(x) v d x \\
\leq & \frac{t^{2}}{2}\|v\|^{2}+C_{1} \frac{t^{4}}{4}\|v\|^{4}-\frac{\lambda t^{\sigma}}{\sigma} \int_{\mathbb{R}^{3}} h_{1}(x)|v|^{\sigma} d x \\
& -\frac{t^{p}}{p} \int_{\mathbb{R}^{3}} h_{2}(x)|v|^{p} d x-t \int_{\mathbb{R}^{3}} g(x) v d x<0,
\end{aligned}
$$

for $t>0$ small enough, since $1<\sigma<2$ and $p>4$. Hence we deduce that $\inf \{J(u): u \in$ $\left.\bar{B}_{\rho}\right\}<0$, where $\rho>0$ is given by Lemma 3.4. In addition, by (3.15) we have

$$
\begin{aligned}
J(u) & \geq \frac{1}{4}\|u\|^{2}-\lambda \beta_{1}\|u\|^{\sigma}-\beta_{2}\|u\|^{p}-V_{0}^{-1}\|g\|_{2}^{2} \\
& \geq-\lambda \beta_{1}\|u\|^{\sigma}-\beta_{2}\|u\|^{p}-V_{0}^{-1}\|g\|_{2}^{2}
\end{aligned}
$$

which implies that $J$ is bounded below in $\bar{B}_{\rho}$. Furthermore, we have

$$
-\infty<c_{0}=\inf \left\{J(u): u \in \bar{B}_{\rho}\right\}<0 .
$$

Therefore, the Ekleland's variational principle implies that there exists a sequence $\left\{u_{n}\right\} \subset \bar{B}_{\rho}$ such that

$$
c_{0} \leq J\left(u_{n}\right) \leq c_{0}+\frac{1}{n},
$$

and

$$
J\left(u_{n}\right) \leq J(w)+\frac{1}{n}\left\|u_{n}-w\right\|, \quad \forall w \in \bar{B}_{\rho} .
$$

Then, arguing as the proof Step 1. in Theorem 1.2, we conclude that there exists $u_{0} \in E$ such that $J\left(u_{0}\right)=\inf _{u \in \bar{B}_{\rho}} J(u)=c_{0}<0$ and $J^{\prime}\left(u_{0}\right)=0$.
Step 2. Now, we apply Proposition 2.4 to obtain the positive energy solution. Evidently, $J \in C^{1}(E, \mathbb{R})$ and $J(0)=0$. By Lemma $3.4 J$ satisfies (i) whenever $\|g\|_{2} \leq m_{0}$. Moreover, Lemma 3.5 implies that $J$ satisfies (ii), and $J$ satisfies the (PS)-condition by Lemma 3.6. Hence, Proposition 2.4 implies that there exists a function $\bar{u}_{0} \in E$ such that $J\left(\bar{u}_{0}\right)=\bar{c}_{0} \geq \alpha>0$ and $J^{\prime}\left(\bar{u}_{0}\right)=0$.

The proof is completed.

## Acknowledgements

The authors would like to express sincere thanks to the anonymous referee for his/her carefully reading the manuscript and valuable comments and suggestions. This work was supported by Natural Science Foundation of China (NSFC 11671403) and the Mathematics Interdisciplinary Science project of CSU.

## References

[1] A. Ambrosetti, On Schrödinger-Poisson systems, Milan J. Math. 76(2008), 257-274. MR2465993; url
[2] A. Azzollini, P. d'Avenia, A. Pomponio, On the Schrödinger-Maxwell equations under the effect of a general nonlinear term, Ann. Inst. H. Poincaré Anal. Non Linéaire 27(2010), 779-791. MR2595202; url
[3] T. D'Aprile, D. Mugnai, Solitary waves for nonlinear Klein-Gordon-Maxwell and Schrödinger-Maxwell equations, Proc. Roy. Soc. Edinburgh Sect. A 134(2004), 893-906. MR2099569; url
[4] T. Bartsch, Z. Q. Wang, M. Willem, The Dirichlet problem for superlinear elliptic equations, in: Stationary partial differential equations, Vol. II, Handbook of Differential Equations, Elsevier/North-Holland, Amsterdam, 2005, pp. 1-55. MR2181481; url
[5] V. Benci, F. Donato, An eigenvalue problem for the Schrödinger-Maxwell equations, Topol. Methods Nonlinear 11(1998), 283-293. MR1659454
[6] G. Cerami, G. Vaira, Positive solutions for some non-autonomous Schrödinger-Poisson systems, J. Differential Equations 248(2010), 521-543. MR2557904; url
[7] S. J. Chen, C. L. Tang, High energy solutions for the superlinear Schrödinger-Maxwell equations, Nonlinear Anal. 71(2009), 4927-4934. MR2548724; url
[8] S. J. Chen, C. L. Tang, Multiple solutions for nonhomogeneous Schrödinger-Maxwell and Klein-Gordon-Maxwell equations on $\mathbb{R}^{3}, N o D E A$ Nonlinear Differential Equations Appl. 17(2010), 559-574. MR2728537; url
[9] P. Chen, C. Tian, Infinitely many solutions for Schrödinger-Maxwell equations with indefinite sign subquadratic potentials, Appl. Math. Comput. 226(2014), 492-502. MR3144327; url
[10] S. Chen, C. Wang, Existence of multiple nontrivial solutions for a Schrödinger-Poisson system, J. Math. Anal. Appl. 441(2014), 787-793. MR3128432; url
[11] B. Cheng, X. Wu, J. Liu, Multiple solutions for a class of Kirchhoff type problems with concave nonlinearity, NoDEA Nonlinear Differential Equations Appl. 19(2012), 521-537. MR2984595; url
[12] L. Ding, L. Li, J. L. Zhang, Multiple solutions for nonhomogeneous Schrödinger-Poisson systems with the asymptotical nonlinearity in $\mathbb{R}^{3}$, Taiwanese J. Math. 17(2013), 1627-1650. MR3106034; url
[13] X. Chang, Ground state solutions of asymptotically linear fractional Schrödinger equations, J. Math. Phys. 54(2013), 061504, 1-10. url
[14] W. N. Huang, X. H. Tang, The existence of infinitely many solutions for the nonlinear Schrödinger-Maxwell equations, Results Math. 65(2014), 223-234. MR3162440; url
[15] Y. Jiang, Z. Wang, H. S. Zhou, Multiple solutions for a nonhomogeneous SchrödingerMaxwell system in $\mathbb{R}^{3}$, Nonlinear Anal. 83(2013), 50-57. MR3021537; url
[16] Q. Li, H. Su, Z. Wei, Existence of infinitely many large solutions for the nonlinear Schrödinger-Maxwell equations, Nonlinear Anal. 72(2010), 4264-4270. MR2606783; url
[17] Z. Liang, J. $\mathrm{Xu}_{\mathrm{u}}, \mathrm{X} . \mathrm{Z}_{\mathrm{Hu}}$, Revisit to sign-changing solutions for the nonlinear SchrödingerPoisson system in $\mathbb{R}^{3}$, J. Math. Anal. Appl. 435(2016), 783-799. MR3423428; url
[18] H. Liu, H. Chen, G. Wang, Multiplicity for a 4-sublinear Schrödinger-Poisson system with sign-changing potential via Morse theory, Compt. Rendus Math. 354(2016), 75-80. MR3439728; url
[19] J. Mawhin, M. Willem, Critical point theory and Hamiltonian systems, Springer-Verlag, New York, 1989. MR0982267; url
[20] P. H. Rabinowitz, Minimax methods in critical point theory with applications to differential equations, in: CBMS Regional Conference Series in Mathematics, Vol. 65, American Mathematical Society, Providence, RI, 1986, pp. 257-274. MR0845785; url
[21] D. Ruiz, The Schrödinger-Poisson equation under the effect of a nonlinear local term, J. Funct. Anal. 237(2006), 655-674. MR2230354; url
[22] A. Salvatore, Multiple solitary waves for a non-homogeneous Schrödinger-Maxwell system in $\mathbb{R}^{3}$, Adv. Nonlinear Stud. 6(2006), 157-169. MR2219833; url
[23] O. Sánchez, J. Soler, Long-time dynamics of the Schrödinger-Poisson-Slater system, J. Stat. Phys. 114(2004), 179-204. MR2032129; url
[24] W. Shuai, Q. Wang, Existence and asymptotic behavior of sign-changing solutions for the nonlinear Schrödinger-Poisson system in $\mathbb{R}^{3}$, Z. Angew. Math. Phys. 66(2015), 3267-3282. MR3428464; url
[25] J. Sun, Infinitely many solutions for a class of sublinear Schrödinger-Maxwell equations, J. Math. Anal. Appl. 390(2012), 514-522. MR2890533; url
[26] J. Sun, H. Chen, L. Yang, Positive solutions of asymptotically linear Schrödinger-Poisson systems with a radial potential vanishing at infinity, Nonlinear Anal. 74(2011), 413-423. MR2733219; url
[27] X. TANG, Infinitely many solutions for semilinear Schrödinger equations with signchanging potential and nonlinearity, J. Math. Anal. Appl. 401(2013), 407-415. MR3011282; url
[28] L. Wang, S. Ma, X. Wang, On the existence of solutions for nonhomogeneous Schrödinger-Poisson system, Bound. Value Probl. 2016, 2016:76, 11 pp. MR3484027; url
[29] M. Willem, Minimax theorems, Birkhäuser Boston Inc, Boston, 1996. MR1400007; url
[30] L. Xu, H. Chen, Multiplicity of small negative-energy solutions for a class of nonlinear Schrödinger-Poisson systems, Appl. Math. Comput. 243(2014), 817-824. MR3244529; url
[31] L. $\mathrm{X}_{\mathrm{u}}, \mathrm{H}$. CHEN, Existence of infinitely many solutions for generalized SchrödingerPoisson system, Bound. Value Probl. 2014, 2014:196, 12 pp. MR3286117; url
[32] M. H. Yang, Z.Q. Han, Existence and multiplicity results for the nonlinear SchrödingerPoisson systems, Nonlinear Anal. Real World Appl. 13(2012), 1093-1101. MR2863939; url
[33] Y. Ye, C. L. Tang, Multiple solutions for Kirchhoff-type equations in $\mathbb{R}^{N}$, J. Math. Phys. 54(2013), 081508, 16 pp. MR3135452; url
[34] J. Zhang, On ground state and nodal solutions of Schrödinger-Poisson equations with critical growth, J. Math. Anal. Appl. 428(2015), 387-404. MR3326993; url
[35] Q. Zhang, F. Li, Z. Liang, Existence of multiple positive solutions to nonhomogeneous Schrödinger-Poisson system, Appl. Math. Comput. 259(2015), 353-363. MR3338372; url


[^0]:    ${ }^{\boxtimes}$ Corresponding author. Email: sofiane_math@csu.edu.cn

