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Neuron model with a period three internal decay rate

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Abstract. In this paper we will study a non-autonomous piecewise linear difference equation that describes a discrete version of a single neuron model. We will investigate the periodic behavior of solutions relative to the sequence periodic with period three internal decay rate. In fact, we will show that only periodic cycles with period $3k$, $k = 1, 2, 3, \dots$ can exist and also show their stability character.

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1 Introduction

The basic model of our investigation is the delayed differential equation

$$x'(t) = -g(x(t-\tau)) \tag{1.1}$$

that was used as a model for a single neuron with no internal decay [12]; where $g : \mathbf{R} \to \mathbf{R}$ is either a sigmoid or a piecewise linear signal function and $\tau \le 0$ is a synaptic transmission delay.

From (1.1) we obtain a model for a single neuron with no internal decay as the following equation

$$x'(t) = -g(x([t])), \tag{1.2}$$

where $[t]$ denotes a greatest integer function. When we integrate (1.2) from n to $t \in [n,\ n+1[$ we get

$$x(t) = x(n) - \int_n^t g(x([s]))\, ds = x(n) - g(x(n))(t-n).$$

By letting $t \to n+1$ and denoting $x(n) = x_n$, we obtain a difference equation

$$x_{n+1} = x_n - g(x_n).$$

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This equation is generalized for a discrete-time network of a single neuron model as follows:

$$x_{n+1} = \beta x_n - g(x_n), \qquad n = 0, 1, 2, \ldots, \tag{1.3}$$

where $\beta > 0$ is an internal decay rate and g is a signal function. This model (1.3) is mentioned in many articles [3,5,11,13–18]. In several papers it is analyzed as a single neuron model where a signal function g is the following piecewise constant with McCulloch–Pitts non-linearity:

$$g(x) = \begin{cases} 1, & x \geq 0, \\ -1, & x < 0, \end{cases} \tag{1.4}$$

here 0 is referred as the threshold.

In [1, 2] models with a different signal function (with more than one threshold) were studied. In [10] is considered another discrete neuron model where periodic solutions exist. Piecewise linear difference equations have been used as mathematical models for various applications, including neurons (see [6]).

In this paper we investigate the boundedness nature and the periodic character of solutions relative to the periodic internal decay rate; in particular, we study the following non-autonomous piecewise linear difference equation:

$$x_{n+1} = \beta_n x_n - g(x_n), \tag{1.5}$$

where

$$\beta_n = \begin{cases} \beta_0, & \text{if } n = 3k, \\ \beta_1, & \text{if } n = 3k+1, \\ \beta_2, & \text{if } n = 3k+2, \end{cases} \qquad k = 0, 1, 2, \ldots$$

$\beta_n > 0$, $n = 0, 1, 2, \ldots$, where at least two of coefficients are different and $g(x)$ is in the form (1.4).

In [4] we studied the model where $(\beta_n)_{n=0}^{\infty}$ is a periodic sequence with period two. So far the neuron model (1.3) has not been studied with a periodic internal decay rate β. The goal of this paper is to investigate the boundedness nature and the periodic character of solutions. Furthermore, we determine the relationship of periodic cycles relative to the period of parameters and relative to the relationship between parameters as well. We remark that we investigate a non-linear process and a discontinuous process as well. For understanding behavior of our model we have considered a lot of solutions of equation (1.3) with different values of $\beta_n > 0$, $n = 0, 1, 2, \ldots$, and initial conditions. Obviously, this "invisible work" we have made as computer experiments.

We organize our manuscript as follows. In the next section we provide important concepts of difference equations; then we analyze (1.5) and formulate results about the periodicity and stability. At the end we give some concluding remarks and future ideas.

2 Basic concepts and definitions of difference equations

To analyze the behavior of (1.5), it is essential to review some basic theory of difference equations (see [8,9,17]).

Consider a first order difference equation in the form

$$x_{n+1} = f(x_n), \qquad n = 0, 1, \ldots, \tag{2.1}$$

where $f : \mathbf{R} \to \mathbf{R}$ is a given function. A solution of (2.1) is a sequence $(x_n)_{n\in\mathbf{N}}$ that satisfies (2.1) for all $n = 0, 1, ...$ If an initial condition $x_0 \in \mathbf{R}$ is given, then the *orbit* $O(x_0)$ of a point x_0 is defined as a set of points

$$O(x_0) = \{x_0, x_1 = f(x_0), x_2 = f(x_1) = f^2(x_0), x_3 = f(x_2) = f^3(x_0), \dots\}.$$

Definition 2.1. A point x_s is said to be a *equilibrium point* of the map f, an *fixed point* or a *stationary state* of (2.1) if $f(x_s) = x_s$.

Definition 2.2. An *equilibrium point* of (2.1) is *stable* if

$$\forall \varepsilon > 0\ \exists \delta > 0\ \forall x_0 \in \mathbf{R}\ \forall n \in \mathbf{N}\ |x_0 - x_s| < \delta \ \Rightarrow\ |f^n(x_0) - x_s| < \varepsilon.$$

Otherwise, the equilibrium point x_s is *unstable*.

Definition 2.3. An orbit $O(x_0)$ of the initial point x_0 of equation (2.1) is said to be *periodic of period* $p \geq 2$ if

$$x_p = x_0 \quad \text{and} \quad x_i \neq x_0, \qquad 1 \leq i \leq p-1.$$

In this case we can say that the orbit $O(x_0)$ of the point x_0 is a *cycle* with period p too.

Definition 2.4. A *periodic orbit* $\{x_0, x_1, x_2, \dots, x_{p-1}, \dots\}$ of period p is *stable* if each point x_i, $i = 0, 1, \dots, p-1$, is a stable equilibrium point of the difference equation $x_{n+1} = f^p(x_n)$. A *periodic orbit* of period p which is not stable is said to be *unstable*.

The next theorem [7] will be a vital tool for the stability analysis.

Theorem 2.5. *Let $O(x_0)$ be a periodic orbit of period p of (2.1), where f is continuously differentiable at all points of the orbit. Then the following statements hold true.*

1. *If $|f'(x_0) \cdot f'(x_1) \cdot \dots \cdot f'(x_{p-1})| < 1$, then the orbit $O(x_0)$ is stable,*
2. *If $|f'(x_0) \cdot f'(x_1) \cdot \dots \cdot f'(x_{p-1})| > 1$, then the orbit $O(x_0)$ is unstable.*

3 What cycle does not exists for all $\beta_0 > 0$, $\beta_1 > 0$ and $\beta_2 > 0$

First of all, we assume that coefficients β_0, β_1 and β_2 are positive real numbers and at least two from them are different (not equal to teach other).

Second of all, we remark that

$$\begin{aligned} x_{3n+1} &= \beta_0 x_{3n}, \\ x_{3n+2} &= \beta_1 x_{3n+1}, \qquad n = 0, 1, 2, \dots \\ x_{3n+3} &= \beta_2 x_{3n+2}, \end{aligned} \tag{3.1}$$

Furthermore, we remark that equation (1.5) has no equilibrium points; moreover, equation (1.5) has no periodic points with periods $3n+1$ and $3n+2$, $n = 0, 1, 2, \dots$

Theorem 3.1. *Equation* (1.5) *has no equilibrium points and has no periodic orbits with periods* $3n+1$ *and* $3n+2$, $n = 0, 1, 2, \dots$

Proof. Observe that if $x_0 = 0$, then $x_1 = -1$; therefore 0 is not an equilibrium point.

Now if we assume that $x_0 > 0$ is an equilibrium point, then by computation we get:

$$x_1 = \beta_0 x_0 - 1 = x_0 \Rightarrow x_0 = \frac{1}{\beta_0 - 1} \quad \text{and}$$
$$x_2 = \beta_1 x_1 - 1 = \beta_1 x_0 - 1 = x_0 \Rightarrow x_0 = \frac{1}{\beta_1 - 1} \quad \text{and}$$
$$x_3 = \beta_2 x_2 - 1 = \beta_2 x_0 - 1 = x_0 \Rightarrow x_0 = \frac{1}{\beta_2 - 1}$$

and therefore we see that $\beta_0 = \beta_1 = \beta_2$.

Similarly, if we assume that $x_0 < 0$, then it is an equilibrium point only in the case if all the coefficients are equal.

Now we will assume that equation (1.5) has a periodic orbit of period $3n+1$, $n = 1, 2, 3, \ldots$

Suppose that $x_0 = 0$. Then $x_1 = \beta_0 x_0 - 1 = -1$, $x_2 = \beta_1 x_0 + 1 = -\beta_1 + 1$ and $x_{3n+1} = x_0 = 0$, $x_{3n+2} = \beta_1 x_0 - 1 = -1 = x_1$. But

$$x_{3n+3} = \beta_2 x_1 + 1 = -\beta_2 + 1 = x_2 = -\beta_1 + 1$$

and therefore $\beta_1 = \beta_2$. Now notice that the following statements hold true.

If $x_2 > 0$, then $x_{3n+4} = \beta_0 x_2 - 1 = x_3 = \beta_2 x_2 - 1 \Rightarrow \beta_0 = \beta_2$.

If $x_2 < 0$, then $x_{3n+4} = \beta_0 x_2 + 1 = x_3 = \beta_2 x_2 + 1 \Rightarrow \beta_0 = \beta_2$.

If $x_2 = 0$, then $x_{3n+4} = \beta_0 x_2 - 1 = -1 = x_3$ and then

$$x_{3n+5} = \beta_1 \cdot (-1) + 1 = x_4 = \beta_0 \cdot (-1) + 1 \quad \Rightarrow \quad \beta_1 = \beta_0.$$

We conclude that in all possible cases the three coefficients are equal.

Suppose that $x_0 > 0$. Then $x_1 = \beta_0 x_0 - 1$ and $x_{3n+1} = x_0 > 0$. Therefore we get

$$x_{3n+2} = \beta_1 x_0 - 1 = x_1 = \beta_0 x_0 - 1.$$

Since $x_0 > 0$ then we conclude that $\beta_1 = \beta_0$. Now notice that the following statements hold true.

If $x_1 > 0$, then $x_{3n+3} = \beta_2 x_1 - 1 = x_2 = \beta_1 x_1 - 1 \Rightarrow \beta_2 = \beta_1$.

If $x_1 < 0$, then $x_{3n+3} = \beta_2 x_1 + 1 = x_2 = \beta_1 x_1 + 1 \Rightarrow \beta_2 = \beta_1$.

If $x_1 = 0$, then $x_{3n+3} = \beta_2 x_1 - 1 = -1 = x_2$ and then

$$x_{3n+4} = \beta_0 \cdot (-1) + 1 = x_3 = \beta_2 \cdot (-1) + 1 \quad \Rightarrow \quad \beta_0 = \beta_2.$$

We conclude that in all possible cases the three coefficients are equal.

Suppose that $x_0 < 0$. Then $x_1 = \beta_0 x_0 + 1$ and $x_{3n+1} = x_0 < 0$. Therefore we get

$$x_{3n+2} = \beta_1 x_0 + 1 = x_1 = \beta_0 x_0 + 1.$$

Since $x_0 < 0$ then we conclude that $\beta_1 = \beta_0$.

If $x_1 > 0$, then $x_{3n+3} = \beta_2 x_1 - 1 = x_2 = \beta_1 x_1 - 1 \Rightarrow \beta_2 = \beta_1$.

If $x_1 < 0$, then $x_{3n+3} = \beta_2 x_1 + 1 = x_2 = \beta_1 x_1 + 1 \Rightarrow \beta_2 = \beta_1$.

If $x_1 = 0$, then $x_{3n+3} = \beta_2 x_1 - 1 = -1 = x_2$ and then

$$x_{3n+4} = \beta_0 \cdot (-1) + 1 = x_3 = \beta_2 \cdot (-1) + 1 \quad \Rightarrow \quad \beta_0 = \beta_2.$$

We conclude that in all possible cases the three coefficients are equal.

The proof that equation (1.5) has no periodic orbits with period $3n+2$, $n = 0, 1, 2, \ldots$, is similar. □

The number of periodic orbits depends on the relationship between the parameters β_0, β_1 and β_2. The number of possible situations are much more complicated with three periodic coefficient compared to the case where we had two periodic coefficients what we have studied in [4]. For example, in the case with coefficients with period two when both are less than 1; there exist only periodic solution with period two. However, in the case we considered in this paper if all three periodic coefficients are less than 1, then there are no periodic solutions with period three and instead we have periodic solutions with period 6! Such conclusion we obtained considering many different pictures like in Figure 3.1.

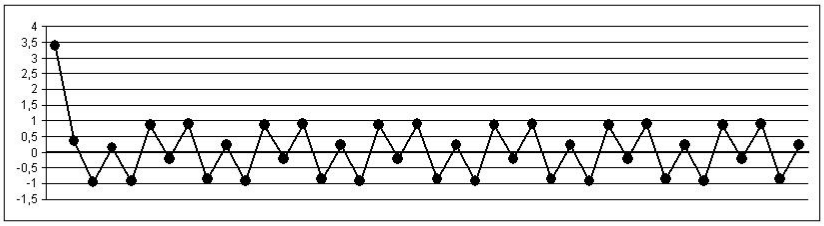

Figure 3.1: Solution of difference equation (1.5), if $\beta_0 = 0.4$, $\beta_1 = 0.15$, $\beta_2 = 0.9$ and $x_0 = 3.4$.

4 Existence of period three solutions

Theorem 4.1. *If $0 < \beta_0 < 1$, $0 < \beta_1 < 1$ and $0 < \beta_2 < 1$ then equation (1.5) has no periodic orbits with periods three.*

Proof. We seek period three solutions in all possible situations. We will start with a special case by letting $x_0 = 0$. Then it follows that $x_1 = \beta_0 \cdot 0 - 1 = -1 < 0$, and therefore $x_2 = \beta_1 \cdot (-1) + 1$. If $x_2 < 0$ then $\beta_1 > 1$ which contradict with assumptions of theorem. If $x_2 > 0$ (situation with $x_2 = -\beta_1 + 1 = 0$ or $\beta_1 = 1$ contradicts with the assumptions of theorem too) then $x_3 = -\beta_1\beta_2 + \beta_2 - 1 = x_0 = 0$ and since $0 < \beta_1 < 1$ it follows that $\beta_2 = \frac{1}{1-\beta_1} > 1$ which contradict with assumptions of theorem.

If $x_0 \neq 0$ then here the solutions with period three are possible in eight different situations: four possibilities with $x_0 > 0$ and 1) $x_1 \geq 0$, $x_2 \geq 0$ or 2) $x_1 \geq 0$, $x_2 < 0$, or 3) $x_1 < 0$, $x_2 \geq 0$, or 4) $x_1 < 0$, $x_2 < 0$ and similar four possibilities with $x_0 < 0$. We consider only four possibilities with $x_0 > 0$ while another cases with $x_0 < 0$ are symmetric.

If $x_0 > 0$ then $x_1 = \beta_0 x_0 - 1$.

Case 1: $x_1 \geq 0$, $x_2 \geq 0$. If $x_1 = 0$ then $x_2 = -1$ and it is not Case 1. If $x_1 > 0$ then $x_2 = \beta_0\beta_1 x_0 - \beta_1 - 1$. If $x_2 = 0$ then $x_3 = -1 = x_0$ and it is not Case 1. If $x_2 > 0$ then $x_3 = \beta_0\beta_1\beta_2 x_0 - \beta_1\beta_2 - \beta_2 - 1 = x_0$. It follows that

$$x_0 = \frac{-1 - \beta_2 - \beta_1\beta_2}{1 - \beta_0\beta_1\beta_2}.$$

Since $\beta_0\beta_1\beta_2 < 1$ then $x_0 < 0$ which is contradiction with assumption above.

Case 2: $x_1 \geq 0$, $x_2 < 0$. If $x_2 < 0$ then $x_3 = \beta_0\beta_1\beta_2 x_0 - \beta_1\beta_2 - \beta_2 + 1 = x_0$. It follows that

$$x_0 = \frac{1 - \beta_2 - \beta_1\beta_2}{1 - \beta_0\beta_1\beta_2}.$$

Since $x_1 \geq 0$ then $x_0 \geq \frac{1}{\beta_0}$ and therefore

$$\frac{1}{\beta_0} \leq \frac{1-\beta_2-\beta_1\beta_2}{1-\beta_0\beta_1\beta_2}.$$

From this inequality it follows

$$0 \leq \frac{1-\beta_2-\beta_1\beta_2}{1-\beta_0\beta_1\beta_2} - \frac{1}{\beta_0} = \frac{\beta_0-\beta_0\beta_2-1}{\beta_0(1-\beta_0\beta_1\beta_2)}.$$

The previous expression is strictly less than 0 since $\beta_0 < 1$ and we have a contradiction.

Case 3: $x_1 < 0$, $x_2 \geq 0$. If $x_1 < 0$ then $x_2 = \beta_0\beta_1 x_0 - \beta_1 + 1 > 0$ (if $x_2 = 0$ then $x_3 = -1$, which is a contradiction) therefore $x_3 = \beta_0\beta_1\beta_2 x_0 - \beta_1\beta_2 + \beta_2 - 1 = x_0$. It follows that

$$x_0 = \frac{-1+\beta_2-\beta_1\beta_2}{1-\beta_0\beta_1\beta_2} = \frac{\beta_2(1-\beta_1)-1}{1-\beta_0\beta_1\beta_2}.$$

Since $\beta_2 < 1$ and $\beta_1 < 1$ then $1-\beta_1 < 1$ and therefore product $\beta_2(1-\beta_1) < 1$ and we have a contradiction with $x_0 > 0$.

Case 4: $x_1 < 0$, $x_2 < 0$. This case is not possible because as $x_2 = \beta_0\beta_1 x_0 - \beta_1 + 1 < 0$ implies that $x_0 < \frac{\beta_1 - 1}{\beta_0\beta_1} < 0$. □

However, period three solutions of equation (1.5) exist in other cases too. We will prove two theorems but these theorems do not examine all the possible cases. First of all, we conclude that if the product of the coefficients $\beta_1\beta_2\beta_3$ is strictly greater than 1, then solutions with period three always exist.

Theorem 4.2. *If $\beta_0\beta_1\beta_2 > 1$ then initial conditions*

$$x_0 = \frac{\beta_1\beta_2+\beta_2+1}{\beta_0\beta_1\beta_2-1} \quad \text{and} \quad x_0 = -\frac{\beta_1\beta_2+\beta_2+1}{\beta_0\beta_1\beta_2-1}$$

form periodic solutions of equation (1.5) with period three; all points of orbit are positive in first case and negative in second case and both orbits are unstable.

Proof. We can simply verify that the first initial condition forms periodic solutions of equation (1.5) with period three and all points are positive. Since $\beta_0\beta_1\beta_2 > 1$ then $x_0 = \frac{\beta_1\beta_2+\beta_2+1}{\beta_0\beta_1\beta_2-1} > 0$. Then

$$\begin{aligned} x_1 = \beta_0 x_0 - 1 &= \frac{\beta_0+\beta_0\beta_2+\beta_0\beta_1\beta_2}{\beta_0\beta_1\beta_2-1} - 1 \\ &= \frac{\beta_0+\beta_0\beta_2+\beta_0\beta_1\beta_2-\beta_0\beta_1\beta_2+1}{\beta_0\beta_1\beta_2-1} = \frac{\beta_0+\beta_0\beta_2+1}{\beta_0\beta_1\beta_2-1} > 0. \end{aligned}$$

Then

$$x_2 = \beta_1 x_1 - 1 = \frac{\beta_0\beta_1+\beta_0\beta_1\beta_2+\beta_1-\beta_0\beta_1\beta_2+1}{\beta_0\beta_1\beta_2-1} = \frac{\beta_0\beta_1+\beta_1+1}{\beta_0\beta_1\beta_2-1} > 0.$$

Therefore we see that

$$x_3 = \beta_2 x_2 - 1 = \frac{\beta_0\beta_1\beta_2+\beta_1\beta_2+\beta_2-\beta_0\beta_1\beta_2+1}{\beta_0\beta_1\beta_2-1} = \frac{\beta_1\beta_2+\beta_2+1}{\beta_0\beta_1\beta_2-1} = x_0.$$

In the second case we will show the existence of a negative solution with period three.

Instability follows from Theorem 2.5. □

In next theorem we will consider cases when period three solution forms cycle with points $x_0 > 0$, $x_1 < 0$, $x_2 < 0$ (or $x_0 < 0$, $x_1 > 0$, $x_2 > 0$).

Theorem 4.3. *If $x_0 = \frac{\beta_1\beta_2-\beta_2-1}{\beta_0\beta_1\beta_2-1}$ or $x_0 = -\frac{\beta_1\beta_2-\beta_2-1}{\beta_0\beta_1\beta_2-1}$, then the following statements hold true:*

1) if $\beta_0\beta_1\beta_2 > 1$ and $\frac{1+\beta_2}{\beta_2} < \beta_1 < \frac{1}{1-\beta_0}$ (consequently $\beta_1 > 1$ and $0 < \beta_0 < 1$), then x_0 forms unstable periodic solutions of equation (1.5) with period three;

2) if $\beta_0\beta_1\beta_2 < 1$ and $0 < \frac{1}{1-\beta_0} < \beta_1 < \frac{1+\beta_2}{\beta_2}$ (consequently $0 < \beta_0 < 1$ and $\beta_1 > 1$), then x_0 forms stable periodic solutions of equation (1.5) with period three.

Proof. Similarly as in the previous theorem we can simply verify that the given initial conditions form periodic solutions of equation (1.5) with period three. Assertions about the stability or the instability follows from Theorem 2.5. □

The case when $\beta_0\beta_1\beta_2 = 1$ is very interesting and a different case. In fact, in this situation there exist segments of initial conditions which form cycles with period three.

At first we remark that cycle with period three is possible in 8 combinations with dependence of sign of points. If $\beta_0\beta_1\beta_2 = 1$ then two cases are impossible, that is, cases with all positive or all negative points. If we assume that $x_0 > 0$, then $x_1 = \beta_0 x_0 - 1$. If $x_1 > 0$ then $x_2 = \beta_0\beta_1 x_0 - \beta_1 - 1$. And if $x_2 > 0$ then $x_3 = \beta_0\beta_1\beta_2 x_0 - \beta_1\beta_2 - \beta_2 - 1 = x_0 - \beta_1\beta_2 - \beta_2 - 1 = x_0$. From this last equality follows that $\beta_1\beta_2 + \beta_2 + 1 = 0$, but $\beta_1 > 0$ and $\beta_2 > 0$; therefore it is impossible. Obviously the symmetrical case with negative points is similar and impossible. Other cases are considered in the theorem.

Theorem 4.4. *Let $\beta_0\beta_1\beta_2 = 1$. Then the following statements are true:*

1) if $1 - \beta_1\beta_2 - \beta_2 = 0$ (this equality holds with the requirements that $\beta_0 > 1$, $\beta_1 = \frac{1}{\beta_0-1}$ and $\beta_2 = \frac{\beta_0-1}{\beta_0}$) then every initial condition $x_0 \in [-1, -\frac{1}{\beta_0}[\cup [\frac{1}{\beta_0}, 1[$ forms cycles with period three which are stable periodic orbits except $\frac{1}{\beta_0}$ and -1;

2) if $\beta_2 - \beta_1\beta_2 - 1 = 0$ (this equality holds with the requirements that $\beta_0 > 0$, $\beta_1 = \frac{1}{\beta_0+1}$ and $\beta_2 = \frac{\beta_0+1}{\beta_0}$) then every initial condition $x_0 \in [-\frac{1}{\beta_0}, \frac{1}{\beta_0}[$ forms cycles with period three which are stable periodic orbits except 0 and $-\frac{1}{\beta_0}$;

3) if $1 + \beta_2 - \beta_1\beta_2 = 0$ (this equality holds with three requirements: $0 < \beta_0 < 1$, $\beta_1 = \frac{1}{1-\beta_0}$ and $\beta_2 = \frac{1-\beta_0}{\beta_0}$) then every initial condition $x_0 \in [-1, 1[$ forms cycles with period three witch are stable periodic orbits except 0 and -1.

Proof. We will prove only the first statement; proofs of other statements are similar and will be omitted.

In the first statement we consider the situation with $x_0 > 0$, $x_1 > 0$ and $x_1 < 0$. Then

$$\begin{aligned} & x_0 > 0, \\ & x_1 = \beta_0 x_0 - 1 > 0 \Rightarrow x_0 > \frac{1}{\beta_0}, \\ & x_2 = \beta_0\beta_1 x_0 - \beta_1 - 1 < 0 \Rightarrow x_0 < \frac{1+\beta_1}{\beta_0\beta_1} > \frac{1}{\beta_0}, \\ & x_3 = \beta_0\beta_1\beta_2 x_0 - \beta_1\beta_2 - \beta_2 + 1 = x_0 - \beta_1\beta_2 - \beta_2 + 1 = x_0 \Rightarrow \\ & \quad \Rightarrow 1 - \beta_1\beta_2 - \beta_2 = 0. \end{aligned} \tag{4.1}$$

From the equalities $\beta_0\beta_1\beta_2 = 1$ and $1 - \beta_1\beta_2 - \beta_2 = 0$ we obtain $\beta_2 = \frac{1}{\beta_0\beta_1} = \frac{1}{1+\beta_1}$. Then we see that $\beta_1 = \frac{1}{\beta_0 - 1}$ and therefore $\beta_2 = \frac{1}{\beta_0\beta_1} = \frac{\beta_0 - 1}{\beta_0}$. From this follows that $\beta_0 > 1$.

From (4.1) we have a double inequality

$$\frac{1}{\beta_0} < x_0 < \frac{1+\beta_1}{\beta_0\beta_1} = \frac{1}{\beta_0} \cdot \frac{1 + \frac{1}{\beta_0 - 1}}{\frac{1}{\beta_0 - 1}} = 1.$$

If we consider a symmetric situation $x_0 < 0$, $x_1 < 0$ and $x_1 > 0$ then we obtain the inequality

$$-1 < x_0 < \frac{1}{\beta_0}.$$

Boundaries of segments we will verify individually (some of these form period three solution but not all points of cycle belong to the segments!):

$$x_0 = -1 < 0,$$
$$x_1 = -\beta_0 + 1 < 0,$$
$$x_2 = -\beta_0\beta_1 + \beta_1 + 1 = -\beta_0 \cdot \frac{1}{\beta_0 - 1} + \frac{1}{\beta_0 - 1} + 1 = 0,$$
$$x_3 = -1 = x_0 \quad \Rightarrow \quad -1 \text{ belongs to segment;}$$
$$x_0 = -\frac{1}{\beta_0} < 0,$$
$$x_1 = \beta_0 \cdot \left(-\frac{1}{\beta_0}\right) + 1 = 0,$$
$$x_2 = -1,$$
$$x_3 = -\beta_2 + 1 = -\frac{\beta_0 - 1}{\beta_0} + 1 = \frac{1}{\beta_0} \neq x_0 \quad \Rightarrow \quad -\frac{1}{\beta_0} \text{ does not belong to segment;}$$
$$x_0 = \frac{1}{\beta_0} > 0,$$
$$x_1 = \beta_0 \cdot \left(\frac{1}{\beta_0}\right) - 1 = 0,$$
$$x_2 = -1,$$
$$x_3 = -\beta_2 + 1 = \frac{1}{\beta_0} = x_0 \quad \Rightarrow \quad \frac{1}{\beta_0} \text{ belongs to segment;}$$
$$x_0 = 1 > 0,$$
$$x_1 = \beta_0 - 1 > 0,$$
$$x_2 = \beta_1(\beta_0 - 1) - 1 = \frac{1}{\beta_0 - 1} \cdot (\beta_0 - 1) - 1 = 0,$$
$$x_3 = -1 \neq x_0 \quad \Rightarrow \quad 1 \text{ does not belong to segment.}$$

Stability follows directly from the Definition 2.4 of stability. □

5 Existence of period six solutions

The solutions with period six are possible in $2^6 = 64$ different situations; we see that there are too many different cases to analyze. First of all, we will clarify what period six cycles exist if all coefficients are less than 1. Indeed, if all the coefficients are less than 1, then there exist at least two cycles with period six.

Theorem 5.1. *If* $0 < \beta_0 < 1$, $0 < \beta_1 < 1$ *and* $0 < \beta_2 < \frac{1}{1-\beta_1}$ *then initial conditions*

$$x_0 = \frac{\beta_1\beta_2 - \beta_2 + 1}{1 + \beta_0\beta_1\beta_2} \quad \text{and} \quad x_0 = -\frac{\beta_1\beta_2 - \beta_2 + 1}{1 + \beta_0\beta_1\beta_2}$$

form periodic solutions of equation (1.5) with period six and there are stable if $\beta_0\beta_1\beta_2 < 1$.

Proof. We will find an orbit with period six and we will show that every two neighbors have different sign.

We will start with the case when $x_0 > 0$. Then we see that

$$x_1 = \beta_0 x_0 - 1 < 0 \quad \Rightarrow \quad x_0 < \frac{1}{\beta_0}, \tag{5.1}$$

$$x_2 = \beta_0\beta_1 x_0 - \beta_1 + 1 \geq 0 \quad \Rightarrow \quad x_0 \geq \frac{\beta_1 - 1}{\beta_0\beta_1}, \tag{5.2}$$

$$x_3 = \beta_0\beta_1\beta_2 x_0 - \beta_1\beta_2 + \beta_2 - 1 < 0 \quad \Rightarrow \quad x_0 < \frac{\beta_1\beta_2 - \beta_2 + 1}{\beta_0\beta_1\beta_2}, \tag{5.3}$$

$$x_4 = \beta_0^2\beta_1\beta_2 x_0 - \beta_0\beta_1\beta_2 + \beta_0\beta_2 - \beta_0 + 1 \geq 0 \quad \Rightarrow \quad x_0 \geq \frac{\beta_0\beta_1\beta_2 - \beta_0\beta_2 + \beta_0 - 1}{\beta_0^2\beta_1\beta_2}, \tag{5.4}$$

$$\begin{aligned} &x_5 = \beta_0^2\beta_1^2\beta_2 x_0 - \beta_0\beta_1^2\beta_2 + \beta_0\beta_1\beta_2 - \beta_0\beta_1 + \beta_1 - 1 < 0 \quad \Rightarrow \\ &x_0 < \frac{\beta_0\beta_1^2\beta_2 - \beta_0\beta_1\beta_2 + \beta_0\beta_1 - \beta_1 + 1}{\beta_0^2\beta_1^2\beta_2}, \end{aligned} \tag{5.5}$$

$$\begin{aligned} x_6 &= \beta_0^2\beta_1^2\beta_2^2 x_0 - \beta_0\beta_1^2\beta_2^2 + \beta_0\beta_1\beta_2^2 - \beta_0\beta_1\beta_2 + \beta_1\beta_2 - \beta_2 + 1 = x_0 \quad \Rightarrow \\ x_0 &= \frac{-\beta_0\beta_1^2\beta_2^2 + \beta_0\beta_1\beta_2^2 - \beta_0\beta_1\beta_2 + \beta_1\beta_2 - \beta_2 + 1}{1 - \beta_0^2\beta_1^2\beta_2^2} \\ &= \frac{\beta_1\beta_2(1 - \beta_0\beta_1\beta_2) - \beta_2(1 - \beta_0\beta_1\beta_2) + (1 - \beta_0\beta_1\beta_2)}{(1 - \beta_0\beta_1\beta_2)(1 + \beta_0\beta_1\beta_2)} \\ &= \frac{\beta_1\beta_2 - \beta_2 + 1}{1 + \beta_0\beta_1\beta_2} = \frac{\beta_2(\beta_1 - 1) + 1}{1 + \beta_0\beta_1\beta_2}. \end{aligned} \tag{5.6}$$

Now we will verify that all the inequalities (5.1)–(5.5) hold true.

The inequality (5.1) $x_0 = \frac{\beta_1\beta_2 - \beta_2 + 1}{1 + \beta_0\beta_1\beta_2} < \frac{1}{\beta_0}$ holds since

$$\frac{\beta_1\beta_2 - \beta_2 + 1}{1 + \beta_0\beta_1\beta_2} - \frac{1}{\beta_0} = \frac{\beta_0\beta_1\beta_2 - \beta_0\beta_2 + \beta_0 - 1 - \beta_0\beta_1\beta_2}{(1 + \beta_0\beta_1\beta_2)\beta_0} = \frac{-\beta_0(\beta_2 + 1) - 1}{(1 + \beta_0\beta_1\beta_2)\beta_0} < 0.$$

We remark that if $0 < \beta_1 < 1$ and $0 < \beta_2 < \frac{1}{1-\beta_1}$ then from (5.6) we see that $x_0 > 0$. If $0 < \beta_1 < 1$ then the inequality (5.2) $x_0 \geq \frac{\beta_1 - 1}{\beta_0\beta_1}$ holds since $x_0 > 0$ and $\frac{\beta_1 - 1}{\beta_0\beta_1} < 0$.

The third inequality (5.3) $x_0 = \frac{\beta_1\beta_2-\beta_2+1}{1+\beta_0\beta_1\beta_2} < \frac{\beta_1\beta_2-\beta_2+1}{\beta_0\beta_1\beta_2}$ holds always while $1+\beta_0\beta_1\beta_2 > \beta_0\beta_1\beta_2$.

The fourth inequality (5.4) $x_0 \geq \frac{\beta_0\beta_1\beta_2-\beta_0\beta_2+\beta_0-1}{\beta_0^2\beta_1\beta_2}$ holds since the right side is equal with $\frac{\beta_0\beta_2(\beta_1-1)+\beta_0-1}{\beta_0^2\beta_1\beta_2}$ and it is negative.

The last inequality (5.5)

$$x_0 = \frac{\beta_1\beta_2-\beta_2+1}{1+\beta_0\beta_1\beta_2} < \frac{\beta_0\beta_1^2\beta_2-\beta_0\beta_1\beta_2+\beta_0\beta_1-\beta_1+1}{\beta_0^2\beta_1^2\beta_2}$$

while the right side obviously is greater so long as $\beta_1 < 1$

$$\begin{aligned}\frac{\beta_0\beta_1^2\beta_2-\beta_0\beta_1\beta_2+\beta_0\beta_1-\beta_1+1}{\beta_0^2\beta_1^2\beta_2} &= \frac{\beta_0\beta_1(\beta_1\beta_2-\beta_2+1)+1-\beta_1}{\beta_0\beta_1\cdot\beta_0\beta_1\beta_2}\\ &= \frac{\beta_1\beta_2-\beta_2+1}{\beta_0\beta_1\beta_2}+\frac{1-\beta_1}{\beta_0^2\beta_1^2\beta_2}.\end{aligned}$$

□

Corollary 5.2. *If $0 < \beta_0 < 1$, $0 < \beta_1 < 1$ and $0 < \beta_2 < 1$ then initial conditions*

$$x_0 = \frac{\beta_1\beta_2-\beta_2+1}{1+\beta_0\beta_1\beta_2} \quad \textit{and} \quad x_0 = -\frac{\beta_1\beta_2-\beta_2+1}{1+\beta_0\beta_1\beta_2}$$

form stable periodic solutions of equation (1.5) *with period six.*

The existence of period six solutions are possible in many other cases if $\beta_0\beta_1\beta_2 > 1$. Here we show one of possibilities if first three points are positive and second three points are negative.

Theorem 5.3. *If $\beta_0\beta_1\beta_2 > 1$ and*

$$\frac{1+\beta_1}{\beta_0\beta_1} < x_0 = \frac{1+\beta_2+\beta_1\beta_2}{\beta_0\beta_1\beta_2+1} < \frac{-1-\beta_1+\beta_0\beta_1+\beta_0\beta_1\beta_2+\beta_0\beta_1^2\beta_2}{\beta_0^2\beta_1^2\beta_2},$$

then x_0 (also $-x_0$) forms unstable periodic solutions of equation (1.5) *with period six.*

Proof. We construct a period six solution with the first three positive points and second three negative points. From the first three steps we obtain:

$$\begin{aligned}&x_0 > 0,\\ &x_1 = \beta_0x_0-1 > 0, \qquad \Rightarrow \quad x_0 > \frac{1+\beta_1}{\beta_0\beta_1} > \frac{1}{\beta_0}.\\ &x_2 = \beta_0\beta_1x_0-\beta_1-1 > 0\end{aligned}$$

Next three iterations are in the form

$$\begin{aligned}x_3 &= \beta_0\beta_1\beta_2x_0-\beta_1\beta_2-\beta_2-1 < 0,\\ x_4 &= \beta_0^2\beta_1\beta_2x_0-\beta_0\beta_1\beta_2-\beta_0\beta_2-\beta_0+1 < 0,\\ x_5 &= \beta_0^2\beta_1^2\beta_2x_0-\beta_0\beta_1^2\beta_2-\beta_0\beta_1\beta_2-\beta_0\beta_1+\beta_1+1 < 0,\end{aligned}$$

From the inequalities $x_3 < 0$, $x_4 < 0$, $x_5 < 0$ we obtain

$$x_0 < \frac{-1-\beta_1+\beta_0\beta_1+\beta_0\beta_1\beta_2+\beta_0\beta_1^2\beta_2}{\beta_0^2\beta_1^2\beta_2}.$$

The sixth iteration gives the formula for the pattern of the period six solution:

$$\begin{aligned} x_6 &= \beta_0^2\beta_1^2\beta_2^2 x_0 - \beta_0\beta_1^2\beta_2^2 - \beta_0\beta_1\beta_2^2 - \beta_0\beta_1\beta_2 + \beta_1\beta_2 + \beta_2 + 1 = x_0 \quad \Rightarrow \\ x_0 &= \frac{\beta_0\beta_1^2\beta_2^2 + \beta_0\beta_1\beta_2^2 + \beta_0\beta_1\beta_2 - \beta_1\beta_2 - \beta_2 - 1}{(\beta_0\beta_1\beta_2)^2 - 1} \\ &= \frac{(1+\beta_2+\beta_1\beta_2)(\beta_0\beta_1\beta_2 - 1)}{(\beta_0\beta_1\beta_2)^2 - 1} = \frac{1+\beta_2+\beta_1\beta_2}{\beta_0\beta_1\beta_2 + 1}. \end{aligned}$$

The instability follows from Theorem 2.5. □

If $\beta_0\beta_1\beta_2 = 1$ then the similar result as in Theorem 4.4 is possible for the existence of cycle with period six (but here we do not consider all the possible cases).

Theorem 5.4. *If $\beta_0 > 1$, $\beta_1 > \frac{1}{\beta_0 - 1}$ and $\beta_2 = \frac{1}{\beta_0\beta_1}$, then every initial condition $x_0 \in \left[-1, -\frac{1+\beta_1}{\beta_0\beta_1}\right[\cup \left[\frac{1+\beta_1}{\beta_0\beta_1}, 1\right[$ forms cycles with period six witch are stable periodic orbits except -1 and $\frac{1+\beta_1}{\beta_0\beta_1}$.*

Proof. We construct period six solution with the first three positive points and second three negative points with the condition that $\beta_0\beta_1\beta_2 = 1$. We obtain the following double inequality:

$$\frac{1}{\beta_0} \cdot \frac{1+\beta_1}{\beta_1} < x_0 < 1.$$

This must hold true as the following system holds true:

$$\begin{cases} \beta_0\beta_1\beta_2 = 1, \\ \frac{1}{\beta_0} \cdot \frac{1+\beta_1}{\beta_1} < 1. \end{cases}$$

Since $\frac{1+\beta_1}{\beta_1} > 1$ then $\beta_0 > 1$, $\beta_1 > \frac{1}{\beta_0 - 1}$ and $\beta_2 = \frac{1}{\beta_0\beta_1}$. Stability follows directly from the Definition 2.4 of stability. □

6 Existence of solutions with period greater than six

If we consider the situation with $\beta_0\beta_1\beta_2 > 1$ then at first we remark that there do not exist periodic solutions where all points are positive (or negative) other than period three solutions. If we want to find positive initial condition for solution of equation (1.5) what is periodic with period n (where n is a multiple of 3) and all the elements are positive, then it is in the following form:

$$x_0 = \frac{\beta_0^{n-1}\beta_1^n\beta_2^n + \beta_0^{n-1}\beta_1^{n-1}\beta_2^n + \beta_0^{n-1}\beta_1^{n-1}\beta_2^{n-1} + \cdots + \beta_0\beta_1\beta_2 + \beta_1\beta_2 + \beta_2 + 1}{(\beta_0\beta_1\beta_2)^n - 1}. \tag{6.1}$$

Observe that it is possible to reduce this formula since

$$(\beta_0\beta_1\beta_2)^n - 1 = (\beta_0\beta_1\beta_2 - 1)((\beta_0\beta_1\beta_2)^{n-1} + (\beta_0\beta_1\beta_2)^{n-2} + \cdots + 1)$$

and

$$\begin{aligned} &\beta_0^{n-1}\beta_1^n\beta_2^n + \beta_0^{n-1}\beta_1^{n-1}\beta_2^n + \beta_0^{n-1}\beta_1^{n-1}\beta_2^{n-1} + \cdots + \beta_0\beta_1\beta_2 + \beta_1\beta_2 + \beta_2 + 1 \\ &\quad = ((\beta_0\beta_1\beta_2)^{n-1} + (\beta_0\beta_1\beta_2)^{n-2} + \cdots + 1)(\beta_1\beta_2 + \beta_2 + 1). \end{aligned}$$

Therefore we have

$$x_0 = \frac{\beta_1\beta_2 + \beta_2 + 1}{\beta_0\beta_1\beta_2 - 1},$$

and it is an initial condition of period three solution (Theorem 4.2).

On the other hand, if in the numerator of formula (6.1) there is at least one $-$ sign (but not all the elements with $-$ sign), then there is no possibility to reduce the formula and we obtain initial conditions for the solution with period n (where n is a multiple of 3). Observe that in this case these are necessary conditions to determine what segment x_0 belongs in the order of β_0, β_1, β_2. Then we obtain double inequalities that are applicable only for the considered periodic cycle. With use of formula (6.1) we can find the initial condition of cycle with an arbitrary period (that is a multiple of 3). Of course we see that all the cycles are unstable by Theorem 2.5 since $\beta_0\beta_1\beta_2 > 1$.

For example, in the next theorem we show one of the possibilities for the existence of solution with period nine. In this case of theorem the cycle forms points such that the first three points are negative and the next six are positive. But in general case other combinations of negative points and positive points of the cycle are possible.

Theorem 6.1. *If* $\beta_0\beta_1\beta_2 > \frac{1+\sqrt{5}}{2}$ *and*

$$\frac{1+\beta_1+\beta_0\beta_1+\beta_0\beta_1\beta_2+\beta_0\beta_1^2\beta_2-\beta_0^2\beta_1^2\beta_2-\beta_0^2\beta_1^2\beta_2^2-\beta_0^2\beta_1^3\beta_2^2}{\beta_0^3\beta_1^3\beta_2^2} < x_0 = \frac{(1+\beta_2+\beta_1\beta_2)(1+\beta_0\beta_1\beta_2-(\beta_0\beta_1\beta_2)^2)}{(\beta_0\beta_1\beta_2)^3-1} < -\frac{1+\beta_1}{\beta_0\beta_1}$$

then x_0 *(also* $-x_0$*) forms unstable periodic solutions of equation* (1.5) *with period nine.*

Proof. We construct a periodic solution with period nine in fixed form; first three points are negative and next six points are positive.

We start with $x_0 < 0$. Then

$$x_1 = \beta_0 x_0 + 1 < 0 \quad \Rightarrow \quad x_0 < -\frac{1}{\beta_0},$$

$$x_2 = \beta_1 x_1 + 1 = \beta_0\beta_1 x_0 + \beta_1 + 1 < 0 \quad \Rightarrow \quad x_0 < -\frac{1+\beta_1}{\beta_0\beta_1} < -\frac{1}{\beta_0},$$

and so forth we obtain inequalities

$$x_3 = \beta_2 x_2 + 1 > 0 \quad \Rightarrow \quad x_0 > -\frac{1+\beta_2+\beta_1\beta_2}{\beta_0\beta_1\beta_2} = -\frac{1+\beta_1}{\beta_0\beta_1} - \frac{1}{\beta_0\beta_1\beta_2},$$

$$x_4 = \beta_0 x_3 - 1 > 0 \quad \Rightarrow \quad x_0 > \frac{1-\beta_0+\beta_0\beta_2-\beta_0\beta_1\beta_2}{\beta_0^2\beta_1\beta_2} = -\frac{1+\beta_2+\beta_1\beta_2}{\beta_0\beta_1\beta_2} + \frac{1}{\beta_0^2\beta_1\beta_2},$$

$$x_5 = \beta_1 x_4 - 1 > 0 \quad \Rightarrow \quad \begin{cases} x_0 > \frac{1+\beta_1-\beta_0\beta_1-\beta_0\beta_1\beta_2-\beta_0\beta_1^2\beta_2}{\beta_0^2\beta_1^2\beta_2} \\ \quad = \frac{1-\beta_0+\beta_0\beta_2-\beta_0\beta_1\beta_2}{\beta_0^2\beta_1\beta_2} + \frac{1}{\beta_0^2\beta_1^2\beta_2}, \end{cases}$$

$$x_6 = \beta_2 x_5 - 1 > 0 \quad \Rightarrow \quad \begin{cases} x_0 > \frac{1+\beta_2+\beta_1\beta_2-\beta_0\beta_1\beta_2-\beta_0\beta_1\beta_2^2-\beta_0\beta_1^2\beta_2^2}{\beta_0^2\beta_1^2\beta_2^2} \\ \quad = \frac{1+\beta_1-\beta_0\beta_1-\beta_0\beta_1\beta_2-\beta_0\beta_1^2\beta_2}{\beta_0^2\beta_1^2\beta_2} + \frac{1}{\beta_0^2\beta_1^2\beta_2^2}, \end{cases}$$

$$x_7 = \beta_0 x_6 - 1 > 0 \quad \Rightarrow \quad \begin{cases} x_0 > \frac{1+\beta_0+\beta_0\beta_2+\beta_0\beta_1\beta_2-\beta_0^2\beta_1\beta_2-\beta_0^2\beta_1\beta_2^2-\beta_0^2\beta_1^2\beta_2^2}{\beta_0^3\beta_1^2\beta_2^2} \\ \quad = \frac{1+\beta_2+\beta_1\beta_2-\beta_0\beta_1\beta_2-\beta_0\beta_1\beta_2^2-\beta_0\beta_1^2\beta_2^2}{\beta_0^2\beta_1^2\beta_2^2} + \frac{1}{\beta_0^3\beta_1^2\beta_2^2}, \end{cases}$$

$$x_8 = \beta_1 x_7 - 1 > 0 \quad \Rightarrow \quad \begin{cases} x_0 > \frac{1+\beta_1+\beta_0\beta_1+\beta_0\beta_1\beta_2+\beta_0\beta_1^2\beta_2-\beta_0^2\beta_1^2\beta_2-\beta_0^2\beta_1^2\beta_2^2-\beta_0^2\beta_1^3\beta_2^2}{\beta_0^3\beta_1^3\beta_2^2} \\ \quad = \frac{1+\beta_0+\beta_0\beta_2+\beta_0\beta_1\beta_2-\beta_0^2\beta_1\beta_2-\beta_0^2\beta_1\beta_2^2-\beta_0^2\beta_1^2\beta_2^2}{\beta_0^3\beta_1^2\beta_2^2} + \frac{1}{\beta_0^3\beta_1^3\beta_2^2}, \end{cases}$$

$$x_9 = \beta_2 x_8 - 1 = x_0 < 0 \quad \Rightarrow \quad x_0 = \frac{(1+\beta_2+\beta_1\beta_2)(1+\beta_0\beta_1\beta_2-\beta_0^2\beta_1^2\beta_2^2)}{\beta_0^3\beta_1^3\beta_2^3 - 1} < 0.$$

Consequently we obtain a double inequality

$$\frac{1+\beta_1+\beta_0\beta_1+\beta_0\beta_1\beta_2+\beta_0\beta_1^2\beta_2-\beta_0^2\beta_1^2\beta_2-\beta_0^2\beta_1^2\beta_2^2-\beta_0^2\beta_1^3\beta_2^2}{\beta_0^3\beta_1^3\beta_2^2}$$
$$< x_0 = \frac{(1+\beta_2+\beta_1\beta_2)(1+\beta_0\beta_1\beta_2-(\beta_0\beta_1\beta_2)^2)}{(\beta_0\beta_1\beta_2)^3 - 1} < -\frac{1+\beta_1}{\beta_0\beta_1}.$$

First we set the inequality $\frac{1+\beta_0\beta_1\beta_2-(\beta_0\beta_1\beta_2)^2}{(\beta_0\beta_1\beta_2)^3-1} < 0$. If we assume that $\beta_0\beta_1\beta_2 > 1$, then $(\beta_0\beta_1\beta_2)^2 - \beta_0\beta_1\beta_2 - 1 > 0$. From the last inequality we obtain $\beta_0\beta_1\beta_2 > \frac{1+\sqrt{5}}{2}$. Instability follows from Theorem 2.5. □

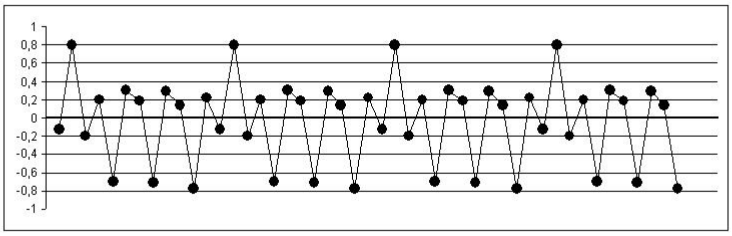

Figure 6.1: Solution of difference equation (1.5), if $\beta_0 = 1.5$, $\beta_1 = 1$, $\beta_2 = 4$ and $x_0 \approx -0.13359$.

For example, in Figure 6.1 we see a solution with period twelve. In this case the formula for initial condition is

$$x_0 = \frac{(1-\beta_2+\beta_1\beta_2)(1+\beta_0\beta_1\beta_2+(\beta_0\beta_1\beta_2)^2-(\beta_0\beta_1\beta_2)^3)}{(\beta_0\beta_1\beta_2)^4 - 1}.$$

7 Unbounded solutions

If $\beta_0\beta_1\beta_2 > 1$, then we have observed that our difference equation exhibits unbounded solutions. The next theorem gives some conditions for x_0 that form unbounded solutions but it is possible to find other cases when unbounded solutions exist.

Theorem 7.1. *If $\beta_0\beta_1\beta_2 > 1$ and*

$$x_0 > \frac{1+\beta_2+\beta_1\beta_2}{\beta_0\beta_1\beta_2 - 1},$$

then x_0 forms unbounded solutions of equation (1.5) – going to $+\infty$.

If $\beta_0\beta_1\beta_2 > 1$ and

$$x_0 < -\frac{1+\beta_2+\beta_1\beta_2}{\beta_0\beta_1\beta_2 - 1},$$

then x_0 forms unbounded solutions of equation (1.5) – going to $-\infty$.

Proof. Now we will find a solution which tends to $+\infty$ – this means that there exists N such that for $n > N$ all points x_n of solution are positive. We start with $x_0 > 0$. Then we see that

$$
\begin{aligned}
&x_1 = \beta_0 x_0 - 1 > 0 \quad \Rightarrow \quad x_0 > \frac{1}{\beta_0}.\\
&x_2 = \beta_1 x_1 - 1 = \beta_0\beta_1 x_0 - \beta_1 - 1 > 0 \quad \Rightarrow \quad x_0 > \frac{\beta_1+1}{\beta_0\beta_1} = \frac{1}{\beta_0}\frac{\beta_1+1}{\beta_1} > \frac{1}{\beta_0},\\
&x_3 = \beta_2 x_2 - 1 = \beta_0\beta_1\beta_2 x_0 - \beta_1\beta_2 - \beta_2 - 1 > 0 \quad \Rightarrow\\
&x_0 > \frac{1+\beta_2+\beta_1\beta_2}{\beta_0\beta_1\beta_2} = \frac{1}{\beta_0}\left(\frac{1}{\beta_1\beta_2} + \frac{1+\beta_1}{\beta_1}\right) > \frac{1}{\beta_0}\frac{\beta_1+1}{\beta_1},\\
&x_4 = \beta_3 x_3 - 1 = \beta_0^2\beta_1\beta_2 x_0 - \beta_0\beta_1\beta_2 - \beta_0\beta_2 - \beta_0 - 1 > 0 \quad \Rightarrow\\
&x_0 > \frac{1+\beta_0+\beta_0\beta_2+\beta_0\beta_1\beta_2}{\beta_0^2\beta_1\beta_2} = \frac{1+\beta_0(1+\beta_2+\beta_1\beta_2)}{\beta_0^2\beta_1\beta_2} > \frac{1+\beta_2+\beta_1\beta_2}{\beta_0\beta_1\beta_2},\\
&x_5 = \beta_4 x_4 - 1 = \beta_0^2\beta_1^2\beta_2 x_0 - \beta_0\beta_1^2\beta_2 - \beta_0\beta_1\beta_2 - \beta_0\beta_1 - \beta_1 - 1 > 0 \quad \Rightarrow\\
&x_0 > \frac{1+\beta_1+\beta_0\beta_1+\beta_0\beta_1\beta_2+\beta_0\beta_1^2\beta_2}{\beta_0^2\beta_1^2\beta_2} = \frac{1+\beta_1(1+\beta_0+\beta_0\beta_2+\beta_0\beta_1\beta_2)}{\beta_0^2\beta_1^2\beta_2}\\
&\quad > \frac{1+\beta_0+\beta_0\beta_2+\beta_0\beta_1\beta_2}{\beta_0^2\beta_1\beta_2},\\
&x_6 = \beta_5 x_5 - 1 = \beta_0^2\beta_1^2\beta_2^2 x_0 - \beta_0\beta_1^2\beta_2^2 - \beta_0\beta_1\beta_2^2 - \beta_0\beta_1\beta_2 - \beta_1\beta_2 - \beta_2 - 1 > 0 \quad \Rightarrow\\
&x_0 > \frac{1+\beta_2+\beta_1\beta_2+\beta_0\beta_1\beta_2+\beta_0\beta_1\beta_2^2+\beta_0\beta_1^2\beta_2^2}{\beta_0^2\beta_1^2\beta_2^2} = \frac{(1+\beta_2+\beta_1\beta_2)+\beta_0\beta_1\beta_2(1+\beta_2+\beta_1\beta_2)}{(\beta_0\beta_1\beta_2)^2}\\
&\quad = \frac{(1+\beta_2+\beta_1\beta_2)(1+\beta_0\beta_1\beta_2)}{(\beta_0\beta_1\beta_2)^2} > \frac{1+\beta_1+\beta_0\beta_1+\beta_0\beta_1\beta_2+\beta_0\beta_1^2\beta_2}{\beta_0^2\beta_1^2\beta_2},\\
&x_7 = \beta_0 x_6 - 1 = \beta_0^3\beta_1^2\beta_2^2 x_0 - \beta_0^2\beta_1^2\beta_2^2 - \beta_0^2\beta_1\beta_2^2 - \beta_0^2\beta_1\beta_2 - \beta_0\beta_1\beta_2 - \beta_0\beta_2 - \beta_0 - 1 > 0 \quad \Rightarrow\\
&x_0 > \frac{1+\beta_0+\beta_0\beta_2+\beta_0\beta_1\beta_2+\beta_0^2\beta_1\beta_2+\beta_0^2\beta_1\beta_2^2+\beta_0^2\beta_1^2\beta_2^2}{\beta_0^3\beta_1^2\beta_2^2}\\
&\quad = \frac{1}{\beta_0} + \frac{(1+\beta_0+\beta_0\beta_2)(1+\beta_0\beta_1\beta_2)}{\beta_0^3\beta_1^2\beta_2^2},\\
&x_8 = \beta_1 x_7 - 1 = \beta_0^3\beta_1^3\beta_2^2 x_0 - \beta_0^2\beta_1^3\beta_2^2 - \beta_0^2\beta_1^2\beta_2^2 - \beta_0^2\beta_1^2\beta_2\\
&\qquad - \beta_0\beta_1^2\beta_2 - \beta_0\beta_1\beta_2 - \beta_0\beta_1 - \beta_1 - 1 > 0 \quad \Rightarrow\\
&x_0 > \frac{1+\beta_1+\beta_0\beta_1+\beta_0\beta_1\beta_2+\beta_0\beta_1^2\beta_2+\beta_0^2\beta_1^2\beta_2+\beta_0^2\beta_1^2\beta_2^2+\beta_0^2\beta_1^3\beta_2^2}{\beta_0^3\beta_1^3\beta_2^2}\\
&\quad = \frac{\beta_1+1}{\beta_0\beta_1} + \frac{(1+\beta_1+\beta_0\beta_1)(1+\beta_0\beta_1\beta_2)}{\beta_0^3\beta_1^3\beta_2^2},\\
&\quad \vdots
\end{aligned}
$$

And consequently we obtain the three inequalities

$$\begin{aligned} x_{3k} &= \beta_2 x_{3k-1} - 1 > 0 \quad \Rightarrow \\ x_0 &> \frac{(1+\beta_2+\beta_1\beta_2)(1+\beta_0\beta_1\beta_2+\cdots+(\beta_0\beta_1\beta_2)^{k-1})}{(\beta_0\beta_1\beta_2)^k} \\ &= \frac{(1+\beta_2+\beta_1\beta_2)((\beta_0\beta_1\beta_2)^k-1)}{(\beta_0\beta_1\beta_2)^k(\beta_0\beta_1\beta_2-1)}, \end{aligned} \tag{7.1}$$

$$\begin{aligned} x_{3k+1} &= \beta_0 x_{3k} - 1 > 0 \quad \Rightarrow \\ x_0 &> \frac{1}{\beta_0} + \frac{(1+\beta_0+\beta_0\beta_2)(1+\beta_0\beta_1\beta_2+\cdots+(\beta_0\beta_1\beta_2)^{k-1})}{\beta_0(\beta_0\beta_1\beta_2)^k} \\ &= \frac{1}{\beta_0} + \frac{(1+\beta_0+\beta_0\beta_2)((\beta_0\beta_1\beta_2)^k-1)}{\beta_0(\beta_0\beta_1\beta_2)^k(\beta_0\beta_1\beta_2-1)}, \end{aligned} \tag{7.2}$$

$$\begin{aligned} x_{3k+2} &= \beta_1 x_{3k+1} - 1 > 0 \quad \Rightarrow \\ x_0 &> \frac{\beta_1+1}{\beta_0\beta_1} + \frac{(1+\beta_1+\beta_0\beta_1)(1+\beta_0\beta_1\beta_2+\cdots+(\beta_0\beta_1\beta_2)^{k-1})}{\beta_0\beta_1(\beta_0\beta_1\beta_2)^k} \\ &= \frac{\beta_1+1}{\beta_0\beta_1} + \frac{(1+\beta_1+\beta_0\beta_1)((\beta_0\beta_1\beta_2)^k-1)}{\beta_0\beta_1(\beta_0\beta_1\beta_2)^k(\beta_0\beta_1\beta_2-1)}. \end{aligned} \tag{7.3}$$

Since $\beta_0\beta_1\beta_2 > 1$ then

$$\lim_{k\to\infty} \frac{(\beta_0\beta_1\beta_2)^k-1}{(\beta_0\beta_1\beta_2)^k(\beta_0\beta_1\beta_2-1)} = \lim_{k\to\infty} \frac{1-\frac{1}{(\beta_0\beta_1\beta_2)^k}}{\beta_0\beta_1\beta_2-1} = \frac{1}{\beta_0\beta_1\beta_2-1}$$

Therefore the inequality (7.1) is in form

$$x_0 \geq \frac{1+\beta_2+\beta_1\beta_2}{\beta_0\beta_1\beta_2-1}.$$

Therefore the inequality (7.2) is in form

$$x_0 \geq \frac{1}{\beta_0} + \frac{1+\beta_0+\beta_0\beta_2}{\beta_0(\beta_0\beta_1\beta_2-1)} = \frac{\beta_0\beta_1\beta_2-1+1+\beta_0+\beta_0\beta_2}{\beta_0(\beta_0\beta_1\beta_2-1)} = \frac{1+\beta_2+\beta_1\beta_2}{\beta_0\beta_1\beta_2-1}.$$

Therefore the inequality (7.3) is in form

$$\begin{aligned} x_0 &\geq \frac{\beta_1+1}{\beta_0\beta_1} + \frac{1+\beta_1+\beta_0\beta_1}{\beta_0\beta_1(\beta_0\beta_1\beta_2-1)} \frac{\beta_0\beta_1^2\beta_2-\beta_1+\beta_0\beta_1\beta_2-1+1+\beta_1+\beta_0\beta_1}{\beta_0\beta_1(\beta_0\beta_1\beta_2-1)} \\ &= \frac{1+\beta_2+\beta_1\beta_2}{\beta_0\beta_1\beta_2-1}. \end{aligned}$$

From Theorem 4.2 it follows that the initial condition $x_0 = \frac{1+\beta_2+\beta_1\beta_2}{\beta_0\beta_1\beta_2-1}$ forms a positive solution with period three. Therefore now we conclude that if $x_0 > \frac{1+\beta_2+\beta_1\beta_2}{\beta_0\beta_1\beta_2-1}$, then the solution is unbounded. □

8 Conclusion

In this article we have considered three different situations.

If $\beta_0\beta_1\beta_2 < 1$ and all the coefficients are strictly less than 1, then there are no periodic solution with period three (period three is the smallest period what theoretically here is possible), but always there exists a stable periodic solution with period six. If all coefficients are not strictly less than 1, then for some combinations of coefficients are possible periodic solutions with period three and period six.

If $\beta_0\beta_1\beta_2 = 1$, then for some combinations of the coefficients exist segments with initial conditions that forms periodic solutions with period three and period six. We have not investigated other periods.

If $\beta_0\beta_1\beta_2 > 1$, then there always exist unstable positive periodic solutions and negative periodic solutions with period three. In fact, if all elements of the cycle are nonpositive then for some combinations of coefficients it is possible to have periodic solutions with period three, six, nine and in general with period $3k$, $k = 1, 2, 3, \ldots$ Furthermore, we have proven that there always exist unbounded solutions too.

Now we will consider an example where $\beta_0 = 1$, $\beta_1 = 2$ and $\beta_2 = 3$. Then equation (1.5) is in the form

$$x_{n+1} = \begin{cases} h_0(x_n) = \begin{cases} x_n - 1, & x_n \geq 0 \\ x_n + 1, & x_n < 0 \end{cases}, & \text{if } n = 3k, \\ h_1(x_n) = \begin{cases} 2x_n - 1, & x_n \geq 0 \\ 2x_n + 1, & x_n < 0 \end{cases}, & \text{if } n = 3k+1, \\ h_2(x_n) = \begin{cases} 3x_n - 1, & x_n \geq 0 \\ 3x_n + 1, & x_n < 0 \end{cases}, & \text{if } n = 3k+2. \end{cases}$$

Figure 8.1 shows graphs of functions h_0, h_1 and h_2. The dotted line is the bisector $y = x$. In this case the initial condition

$$x_0 = \frac{1 + \beta_2 + \beta_1\beta_2}{\beta_0\beta_1\beta_2 - 1} = \frac{1 + 3 + 6}{6 - 1} = 2$$

forms a period three solution $\{2, 1, 1\}$, also $\{-2, -1, -1\}$. Here, there exist other cycles with period three $\{0.8, -0.2, 0.6\}$, $\{0.4, -0.6, -0.2\}$ and symmetric cycles $\{-0.8, 0.2, -0.6\}$, $\{-0.4, 0.6, 0.2\}$, but there does not exist cycles with $x_0 > 0$, $x_1 > 0$, $x_2 < 0$ – it is not possible. The one of possible cycle with period six is $\{\frac{4}{7}, -\frac{3}{7}, \frac{1}{7}, -\frac{4}{7}, \frac{3}{7}, -\frac{1}{7}\}$. If $x_0 > 2$, then all solutions with these initial points are unbounded; for example,

$$\{2.2, 1.2, 1.4, 3.2, 2.2, 3.4, 9.2, 8.2, 15.4, \ldots\}$$

(see the beginning of this solution in Figure 8.1). If $-2 < x_0 < 2$, then graphical analysis show that all orbits belong in the segment $[-2, 2]$. But, for example, if $x_0 = 1.224$ (see Figure 8.2) then we can not directly say that solution is periodic or not – it is possible that period is very large or indeed it is a chaotic orbit.

On the other hand, even if we consider unstable periodic orbit with large period then we have problem with visualization of such orbit. For example, in Figure 8.3 is considered solution of equation (1.5) with period eighteen. The rounding-off of numbers of computer create mistakes such that there might exist iteration which do not belong to the considered orbit. Essentially there is a possibility that initial condition is not correct on this account.

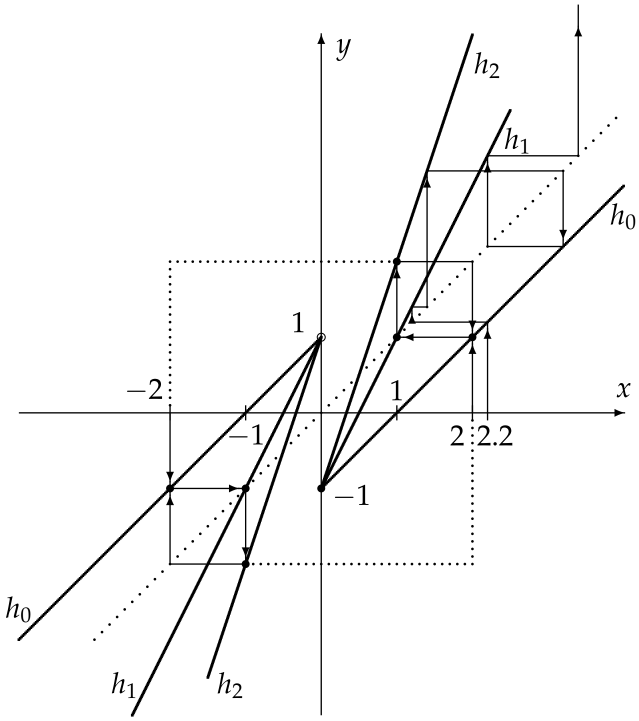

Figure 8.1: Graphs of functions h_0, when $\beta_0 = 1$, h_1, when $\beta_1 = 2$ and h_2, when $\beta_2 = 3$. Cycles $\{2, 1, 1\}$ and $\{-2, -1, -1\}$ and the graphical analysis for unbounded solution with $x_0 = 2.2$.

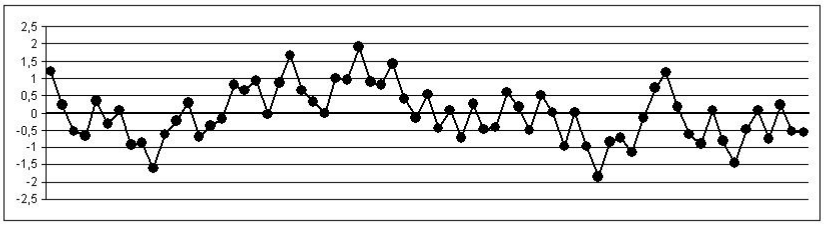

Figure 8.2: Solution of equation (1.5), if $\beta_0 = 1$, $\beta_1 = 2$, $\beta_2 = 3$ and $x_0 = 1.224$.

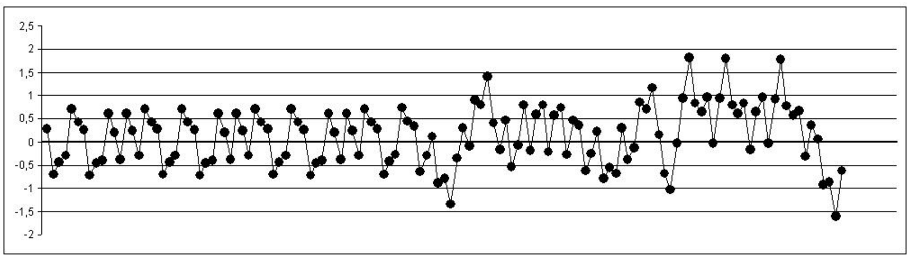

Figure 8.3: Solution of difference equation (1.5), if $\beta_0 = 1$, $\beta_1 = 2$, $\beta_2 = 3$ and $x_0 \approx 20.28519987$.

Unfortunately for another coefficients such as $\beta_0 = 2$, $\beta_1 = 5$ and $\beta_2 = 3$, we cannot say that there exists an invariant interval. This is certainly a question for future work to address.

It is our goal to continue the investigation of periodic nature (and chaotic too) of the solutions of equation (1.5). Obviously, the possible behavior of solutions of equation (1.5) are complicated if the sequence of coefficients $(\beta_n)_{n=0}^{+\infty}$ is periodic with period grater than 2 (see [4]). We think that periodicity of coefficients greater than 3 makes the future work yet more complicated but definitely much more interesting.

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