Electronic Journal of Qualitative Theory of Differential Equations
2017, No. **54**, 1–11; doi: 10.14232/ejqtde.2017.1.54 http://www.math.u-szeged.hu/ejqtde/

# Subharmonic solutions with prescribed minimal period for a class of second order impulsive systems

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Received 23 September 2016, appeared 4 July 2017
Communicated by Gabriele Bonanno

**Abstract.** Based on variational methods and critical point theory, the existence of subharmonic solutions with prescribed minimal period for a class of second-order impulsive systems is derived by estimating the energy of the solution. An example is presented to illustrate the result.

**Keywords:** subharmonic solution, impulsive system, minimal period, variational method.

**2010 Mathematics Subject Classification:** 34B37, 58E30.

## 1 Introduction

This paper is devoted to the existence of subharmonic solutions with prescribed minimal period $pT$ for the following second-order impulsive system

$$\begin{cases} \ddot{v}(t) + Dv(t) + \nabla F(t, v(t)) = 0, \quad \text{a.e. } t \in \mathbb{R}, & (1.1a) \\ \Delta(\dot{v}^i(t_j)) := \dot{v}^i(t_j^+) - \dot{v}^i(t_j^-) = I_{ij}(v^i(t_j)), \quad i = 1, 2, \ldots, N, \ j \in \mathbb{Z}_0, & (1.1b) \end{cases}$$

where $p > 1$ is an integer, $T > 0$, $\mathbb{Z}_0 := \mathbb{Z}^+ \cup \mathbb{Z}^-$, $\nabla F(t,x)$ is the gradient of $F(t,x)$ with respect to $x$ and $F(t,x) \in C^1(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$, $D$ is an $N \times N$ real symmetric constant matrix with $\lambda < 0$ as its eigenvalues, $\dot{v}^i(t_j^{\pm}) = \lim_{t \to t_j^{\pm}} \dot{v}^i(t)$, $0 < t_1 < t_2 < \cdots < t_l < T$, $t_j (j \in \mathbb{Z}_0)$ is a $T$-periodic extension of $t_j (j \in \{1, 2, \ldots, l\})$. And for each $i$, $I_{ij}$ is $l$-periodic with respect to $j$, where $I_{ij} \in C(\mathbb{R}, \mathbb{R})$.

Impulsive effects exist widely in many evolution processes in which their states are changed abruptly at certain moments of time. There have been many approaches to study impulsive problems, such as method of upper and lower solutions with the monotone iterative technique, fixed point theory and topological degree theory. In recent years, variational method was employed to consider the existence of solutions for impulsive problems (see e.g. [1–5, 8, 9, 11, 13, 14]).

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When $D = 0$ and all $I_{ij} \equiv 0$, (1.1) is reduced to the Hamiltonian system, which has been studied extensively on subharmonic solutions (see e.g. [10, 15, 17, 18]). Recently, Luo, Xiao and Xu [6] established the conditions for the existence of subharmonic solutions for the following impulsive differential equation

$$\begin{cases} \ddot{u}(t) + f(t, u(t)) = 0, \\ \Delta(\dot{u}(t_k)) = I_k(u(t_k)), \end{cases}$$

where $f(t, x) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $I_k \in C(\mathbb{R}, \mathbb{R}^+ \cup \{0\})$. After that, Xie and Luo [16] investigated subharmonic solutions for the following forced pendulum equation with impulsive effects

$$\begin{cases} \ddot{x}(t) + A \sin x(t) = f(t), \\ \Delta(\dot{x}(t_k)) = I_k(x(t_k)), \end{cases}$$

where $f : \mathbb{R} \to \mathbb{R}$ and

$$0 \leq \sum_{k=1}^{pm-1} \int_0^{x(t_k)} I_k(s) ds \quad \text{for each } x \in H^1_{pT}. \tag{1.2}$$

However there are cases which are not possible to satisfy $I_k \geq 0$ or (1.2). For example, impulsive functions $I_k(s) = -s/9$. Thus it is valuable to further improve conditions on impulsive functions. One thing to be noted is that a problem with impulsive functions $-s/9$ is considered in this paper (see Example 4.1 in Section 4). What is more, to the best of our knowledge, the existence of subharmonic solutions for impulsive systems has received considerably less attention.

Inspired by the aforementioned facts, we consider the impulsive system (1.1) under different assumptions on the impulsive function from [6] and [16]. It will be shown that $v$ satisfies (1.1) if and only if $u = Q^{-1}v$ satisfies

$$\begin{cases} \ddot{u}(t) + \lambda u(t) + Q^{-1} \nabla F(t, Qu(t)) = 0, \quad \text{a.e. } t \in \mathbb{R}, & \text{(1.3a)} \\ \Delta(\dot{u}^i(t_j)) = \sum_{r=1}^{N} q_{ri} I_{rj} \left( \sum_{k=1}^{N} q_{rk} u^k(t_j) \right), \quad i = 1, 2, \ldots, N, \ j \in \mathbb{Z}_0. & \text{(1.3b)} \end{cases}$$

After that, subharmonic solutions of (1.3) will be obtained by estimating the energy of the solution in terms of minimal period. Finally, an example is given to illustrate the result, and a corollary concerning the equation (1.1a) is presented.

## 2 Preliminaries

Let us recall some basic concepts.

$$H^1_{pT} := \left\{ u : [0, pT] \to \mathbb{R}^N \;\middle|\; \begin{array}{l} u \text{ is absolutely continuous,} \\ u(0) = u(pT) \text{ and } \dot{u} \in L^2(0, pT; \mathbb{R}^N) \end{array} \right\}$$

is a Hilbert space with the inner product

$$\langle u, v \rangle = \int_0^{pT} (\dot{u}(t), \dot{v}(t)) dt + \int_0^{pT} (u(t), v(t)) dt, \quad \forall u, v \in H^1_{pT},$$

where $(\cdot,\cdot)$ denotes the inner product in $\mathbb{R}^N$, and the corresponding norm is

$$\|u\| = \left(\|\dot{u}\|_{L^2}^2 + \|u\|_{L^2}^2\right)^{\frac{1}{2}},$$

where $\|\cdot\|_{L^2}$ is the norm of $L^2(0, pT; \mathbb{R}^N)$. Assume that orthogonal matrix $Q = (q_{rj})_N$ satisfies $Q^{-1}DQ = Q^T DQ = \lambda I$ and $|\cdot|$ denotes the norm in $\mathbb{R}^N$, the orthogonality of $Q$ implies that

$$|Qu(t)| = |u(t)| \quad \text{and} \quad \sum_{r=1}^{N} q_{rk}^2 = \sum_{j=1}^{N} q_{kj}^2 = 1, \quad k = 1, 2, \ldots, N. \tag{2.1}$$

**Lemma 2.1.** *$v$ satisfies the impulsive system (1.1) if and only if $u = Q^{-1}v$ satisfies the impulsive system (1.3).*

*Proof.* Multiplying both sides of (1.1a) by $Q^{-1}$ results (1.3a). In view of (1.1b) and

$$\Delta(\dot{v}^i(t_j)) = \sum_{k=1}^{N} q_{ik}\left[\lim_{t\to t_j^+} \dot{u}^k(t) - \lim_{t\to t_j^-} \dot{u}^k(t)\right] = \sum_{k=1}^{N} q_{ik}\Delta(\dot{u}^k(t_j)),$$

we have

$$\sum_{k=1}^{N} q_{ik}\Delta(\dot{u}^k(t_j)) = I_{ij}\left(\sum_{k=1}^{N} q_{ik}u^k(t_j)\right), \quad i = 1, 2, \ldots, N,\ j \in \mathbb{Z}_0.$$

Solutions of the above nonhomogeneous linear equations are (1.3b). □

If $u(t)$ is a $pT$-periodic solution of (1.3), following the ideas of [9], we have

$$\int_0^{pT} \left(\ddot{u}(t) + \lambda u(t) + Q^{-1}\nabla F(t, Qu(t)), v(t)\right) dt = 0, \quad \text{for } v \in H^1_{pT}. \tag{2.2}$$

By (1.3b), the first term of the above equation is

$$\begin{aligned}
\int_0^{pT} (\ddot{u}(t), v(t))\, dt &= \int_0^{t_1} (\ddot{u}(t), v(t))\, dt + \sum_{j=1}^{pl-1} \int_{t_j}^{t_{j+1}} (\ddot{u}(t), v(t))\, dt + \int_{pl}^{pT} (\ddot{u}(t), v(t))\, dt \\
&= -\sum_{j=1}^{pl}\sum_{i=1}^{N} \left[\left(\dot{u}^i(t_j^+) - \dot{u}^i(t_j^-)\right) v^i(t_j)\right] - \int_0^{pT} (\dot{u}(t), \dot{v}(t))dt \\
&= -\sum_{j=1}^{pl}\sum_{r=1}^{N} \left[\sum_{i=1}^{N} q_{ri}v^i(t_j)\right] I_{rj}\left(\sum_{k=1}^{N} q_{rk}u^k(t_j)\right) - \int_0^{pT} (\dot{u}(t), \dot{v}(t))dt.
\end{aligned} \tag{2.3}$$

It follows from (2.2) and (2.3) that

$$\begin{aligned}
&\int_0^{pT} (\dot{u}(t), \dot{v}(t))dt + \sum_{j=1}^{pl}\sum_{r=1}^{N} \left\{\left[\sum_{i=1}^{N} q_{ri}v^i(t_j)\right] I_{rj}\left(\sum_{k=1}^{N} q_{rk}u^k(t_j)\right)\right\} \\
&\qquad = \lambda \int_0^{pT} (u(t), v(t))\, dt + \int_0^{pT} \left(Q^{-1}\nabla F(t, Qu(t)), v(t)\right) dt.
\end{aligned} \tag{2.4}$$

**Definition 2.2.** A function $u \in H^1_{pT}$ is a weak $pT$-periodic solution of (1.3) if (2.4) holds for any $v \in H^1_{pT}$.

Consider the functional $\Phi : H^1_{pT} \to \mathbb{R}$ defined by

$$\Phi(u) := \frac{1}{2}\int_0^{pT} |\dot{u}(t)|^2 dt - \frac{1}{2}\lambda \int_0^{pT} |u(t)|^2 dt - \int_0^{pT} F(t, Qu(t))dt + \phi(u),$$

where

$$\phi(u) := \sum_{j=1}^{pl}\sum_{r=1}^{N} \int_0^{\sum_{k=1}^{N} q_{rk}u^k(t_j)} I_{rj}(y)dy.$$

Thanks to $Q^T = Q^{-1}$, for any $u, v \in H^1_{pT}$, we have

$$\begin{aligned}\langle \Phi'(u), v\rangle = {} & \int_0^{pT} (\dot{u}(t), \dot{v}(t))dt - \lambda \int_0^{pT} (u(t), v(t))\, dt - \int_0^{pT} \left(Q^{-1}\nabla F(t, Qu(t)), v(t)\right) dt \\ & + \sum_{j=1}^{pl}\sum_{r=1}^{N} \left\{ \left[\sum_{i=1}^{N} q_{ri}v^i(t_j)\right] I_{rj}\left(\sum_{k=1}^{N} q_{rk}u^k(t_j)\right)\right\}. \end{aligned} \tag{2.5}$$

So critical points of $\Phi$ correspond to weak $pT$-periodic solutions of (1.3) (but $pT$ might not be the minimal period). It follows from (2.3) and (2.5) that

$$\begin{aligned}\langle \Phi'(u), v\rangle = {} & -\int_0^{pT} (\ddot{u}(t), v(t))dt - \lambda \int_0^{pT} (u(t), v(t))\, dt \\ & - \int_0^{pT} \left(Q^{-1}\nabla F(t, Qu(t)), v(t)\right) dt. \end{aligned} \tag{2.6}$$

Consider the restriction of $\Phi$ on a subspace $X$ of $H^1_{pT}$, where

$$X := \left\{u \in H^1_{pT}\, | u(-t) = -u(t)\right\}.$$

For any $u \in X$, by Wirtinger's inequality,

$$\|u\|^2_{L^2} \le \frac{p^2T^2}{4\pi^2}\|\dot{u}\|^2_{L^2} = \frac{p^2}{\omega^2}\|\dot{u}\|^2_{L^2} \quad \text{and} \quad \|u\|^2 \le \frac{p^2+\omega^2}{\omega^2}\|\dot{u}\|^2_{L^2}, \tag{2.7}$$

where $\omega = 2\pi/T$. For convenience, we introduce some assumptions.

(H1) $\nabla F(t, x)$ has minimal period $T$ in $t$, and $F(-t, -x) = F(t, x)$.

(H2) There exist constants $A \ge \overline{A} > -\lambda$ and $B > 0$ such that

$$F(t, x) - \nabla F(t, 0)x \le \frac{A}{2}|x|^2, \quad \text{for } t \in \mathbb{R}, x \in \mathbb{R}^N$$

and

$$F(t, x) - \nabla F(t, 0)x \ge \frac{\overline{A}}{2}|x|^2, \quad \text{for } t \in \mathbb{R}, |x| \le B.$$

(H3) For each $i = 1, 2, \dots, N$, $j = 1, 2, \dots, l$, there exist constants $a_{ij} \ge 0$ such that

$$|I_{ij}(y)| \le a_{ij}|y|, \quad \text{for every } y \in \mathbb{R}.$$

Thanks to (2.1) and Hölder's inequality, we have

$$\left|\sum_{k=1}^{N} q_{rk}u^k(t_j)\right| \le \left(\sum_{k=1}^{N} q_{rk}^2\right)^{\frac{1}{2}} \left(\sum_{k=1}^{N}(u^k(t_j))^2\right)^{\frac{1}{2}} = |u(t_j)|,$$

which combined with (H3) yields to

$$\left|\int_0^{\sum_{k=1}^{N} q_{rk}u^k(t_j)} I_{rj}(y)dy\right| \le \frac{a_{rj}}{2}\left|\sum_{k=1}^{N} q_{rk}u^k(t_j)\right|^2 \le \frac{a_{rj}}{2}\left|u(t_j)\right|^2. \tag{2.8}$$

For any $u \in H^1_{pT}$ and each $k = 1, 2, \dots, N$, it follows from the mean value theorem that

$$\frac{1}{pT}\int_0^{pT} u^k(s)ds = u^k(\tau)$$

for some $\tau \in (0, pT)$. Hence, for $t \in [0, pT]$, using Hölder's inequality,

$$\begin{aligned}|u^k(t)| &= \left|u^k(\tau) + \int_\tau^t \dot{u}^k(s)ds\right| \le \frac{1}{pT}\int_0^{pT}\left|u^k(s)\right|ds + \int_0^{pT}\left|\dot{u}^k(s)\right|ds \\ &\le (pT)^{-\frac{1}{2}}\|u^k\|_{L^2} + (pT)^{\frac{1}{2}}\|\dot{u}^k\|_{L^2},\end{aligned}$$

which combined with the discrete version of Minkowski's inequality yields to

$$\begin{aligned}|u(t)| &= \left(\sum_{k=1}^{N}|u^k(t)|^2\right)^{\frac{1}{2}} \le \left(\sum_{k=1}^{N}\left[(pT)^{-\frac{1}{2}}\|u^k\|_{L^2} + (pT)^{\frac{1}{2}}\|\dot{u}^k\|_{L^2}\right]^2\right)^{\frac{1}{2}} \\ &\le (pT)^{-\frac{1}{2}}\|u\|_{L^2} + (pT)^{\frac{1}{2}}\|\dot{u}\|_{L^2}.\end{aligned}$$

In view of this inequality and (2.8), we find

$$\begin{aligned}|\phi(u)| &\le \sum_{j=1}^{pl}\sum_{r=1}^{N}\frac{a_{rj}}{2}\left|u(t_j)\right|^2 \le \sum_{j=1}^{pl}\sum_{r=1}^{N} a_{rj}\left((pT)^{-1}\|u\|^2_{L^2} + pT\|\dot{u}\|^2_{L^2}\right) \\ &\le \varrho T^{-1}\|u\|^2_{L^2} + \varrho p^2 T\|\dot{u}\|^2_{L^2},\end{aligned} \tag{2.9}$$

where $\varrho := \sum_{j=1}^{l}\sum_{r=1}^{N} a_{rj}$.

The following fact is important in the proof of our main result.

**Lemma 2.3** ([12, Theorem 1.2])**.** *Suppose $V$ is a reflexive Banach space with norm $\|\cdot\|$, and let $M \subset V$ be a weakly closed subset of $V$. Suppose $E : M \to \mathbb{R}\cup\{+\infty\}$ is coercive and (sequentially) weakly lower semi-continuous on $M$ with respect to $V$. Then $E$ is bounded from below on $M$ and attains its infimum in $M$.*

**Lemma 2.4.** *Suppose the assumption (H1) holds. If $u$ is a critical point of $\Phi$ on $X$, then $u$ is also a critical point of $\Phi$ on $H^1_{pT}$. And the minimal period of $u$ is an integer multiple of $T$.*

*Proof.* If $u$ is a critical point of $\Phi$ on $X$, that is, $\langle\Phi'(u), v\rangle = 0$ holds for any $v \in X$ and $u$ is odd, then $Q^{-1}\nabla F(t, Qu(t))$ is $pT$-periodic and odd in $t$ by (H1). Thus for any even $w \in H^1_{pT}$, we have

$$\int_0^{pT}\left(Q^{-1}\nabla F(t, Qu(t)), w(t)\right)dt = 0.$$

So, in view of (2.6), we have $\langle \Phi'(u), w\rangle = 0$. That gives us that $\langle \Phi'(u), v\rangle = 0$ holds for any $v \in H^1_{pT}$, which implies that the equation (1.3a) holds by (2.6). Assume that the minimal period of $u$ is $pT/q$ for some integer $q > 1$, it follows from (1.3a) that $\nabla F(t, Qu(t))$ is $pT/q$-periodic, then

$$\nabla F(t, Qu(t)) = \nabla F\left(t + \frac{pT}{q}, Qu\left(t + \frac{pT}{q}\right)\right) = \nabla F\left(t + \frac{pT}{q}, Qu(t)\right).$$

Thus $p/q$ must be an integer by (H1), which completes the proof. □

## 3 Main results

In this section, main results of this paper are obtained.

**Theorem 3.1.** *If (H1), (H2) and (H3) hold, and there exists an integer $p > 1$ such that $2\varrho p^2 T < 1$,*

$$\frac{\omega^2}{\overline{A} + \lambda} < p^2 < \frac{\omega^2 s_p^2}{2\varrho T \omega^2 s_p^2 + A + \lambda + 2\varrho/T}, \tag{3.1}$$

$$\limsup_{|x|\to\infty} \frac{F(t,x)}{|x|^2} < \left(\frac{1}{2} - \varrho p^2 T\right)\frac{\omega^2}{p^2} - \frac{\varrho}{T} - \frac{\lambda}{2}, \quad \textit{uniformly for } t \in \mathbb{R} \tag{3.2}$$

*and*

$$\int_0^T |\nabla F(t,0)|^2 dt < K\left[(1 - 2\varrho p^2 T)\left(\frac{\omega s_p}{p}\right)^2 - \lambda - A - \frac{2\varrho}{T}\right], \tag{3.3}$$

*where $s_p$ is the least prime factor of $p$, $\varrho := \sum_{j=1}^{l}\sum_{r=1}^{N} a_{rj}$ and*

$$K := \frac{\pi B^2}{\omega}\left[\overline{A} + \lambda - \left(\frac{\omega}{p}\right)^2\right] - 2B^2 \sum_{j=1}^{l}\left(\sum_{r=1}^{N} a_{rj}^2\right)^{\frac{1}{2}}.$$

*Then the impulsive system (1.1) has at least one weak periodic solution with minimal period $pT$.*

*Proof.* We will complete the proof in three steps.

*Step 1.* $\Phi$ has a critical point $u^*$ on $X$ with $\inf_{u\in X}\Phi(u) = \Phi(u^*)$.

Let $\{u_n\}$ be a weakly convergent sequence to $u_0$ in $H^1_{pT}$, then $\{u_n\}$ converges uniformly to $u_0$ on $[0, pT]$ (see Proposition 1.2 in [7]) and there exists a constant $C_1 > 0$ such that $\|u_s\|_\infty \le C_1$, $s = 0, 1, 2, \dots$, where $\|u\|_\infty := \max_{t\in[0,pT]}|u(t)|$. Then $F(t,x) \in C^1(\mathbb{R}\times\mathbb{R}^N, \mathbb{R})$ implies that $F(t, Qu_n(t))$ converges uniformly to $F(t, Qu_0(t))$ on $[0, pT]$. It follows from the continuity of $I_{ij}$ and (2.1) that

$$\begin{aligned}|\phi(u_n) - \phi(u_0)| &\le \sum_{j=1}^{pl}\sum_{r=1}^{N}\left|\int_{\sum_{k=1}^N q_{rk}u_0^k(t_j)}^{\sum_{k=1}^N q_{rk}u_n^k(t_j)} I_{rj}(y)dy\right| \\ &\le plNC_2\sum_{k=1}^{N}|q_{rk}|\left|u_n^k(t_j) - u_0^k(t_j)\right| \le plNC_2\|u_n - u_0\|_\infty \to 0,\end{aligned}$$

as $n\to\infty$, where $C_2 = \max\{|I_{rj}(y)| : |y| \le C_1,\ r = 1,2,\dots,N,\ j = 1,2,\dots,l\}$. Hence

$$-\int_0^{pT} F(t, Qu(t))dt + \phi(u)$$

is weakly continuous on $H^1_{pT}$. Moreover, it is clear that

$$\frac{1}{2}\int_0^{pT} |\dot{u}(t)|^2 dt \quad \text{and} \quad -\frac{1}{2}\lambda\int_0^{pT} |u(t)|^2 dt$$

are lower semi-continuous and convex. Therefore $\Phi$ is weakly lower semi-continuous on $H^1_{pT}$.

Let $\{u_n\} \in X$ and $u_n \rightharpoonup u$ as $n \to \infty$. By the Mazur theorem [7, p. 4], there exists a sequence of convex combinations $\{v_k\}$ such that $v_k \to u$ in $H^1_{pT}$. It follows from $X$ is a closed convex space that $\{v_k\} \in X$ and $u \in X$. Thus, $X$ is a weakly closed subset of $H^1_{pT}$.

By (3.2), there exist $0 < \varepsilon_0 < (0.5 - \varrho p^2 T)\omega^2/p^2$ and $W > 0$ such that

$$\frac{F(t,x)}{|x|^2} < \left(\frac{1}{2} - \varrho p^2 T\right)\frac{\omega^2}{p^2} - \frac{\varrho}{T} - \frac{\lambda}{2} - \varepsilon_0, \quad \text{for } |x| > W,\ t \in \mathbb{R},$$

which combined with (2.1) yields to

$$\begin{aligned}\int_0^{pT} F(t, Qu(t))dt &= \int_{\Omega_1} F(t, Qu(t))dt + \int_{\Omega_2} F(t, Qu(t))dt \\ &\le MpT + \left(\left(\frac{1}{2} - \varrho p^2 T\right)\frac{\omega^2}{p^2} - \frac{\varrho}{T} - \frac{\lambda}{2} - \varepsilon_0\right)\|u\|^2_{L^2},\end{aligned} \tag{3.4}$$

where $\Omega_1 := \{t \in [0, pT] \mid |u(t)| \le W\}$, $\Omega_2 := \{t \in [0, pT] \mid |u(t)| > W\}$ and

$$M := \sup_{t\in[0,pT], |x|\le W} F(t,x).$$

By (2.7), (2.9) and (3.4), we have

$$\begin{aligned}\Phi(u) &\ge \frac{1}{2}\|\dot{u}\|^2_{L^2} - \left(\left(\frac{1}{2} - \varrho p^2 T\right)\frac{\omega^2}{p^2} - \frac{\varrho}{T} - \varepsilon_0\right)\|u\|^2_{L^2} - MpT - \varrho T^{-1}\|u\|^2_{L^2} - \varrho p^2 T\|\dot{u}\|^2_{L^2} \\ &\ge \left(\frac{1}{2} - \varrho p^2 T\right)\|\dot{u}\|^2_{L^2} - \left(\left(\frac{1}{2} - \varrho p^2 T\right)\frac{\omega^2}{p^2} - \varepsilon_0\right)\frac{p^2}{\omega^2}\|\dot{u}\|^2_{L^2} - MpT \\ &\ge \frac{\varepsilon_0 p^2}{\omega^2 + p^2}\|u\|^2 - MpT, \quad \text{for any } u \in X.\end{aligned}$$

So for any $u \in X$, $\Phi(u) \to +\infty$ as $\|u\| \to \infty$. Thus it follows from Lemma 2.3 that the result of Step 1 holds.

*Step 2.* Under the assumptions of Theorem 3.1, we have

$$\inf_{u\in X} \Phi(u) < B_{s_p}, \tag{3.5}$$

where

$$B_q := -\frac{p}{2}\left[(1 - 2\varrho p^2 T)\left(\frac{\omega q}{p}\right)^2 - \lambda - A - \frac{2\varrho}{T}\right]^{-1}\int_0^T |\nabla F(t,0)|^2 dt,$$

for integer $q \ge 1$.

In fact, taking $\overline{u}(t) = (B\sin(\omega t/p), 0, \ldots, 0)^T$, it is clear that $\overline{u} \in X$. Since $\nabla F(t,0) \in C(\mathbb{R}, \mathbb{R}^N)$ is $T$-periodic and $p > 1$, by Fourier expansion, we have $\int_0^{pT} \nabla F(t,0) Q\overline{u}(t)dt = 0$. The continuity of $I_{rj}(y)$, (2.1) and (H3) imply that

$$|\phi(\overline{u})| \le \sum_{j=1}^{pl}\sum_{r=1}^{N} \left|I_{rj}(\theta) q_{r1} B\sin(\omega t_j/p)\right| \le pB^2 \sum_{j=1}^{l}\left(\sum_{r=1}^{N} a^2_{rj}\right)^{\frac{1}{2}}, \tag{3.6}$$

where $\theta$ lies between 0 and $q_{r1}B\sin(\omega t_j/p)$. By (H2), we have

$$\int_0^{pT} F(t,Q\overline{u}(t))dt = \int_0^{pT} F(t,Q\overline{u}(t)) - \nabla F(t,0)Q\overline{u}(t)dt + \int_0^{pT} \nabla F(t,0)Q\overline{u}(t)dt \geq \frac{\overline{A}}{2}\|\overline{u}\|_{L^2}^2,$$

which combined with (3.6) yields to

$$\begin{aligned}
\Phi(\overline{u}) &\leq \frac{1}{2}\|\dot{\overline{u}}\|_{L^2}^2 - \frac{1}{2}\lambda\|\overline{u}\|_{L^2}^2 - \frac{\overline{A}}{2}\|\overline{u}\|_{L^2}^2 + pB^2\sum_{j=1}^{l}\left(\sum_{r=1}^{N}a_{rj}^2\right)^{\frac{1}{2}} \\
&= \frac{1}{2}B^2\left(\frac{\omega}{p}\right)^2\frac{pT}{2} - \frac{1}{2}(\lambda+\overline{A})B^2\frac{pT}{2} + pB^2\sum_{j=1}^{l}\left(\sum_{r=1}^{N}a_{rj}^2\right)^{\frac{1}{2}} \\
&= \frac{p\pi B^2}{2\omega}\left[\left(\frac{\omega}{p}\right)^2 - \lambda - \overline{A}\right] + pB^2\sum_{j=1}^{l}\left(\sum_{r=1}^{N}a_{rj}^2\right)^{\frac{1}{2}}.
\end{aligned} \tag{3.7}$$

It follows from (3.1) that

$$(1-2\varrho p^2T)\left(\frac{\omega s_p}{p}\right)^2 - \lambda - A - \frac{2\varrho}{T} > 0,$$

thus (3.3) implies that

$$\begin{aligned}
&\frac{p\pi B^2}{2\omega}\left[\left(\frac{\omega}{p}\right)^2 - \lambda - \overline{A}\right] + pB^2\sum_{j=1}^{l}\left(\sum_{r=1}^{N}a_{rj}^2\right)^{\frac{1}{2}} \\
&\quad < -\frac{p}{2}\left[(1-2\varrho p^2T)\left(\frac{\omega s_p}{p}\right)^2 - \lambda - A - \frac{2\varrho}{T}\right]^{-1}\int_0^T |\nabla F(t,0)|^2 dt = B_{s_p},
\end{aligned}$$

which combined with (3.7) yields to (3.5).

*Step 3.* The critical point $u^*$ has minimal period $pT$.

Assume the contrary; minimal period of $u^*$ is $pT/q$ for some integer $q > 1$. By Lemma 2.4, $q$ is a factor of $p$ and $q \geq s_p$. By Fourier expansion,

$$u^*(t) = \sum_{i=1}^{N}\left[\sum_{k=1}^{+\infty} b_{ki}\sin\frac{\omega qk}{p}t\right]e_i,$$

where $\{e_1, e_2, \ldots, e_N\}$ denotes the canonical orthogonal basis in $\mathbb{R}^N$. By (H2), (2.1) and Hölder's inequality,

$$\begin{aligned}
\int_0^{pT} F(t,Qu^*(t))dt &= \int_0^{pT} F(t,Qu^*(t)) - \nabla F(t,0)Qu^*(t)dt + \int_0^{pT} \nabla F(t,0)Qu^*(t)dt \\
&\leq \frac{A}{2}\|u^*\|_{L^2}^2 + \|\nabla F(t,0)\|_{L^2}\|u^*\|_{L^2},
\end{aligned}$$

which combined with (2.9) yields to

$$\begin{aligned}
\Phi(u^*) &\geq \frac{1}{2}\|\dot{u}^*\|_{L^2}^2 - \frac{1}{2}\lambda\|u^*\|_{L^2}^2 - \frac{A}{2}\|u^*\|_{L^2}^2 - \|\nabla F(t,0)\|_{L^2}\|u^*\|_{L^2} - \frac{\varrho}{T}\|u^*\|_{L^2}^2 - \varrho p^2T\|\dot{u}^*\|_{L^2}^2 \\
&\geq \left(\frac{1}{2} - \varrho p^2T\right)\left(\frac{\omega q}{p}\right)^2\|u^*\|_{L^2}^2 - \frac{1}{2}(\lambda + A + \frac{2\varrho}{T})\|u^*\|_{L^2}^2 - \|\nabla F(t,0)\|_{L^2}\|u^*\|_{L^2} \\
&= \left[(1-2\varrho p^2T)\left(\frac{\omega q}{p}\right)^2 - \lambda - A - \frac{2\varrho}{T}\right]\frac{\|u^*\|_{L^2}^2}{2} - \left(p\int_0^T |\nabla F(t,0)|^2 dt\right)^{\frac{1}{2}}\|u^*\|_{L^2} \geq B_q,
\end{aligned}$$

as we find by minimizing with respect to $\|u^*\|_{L^2}$. This contradicts with (3.5) since $B_q \geq B_{s_p}$ for $q \geq s_p$.

Thus it follows from Lemma 2.4 that $\Phi$ has a critical point $u^*$ on $H^1_{pT}$ and $u^*$ has minimal period $pT$. Therefore $Qu^*$ is a weak periodic solution of (1.1) with minimal period $pT$ by Lemma 2.1. $\square$

## 4 Examples and corollaries

In this section, an example is given to illustrate Theorem 3.1, and a corollary of Theorem 3.1 concerning the equations (1.1a) is presented.

**Example 4.1.** Consider the impulsive system (1.1) with $\lambda = -1$, $N = 3$, $l = 1$, impulsive functions $I_{i1}(s) = -s/9$ for $i = 1, 2, 3$ and

$$F(t,x) = \frac{1}{2}(25\cos(10\pi t) + 425)\sum_{i=1}^{3} x_i \sin x_i.$$

Since $x \sin x \leq x^2$, $400 \leq (25\cos(10\pi t) + 425) \leq 450$ and

$$\sin x \geq \frac{2}{\pi}x, \quad \text{for } 0 \leq x \leq \frac{\pi}{2},$$

we have (H1), (H2) and (H3) hold with $T = 0.2$, $A = 450$, $\overline{A} = 800/\pi$, $B = \pi/2$ and $a_{i1} = 1/9$ for $i = 1, 2, 3$. In view of

$$\lim_{|x|\to\infty} \frac{F(t,x)}{|x|^2} = 0 \quad \text{and} \quad |\nabla F(t,0)| = 0,$$

it could be verified directly that all the assumptions of Theorem 3.1 hold with $p = 2$. Thus Example 4.1 has at least one weak periodic solution with minimal period 0.4.

When $I_{ij} \equiv 0$, assumption (H3) holds with $a_{ij} = 0$. The following result concerning the equation (1.1a) could be deserved by Theorem 3.1.

**Corollary 4.2.** *Assume that $F$ satisfies (H1), (H2) and there exists an integer $p > 1$ such that*

$$\frac{\omega^2}{\overline{A} + \lambda} < p^2 < \frac{\omega^2 s_p^2}{A + \lambda},$$

$$\limsup_{|x|\to\infty} \frac{F(t,x)}{|x|^2} < \frac{\omega^2}{2p^2} - \frac{\lambda}{2}, \quad \textit{uniformly for } t \in \mathbb{R}$$

*and*

$$\int_0^T |\nabla F(t,0)|^2 dt < \frac{\pi B^2}{\omega}\left[\overline{A} + \lambda - \left(\frac{\omega}{p}\right)^2\right]\left[\left(\frac{\omega s_p}{p}\right)^2 - \lambda - A\right],$$

*where $s_p$ is the least prime factor of $p$. Then the equation (1.1a) has at least one weak periodic solution with minimal period $pT$.*

**Remark 4.3.** When prime integer $p \to \infty$, the following is deserved by Corollary 4.2.

Assume that $F$ satisfies (H1), (H2) and

$$\limsup_{|x|\to\infty} \frac{F(t,x)}{|x|^2} \leq -\frac{\lambda}{2}, \quad \text{uniformly for } t \in \mathbb{R}.$$

If
$$\int_0^T |\nabla F(t,0)|^2 dt < \frac{\pi B^2}{\omega} \left(\overline{A} + \lambda\right) (\omega^2 - \lambda - A),$$
then there exists $P > 0$ such that, for any prime integer $p > P$, the equation (1.1a) has at least one weak periodic solution with minimal period $pT$.

## Acknowledgements

The authors would like to present their sincere thanks to the anonymous reviewer for his/her valuable and helpful comments and suggestions, which greatly improved this paper.

This work has been partially supported by National Natural Science Foundation of China (No. 11401420, No. 11326117, No. 11626182), Natural Science Foundation of Shanxi (No. 2013021001-2, No. 201601D102002), YFTUT (No. 2013T062) and TFTUT (No. tyut-rc201212a).

## References

[1] G. Bonanno, B. D. Bella, J. Henderson, Existence of solutions to second-order boundary-value problems with small perturbations of impulses, *Electron. J. Differ. Equ.* **2013**, No. 126, 1–14. MR3065079

[2] G. Bonanno, R. Rodriguez-Lopez, S. Tersian, Existence of solutions to boundary value problem for impulsive fractional differential equations, *Fract. Calc. Appl. Anal.* **17**(2014), No. 3, 717–744. MR3260304; url

[3] P. Chen, X. Tang, Existence and multiplicity of solutions for second-order impulsive differential equations with Dirichlet problems, *Appl. Math. Comput.* **218**(2012), No. 24, 11775–11789. MR2945181; url

[4] B. Dai, D. Zhang, The existence and multiplicity of solutions for second-order impulsive differential equations on the half-line, *Results Math.* **63**(2013), No. 1-2, 135–149. MR3009677; url

[5] X. He, P. Chen, Existence of solutions for a class of second-order sublinear and linear Hamiltonian systems with impulsive effects, *Electron. J. Qual. Theory Differ. Equ.* **2011**, No. 78, 1–15. MR2838506; url

[6] Z. Luo, J. Xiao, Y. Xu, Subharmonic solutions with prescribed minimal period for some second-order impulsive differential equations, *Nonlinear Anal.* **75**(2012), No. 4, 2249–2255. MR2870914; url

[7] J. Mawhin, M. Willem, *Critical point theory and Hamiltonian systems*, Springer-Verlag, New York, 1989. MR0982267; url

[8] J. J. Nieto, Variational formulation of a damped Dirichlet impulsive problem, *Appl. Math. Lett.* **23**(2010), No. 8, 940–942. MR2651478; url

[9] J. J. Nieto, D. O'Regan, Variational approach to impulsive differential equations, *Nonlinear Anal. Real World Appl.* **10**(2009), No. 2, 680–690. MR2474254; url

[10] P. H. Rabinowitz, On subharmonic solutions of Hamiltonian systems, *Comm. Pure Appl. Math.* **33**(1980), No. 5, 609–633. MR0586414; url

[11] T. Shen, W. Liu, Multiplicity of solutions for Dirichlet boundary conditions of second-order quasilinear equations with impulsive effects, *Electron. J. Qual. Theory Differ. Equ.* **2015**, No. 97, 1–10. MR3438737; url

[12] M. Struwe, *Variational methods. Applications to nonlinear partial differential equations and Hamiltonian systems*, 4th ed., Springer, Berlin, 2008. MR2431434; url

[13] J. Sun, H. Chen, J. J. Nieto, M. Otero-Novoa, The multiplicity of solutions for perturbed second-order Hamiltonian systems with impulsive effects, *Nonlinear Anal.* **72**(2010), No. 12, 4575–4586. MR2639205; url

[14] Y. Tian, W. Ge, Applications of variational methods to boundary-value problem for impulsive differential equations, *Proc. Edinburgh Math. Soc. (2)* **51**(2008), No. 2, 509–528. MR2465922; url

[15] Q. Wang, Z.-Q. Wang, J.-Y. Shi, Subharmonic oscillations with prescribed minimal period for a class of Hamiltonian systems, *Nonlinear Anal.* **28**(1997), No. 7, 1273–1282. MR1422815; url

[16] J. Xie, Z. Luo, Subharmonic solutions with prescribed minimal period of an impulsive forced pendulum equation, *Appl. Math. Lett.* **52**(2016), 169–175. MR3416402; url

[17] J. Yu, Subharmonic solutions with prescribed minimal period of a class of nonautonomous Hamiltonian systems, *J. Dynam. Differential Equations* **20**(2008), No. 4, 787–796. MR2448212; url

[18] X. Zhang, X. Tang, Subharmonic solutions for a class of non-quadratic second order Hamiltonian systems, *Nonlinear Anal. Real World Appl.* **13**(2012), No. 1, 113–130. MR2846824; url