Turán type oscillation inequalities in L^q norm on the boundary of convex domains

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Abstract

Some 76 years ago Paul Turán was the first to establish lower estimations of the ratio of the maximum norm of the derivatives of polynomials and the maximum norm of the polynomials themselves on the interval $\mathbb{I} := [-1, 1]$ and on the unit disk $\mathbb{D} := \{z \in \mathbb{C} \ : \ |z| \leq 1\}$ under the normalization condition that the zeroes of the polynomial all lie in the interval or in the disk, respectively. He proved that with $n := \deg p$ tending to infinity, the precise growth order of the minimal possible ratio of the derivative norm and the norm is $\sqrt{n}$ for $\mathbb{I}$ and n for $\mathbb{D}$.

Erőd continued the work of Turán and extended his results to several other domains. The growth of the minimal possible ratio of the ∞-norm of the derivative and the polynomial itself was proved to be of order n for all compact convex domains a decade ago.

Although Turán himself gave comments about the above oscillation question in L^q norms, till recently results were known only for $\mathbb{D}$ and $\mathbb{I}$. Here we prove that in L^q norm the oscillation order is again n for a certain class of convex domains, including all smooth convex domains and also convex polygonal domains having no acute angles at their vertices.

Keywords: Bernstein-Markov Inequalities, Turán's lower estimate of derivative norm, logarithmic derivative, Chebyshev constant, transfinite diameter, capacity, outer angle, convex domains, width of a convex domain, depth of a convex domain, Blaschke Rolling Ball Theorems
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1. Introduction

1.1. The oscillation of a polynomial in maximum norm

At the turn of the 19th and 20th centuries, the first estimates of the derivative of a polynomial via the maximum of its values appeared. They were obtain by V.Markov in 1889, for algebraic polynomials on an interval, by Bernstein and M. Riesz in 1914, for trigonometric polynomials on $[0, 2\pi]$ and algebraic polynomials on the unit circle. In 1923, Szegő [52] obtained an estimate for a large class of (not necessarily convex, but piecewise smooth) domains. Namely, if $K \subset \mathbb{C}$ is a piecewise smooth simply connected domain, with its boundary consisting of finitely many analytic Jordan arcs, and if the maximum of the outer angles at the joining vertices of these arcs is[1] $\beta \in [\pi, 2\pi]$, then the domain admits a Markov type inequality of the form $\|p'\|_K \le c_K n^{\beta/2\pi} \|p\|_K$ for any polynomial p of degree n. Here the norm $\|\cdot\| := \|\cdot\|_K$ denotes sup norm over values attained on K. This inequality is essentially sharp for all such domains. In particular, this immediately implies that for *analytically smooth* convex domains the Markov factor is $O(n)$. For the unit disk

$$\mathbb{D} := \{z \in \mathbb{C} \ : \ |z| \le 1\}$$

even the exact inequality is well-known:

$$\|p'\|_{\mathbb{D}} \le n \|p\|_{\mathbb{D}}. \tag{1}$$

This was conjectured, and almost proved, by Bernstein [9, 10]; for the first published proof see [44]. Similarly, the precise result is also classical for the unit interval

$$\mathbb{I} := [-1, 1]$$

then we have Markov's Inequality $\|p'\|_{\mathbb{I}} \le n^2 \|p\|_{\mathbb{I}}$, which is sharp[2], see [34].

In 1939 Paul Turán started to study converse inequalities of the form

$$\|p'\|_K \ge c_K n^A \|p\|_K.$$

Clearly such a converse can only hold if further restrictions are imposed on the occurring polynomials p. Turán assumed that all zeroes of the polynomials belong to K. So denote the set of complex (algebraic) polynomials of degree (exactly) n as $\mathcal{P}_n$, and the subset with all the n (complex) roots in some set $K \subset \mathbb{C}$ by $\mathcal{P}_n(K)$. Denote by Γ the boundary of K. The (normalized) quantity under our study is the "inverse Markov factor" or "oscillation factor"

$$M_{n,q}(K) := \inf_{p \in \mathcal{P}_n(K)} M_q(p) \qquad \text{with} \qquad M_q(p) := \frac{\|p'\|_{L^q(\Gamma)}}{\|p\|_{L^q(\Gamma)}}, \tag{2}$$

[1] If the domain is bounded, then for all directions it has supporting lines, whence there are points where the outer angle is at least π.

[2] Note that in this case the outer angles at the break-points of the piecewise smooth boundary are exactly 2π at each end.

where, as usual,

$$\|p\|_q := \|p\|_{L^q(\Gamma)} := \left(\int_\Gamma |p(z)|^q |dz| \right)^{1/q}, \quad (0 < q < \infty)$$
$$\|p\|_K := \|p\|_{L^\infty(\Gamma)} = \sup_{z\in\Gamma} |p(z)| = \sup_{z\in K} |p(z)|.$$

Theorem A (Turán, [56, p. 90]). *If $p \in \mathcal{P}_n(\mathbb{D})$, then we have*

$$\|p'\|_{\mathbb{D}} \geq \frac{n}{2} \|p\|_{\mathbb{D}} . \tag{3}$$

Theorem B (Turán, [56, p. 91]). *If $p \in \mathcal{P}_n(\mathbb{I})$, then we have*

$$\|p'\|_{\mathbb{I}} \geq \frac{\sqrt{n}}{6} \|p\|_{\mathbb{I}} .$$

Inequality (3) of Theorem A is best possible. Regarding Theorem B, Turán pointed out by example of $(1-x^2)^n$ that the $\sqrt{n}$ order is sharp. Some slightly improved constants can be found in [6] and [31], however, the exact value of the constants and the corresponding extremal polynomials were already computed for all fixed n by Erőd in [21].

Now we are going to describe results concerning Turán-type inequalities (2) for general convex sets. To study (2) some geometric parameters of the convex domain K are involved naturally. We write $d := d(K) := \operatorname{diam}(K)$ for the *diameter* of K, and $w := w(K) := \operatorname{width}(K)$ for the *minimal width* of K. That is,

$$d(K) := \max_{z', z'' \in K} |z' - z''|,$$

$$w(K) := \min_{\gamma\in[-\pi,\pi]} \left(\max_{z\in K} \Re(ze^{i\gamma}) - \min_{z\in K} \Re(ze^{i\gamma}) \right).$$

Note that a (closed) convex domain is a (closed), bounded, convex set $K \subset \mathbb{C}$ with nonempty interior, hence $0 < w(K) \leq d(K) < \infty$.

The key to Theorem A was the following observation, which had already been present implicitly in [56, the footnote on p. 93] and [21] and was later formulated explicitly by Levenberg and Poletsky in [31, Proposition 2.1].

Lemma C (Turán). *Assume that $z \in \partial K$ and that there exists a disc $D_R = \{\zeta \in \mathbb{C} \ : \ |\zeta - z_0| = R\}$ of radius R so that $z \in \partial D_R$ and $K \subset D_R$. Then for all $p \in \mathcal{P}_n(K)$ we have*

$$|p'(z)| \geq \frac{n}{2R} |p(z)|. \tag{4}$$

The proof of this is really easy, so let us recall it for completeness.

PROOF. As $\dfrac{p'}{p}(z) = \sum_j \dfrac{1}{z - z_j}$ and $\Re \dfrac{1}{1-\zeta} \ge 1/2 \quad (\forall |\zeta| < 1)$, if all $z_j \in D_R$, then

$$R \cdot \left| \frac{p'}{p}(z) \right| \ge \Re \left\{ (z - z_0) \sum_j \frac{1}{z - z_j} \right\}$$
$$= \sum_j \Re \frac{z - z_0}{(z - z_0) - (z_j - z_0)} = \sum_j \Re \frac{1}{1 - \frac{z_j - z_0}{z - z_0}} \ge \frac{n}{2}.$$

Given this elementary observation, Levenberg and Poletsky found it worthwhile to formally define the crucial property of convex sets, necessary for drawing such an easy and direct conclusion on the inverse Markov factors.

Definition 1. *A set $K \Subset \mathbb{C}$ is called* R-circular*, if for any $z \in \partial K$ there exists a disk D_R of radius R, such that $z \in \partial D_R$ and $D_R \supset K$.*

Thus for any R-circular K and $p \in \mathcal{P}_n(K)$ at the boundary point $z \in \partial K$ with $\|p\|_K = |p(z)|$ we can draw the disk D_R and it follows

Theorem D (Erőd; Levenberg-Poletsky). *For an R-circular K we have*

$$\|p'\|_K \ge \frac{n}{2R} \|p\|_K \quad (\forall p \in \mathcal{P}_n(K)) \qquad \text{that is} \qquad M_n(K) \ge \frac{n}{2R} \ .$$

There are many important examples of R-circular compact sets and domains. E.g. a union of two circular arcs, joining at a vertex of angle less than π, is always R-circular with some R. Smooth convex closed curves, together with the encircled convex domain K, with curvature exceeding $\kappa > 0$ are always R-circular with $R = 1/\kappa$ according to a classical theorem of Blaschke [11]. Further extensions of the Blaschke Rolling Ball Theorem allows to realize R-circularity of much more general convex curves γ.

Lemma E (Strantzen). *Let the convex domain K have boundary curve $\Gamma = \partial K$ and let $\kappa > 0$ be a fixed constant. Assume that the convex boundary curve Γ (which is twice differentiable linearly almost everywhere) satisfies the curvature condition $\ddot{\Gamma} \ge \kappa$ almost everywhere. Then to each boundary point $\zeta \in \partial K$ there exists a disk D_R of radius $R = 1/\kappa$, such that $\zeta \in \partial D_R$, and $K \subset D_R$. That is, K is $R = 1/\kappa$-circular.*

PROOF. This result is essentially the far-reaching, relatively recent generalization of Blaschke's Rolling Ball Theorem by Strantzen. A reference for it is Lemma 9.11 on p. 83 of [16]. For more details on this, as well as for some new approaches to the proof of this generalization of the classical Blaschke Rolling Ball Theorem, see [43].

Obviously, the above entails an order n oscillation result for all convex domains with this a.e. condition on the curvature of the boundary curve. This leads us to the topic of Turán type oscillation problems for more general sets and domains.

Drawing from the work of Turán, Erőd [21, p. 74] already addressed the question: "For what kind of domains does he method of Turán apply?" Clearly, by "applies" he meant that it provides order n oscillation for the derivative. Moreover, he introduced new ideas into the investigation – including the application of Chebyshev's Inequality (6) below – so clearly he did not simply pursue the effect of Turán's original methods, but was indeed after the right oscillation order of general domains. In particular, he showed

Theorem F (Erőd, [21, p. 73]). *Let $0 < b < 1$ and let E_b denote the ellipse domain with major axes $[-1, 1]$ and minor axes $[-ib, ib]$. Then for all $p \in \mathcal{P}_n(E_b)$ we have*

$$\|p'\| \geq \frac{b}{2} n \|p\|.$$

Moreover, he elaborated on the inverse Markov factors belonging to domains with some favorable geometric properties.The most general domains with $M(K) \gg n$, found by Erőd, were described on p. 77 of [21].

Theorem G (Erőd). *Let K be any convex domain bounded by finitely many Jordan arcs, joining at vertices with angles $< \pi$, with all the arcs being C^2-smooth and being either straight lines of length $< \Delta(K)$, where $\Delta(K)$ stands for the transfinite diameter of K, or having positive curvature bounded away from 0 by a fixed constant $\kappa > 0$. Then there is a constant $c(K)$, such that $M_n(K) \geq c(K)n$ for all $n \in \mathbb{N}$.*

Note that this latter result of Erőd incorporates regular k-gons G_k for large enough k, but not the square $Q = G_4$, because the side length h of a square is larger than the quarter of the transfinite diameter Δ: actually, $\Delta(Q) \approx 0.59017\ldots h$, while for the regular k-gon of side length h we have

$$\Delta(G_k) = \frac{\Gamma(1/k)}{\sqrt{\pi} 2^{1+2/k} \Gamma(1/2 + 1/k)} h$$

(see e.g. [38, p. 135]), so $\Delta(G_k) > h$ iff $k \geq 7$. This implies $M_n(G_k) \geq c_k n$ for $k \geq 7$.

In [19], Erdélyi proved order n oscillation for the square[3] $Q = G_4$, too. A result of [40] also implied $M_n(G_k) \geq c_k n$ for $k \geq 4$, but still not for a triangle.

To deal with the flat case of straight line boundary arcs, Erőd involved another approach, cf. [21, p. 76], appearing later to be essential for obtaining a general answer formulated in Theorem I below, and playing an essential role in many further developments, including ours here. Namely, Erőd quoted Faber

[3]Erdélyi also proves similar results on rhombuses, under the further condition of some symmetry of the polynomials in consideration – e.g. if the polynomials are real, or odd. Note also that his work on the topic preceded [41] and apparently was accomplished without being aware of details of [21].

[22] for the fundamental result of Chebyshev on the monic polynomial of minimal norm on an interval. Since this approach will be extensively applied also in our work, we summarized basic facts regarding this in the below Section 3.

For a few further examples, remarks and open problems regarding inverse Markov factors for various classes of compact sets which are not necessarily convex, see [31, 41, 42].

A lower estimate of the inverse Markov factor for any compact convex set (and of the same order as was known for the interval) was obtained in full generality only in 2002, see [31, Theorem 3.2].

Theorem H (Levenberg-Poletsky). *If $K \subset \mathbb{C}$ is a compact, convex set, $d := \operatorname{diam} K$ and $p \in \mathcal{P}_n(K)$, then we have*

$$\|p'\|_K \geq \frac{\sqrt{n}}{20 \operatorname{diam}(K)} \|p\|_K \ .$$

Clearly, assuming boundedness is natural, since all polynomials have $\|p_n\|_K = \infty$ when the set K is unbounded. Also, restricting ourselves to *closed* bounded sets – i.e., to compact sets – does not change the sup norm of polynomials under study, as all polynomials are continuous.

Recall that the term *convex domain* stands for a compact, convex subset of $\mathbb{C}$ having nonempty interior. That is, assuming that K is a (bounded, closed) convex *domain*, not just a compact convex *set*, means that we exclude only the case of the interval, for which already Turán clarified that the order of oscillation is precisely $\sqrt{n}$.

So in order to clarify the order of oscillation for all compact convex sets it remains to clarify the order of oscillation for compact convex domains. The solution of this general problem[4] has been published in 2006, see [41].

Theorem I (Halász–Révész). *Let $K \subset \mathbb{C}$ be any bounded convex domain. Then for all $p \in \mathcal{P}_n(K)$ we have*

$$\|p'\|_K \geq 0.0003 \frac{w(K)}{d^2(K)} n \|p\|_K \ .$$

Remark 1. This indeed provides the precise order, for an even larger order than n cannot occur, not for any particular compact set. Namely, let $K \subset \mathbb{C}$ be any compact set with diameter $d := \operatorname{diam}(K)$. Then for all n there exists a polynomial $p \in \mathcal{P}_n(K)$ of degree exactly n satisfying

$$\|p'\| \leq \ C'(K) \ n \ \|p\| \qquad \text{with} \qquad C'(K) := 1/\operatorname{diam}(K).$$

[4] Preceding this, an intermediate result of order $n^{2/3}$ oscillation for all compact convex domains has been worked out in [40] – in view of the later developments, this has not been published in a journal.

Indeed, considering a diameter $[z_0, w_0]$ and the polynomial $p(z) = (z - z_0)^d$, the respective norm is $\|p\|_\infty = d^n$ while the derivative norm becomes $\|p'\|_\infty = nd^{n-1}$, both attained at $w_0 \in K$.

So, this settles the question of the *order*, but not the precise *dependence on the geometry*. However, up to an *absolute constant factor*, even the dependence on the geometrical features of the domain was also clarified in [41].

Theorem J (Révész). *Let $K \subset \mathbb{C}$ be any compact, connected set with diameter d and minimal width w. Then for all $n > n_0 := n_0(K) := 2(d/16w)^2 \log(d/16w)$ there exists a polynomial $p \in \mathcal{P}_n(K)$ of degree exactly n satisfying*

$$\|p'\|_K \le C'(K) n \|p\|_K \qquad \text{with} \qquad C'(K) := 600\, \frac{w(K)}{d^2(K)} \ .$$

So, interestingly, it turned out that among all convex compacta only intervals can have an inverse Markov constant of order $\sqrt{n}$, while domains with nonempty interior have oscillation order n.

One may ask, how useful, how general these results are? One fundamental area of potential applications is the theory of orthogonal polynomials. Badkov [7, 8] applied Turán's inequality (3) for estimations of polynomials orthogonal on the circle with respect to a weight. It is well known that polynomials orthogonal on a circle or on an interval (with respect to some weight there) have all their zeros on the interval or inside the circle, respectively. That is, if P_n is the n^{th} orthogonal polynomail, then certainly we have $P_n \in \mathcal{P}_n(K)$ and the above oscillation results apply.

Analogous phenomenon takes place in the case of a rectifiable curve or, more generally, a compact set with a measure. The precise statements can be found in [23, Satz III], [46], [18, §10.2], [54, Ch XVI, 16.2, (6)], [51].

The respective upper estimations, i.e. Bernstein-Markov type inequalities were extensively studied under analogous constraints for the zeroes [20]. Since the oscillation results of Turán type are also formulated under zero restrictions, it is of interest to compare the upper and lower estimations of these derivative norm estimates.

The first relevant result were asked about by Erdős and solved by Lax [30]: this states that if the zeroes of a polynomial are all *outside* the unit circle, then the classical Bernstein Inequality (1) can be improved to $\|p'\|_{\mathbb{D}} \le \frac{n}{2}\|p\|_{\mathbb{D}}$. For further study of the topic of constrained Bernstein-Markov Inequalities we refer the reader to [32, Theorem, p. 58], [20, 3] and the references therein.

This improvement by the factor 1/2 reminds us the Turán inequality: the common extremal polynomial is indeed the boundary case $p_n(z) := z^n - 1$ (and rotations of thereof). Note that here the Turán type restriction of $p \in \mathcal{P}_n(K)$ does not allow any improvement: the extremal z^n provides an oscillation of exactly n in regard of the Bernstein Inequality. Therefore, this very first result in constrained Bernstein-Markov Inequalities already prompts us to consider classes of polynomials with taking the norm on one set, while restricting the location of zeroes to another one. Concretely, the above Lax result talks about the class $\mathcal{P}_n(\mathbb{C}\setminus\mathbb{D})$, under the (maximum) norm on $\mathbb{D}$ (or on the boundary circle $\partial\mathbb{D}$). Analogously, we can consider $\mathcal{P}_n(K)$ under the norm on another set L: the respective oscillation factors we may denote by

$$M_{n,L}(K) := \inf_{P\in\mathcal{P}_n(K)} M_{\|\cdot\|_L}(P) \qquad \text{with} \qquad M_{\|\cdot\|_L}(P) := \frac{\|P'\|_L}{\|P\|_L},$$

with the norm $\|\cdot\|_L$ being the maximum norm[5] taken on the set L.

In connection to the Turán topic, this has also been investigated at least when the set L is a disk: $L = D_R = \{z \ : \ |z| \le R\}$.

Malik [32] ($R<1$) and Govil [28] ($R>1$) showed that

Theorem K (Malik, Govil). *For any $R \ge 0$ we have*

$$M_{n,D_R}(K) = \begin{cases} \frac{n}{1+R}, & R \le 1, \\ \frac{n}{1+R^n}, & R \ge 1. \end{cases}$$

See also [1, 4, 36] and the references therein. However, apart from these rather special choices of concentric disks, we have not found any result in the literature for more general situations.

1.2. Pointwise and integral mean estimates of oscillation

There are many papers dealing with the L^q-versions of Turán's inequality for the (unit) disk $\mathbb{D}$, the (unit) interval $\mathbb{I}$, or for the period (one dimensional torus or circle) $\mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$ (here with the understanding that we consider real trigonometric polynomials, not complex polynomials). A nice review of the results obtained before 1994 is given in [35, Ch. 6, 6.2.6, 6.3.1].

The story started by an obvious observation. Namely, already Turán himself mentioned in [56] that on the perimeter of the disk $\mathbb{D}$ – and, as is easily observed, the same way under conditions of Theorem A – actually a more general pointwise inequality holds *at all points* of $\partial\mathbb{D}$. Namely, for $p \in \mathcal{P}_n(K)$ we have

$$|p'(z)| \ge \frac{n}{2}|p(z)|, \quad |z| = 1, \tag{5}$$

and as a corollary, for any $q > 0$,

$$\left(\int_{|z|=1} |p'(z)|^q |dz| \right)^{1/q} \ge \frac{n}{2} \left(\int_{|z|=1} |p(z)|^q |dz| \right)^{1/q}.$$

[5] Or, more generally, one may even consider $M_{n,\|\cdot\|}(K) := \inf_{P\in\mathcal{P}_n(K)} M_{\|\cdot\|}(P)$ with $M_{\|\cdot\|}(P) := \frac{\|P'\|}{\|P\|}$ being an arbitrarily fixed norm $\|\cdot\|$.

In other words, the Turán result Theorem A extends to all norms on the perimeter, including all norms $L^q(\partial\mathbb{D})$, and we have for all polynomials $p \in \mathcal{P}_n(K)$

$$\|p'\|_{L^q(\partial\mathbb{D})} \geq \frac{n}{2}\|p\|_{L^q(\partial\mathbb{D})}, \qquad M_{n,q}(\mathbb{D}) \geq \frac{n}{2} .$$

The same way, for R-circular domains the result of Theorem D extends as

$$\|p'\|_{L^q(\partial K)} \geq \frac{n}{2R}\|p\|_{L^q(\partial K)}, \qquad M_{n,q}(K) \geq \frac{n}{2R} .$$

However, calculating the respective quantities for the function $p_n(z) := z^n - 1$, it turns out that this inequality is not the best (at least not for the most symmetric choice p_n), and the determination of the value of the precise constant, as well as identifying the extremal functions, remains to be done.

A related question is to compare the maximum norm of p' to the L^q norm of p itself. In this regard, the exact constant is known from the work of Malik in [33].

Theorem L (Malik). *For the unit disc $\mathbb{D}$ and any $0 < q < \infty$ we have*

$$\|p'\|_{\mathbb{D}} \geq \left(\frac{\Gamma(q/2+1)}{2\sqrt{\pi}\Gamma(q/2+1/2)}\right)^{1/q} \frac{n}{2} \|p\|_{L^q(\partial\mathbb{D})},$$

moreover, the inequality is the best possible, equality occurring precisely for all $az^n + b$ with $|a| = |b|$.

Some other results comparing different norms were obtained by Saff and Sheil-Small [47], Rubinstein [45], Aziz and Ahemad [4], Paul, Shah and Singh [36].

In the paper [7, Cor. 11.1], another proof of the pointwise Turán inequality (5) is given. The proof is based on the properties of orthogonal polynomials and the Christoffel function.

The classical inequalities of Bernstein and Markov are generalized for various differential operators, too, see [2]. In this context, also Turán type converses have been already investigated: namely, Akopyan [1] studied Turán-type inequalities in L^2-norm on the circle for some generalization of the operator of differentiation.

The estimation of the L^q norm, or of any norm including e.g. any weighted L^q-norms, goes the same way if we have a pointwise estimation for all, (or for linearly almost all), boundary points. Therefore, as above, we can formulate e.g. the next result, see [42].

Theorem M . *Assume that the boundary curve $\gamma : [0, L] \to \Gamma := \partial K$ of the convex domain K satisfies at (linearly) almost all points the condition that it has a curvature, not smaller than a given positive constant κ, i.e. $\ddot{\gamma} \geq \kappa (> 0)$ a.e.*

Then for any norm $\| \cdot \|$, defined for some class of functions (including the polynomials) on the boundary curve, we have $\|p'\| \geq \frac{n}{2\kappa}\|p\|$. In particular, $M_{n,q}(K) \geq \frac{n}{2\kappa}$.

In case we discuss maximum norms, one can assume that $p(z)$ is maximal, and it suffices to obtain a lower estimation of $p'(z)$ only at such a special point – for general norms, however, this is not sufficient. The above results work only for we have a pointwise inequality of the same strength *everywhere*, or almost everywhere. The situation becomes considerably more difficult, when such a statement cannot be proved. E.g. if the domain in question is not strictly convex, i.e. if there is a line segment on the boundary, then the zeroes of the polynomial can be arranged so that even some zeroes of the derivative lie on the boundary, and at such points $p'(z)$ – even $p'(z)/p(z)$ – can vanish. As a result, at such points no fixed lower estimation can be guaranteed, and lacking a uniformly valid pointwise comparision of p' and p, a direct conclusion cannot be drawn either.

This explains why the case of the interval $\mathbb{I}$ already proved to be much more complicated for the integral mean norms.

In a series of papers [66, 67, 68, 69, 70], Zhou proved the inequality

$$\left(\int_{-1}^{1} |p^{(k)}(x)|^p dx\right)^{1/p} \geq C_{p,q}^{(k)}(n) \left(\int_{-1}^{1} |p(x)|^q dx\right)^{1/q},$$

in the case of $k = 1$ and $0 < p \leq q \leq \infty$, $1 - 1/p + 1/q \geq 0$ with the constant $C_{p,q}^{(1)}(n) = c_{p,q}\left(\sqrt{n}\right)^{1-1/p+1/q}$.

The best possible constants $C_{p,q}^{(k)}(n)$ were found by Babenko and Pichugov [5] for $p = q = \infty$, $k = 2$, by Bojanov [12] for $1 \leq p \leq \infty$, $q = \infty$, $1 \leq k \leq n$, and by Varma [60] in the case of $p = q = 2$, $k = 1$.

Exact Turán-type inequalities for trigonometric polynomials in different L^q-metrics on $\mathbb{T}$ were proved in [5, 6, 13, 57, 58, 29].

Other inequalities on $\mathbb{I}$, $\mathbb{D}$, the positive semiaxes, or on $\mathbb{T}$ in various weighted L^q-metrics can be found in [59, 61, 63, 71, 29, 62].

As said above, we also have a direct result for R-circular domains, and R-circularity could be ascertained by some conditions on the curvature. However, apart from these, for general domains, the situation was much less clear. Here is the only result, known for convex domains in general, and formulated in [31].

Theorem N (Levenberg-Poletsky). *Asume that K is a compact convex subset of the complex plane. Let $z \in \partial K$ and $p \in \mathcal{P}_n(K)$. Then there exists another point $\zeta \in K$, of distance $|z - \zeta| \leq c_K/\sqrt{n}$, such that $|p'(\zeta)| \geq c_K'\sqrt{n}|p(z)|$.*

Clearly, this is too weak for proving anything on $\|p'\|_{L^q(\partial K)}$, for in general $\zeta \notin \partial K$, and in any case the full set of such ζ points can well be just a finite point set (some $c_k/\sqrt{n}$ net of the boundary).

To obtain something in the $L^q(\partial K)$ norm, we need to prove pointwise estimates for much more points, essentially for the "majority" of the points, with the respective ζ points strictly lying on the boundary ∂K, and in fact essentially we cannot allow ζ be different from z (and so perhaps coincide for a large set of points z). We undertake this, also aiming at stronger inequalities, than the $\sqrt{n}$ order in the above result.

2. Statement of new results

In the later parts of our paper we will prove a rather general main result, the formulation of which, however, requires some preparations, i.e. certain geometrical notions and definitions, to be developed first. Therefore, here we give only the two main corollaries of the below Theorem 3, which provide us the main motivation for the whole study.

Theorem 1. *Let $K \Subset \mathbb{C}$ be any smooth convex domain on the plane. Then there exists a positive constant $C = C_K$, such that we have $M_{n,q}(K) > C_K n$ for all $n \in \mathbb{N}$.*

Recall that we use the terminology of being *smooth* in the sense that the boundary curve $\gamma : \mathbb{R} \to \partial K$ is differentiable – i.e. it has a (unique) tangent at each points. In case of convex domains this also implies that $\gamma \in C^1(\mathbb{R})$, but still we did not assume γ to be $C^2(\mathbb{R})$, as is quite usual in (classical) convex geometry.

That the very nature of this result does not really depend on smoothness, is well seen from our next result.

Theorem 2. *Let $K \Subset \mathbb{C}$ be any convex (non-degenerate, i.e. bounded and with nonempty interior) polygon on the plane, with no acute angles at its vertices. Then there exists a positive constant $C = C_K$, such that we have $M_{n,q}(K) > C_K n$ for all $n \in \mathbb{N}$.*

Remark 2. For the infinity norm case, clarifying the situation (the order of growth of $M_n(G_k)$) for regular k-gons took long. Here we see that for any $k \geq 4$ the regular k-gon has $M_n(G_k) > C_k n$. However, the regular (and any other) triangle, necessarily having some acute angles, turns to be an entirely different case, the treatment of which requires further ideas, too. We hope to return to that in a subsequent paper.

The organization of the material in the current work is as follows. Next we summarize classical results on the Chebyshev constant and transfinite diameter, and then we continue in the subsequent section by introducing a few geometrical notions, necessary to the formulation of our further results.

In Section 5 we formulate and prove our main result, together with a certain pointwise result of independent interest. The below Theorem 3 directly implies both the above stated Theorems 1 and 2 as corollaries (which is seen from Proposition 1 and Corollary 1), whence we may indeed say that Theorem 3 is our main result in this paper.

Next, in Section 6 we prove that in rather general situations–and surely for all convex domains–the investigated oscillation factor is *at most* a constant times n, thus clarifying at least the order of growth (with the degree n) of this factor for the domains discussed in our results.

Finally, in the concluding section we formulate our conjecture about the general situation, and offer some further comments.

3. Chebyshev estimates and transfinite diameter

In this section we summarize classical, yet powerful results going back to Chebyshev.

Lemma O (Chebyshev). *Let $J = [u, v]$ be any interval on the complex plane with $u \neq v$. Then for all $k \in \mathbb{N}$ we have*

$$\min_{w_1,\dots,w_k \in \mathbb{C}} \max_{z \in J} \left| \prod_{j=1}^{k} (z - w_j) \right| \geq 2 \left(\frac{|J|}{4} \right)^k . \tag{6}$$

Proof. Lemma O is essentially the classical result of Chebyshev for a real interval [17], cf. [37, Part 6, problem 66], [15, 35]. In fact, it holds for much more general situations, e.g. it remains valid in exactly the same form for arbitrary $J \Subset \mathbb{R}$. Indeed, according to [49] we have $\min\limits_{w_1,\dots,w_k \in \mathbb{R}} \max\limits_{z \in J} \left| \prod_{j=1}^{k} (z - w_j) \right| \geq 2 \ \mathrm{cap}(J)^k$, which can be combined with the below result of Pólya, see Lemma Q, to get the statement.

Recall the well-known basic facts about the Chebyshev constant, the transfinite diameter $\Delta(K)$, and logarithmic capacity $\mathrm{cap}(K)$, which coincide for all $K \Subset \mathbb{C}$, as is known from Fekete [24] and Szegő [53]. That also means a weak (i.e. logarithmic) asymptotical equality of the quantities on the two sides of (6). However, even the same result as in (6) holds true (perhaps with the unimportant loss of the factor 2) even for complex compacta. We will use such type of estimations in the following form.

Lemma P (Faber, Fekete, Szegő). *Let $M \Subset \mathbb{C}$ be any compact set. Then for all $k \in \mathbb{N}$ we have*

$$\min_{w_1,\dots,w_k \in \mathbb{C}} \max_{z \in M} \left| \prod_{j=1}^{k} (z - w_j) \right| \geq \Delta(M)^k = \mathrm{cap}(M)^k, \tag{7}$$

where $\Delta(M) = \mathrm{cap}(M)$ is the transfinite diameter and the capacity of the set M.

Proof. Regarding the formulation in Lemma P cf. Theorem 5.5.4. (a) in [38] or [48, (3.7) page 46]. Historically, it was first Fekete who proved the inequality and also (with say $R = \mathbb{C}$) that the left and middle quantities are–logarithmically, i.e. after normalizing by taking k^{th} roots–asymptotically equivalent. Moreover, he showed that in cases when $\mathbb{C} \setminus M$ is a simply connected domain (if considered with the point ∞), then the limits of the k^{th} roots also agree to the so-called *conformal radius* $\rho(M)$. Before that, Faber [22] has already proved $\max_{z \in M} \left| \prod_{j=1}^{k} (z - w_j) \right| \geq \rho(M)^k$ for M a Jordan domain bounded by a closed analytic Jordan arc. Following Fekete, Szegő showed that the condition of $\mathbb{C} \setminus M$ being simply connected is not necessary, and that with the so-called Robin constant $\gamma(M)$ (equivalent to capacity), the stated inequalities hold true, moreover, $\gamma(M) = \Delta(M)$ in general for all compacta.

Lemma Q (Pólya, see [26][Ch. VII]). *Let $J \subset \mathbb{R}$ be any compact set, $|J|^*$ be its outer Jordan measure. Then $|J|^* \le 4\Delta(J)$.*

Proof. See [26, Ch. VII])[6]. To reflect back to the above discussion of general forms of Chebyshev's Lemma O, recall that $\Delta(J) = \mathrm{cap}(J)$.

There are several known estimates for capacities and the so-called "Widom factors" (see e.g. [27, 64] and the references therein) between Chebyshev constants and corresponding powers of the transfinite diameter: for us, these more precise estimates are not needed, as (7) suffices. There are some known explicit computations or comparisons and estimates of capacities from other geometric parameters of the respective sets: a few most basic ones can be found e.g. in the survey of Ransford [39] or in his book [38, page 135].

One remarkable fact to be noted also here is that the values of the diameter and the transfinite diameter are within a constant factor: for any compact set $E \subset \mathbb{C}$ we have $\Delta(E) \le \mathrm{diam}(E)/2$, and if E is also connected, then $\Delta(E) \ge \mathrm{diam}(E)/4$, too, the disk $\mathbb{D}$ and the interval $\mathbb{I}$ showing sharpness of both estimates, respectively. See e.g. [39, §1.7.1.][7].

4. Some geometrical notions and definitions

We start with a *convex, compact domain* $K \Subset \mathbb{C}$. Then its interior $\mathrm{int}\, K \ne \emptyset$ and $K = \overline{\mathrm{int}\, K}$, while its boundary $\Gamma := \partial K$ is a convex Jordan curve. More precisely, $\Gamma = \mathcal{R}(\gamma)$ is the *range* of a continuous, convex, closed Jordan curve γ on the complex plane $\mathbb{C}$.

If the parameter interval of the Jordan curve γ is $[0, L]$, then this means, that $\gamma : [0, L] \to \mathbb{C}$ is continuous, convex, and one-to-one on $[0, L)$, while $\gamma(L) = \gamma(0)$. While this is the most used setup for curves, we need the two, essentially equivalent interpretations i.e. this compact interval parametrization and also the periodically extended interpretation with $\gamma(t) := \gamma(t - [t/L]L)$ defined periodically all over $\mathbb{R}$. If we need to distinguish, we will say that $\gamma : \mathbb{R} \to \mathbb{C}$ and $\gamma^* : \mathbb{T} := \mathbb{R}/L\mathbb{Z} \to \mathbb{C}$, or equivalently, $\gamma^* : [0, L] \to \mathbb{C}$ with $\gamma^*(L) = \gamma^*(0)$.

Curves can be parameterized equivalently various ways. However, in this work we will restrict ourselves to parametrization with respect to arc length: as the curves are convex, whence rectifiable curves, they always have finite arc length $L := |\gamma^*|$, and parametrization is possible with respect to arc length. Whence also

$$L := |\gamma^*| = |\Gamma|$$

is the arc length of Γ, i.e. the perimeter of K. The parametrization $\gamma : \mathbb{R} \to \partial K$ defines a unique ordering of points, which we assume to be positive in the counterclockwise direction, as usual.

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[6] Of course, the Lebesgue measure $|J|$ of the compact set $J \Subset \mathbb{R}$ does not exceed its outer Jordan mesure $|J|^*$

[7] However, note a disturbing misprint in this fundamental reference: in §1.7.2. the first two displayed formulas must be corrected to have the opposite direction of the inequality sign.

This has an immediate consequence also regarding the derivative, which must then have $|\dot{\gamma}| = 1$, whenever it exists, i.e. (linearly) a.e. on $[0, L) \sim \mathbb{T}$. Since $\dot{\gamma} : \mathbb{R} \to \partial\mathbb{D}$, we can as well describe the value by its angle or argument: the derivative angle function will be denoted by $\alpha := \arg \dot{\gamma} : \mathbb{R} \to \mathbb{R}$. Since, however, the argument cannot be defined on the unit circle without a jump, we decide to fix one value and then define the extension continuously: this way α will not be periodic, but we will have rotational angles depending on the number of (positive or negative) revolutions, if started from the given point. With this interpretation, α is an a.e. defined nondecreasing real function with $\alpha(t) - \frac{2\pi}{L}t$ periodic (by L) and bounded. With the usual left- and right limits α_- and α_+ are the left- resp. right-continuous extensions of α. The geometrical meaning is that if for a parameter value τ the corresponding boundary point is $\gamma(\tau) = \zeta$, then $[\alpha_-(\tau), \alpha_+(\tau)]$ is precisely the interval of values $\beta \in \mathbb{T}$ such that the straight lines $\{\zeta + e^{i\beta}s \ : \ s \in \mathbb{R}\}$ are supporting lines to K at $\zeta \in \partial K$. We will also talk about half-tangents: the left- resp. right- half-tangents are the half-lines emanating from ζ and progressing towards $-e^{i\alpha_-(\tau)}$ and $e^{i\alpha_+(\tau)}$, resp. The union of the half-lines $\{\zeta + e^{i\beta}s \ : \ s \geq 0\}$ for all $\beta \in [\alpha_+(\tau), \pi + \alpha_-(\tau)]$ is precisely the smallest cone with vertex at ζ and containing K.

We will interpret α as a multi-valued function, assuming all the values in $[\alpha_-(\tau), \alpha_+(\tau)]$ at the point τ. Restricting to the periodic (finite interval) interpretation of $\gamma^* : [0, L) \to \mathbb{C}$, without loss of generality we we may assume that $\alpha^* := \arg(\dot{\gamma^*}) : [0, L] \to [0, 2\pi]$. In this regard, we can say that $\alpha^* : \mathbb{R}/L\mathbb{Z} \to \mathbb{T}$ is of bounded variation, with total variation (i.e. total increase) 2π–the same holds for $\alpha : \mathbb{R} \to \mathbb{R}$ over one period.

The curve γ is differentiable at $\zeta = \gamma(\theta)$ if and only if $\alpha_-(\theta) = \alpha_+(\theta)$; in this case the unique tangent of γ at ζ is $\zeta + e^{i\alpha}\mathbb{R}$ with $\alpha = \alpha_-(\theta) = \alpha_+(\theta)$.

It is clear that interpreting α as a function on the boundary points $\zeta \in \partial K$, we obtain a parametrization-independent function: to be fully precise, we would have to talk about $\widetilde{\gamma}$, $\widetilde{\gamma^*}$, $\widetilde{\alpha}$ and $\widetilde{\alpha^*}$. In line with the above, we consider $\widetilde{\alpha}$, resp $\widetilde{\alpha^*}$ *multivalued functions*, all admissible supporting line directions belonging to $[\alpha_-(\tau), \alpha_+(\tau)]$ at $\zeta = \gamma(\tau) \in \partial K$ being considered as $\widetilde{\alpha}$-function values at ζ. At points of discontinuity $\alpha_\pm$ or $\alpha^*_\pm$ and similarly $\widetilde{\alpha}_\pm$ resp. $\widetilde{\alpha^*}_\pm$ are the left-, or right continuous extensions of the same functions.

A convex domain K is called *smooth*, if it has a unique supporting line at each boundary point of K. This occurs iff $\alpha_\pm := \alpha$ is continuously defined for all values of the parameter. For a supporting line $\zeta + e^{i\beta}\mathbb{R}$ the outer normal vector is $\boldsymbol{\nu}(\zeta) := e^{i\beta - i\pi/2}$, and the (outer) normal vectors are precisely the vectors $\boldsymbol{\nu}$ satisfying $\langle z - \zeta, \boldsymbol{\nu} \rangle \leq 0$ $(\forall z \in K)$ with the usual $\mathbb{R}^2$ scalar product, or equivalently, $\Re\left((z - \zeta)\overline{\boldsymbol{\nu}}\right) \leq 0$.

For obvious geometric reasons we call the jump function $\Omega := \alpha_+ - \alpha_-$ the *supplementary angle* function. This is identically zero almost everywhere (and in fact except for a countable set), and has positive values such that the total sum of the (possibly infinite number of) jumps does not exceed the total variation of α, i.e. 2π.

In the sequel we use some local quantities like the *local depth* $h_K(\zeta)$ at boundary points $\zeta \in \partial K$. Take any boundary point $\zeta \in \partial K$, and a supporting line at ζ to K with corresponding normal vector $\boldsymbol{\nu} = \boldsymbol{\nu}(\zeta)$. It is easy to see[8] that at least for some normal directions $\boldsymbol{\nu}$ we have

$$\zeta + \boldsymbol{\nu}\mathbb{R} \cap K = [\zeta, \zeta - h\boldsymbol{\nu}] \tag{8}$$

with some *positive length* h (unless $\operatorname{int} K = \emptyset$). Further, it is also easy to see that the supremum of all such positive lengths in (8) is actually a maximum. This maximum is denoted as $h := h_K(\zeta)$, and is called the *(local) depth* of K at ζ.

With this, we can as well define the (*global*) depth of the convex domain K.

Definition 2. *A convex body K has* depth h_K *with*

$$h_K := \inf\{h_K(\zeta) \ : \ \zeta \in \partial K\} \ . \tag{9}$$

The convex domain K has fixed depth *or* positive depth*, if $h_K > 0$.*

Note that this quantity is not always a minimum, and it can as well be zero, as e.g. in case of the regular triangle.

It is easy to see that if a convex domain has a boundary point $\zeta \in \partial K$, where $\Omega(\zeta) > \pi/2$, then the convex domain cannot have positive depth. On the other hand there cannot be too many of such boundary points, since the sum of the jumps of $\alpha_\pm$ at these points cannot exceed the total variation 2π of $\alpha_\pm$. So there exist at most three points with obtuse supplementary angles, and at most four points with $\Omega(\zeta) > 2\pi/5$, etc.. It is then clear that the largest supplementary angle exists as the maximum of the nonnegative function Ω over the boundary ∂K. We can thus define

Definition 3. *For any convex domain K the* largest supplementary angle *is*

$$\Omega_K := \max_{\partial K} \Omega = \max_{\partial K} (\alpha_+ - \alpha_-).$$

Note that $\Omega_K = 0$ if and only if K is smooth. Also note that the Szegő type outer angle of a say convex domain is $\Omega_K + \pi$.

Finally, let us introduce a version of the *modulus of continuity* function of the normal direction(s) $\arg \boldsymbol{\nu}(\zeta)$ (with respect to distance, i.e. chord length) on the boundary ∂K. Let $d_{\mathbb{T}}(\theta, \tau)$ denote the usual distance on the circle, i.e. the distance of $\theta - \tau$ from $2\pi\mathbb{Z}$. Then the modulus of continuity of the normal vectors of the boundary is defined the following parametrization–free way.

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[8]For if with some normal $\boldsymbol{\nu}$ and the normal line $\ell := \zeta + \boldsymbol{\nu}\mathbb{R}$ we have $\ell \cap K = \{\zeta\}$, then (due to convexity) there cannot be interior points on both sides of this line; the same being true for the supporting line $t := \zeta + i\boldsymbol{\nu}\mathbb{R}$, we find that $\operatorname{int} K \neq \emptyset$ lies strictly inside one quadrant of the plane, whence by the fatness of K also K is in one closed quadrant, and so to any interior point $w \in \operatorname{int} K$ the direction of $\zeta - w$ is normal to K at ζ.

Definition 4. *The modulus of continuity of the (say: outer) normal directions of the boundary of K is defined for $0 \le t < w$ as*

$$\omega_K(t) := \sup \{ d_{\mathbb{T}}(\arg \boldsymbol{\nu}, \arg \boldsymbol{\mu}) \ : \ \boldsymbol{\nu}, \boldsymbol{\mu} \text{ are normal to } K \text{ at } z, z' \text{ resp.}, \ |z - z'| \le t \} . \tag{10}$$

This is equivalent to the modulus of continuity of the tangent directions $\alpha(\tau)$ with respect to chord length or distance, i.e.

$$\omega_K(t) = \sup\{d_{\mathbb{T}}(\alpha^*(\sigma), \alpha^*(\tau)) \ : \ \tau, \sigma \in \mathbb{R}, \ |\gamma(\sigma) - \gamma(\tau)| \le t\}.$$

Further, ω_K can be expressed the following (also parametrization-free) way, too:

$$\omega_K(t) = \sup\{d_{\mathbb{T}}(\widetilde{\alpha^*}(\zeta), \widetilde{\alpha^*}(\zeta')) \ : \ \zeta, \zeta' \in \partial K, |\zeta - \zeta'| \le t\}.$$

Observe that by the definition of the width $w = w_K$, precisely when the chord length reaches w, then there are points $z, z' \in \partial K$ with parallel supporting lines, i.e. with opposite normals, achieving $d_{\mathbb{T}}(\boldsymbol{\nu}, \boldsymbol{\nu}') = \pi$. From that distance on, any definition of the modulus of continuity can only say that $\omega_K(t) = \pi$ for $t \ge w$.

Note that $\omega_K : [0, w) \to [0, \pi)$ is a nondecreasing function with possible jumps: in particular, if $\omega := \omega_K$ is not continuous, then $\omega_K(0) = \Omega_K$ is the very first jump (compared to 0), and all jumps have to be between zero and Ω_K. Now defining [9]

$$\omega_-(0) := 0, \quad \omega_+(w) = \omega_L(w) = \pi,$$

$$\omega_-(t) := \sup_{s<t} \omega_K(s) = \limsup_{s \to t-0} \omega_K(s) = \lim_{s \to t-0} \omega_K(s),$$

and

$$\omega_+(s) := \inf_{s>t} \omega_K(s) = \liminf_{s \to t+0} \omega_K(s) = \lim_{s \to t+0} \omega(s) \quad (s < w),$$

we can consider the modulus of continuity function a multivalued function with $\omega_K(s) = [\omega_-(s), \omega_+(s)]$. This way ω_K becomes a surjective mapping from $[0, w] \to [0, \pi]$ (the full set of possible distances on $\mathbb{T}$) and, again allowing a multivalued interpretation, its inverse can be defined as $\omega^{-1}(\sigma) := \{s \in [0, w] \ : \ \omega_K(s) = \sigma\}$–again, in general, a multivalued function mapping $[0, \pi]$ surjectively to $[0, w]$.

A version of the modulus of continuity *on the boundary curve* can also be considered according to arc length, or, equivalently, according to parametrization: this will be the ordinary modulus of continuity of the composite function $\boldsymbol{\nu} \circ \gamma$, interpreting $\boldsymbol{\nu}$ as a multi-valued function assuming *all admissible values* of outer normal directions; equivalently, the modulus of continuity of $\alpha = \widetilde{\alpha} \circ \gamma$ on $\mathbb{R}$. Note that–as we parameterize the boundary curve γ with respect to arc length–the arc length of the subarc of γ in the counterclockwise direction between points corresponding to parameter values $\tau < \sigma$ is $\int_\tau^\sigma |d\gamma| = \tau - \sigma$.

[9]Observe that now, because we have defined ω_K with the $\le$ sign, we in fact have $\omega_K(t) = \omega_+(t)$; defining the modulus of continuity with respect to the condition $|\zeta - \zeta'| < t$ would provide $\omega_-(t)$.

Definition 5. *The modulus of continuity of normal directions (or tangent directions)* with respect to arc length on the boundary curve *is*

$$\omega_\gamma(s) := \omega(\boldsymbol{\nu}\circ\gamma; s) = \omega(\widetilde{\alpha}\circ\gamma; s) = \omega(\alpha; s) = \sup\{\alpha(\sigma)-\alpha(\tau)\ :\ \tau \le \sigma \le \tau+s\}.$$

Note that $\omega(\alpha, s) : [0, L) \to [0, 2\pi)$ monotonically. The main difference between the two definitions is lying not in the different measurement of the distances between the points– which is already an essential difference, though– but in the fact that the distance of values is measured not in $\mathbb{T}$, but in $\mathbb{R}$, thus allowing ω_γ to increase all over $[0, \infty)$. Still, after reaching the period L there is not much point to consider this modulus, for there we only add the number of full revolutions times 2π, i.e. we have $\omega_\gamma(t) = \omega_\gamma(t - [t/L]L) + [t/L]2\pi$.

5. A result for domains with positive depth

5.1. Statement of the result

The aim of this section is to prove the following theorem on the order of the oscillation in L^q norm for a class of domains defined by the property of having positive depth.

Theorem 3. *Assume that $K \Subset \mathbb{C}$ is a convex domain with positive depth $h_K > 0$. Then for any $n \in \mathbb{N}$ and $p \in \mathcal{P}_n(K)$ it holds*

$$\|p'\|_{q,K} \ge \frac{h_K^4}{3000 d^5}\ n\ \|p\|_{q,K}\ . \qquad (11)$$

Remark 3. Note the generality of the statement. E.g. a regular k-gon G_k is always of positive depth from $k = 4$ on; the only (and essential) exception being the regular triangle. More generally, a polygonal domain has fixed depth iff it has no acute angles.

The whole section is devoted to the proof of this result. In the course of proof we will work out various intermediate steps and results, which will be used later.

In Section 6 we will show that this order of oscillation is again the best possible, like in the case $q = \infty$, thus settling the question of the order of the oscillation in L^q norms for all these domains.

Before starting the main argument, we first give a geometrical discussion of the property that $h_K > 0$.

5.2. A characterization of fixed depth

We have already remarked that $\Omega_K > \pi/2$ implies that $h_K = 0$. When $\Omega_K = \pi/2$, the situation is a little ambiguous.

Remark 4. Observe that in particular in case $\Omega_K = \pi/2$ a rectangle R has positive depth, but the upper semi-disk $U := \{z \ : \ |z| \leq 1 \quad \text{and} \quad \Re z \geq 0\}$ admits zero depth. This is easy to see directly, observing that local depths of boundary points on the diameter tend to zero as the points approach the vertices at ± 1. In particular, in case $\Omega_K = \pi/2$, both $h_K > 0$ and $h_K = 0$ can occur.

Still, a precise characterization of positive depth is possible. The following must be well-known in geometry, but finding no reference to that, we decided to describe this characterization also here.

Proposition 1. *Let K be any convex domain. Then there are the following cases.*

(i) *If for all boundary points the supplementary angles $\Omega(\zeta)$ admit $\Omega(\zeta) < \pi/2$, – that is, if $\Omega_K < \pi/2$ – then the domain K has a fixed positive depth $h_K > 0$.*

(ii) *If for some boundary point(s) the supplementary angle(s) satisfy $\Omega(\zeta) > \pi/2$, then $h_K = 0$.*

(iii) *If $\Omega_K = \pi/2$, but at each boundary point with $\Omega(\zeta) = \pi/2$ the tangent angle function α is constant $\alpha_-(\zeta)$ resp. $\alpha_+(\zeta)$ in a small left, resp. right neighborhood of ζ – i.e. if the point $\zeta \in \partial K$ is a vertex, with two orthogonal straight line segment pieces of the boundary joining at ζ – then the domain K has a fixed positive depth $h_K > 0$.*

(iv) *If $\Omega_K = \pi/2$, but there exists a maximum point of Ω, in any neighborhood of which either the left or the right neighboring piece of the boundary fails to be a straight line segment, then $h_K = 0$.*

Corollary 1. *If K is smooth–i.e. ∂K is a smooth convex Jordan curve–then $h_K > 0$.*

Proof. First note that Cases (i)–(iv) indeed give a full list of possibilities.

Let us first consider Case (i). We argue by contradiction. Take any boundary points ζ_n with corresponding normal vectors $\boldsymbol{\nu}_n$ and satisfying

$$\zeta_n + \boldsymbol{\nu}_n \mathbb{R} \cap K = [\zeta_n, \zeta_n - h_n \boldsymbol{\nu}_n] \ . \tag{12}$$

with $h_n < 1/n$. Let $\omega_n := \zeta_n - h_n \boldsymbol{\nu}_n$ and any corresponding outer normal vector be $\boldsymbol{\mu}_n$. After selecting a subsequence, if necessary, by compactness we may assume that these points and vectors converge. Hence let $\zeta_n \to \zeta$, $\boldsymbol{\nu}_n \to \boldsymbol{\nu}$ and $\boldsymbol{\mu}_n \to \boldsymbol{\mu}$. Since $h_n \to 0$, it follows that $\omega_n \to \zeta$, too.

Observe that for any point $z \in K$ normality of $\boldsymbol{\nu}_n$ at ζ_n means $\langle \boldsymbol{\nu}_n, z - \zeta_n \rangle \leq 0$, and normality of $\boldsymbol{\mu}_n$ at ω_n means $\langle \boldsymbol{\mu}_n, z - \omega_n \rangle \leq 0$, hence by the above convergence we must have $\langle \boldsymbol{\nu}, z - \zeta \rangle \leq 0$, and also $\langle \boldsymbol{\mu}, z - \zeta \rangle \leq 0$. In other words, both $\boldsymbol{\nu}$ and $\boldsymbol{\mu}$ are normal vectors at $\zeta \in \partial K$.

However, $\zeta_n - \omega_n = h_n \boldsymbol{\nu}_n$ is parallel to $\boldsymbol{\nu}_n$, hence normality of $\boldsymbol{\mu}_n$ at ω_n yields $\langle \boldsymbol{\mu}_n, \boldsymbol{\nu}_n \rangle \leq 0$. Again by continuity this entails $\langle \boldsymbol{\mu}, \boldsymbol{\nu} \rangle \leq 0$, that is, the angle between these two normal vectors is at least $\pi/2$. Clearly then $\Omega(\zeta) \geq \pi/2$, a contradiction to our assumption. This concludes the proof of Case (i).

Case (ii) is obvious, and even (iv) can be proved easily, but we give the proof here. So let us take a point $\zeta \in \partial K$ with the given property, and assume, as we may, that $\zeta = 0$, and $\alpha_-(\zeta) = 3\pi/2$, and $\alpha_+(\zeta) = 0$, i.e., the two extremal half-tangents at ζ are the (positively directed) imaginary axis and the (positively directed) real axis. Also let us assume e.g. that no non-degenerate segment piece of the positive imaginary half-axis (i.e. the left half-tangent) belong to the boundary.

Put $a := \max\{\Re z \ : \ z \in K\}$. It can happen that there are several points of K where this maximum is attained; however, for 0 the condition that the imaginary axis is a tangent and that no straight line piece of it belongs to K, implies that $K \cap i\mathbb{R} = \{0\}$. Consider now any $x \in [0, a]$, and the intersection $\{\Re z = x\} \cap K$. This vertical segment has to be above (not below) the positive real axis, since $\mathbb{R}$ is a tangent with $\alpha_+(0) = 0$, and so we can write $\{\Re z = x\} \cap K = [x + ig(x), x + if(x)]$ with two continuous, nonnegative functions $0 \leq g \leq f$ satisfying $g(0) = f(0) = 0$ and g convex, f concave (i.e. "convex from below"), both continuous and in particular $\lim_{x \to 0+0} f(x) = 0$, moreover, both g and f are nondecreasing in a right neighborhood of 0.

Now let us show that at $z := x + ig(x)$ the normal vector $\boldsymbol{\nu}$ has angle $\arg \boldsymbol{\nu} \in [3\pi/2, 2\pi)$. First, $0 \in K$ implies that $\langle -z, \boldsymbol{\nu} \rangle \leq 0$, entailing $\arg \boldsymbol{\nu} \geq \arg(-z) + \pi/2 = \arg z + 3\pi/2 \geq 3\pi/2$. Then again, $\arg \boldsymbol{\nu}$ achieves 2π only when the boundary point is in rightmost position (i.e. of maximal real part value) in K, and thus $\arg \boldsymbol{\nu} \in [3\pi/2, 2\pi)$ for $0 < x < a$. So these mean that the normal line $z + \boldsymbol{\nu}\mathbb{R}$ to $z = x + ig(x)$ intersects the boundary of K at some point $z' := x' + if(x')$ with some $0 \leq x' \leq x$, and thus $f(x') \leq f(x) \to 0$ implies that the length of intersection of K and this normal line tends to 0 together with x. This completes the proof of $h_K = 0$.

Finally, Case (iii) is again easy to prove. First, a little thought shows that the condition is equivalent to the statement that the modulus of continuity satisfies $\omega_+(0) = \pi/2$ and $\omega_K(t) = \pi/2$ for all $0 \leq t \leq t_0$ with some positive value $0 < t_0 < w$.

So we prove now that if $\omega(t_0) = \pi/2$, or, more generally, if $\pi/2 \in \omega(t_0)$ for some $t_0 > 0$, then for any boundary point $z \in \partial K$ and any normal line $m := z + \boldsymbol{\nu}\mathbb{R}$ to K at z, the length of the intersection of $m \cap K = [z, z']$ is at least t_0 (and so even $h_K \geq t_0 > 0$).

Obviously, the chord vector between z and z' is in inner normal direction, whence has an angle (argument) exactly $\pi/2$ above the angle of the positively directed tangent, orthogonal to $\boldsymbol{\nu}$ at z.

Assume, as we may, that $z = 0$, $z' = iy'$ (with some $y' > 0$) and m is just the imaginary axis. By definition of the width $w = w(K)$, as $w > t_0$, there exists a point $z_0 = x_0 + y_0 \in K$ with $y_0 \geq w$: if $x_0 = 0$, then we would have $z_0 = iy_0 \in [z, z'] = [0, y']$ and we would get $y' \geq w > t_0$, concluding the argument. So it remains to deal with $0 < y' < w$ and $x_0 \neq 0$. Let e.g. $x_0 < 0$: then the three points z, z', z_0 cut the boundary Γ of K into three parts, each of them extending between two of them and not containing the third one, and following in the counterclockwise direction as $z \prec z' \prec z_0$ and $\Gamma(z,z') \prec \Gamma(z',z_0) \prec \Gamma(z_0,z)$. According to these normalizations, $\alpha(z') \leq \arg(z_0 - z') \in (\pi/2, \pi)$, for the latter chord vector is $x_0 + i(y_0 - y')$ with $x_0 < 0$ and $y_0 \geq w > y'$. It follows that for any further points $z^* \in \Gamma(z', z_0)$ we necessarily have $\pi \geq \alpha(z^*) \geq \alpha_+(z')$. Note that (unless $z^* = z'$) we cannot have $z^* \in m$, but only $\Re z^* < 0$ and $\arg(z^* - z) > \arg(z' - z) = \pi/2$. Whence $\pi \geq \alpha(z^*) \geq \arg(z^* - z) > \pi/2$, so that $\omega_K(|z^* - z|) > \pi/2$. As now z^* can be arbitrarily close to z', for any value $t^* > |z' - z| = y'$ we have $\omega_K(t^*) > \pi/2$, i.e., any such t^* must satisfy $t^* > t_0$. So if $t^* > y'$ then we have $t^* > t_0$, whence $|z' - z| = y' \geq t_0$, and $h_K \geq t_0$, as needed.

Proposition 2. *For a convex, compact domain K we have $h_K > 0$ if and only if the modulus of continuity function has $\omega_K(\tau) \leq \pi/2$ for all $0 \leq \tau \leq t$ with some $t > 0$.*

Furthermore, with the above extended interpretation of the inverse function of the modulus of continuity function, we have $h_K \geq \mu_K := \max \omega^{-1}(\pi/2)$.

Proof. Consider the four cases in the above Proposition 1. A little thought shows that Case (i) is equivalent to state $\omega_K(\tau) < \pi/2$ $(0 \leq \tau \leq t)$ for sufficiently small $t > 0$, and (iii) is exactly the case when $\omega_K(\tau) = \pi/2$ $(0 \leq \tau \leq t)$ for sufficiently small $t > 0$. So these cases with $h_K > 0$ are such that $\omega_K(\tau) \leq \pi/2$ for all $0 \leq \tau \leq t$ with some $t > 0$. Moreover, Case (ii) means $\omega_K(0) > \pi/2$, and Case (iv) means that $\omega_K(0) = \pi/2$, but for all $\tau > 0$ already $\omega_K(\tau) > \pi/2$, whence the cases with $h_K = 0$ are the ones with $\omega_K(\tau) > \pi/2$ for all $\tau > 0$. This proves the first assertion of the Proposition.

As for the last assertion, there is nothing to prove for $\mu_K = 0$, so we may take $\mu_K > 0$, meaning that there are points $0 < t_0 \in \omega_K^{-1}(\pi)$. Recalling the last argument of the proof of the previous Proposition 1, we can then see that for any such t_0 also $h_K \geq t_0$ holds. Taking $t_0 := \max \omega_K^{-1}(\pi/2)$ thus provides $h_K \geq \mu_K$, too.

Remark 5. To see that the inequality $h_K > \omega_K^{-1}(\pi/2)$ is possible, it suffices to consider a regular hexagon G_6 of side length h, say. Then $h_K(G_6) = 2h$, while $\omega_-(h) = \pi/3$, $\omega_+(h) = 2\pi/3$ and $\omega_K^{-1}(\pi/2) = \{h\}$, $\mu_K = h$.

5.3. Technical preparations for the investigation of $L^q(\partial K)$ norms

First, we will prove a Nikolskii-type estimate, which is similar to the well-known analogous inequality on the real line, found in the book of Timan [55, 4.9.6 (36)].

Lemma 1. *For any polynomial of degree at most n we have that*

$$\|p\|_{L^q(\partial K)} \geq \left(\frac{d}{2(q+1)} \right)^{1/q} \|p\|_{L^\infty(\partial K)} \; n^{-2/q}. \tag{13}$$

PROOF. We first prove a Bernstein-Markov type estimate in the maximum norm. Let $z \in K$ be arbitrary. Then we always have a chord $J := [z, z^*] \subset K$ of length at least $d/2$, for we can take any diameter I of K, and for z^* take the endpoint of I, which is situated farthest from z–which is of course at least of distance $d/2$ from z. Applying Markov's Inequality on J, we obtain

$$|p'(z)| \le \frac{n^2}{|J|/2} \|p\|_{L^\infty(J)} \le \frac{4n^2}{d} \|p\|_{L^\infty(\partial K)}.$$

Therefore, it holds

$$\|p'\|_{L^\infty(\partial K)} \le \frac{4}{d} n^2 \|p\|_{L^\infty(\partial K)}. \tag{14}$$

Consider now a point $z_0 = \gamma(t_0) \in \partial K$, where $\|p\|_\infty$ is attained. Then – using also convexity of K – at each point $z \in K$ we have

$$\begin{aligned} |p(z)| &= \left| p(z_0) + \int_{z_0}^{z} p'(\zeta) d\zeta \right| \ge \|p\|_\infty - |z - z_0| \|p'\|_\infty \\ &\ge \|p\|_\infty \left(1 - \frac{4|z - z_0|}{d} n^2\right). \end{aligned} \tag{15}$$

Now, from (15) we can estimate the q-integral of p as follows. Take $r_0 = d/(4n^2)$, $U := D(z_0, r_0)$ and $\Gamma_0 := \Gamma \cap U$. More precisely, in case there are several pieces of arcs in this intersection (which can only happen for small n, though) then we take only one arc which passes through the point z_0 and extends to the circumference of U in both directions – and drop the remaining pieces. Let us denote the points, falling on the circumference ∂U right preceding and following $z_0 = \gamma(t_0)$ on γ as $z_\pm := \gamma(t_\pm)$. Recalling that γ is parameterized according to arc length, and writing for the parametrization $\gamma : [t_-, t_+] \to \Gamma_0$, and so in particular $\gamma_- : [t_-, t_0] \to \Gamma_-$ and $\gamma_+ : [t_0, t_+] \to \Gamma_+$, we get

$$\begin{aligned} \int_{\gamma_+} |p|^q |d\gamma| &\ge \int_{t_0}^{t_+} \left(1 - \frac{4|\gamma(t) - z_0|}{d} n^2\right)^q \|p\|_\infty^q dt \\ &\ge \|p\|_\infty^q \int_{t_0}^{t_+} \left(1 - \frac{4|\gamma_{[t_0,t]}|}{d} n^2\right)_+^q dt \\ &= \|p\|_\infty^q \int_{t_0}^{t_0 + \frac{d}{4n^2}} \left(1 - \frac{4(t - t_0)}{d} n^2\right)^q dt = \|p\|_\infty^q \frac{d}{4n^2} \int_0^1 (1 - s)^q\, ds \end{aligned}$$

and similarly for γ_-, whence $\int_{\gamma_0} |p|^q |d\gamma| \ge \|p\|_\infty^q \dfrac{d}{2(q+1)n^2}$. As a result, we get

$$\|p\|_{L^q(\partial K)} \ge \left(\int_{\gamma_0} |p|^q |d\gamma|\right)^{1/q} \ge \left(\frac{d}{2(q+1)}\right)^{1/q} \|p\|_\infty n^{-2/q},$$

and the result follows.

Next, let us define the subset $\mathcal{H} := \mathcal{H}^q_K(p) \subset \partial K$ the following way.

$$\mathcal{H} := \mathcal{H}^q_K(p) := \{\zeta \in \partial K \ : \ |p(\zeta)| > cn^{-2/q}\|p\|_\infty\}, \qquad c := (8\pi(q+1))^{-1/q}. \tag{16}$$

Then we can restrict ourselves to the points of $\mathcal{H}$ and neglect whatever happens for points belonging to

$$\mathcal{H}^c = \Gamma \setminus \mathcal{H}.$$

Indeed, Γ is contained in a disk of radius d around any point of K, whence by the well-known property[10] of convex curves, $L := |\gamma| \leq 2\pi d$, and the above Lemma 1 furnishes

$$\int_{\Gamma\setminus\mathcal{H}} |p(z)|^q|dz| \leq \frac{2\pi d c^q}{n^2}\|p\|^q_\infty \leq 4\pi(q+1)c^q\|p\|^q_q = \frac{1}{2}\|p\|^q_q.$$

That leads to

$$\int_{\mathcal{H}} |p|^q|d\gamma| = \int_\gamma |p|^q|d\gamma| - \int_{\gamma\setminus\mathcal{H}} |p|^q|d\gamma| \geq \|p\|^q_q - \frac{1}{2}\|p\|^q_q \geq \frac{1}{2}\|p\|^q_q.$$

Therefore we can restrict to (lower) estimations of $|p'(\zeta)|$ on the set $\mathcal{H}$ where p is assumed to be relatively large (compared to its maximum norm), so that we can assume that

$$\log\frac{\|p\|_\infty}{|p(\zeta)|} \leq \log(c^{-1}n^{2/q}) = \frac{\log(1+q)}{q} + \frac{\log(8\pi)}{q} + \frac{2}{q}\log n \leq \log(16\pi) + 2\log n.$$

for all $q \geq 1$ and $n \geq \sqrt{16\pi}$, i.e. already for $n \geq 8$. Summing up we have

Lemma 2. *Let $\mathcal{H} \subset \partial K$ be defined according to* (16)*. Then for all $p \in \mathcal{P}_n$ we have*

$$\int_{\mathcal{H}} |p|^q \geq \frac{1}{2}\|p\|^q_{L^q(\partial K)}. \tag{17}$$

Furthermore, for any point $\zeta \in \mathcal{H}$, and for any $p \in \mathcal{P}_n(K)$ we also have

$$\log\frac{\|p\|_\infty}{|p(\zeta)|} \leq \log(16\pi) + 2\log n \quad (\forall n \in \mathbb{N}). \tag{18}$$

For relatively small values of the degree n we may get better constants if using an entirely different argument, not suitable to analyze the order regarding the degree, but yielding better numerical values for small n. We will base our calculation on the following classical result of Gabriel [25, Theorem 5.1].

Lemma R (Gabriel). *If Γ is any convex (closed) curve and C any convex curve inside Γ, and if $F(Z)$ is regular inside and on Γ,*

$$\int_C |F^\lambda(Z)||dZ| \leq (\pi(e+1)+e)\int_\Gamma |F^\lambda(Z)||dZ|, \quad (\lambda \geq 0).$$

[10] A reference is [14, p. 52, Property 5] about surface area, presented as a consequence of the Cauchy Formula for surface area.

With this we can now state the next.

Lemma 3. *For any point $\zeta \in \partial K$ on the boundary of the convex domain K and $q \geq 1$ we have the estimate*

$$|p(\zeta)| \leq d(\pi(\pi(e+1)+e))^{1/q} \|p'\|_q. \tag{19}$$

As a direct consequence, we also have

$$\|p'\|_q > \frac{1}{45.3} \, \frac{1}{d} \, \|p\|_q > 0.022 \, \frac{1}{d} \, \|p\|_q. \tag{20}$$

Proof. Evidently, it is enough to consider the case $1 < q < \infty$. Let $z_0 \in K$ be any zero of p. Then for any $\zeta \in \Gamma := \partial K$ the interval $[z_0, \zeta] \subset K$ (by convexity), and so Hölder's inequality furnishes

$$|p(\zeta)| = \left| \int_{[z_0,\zeta]} p'(z) dz \right| \leq \left(\int_{[z_0,\zeta]} |dz| \right)^{(q-1)/q} \left(\int_{[z_0,\zeta]} |p'(z)|^q |dz| \right)^{1/q}.$$

Noting that $|\zeta - z_0| \leq d$ and applying Lemma R to the integral of $|p'(z)|^q$ over the convex curve $C := [z_0, \zeta] \cup [\zeta, z_0]$ (the degenerate closed convex curve encircling around the points ζ and z_0) we get

$$\begin{aligned} |p(\zeta)| &\leq d^{(q-1)/q} \left(\frac{1}{2} (\pi(e+1)+e) \int_\Gamma |p'(z)|^q |dz| \right)^{1/q} \\ &= d^{(q-1)/q} \left(\frac{\pi(e+1)+e}{2} \right)^{1/q} \|p'\|_q, \end{aligned}$$

proving (19).

Now integrating on q-th power, and using again $L \leq 2\pi d$ leads to

$$\|p\|_q^q \leq d^{q-1} \, \frac{\pi(e+1)+e}{2} \|p'\|_q^q \, L \leq d^q \, \pi(\pi(e+1)+e) \|p'\|_q^q < 45.3 \, d^q \, \|p'\|_q^q.$$

Taking q-th root, a small rearrangement finally furnishes even the last assertion, as from here

$$\|p'\|_q > \left(\frac{1}{45.3} \right)^{1/q} \frac{1}{d} \, \|p\|_q \geq \frac{1}{45.3} \, \frac{1}{d} \, \|p\|_q > 0.022 \, \frac{1}{d} \, \|p\|_q.$$

5.4. Proof of a local result in terms of the local depth

In the following we will work out an unconditional pointwise estimate in the sense that it will provide an estimate locally at points $\zeta \in \mathcal{H}$ in terms of $h = h(\zeta, K)$, not using the assumption that $h_K = \inf_{\zeta \in \partial K} h(\zeta, K)$ stays positive or not. When this happens to hold, the below result will almost immediately imply Theorem 3.

Theorem 4. *Let $p \in \mathcal{P}_n(K)$ and let $\mathcal{H} = \mathcal{H}_K^q(p)$ be defined by* (16)*. Then it holds*

$$|p'(\zeta)| \geq \frac{h^4}{1500 d^5} \; n \; |p(\zeta)| \quad (\text{ if } \;\; \zeta \in \mathcal{H}) \,. \tag{21}$$

PROOF. First of all, we may assume that

$$n \geq n_0 := 32 \; d^4/h^4,$$

for values of n not exceeding this bound we can settle the issue referring to (20) of Lemma 3 providing

$$\|p'\|_q \geq \frac{n}{32 d^4/h^4} \|p'\|_q > \frac{1}{1500} \, \frac{h^4}{d^5} \; n \; \|p\|_q. \tag{22}$$

So from here on let us assume $n \geq 32 \; d^4/h^4$. Without loss of generality we also assume $\zeta = 0$ and that a tangent line at $\zeta = 0$, chosen according to the requirement $h = h(\zeta, K)(\geq h_K)$, is just the real line $\mathbb{R}$ (and thus the normal line is the imaginary axis $i\mathbb{R}$). Then we have $K \subset \mathbb{H} := \{z \in \mathbb{C} \; : \; \Im z \geq 0\}$ and $K \cap i\mathbb{R} = [0, ih]$.

Let us denote the set of zeroes of $p \in \mathcal{P}_n(K)$ as

$$\mathcal{Z} := \{z_j = r_j e^{i\varphi_j} \;\; : \;\; j = 1, \ldots, n\} \subset K$$

(listed with possible repetitions according to their multiplicity). We assume, as we may, that $p(z) = \prod_{j=1}^{n}(z - z_j)$.

In what follows we will use for any interval $[\sigma, \theta]$ (and similarly for $[\sigma, \theta)$ etc.) the following notations for the angular sectors, the zeroes in the angular sectors, and the number of zeroes in the angular sectors:

$$\begin{aligned} S[\sigma, \theta] &:= \{z \in \mathbb{C} \;\; : \;\; \arg z \in [\sigma, \theta]\}, \\ \mathcal{Z}[\sigma, \theta] &:= \{z_j \in \mathcal{Z} \;\; : \;\; \arg z_j \in [\sigma, \theta]\} = \mathcal{Z} \cap S[\sigma, \theta], \\ n[\sigma, \theta] &:= \# \mathcal{Z}[\sigma, \theta]. \end{aligned}$$

Let us fix the angle

$$\varphi := \arcsin \left(\frac{h}{8d} \right) \,.$$

We partition the zero set $\mathcal{Z}$ into two subsets as follows.

$$\mathcal{Z}_+ := \mathcal{Z}[\varphi, \pi - \varphi] \qquad \mathcal{Z}_- := \mathcal{Z} \setminus \mathcal{Z}_+ \,.$$

For the corresponding cardinals we write

$$k := \# \mathcal{Z}_+ = n[\varphi, \pi - \varphi], \qquad m := \# \mathcal{Z}_- = n[0, \varphi) + n(\pi - \varphi, \pi] \,.$$

Observe that for any subset $\mathcal{W} \subset \mathcal{Z}$ we have

$$\left| \frac{p'}{p}(0) \right| \geq -\Im \frac{p'}{p}(0) = \sum_{j=1}^{n} \Im \frac{-1}{z_j} \geq \sum_{z_j \in \mathcal{W}} \Im \frac{-1}{z_j} = \sum_{z_j \in \mathcal{W}} \frac{\sin \varphi_j}{r_j} \,, \tag{23}$$

because all terms in the full sum are nonnegative. We apply inequality (23) with $\mathcal{W} = \mathcal{Z}_+$ to obtain

$$M := \left|\frac{p'}{p}(0)\right| \geq \sum_{z_j \in Z_+} \frac{\sin\varphi_j}{r_j} \geq \sin\varphi \sum_{z_j \in Z_+} \frac{1}{r_j} \geq \frac{h}{8d^2}k\,, \tag{24}$$

since for $z_j \in \mathcal{Z}_+$ we have $\varphi_j \in [\varphi, \pi - \varphi]$, $\sin\varphi_j \geq \sin\varphi = h/(8d)$, and $r_j \leq d$.

Now put

$$J := \left[\frac{3}{4}ih, ih\right] \subset K\ .$$

We estimate the distance of any $z_j = x_j + iy_j \in \mathcal{Z}_-$ from J. In fact, taking any point $z = x + iy = re^{i\psi} \in D_d(0) \cap (S[0,\varphi) \cup S(\pi - \varphi, \pi])$, we necessarily have $|z|^2 = x^2 + y^2 \leq d^2$, $0 \leq y \leq d\sin\varphi = h/8$, and therefore, $\mathrm{dist}(z, J) = |z - i3h/4| = \sqrt{x^2 + (y - 3h/4)^2}$. Clearly, then

$$\frac{\mathrm{dist}(z,J)^2}{|z|^2} = \frac{x^2 + y^2 - 3yh/2 + (3h/4)^2}{x^2 + y^2} \geq 1 + \frac{3h^2}{8d^2}\ . \tag{25}$$

In fact, we can do a little better here, taking into account that the other endpoint of J, ih, also belongs to K, whence the diameter provides an upper bound to $|z - ih|$, too, yielding $x^2 + (y-h)^2 \leq d^2$. Using this in the middle of (25), we may write

$$\begin{aligned}\frac{\mathrm{dist}(z,J)^2}{|z|^2} &= \frac{x^2 + y^2 - 3yh/2 + (3h/4)^2}{x^2 + y^2} \\ &= 1 + \frac{9h^2 - 24yh}{16(x^2 + y^2)} \geq 1 + \frac{9h^2 - 24yh}{16(d^2 - h^2 + 2yh)}\ ,\end{aligned}$$

where the last expression is decreasing in $y \leq h/8$, whence admitting

$$\frac{9h^2 - 24yh}{16(d^2 - h^2 + 2yh)} \geq \frac{9h^2 - 24(h/8)h}{16(d^2 - h^2 + 2(h/8)h)} = \frac{3h^2}{8d^2 - 6h^2}.$$

Introducing the parameter $u := d/h \in [1, \infty)$ we thus obtain

$$\frac{\mathrm{dist}(z,J)^2}{|z|^2} \geq 1 + \frac{3h^2}{8d^2 - 6h^2} = \frac{8u^2 - 3}{8u^2 - 6} \qquad \left(u := \frac{d}{h} \in [1, \infty)\right).$$

From here taking logarithms we get

$$\left|\frac{z_j - \tau}{z_j}\right| \geq \exp\left(\frac{1}{2}\log\left(\frac{8u^2 - 3}{8u^2 - 6}\right)\right) \qquad \left(u := \frac{d}{h} \in [1, \infty)\right)\ . \tag{26}$$

Next consider the set $R := K \cap S[\varphi, \pi - \varphi]$. Applying Lemma O to R and $J \subset R$ we are led to

$$\begin{aligned}\max_{z \in J} \prod_{z_j \in \mathcal{Z}_+} \left|\frac{z_j - z}{z_j}\right| &\geq \frac{1}{d^k} \max_{z \in J} \prod_{z_j \in \mathcal{Z}_+} |z_j - z| \geq \frac{1}{d^k}\left(\frac{|J|}{4}\right)^k \\ &= \exp\left(-k\log\left(\frac{16d}{h}\right)\right).\end{aligned} \tag{27}$$

Taking now the point $z_0 \in J$ where this maximum (i.e. the maximum of $|p(z)|$ on J) is attained, combining (26) and (27) and using $m + k = n$ leads to

$$\begin{aligned}
\left|\frac{p(z_0)}{p(0)}\right| = \prod_{z_j \in \mathcal{Z}} \left|\frac{z_j - z_0}{z_j}\right| &\geq \exp\left(m \frac{1}{2} \log\left(\frac{8u^2-3}{8u^2-6}\right) - k \log(16u)\right) \\
&= \exp\left(n \frac{1}{2} \log\left(\frac{8u^2-3}{8u^2-6}\right) - \left\{\frac{1}{2}\log\left(\frac{8u^2-3}{8u^2-6}\right) + \log(16u)\right\} k\right) \\
&= \exp\left(\frac{n}{2} \log\left(\frac{8u^2-3}{8u^2-6}\right) - \psi(u) k\right),
\end{aligned}$$

where

$$\psi(u) := \frac{1}{2} \log\left(\frac{8u^2-3}{8u^2-6}\right) + \log(16u).$$

Taking logarithm, dividing by $\psi(u)$ and rearranging thus provides

$$\begin{aligned}
k &\geq \frac{1}{\psi(u)} \left(\frac{n}{2} \log\left(\frac{8u^2-3}{8u^2-6}\right) - \log\left|\frac{p(z_0)}{p(0)}\right|\right) \\
&\geq \frac{1}{\psi(u)} \left(\frac{n}{2} \log\left(\frac{8u^2-3}{8u^2-6}\right) - \log\frac{\|p\|_\infty}{|p(0)|}\right),
\end{aligned}$$

which, when combining with (24) yields

$$8 \frac{d^5}{h^4} M \geq u^3 k \geq \frac{u^3}{\psi(u)} \left(\frac{n}{2} \log\left(\frac{8u^2-3}{8u^2-6}\right) - \log\frac{\|p\|_\infty}{|p(0)|}\right).$$

We have already seen in Lemma 2 why it suffices to restrict to points of the set $\mathcal{H}$. So from here on we will consider only points $\zeta \in \mathcal{H}$, for which points we may invoke (18) to get

$$8 \frac{d^5}{h^4} M \geq \frac{u^3}{\psi(u)} \left(\frac{n}{2} \log\left(\frac{8u^2-3}{8u^2-6}\right) - \log(16\pi) - 2\log n\right) \quad (\text{ for } \zeta \in \mathcal{H}).$$

Introducing another parameter $v := n/u^4 = (h/d)^4 n$ and collecting everything from the above, a little rearrangement leads to

$$8 \frac{d^5}{h^4}\ M\ \frac{1}{n} \geq \frac{u^3 \log\left(\frac{8u^2-3}{8u^2-6}\right) - \dfrac{2}{vu}\left\{\log(16\pi) + 2\log v + 8\log u\right\}}{\log\left(\frac{8u^2-3}{8u^2-6}\right) + 2\log(16u)}$$

It is clear that for $v \geq e$ this expression is an increasing function of v, therefore we can as well write in the minimal possible value $v \geq v_0 = 32$ to get

$$8 \frac{d^5}{h^4}\ \frac{M}{n} \geq \frac{u^3 \log\left(\frac{8u^2-3}{8u^2-6}\right) - \frac{2}{32u}\left\{\log(\pi) + 14\log 2 + 8\log u\right\}}{\log\left(\frac{8u^2-3}{8u^2-6}\right) + 2\log(16u)} \quad \left(u := \frac{d}{h} \geq 1\right).$$

Here simultaneously dividing the numerator and the denominator by u yields

$$8\frac{d^5}{h^4}\ \frac{M}{n} \geq \frac{u^2\log\left(\frac{8u^2-3}{8u^2-6}\right) - \frac{1}{16u^2}\left\{\log\pi + 14\log 2 + 8\log u\right\}}{\frac{1}{u}\log\left(\frac{8u^2-3}{8u^2-6}\right) + 2\frac{\log(16u)}{u}} \qquad \left(u := \frac{d}{h} \geq 1\right). \tag{28}$$

As $1/u$, $\log\left(\frac{8u^2-3}{8u^2-6}\right)$ and $2\frac{\log(16u)}{u} = 32\frac{\log(16u)}{16u}$ are all decreasing for $u \geq 1$, the denominator is a decreasing function and its maximal value is at the point $u = 1$. So,

$$\frac{1}{u}\log\left(\frac{8u^2-3}{8u^2-6}\right) + 2\frac{\log(16u)}{u} \leq \log(5/2) + 2\log(16) = \log(640).$$

From this and (28) a computation provides

$$\begin{aligned}\frac{d^5}{h^4}\ \frac{M}{n} &\geq \frac{1}{64\log 640}\left(8u^2\log\left(\frac{8u^2-3}{8u^2-6}\right) - \frac{\log(4\pi)+4\log(8u^2)}{2u^2}\right)\\ &= \frac{1}{413.5339\ldots}\left(8u^2\log\left(\frac{1-\frac{3}{8u^2}}{1-\frac{6}{8u^2}}\right) - \frac{4\log(4\pi)}{8u^2} - 16\frac{\log(8u^2)}{8u^2}\right) \qquad (29)\\ &\geq \frac{1}{414}f(t)\end{aligned}$$

with $f(t) := \dfrac{1}{t}\log\left(\dfrac{1-3t}{1-6t}\right) - 4\log(4\pi)\ t + 16t\log t$ and $t := \dfrac{1}{8u^2} \in (0, 1/8]$. It is easy to see that $f(t)$ is a convex function. Indeed, $t\log t$ is convex (with second derivative $1/t > 0$), the linear term is of course convex, and the first part can be developed into a totally positive Taylor-Maclaurin series: $\frac{1}{t}\log\left(\frac{1-3t}{1-6t}\right) = \sum_{k=1}^{\infty}\frac{6^k-3^k}{k}t^{k-1}$.

Numerical evidence shows that $f(t)$ attains its minimal value somewhere around $0.0786\ldots$, and it stays above 0.7 all over $(0, 1/8]$. To establish a sufficiently good lower estimation of the function all over the interval $(0, 1/8]$, we will use convexity simply in the form of a supporting line argument: with any fixed value τ in $(0, 1/8]$ the tangent of f at $(\tau, f(\tau))$ is a supporting line (from below) to f, i.e. $f(t) \geq L(t) := L(\tau; t) := f(\tau) + f'(\tau)(t-\tau)$.

A computation furnishes

$$f'(t) = -\frac{1}{t^2}\log\left(\frac{1-3t}{1-6t}\right) + \frac{1}{t}\left(\frac{-3}{1-3t} + \frac{6}{1-6t}\right) - 4\log(4\pi) + 16 + 16\log t.$$

So now let us take $\tau := 0.078628$, say. Then $f(\tau) \approx 0.700037\ldots > 0.70003$, and another numerical computation furnishes $f'(\tau) \approx -0.000321\ldots > -0.0004$. Since $f'(\tau) < 0$, we find $f(t) \geq \min_{(0,1/8]} L = L(0.125) > 0.70003 - 0.0004 \cdot (0.125 - 0.07) = 0.700008 > 0.7$.

Substituting this estimate in (29), we conclude

$$M \geq \frac{0.7}{414}\frac{h^4}{d^5}n > \frac{1}{600}\frac{h^4}{d^5}n.$$

5.5. Conclusion of the proof for fixed positive depth

Proof (Proof of Theorem 3). We will use for all points $\zeta \in \mathcal{H}$ the estimate (21) complemented by the lower estimation $h \geq h_K$.

$$\|p'\|_q^q \geq \int_{\mathcal{H}} |p'|^q \geq \left(\frac{1}{1500}\frac{h_K^4}{d^5}\right)^q \int_{\mathcal{H}} |p|^q \geq \left(\frac{1}{1500}\frac{h_K^4}{d^5}\right)^q \frac{1}{2}\|p\|_q^q.$$

Taking q^{th} root and estimating $2^{1/q}$ simply by 2 yields Theorem 3.

6. Upper estimation of the oscillation order of convex domains

Given the results for maximum norm and the above results of Theorem M and Theorem 3, it is in order to clarify if the linear growth with n is indeed the maximal possible order of oscillation in L^q norms. That is settled by the next result.

Theorem 5. *Let $K \Subset \mathbb{C}$ be any compact, connected–not necessarily convex– domain, bounded by a finite or countable number of closed, rectifiable Jordan curves Γ_j $(j = 1, \dots)$ with finite total arc length $\sum_j |\Gamma_j| = L < \infty$. Then for any $n \in \mathbb{N}$ there exists some polynomial $p \in \mathcal{P}_n(K)$ with $\|p'\|_{L^q(\partial K)} \leq C(K)n\|p\|_{L^q(\partial K)}$.*

Note that the rectifiable assumption is necessary to have finite L^q norms, for otherwise most polynomials have infinite L^q norms on the boundary. However, apart from this assumption, the domain K is quite general, including nonconvex, multiply connected domains. For disconnected domains, the analysis may be done separately for connected components, and for compact sets without an interior even a lower order of oscillation is possible, as it has been shown at least for the interval $\mathbb{I}$. Therefore, we may be satisfied with the degree of generality of the above formulated assertion.

Proof. We will provide a simple example. Let $J \subset K$ be any diameter, and chose an endpoint of the diameter J. Without loss of generality we may assume that this endpoint is just the origin 0, and we can as well assume that $J = [0, d]$. Then our polynomial will simply be $p(z) := z^n$.

At points, where $|z| \leq d/2$, we have $|p(z)| \leq (d/2)^n = 2^{-n}d^n = 2^{-n}\|p\|_K$, and $|p'(z)| = n|z^{n-1}| \leq n2^{-(n-1)}d^{n-1}$. On the other hand, for points in the ring domain $R := \{z \in \mathbb{C} \ : \ d/2 \leq |z| \leq d\}$ and belonging to K we have $\left|\frac{p'}{p}(z)\right| = |n/z| < 2n/d$. So we can write

$$\begin{aligned} \|p'\|_q^q &:= \int_\Gamma |p'|^q \leq (2n2^{-n}d^{n-1})^q L + \int_{\Gamma\cap R} \left(\frac{2n}{d}|p|\right)^q \\ &\leq \left(\frac{2n}{d}\right)^q \left\{ \left(2^{-n}\frac{\|p\|_\infty}{\|p\|_q}\right)^q L + 1 \right\} \|p\|_q^q. \end{aligned} \tag{30}$$

By construction, the point d on the other end of the diameter J sits in ∂K, and belongs to some of the boundary curves Γ_j, which boundary curve must have some positive length $|\Gamma_j| = \ell > 0$, say. So parameterizing by arc length and starting the parametrization at the point d, we can write $\gamma_j : [0, \ell] \to \Gamma_j$ with $\gamma_j(0) = d = \gamma_j(\ell)$, and obviously for any parameter value $0 \le t \le d$ $|\gamma_j(t)| \ge d - t$, since the arc $\gamma_j\big|_{[0,t]}$ cannot go farther from the left endpoint at d then its arc length t. It follows that at $z = \gamma_j(t)$ it holds $|p(z)| \ge (d-t)^n$ until $0 \le t \le \lambda := \min(d, \ell)$, and so we have

$$\begin{aligned} \|p\|_q^q &\ge \int_{\Gamma_j} |p(z)|^q |d(z)| \ge \int_0^\lambda (d-t)^{nq} dt = \frac{1}{nq+1} \left[d^{nq+1} - (d-\lambda)^{nq+1} \right] \\ &\ge \frac{d \left[d^{nq} - (d-\lambda)^{nq} \right]}{nq+1} \end{aligned}$$

and

$$\left(\frac{\|p\|_\infty}{\|p\|_q} \right)^q \le \frac{nq+1}{d} \left(\frac{1}{1-(1-\lambda/d)^{nq}} \right) < \frac{nq+1}{d} \left(\frac{1}{1-(1-\lambda/d)} \right) = \frac{nq+1}{\lambda}$$

Finally applying this in (30) leads to

$$\|p'\|_q^q \le \left(\frac{2n}{d} \right)^q \left\{ \frac{nq+1}{\lambda 2^{nq}} L + 1 \right\} \|p\|_q^q.$$

Since $q \ge 1$ and $nq + 1 \ge 2$, it suffices to observe that $x/2^x$ decreases for $x \ge 2$ and thus $(nq+1)2^{-nq} \le 2 \max_{x \ge 2} x 2^{-x} = 1$. We finally obtain

$$\|p'\|_q \le \frac{2}{d} \frac{L + \min(d,\ell)}{\min(d,\ell)} n \|p\|_q. \tag{31}$$

As here the constant depends only on the domain K, we conclude that $\|p'\|_q \le C(K) n \|p\|_q$ holds for the chosen polynomial $p \in \mathcal{P}_n(K)$, whence the assertion.

Remark 6. In case of a convex domain K, the parameters occurring here in $C(K)$ have a simpler meaning. First, the boundary ∂K is connected (consists of only one convex curve), and thus $\ell = L$ and $\min(d, \ell) = \min(d, L) = d$. Second, as we have used several times, *for convex curves* the estimate $L \le 2\pi d$ holds true, always, whence (31) simplifies to $\|p'\|_q \le \frac{4\pi+2}{d} n \|p\|_q < \frac{15}{d} n \|p\|_q$, say.

Remark 7. In [41] a more precise value of the constant $C(K)$ has also been obtained for the case of the infinity norm. As for L^q-norm neither the order (in general), nor the more exact constants are known, it seemed to be well ahead of time to bother with sharper values of the constant $C(K)$ here. Nevertheless, our feeling is that the slightly more involved construction of [41] would indeed provide a better constant, which may be sharp, apart from an absolute constant factor, like in case $q = \infty$. Here we do not pursue this issue any more.

7. Concluding remarks

Above we have seen, that like in case of the maximum norm, also for the $L^q(\partial K)$ norm any compact convex domain K admits polynomials $p \in \mathcal{P}_n(K)$ with oscillation not exceeding $O(n)$. On the other hand we have shown for some classes of convex domains that the order of oscillation indeed reaches $c_K n$.

A natural question–quite resembling to the question posed by Erőd 77 years ago in case of the maximum norm–is to identify those domains which indeed admit order n oscillation even in $L^q(\partial K)$ norm.

It has been clarified that, like in case of the maximum norm, also for L^q norms the interval $\mathbb{I}$ behaves differently: there the order of oscillation may be as low as $\sqrt{n}$. Therefore, it is certainly necessary that *some* conditions are assumed for an order n oscillation. The question is if apart from having a nonempty interior, is there need for any additional assumption? We think that probably not.

Conjecture 1. *For all compact convex domains $K \Subset \mathbb{C}$ there exist $c_K > 0$ such that for any $p \in \mathcal{P}_n(K)$ we have $\|p'\|_{L^q(\partial K)} \geq c_K n \|p\|_{L^q(\partial K)}$.*

We are not really close to this conjecture. Let us point out that a general estimate–however weak–is still missing in the full generality of all convex compact domains. Apart from the cases discussed here, we mentioned that the case of the interval $\mathbb{I}$ is clarified–there the oscillation being of order $\sqrt{n}$. So clearly also here there is a difference between various compact convex sets. However, we do not really know if $\mathbb{I}$ is indeed to be "the worst", i.e. of lowest possible oscillation, as for general domains really nothing–not even some mere $\log n$ e.g.–has been proved to date.

Therefore, posing a more modest goal, we would be interested as well *in any estimate* working for *general compact convex domains*, without further assumptions on the geometrical features of it.

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