# Hybridizable Discontinuous Galerkin Methods for modelling 3D seismic wave propagation in harmonic domain

Marie Bonnasse-Gahot, Henri Calandra, Julien Diaz, Stephane Lanteri


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M. Bonnasse-Gahot[1,2], H. Calandra[3], J. Diaz[1] and S. Lanteri[2]

[1] *INRIA Bordeaux-Sud-Ouest, team-project Magique 3D*
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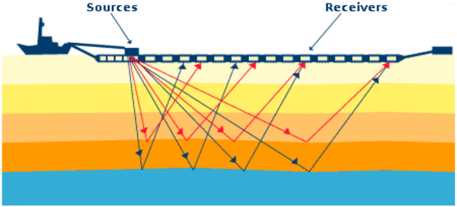
[3] *TOTAL Exploration-Production*

May 15, 2017

# Motivations

## Principles of seismic imaging



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## Examples of seismic imaging campaigns

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## Imaging methods

- Full Wave Inversion (FWI) : **inversion process** requiring to solve many forward problems

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- **Harmonic-domain** : imaging condition simple but huge computational cost

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⚠ **Memory usage** ⚠

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- Elastic wave propagation in the frequency domain : Helmholtz equation

## First order formulation of Helmholtz wave equations

$\mathbf{x} = (x, y, z) \in \Omega \subset \mathbb{R}^3$,

$$\begin{cases} i\omega\rho(\mathbf{x})\mathbf{v}(\mathbf{x}) = \nabla \cdot \underline{\underline{\sigma}}(\mathbf{x}) \\ i\omega\underline{\underline{\sigma}}(\mathbf{x}) = \underline{\underline{C}}(\mathbf{x})\ \underline{\underline{\varepsilon}}(\mathbf{v}(\mathbf{x})) + f_s(\mathbf{x}) \end{cases}$$

- $\mathbf{v}$ : velocity vector
- $\underline{\underline{\sigma}}$ : stress tensor
- $\underline{\underline{\varepsilon}}$ : strain tensor

# Approximation methods

## Discontinuous Galerkin Methods

✓ unstructured tetrahedral meshes
✓ combination between FEM and finite volume method (FVM)
✓ *hp*-adaptivity
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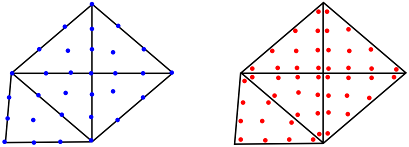

# Approximation methods

## Hybridizable Discontinuous Galerkin Methods

✓ same advantages as DG methods : unstructured tetrahedral meshes, *hp*-adaptivity, easily parallelizable method, discontinuous basis functions
✓ introduction of a new variable defined only on the interfaces
✓ lower number of coupled DOF than classical DG methods

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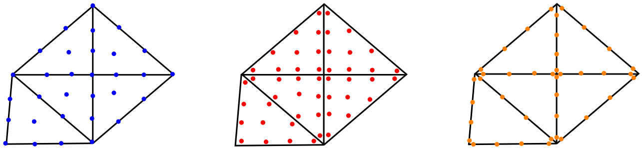

# Approximation methods

## Hybridizable Discontinuous Galerkin Methods


- B. Cockburn, J. Gopalakrishnan and R. Lazarov. Unified hybridization of discontinuous Galerkin, mixed and continuous Galerkin methods for second order elliptic problems. *SIAM Journal on Numerical Analysis,* Vol. 47 :1319-1365, 2009.
- S. Lanteri, L. Li and R. Perrussel. Numerical investigation of a high order hybridizable discontinuous Galerkin method for 2d time-harmonic Maxwell's equations. *COMPEL,* 32(3)1112-1138, 2013.
- N.C. Nguyen, J. Peraire and B. Cockburn. High-order implicit hybridizable discontinuous Galerkin methods for acoustics and elastodynamics. *Journal of Computational Physics,* 230 :7151-7175, 2011
- N.C. Nguyen and B. Cockburn. Hybridizable discontinuous Galerkin methods for partial differential equations in continuum mechanics. *Journal of Computational Physics* 231 :5955–5988, 2012

# Principles of the HDG method

1. Introduction of a Lagrange multiplier $\lambda$

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   - Weak formulation of waves equations

$$\left\{ \begin{array}{l} \displaystyle\int_K i\omega\rho^K \mathbf{v}^K \cdot \mathbf{w} + \int_K \underline{\underline{\sigma}}^K : \nabla\mathbf{w} - \int_{\partial K} \underline{\underline{\sigma}} \cdot \mathbf{n} \cdot \mathbf{w} = 0 \\ \displaystyle\int_K i\omega\underline{\underline{\sigma}}^K : \underline{\underline{\xi}} + \int_K \mathbf{v}^K \cdot \nabla \cdot \left( \underline{\underline{C}}^K \underline{\underline{\xi}} \right) - \int_{\partial K} \mathbf{v} \cdot \underline{\underline{C}}^K \underline{\underline{\xi}} \cdot \mathbf{n} = \int_K {f_S}^K \cdot \underline{\underline{\xi}} \end{array} \right.$$

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$\widehat{\underline{\underline{\sigma}}}^K$ and $\widehat{\mathbf{v}}^K$ are numerical traces of $\underline{\underline{\sigma}}^K$ and $\mathbf{v}^K$ respectively on $\partial K$

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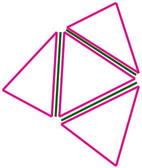
   - 

$$\widehat{\mathbf{v}}^{\partial K} = \lambda^F, \qquad \forall F\in\mathcal{F}_h,$$



# Principles of the HDG method

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   - $$\begin{array}{rcll} \widehat{\mathbf{v}}^{\partial K} & = & \lambda^F, & \forall F \in \mathcal{F}_h, \\ \widehat{\underline{\underline{\sigma}}}^{\partial K} \cdot \mathbf{n} & = & \underline{\underline{\sigma}}^K \cdot \mathbf{n} - \tau \mathbf{I} \left( \mathbf{v}^K - \lambda^F \right), & \text{on } \partial K \end{array}$$

     where $\tau$ is the stabilization parameter ($\tau > 0$)

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   - Local HDG formulation

$$\begin{cases} \displaystyle\int_K i\omega\rho^K \mathbf{v}^K \cdot \mathbf{w} - \int_K \left(\nabla \cdot \underline{\underline{\sigma}}^K\right) \cdot \mathbf{w} + \int_{\partial K} \tau \mathbf{I} \left(\mathbf{v}^K - \lambda^F\right) \cdot \mathbf{w} = 0 \\ \displaystyle\int_K i\omega \underline{\underline{\sigma}}^K : \underline{\underline{\xi}} + \int_K \mathbf{v}^K \cdot \nabla \cdot \left(\underline{\underline{C}}^K \underline{\underline{\xi}}\right) - \int_{\partial K} \lambda^F \cdot \underline{\underline{C}}^K \underline{\underline{\xi}} \cdot \mathbf{n} = 0 \end{cases}$$

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   - Local HDG formulation

$$\mathbb{A}^K \mathbf{W}^K + \mathbb{C}^K \lambda = \mathbf{S}^K$$

with $\mathbf{W}^K = \left( \mathbf{v}^K, \ \underline{\underline{\sigma}}^K \right)^T$ and $\lambda = \left( \lambda^{F_1}, \ \lambda^{F_2}, ..., \ \lambda^{F_{n_f}} \right)^T$, where $n_f = \mathrm{card}(\mathcal{F}_h)$

# Principles of the HDG method

1. Introduction of a Lagrange multiplier $\lambda$
2. Expressing the initial unknowns $\mathbf{W}^K$ as a function of $\lambda$

$$\mathbb{A}^K \mathbf{W}^K + \mathbb{C}^K \lambda = \mathbf{S}^K \longrightarrow \mathbf{W}^K = \left(\mathbb{A}^K\right)^{-1} \mathbf{S}^K - \left(\mathbb{A}^K\right)^{-1} \mathbb{C}^K \lambda$$

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4. $\mathbb{M}\lambda = \mathbf{S}$
5. Computation of the solutions of the initial problem $\mathbf{W}^K$, element by element

# Contents

# Main steps of the algorithms

| | HDG algorithm | | IPDG & FE algorithm |
|---|---|---|---|
| **1.** | Construction of the linear system in $\lambda$ | **1.** | Construction of the linear system in $\mathbf{u}$ |
| **2.** | Resolution of the linear system | **2.** | Resolution of the linear system |
| **3.** | Reconstruction of the initial solution $\mathbf{W}$ | | |

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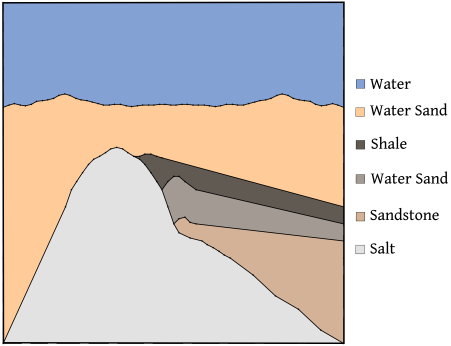
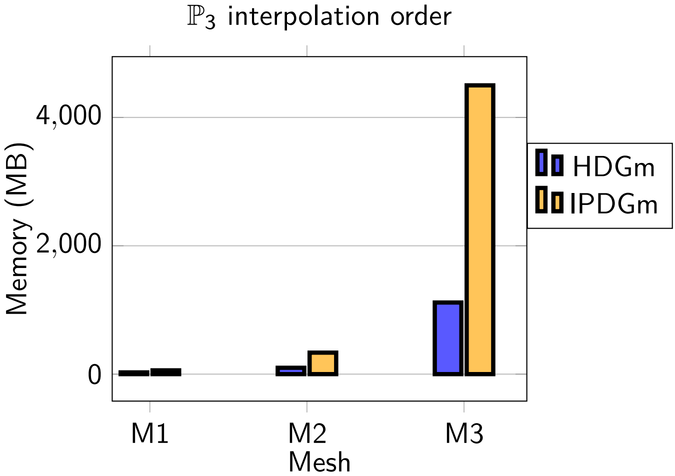
*embarrassingly parallel*

# Anisotropic test case



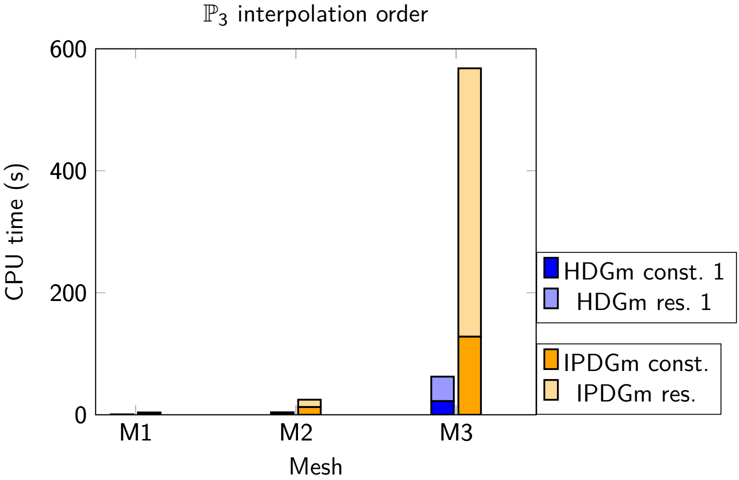

- Three meshes :
  - 600 elements
  - 3000 elements
  - 28000 elements

# Anisotropic case : Memory consumption



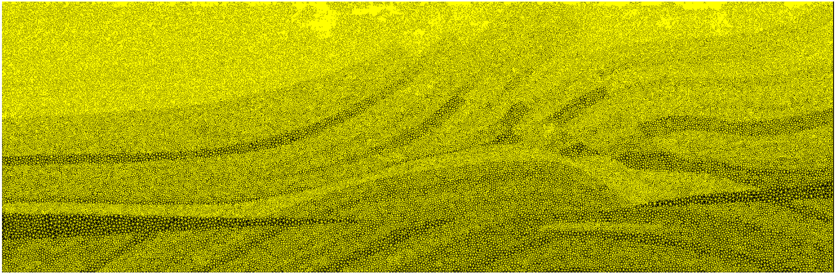

$$\mathbf{IPDG_{memory}} \simeq \mathbf{4} \times \mathbf{HDG_{memory}}$$

# Anisotropic case : CPU time (s)




$$\mathbf{IPDG_{CPUtime}} \simeq \mathbf{9} \times \mathbf{HDG_{CPUtime}}$$

# Marmousi test-case



Computational domain $\Omega$ composed of 235000 triangles

# Cluster configuration

- Hardware specification : 16 nodes, 12 cores by nodes
- Caracteristics of computing nodes :
  - 2 Hexa-core Westmere Intel® Xeon® X5670
  - Frequency : 2,93 GHz
  - Cache L3 : 12 Mo
  - RAM : 96 Go
  - Infiniband DDR : 20Gb/s
  - Ethernet : 1Gb/s

# Efficiency of the parallelism of the construction of the global matrix

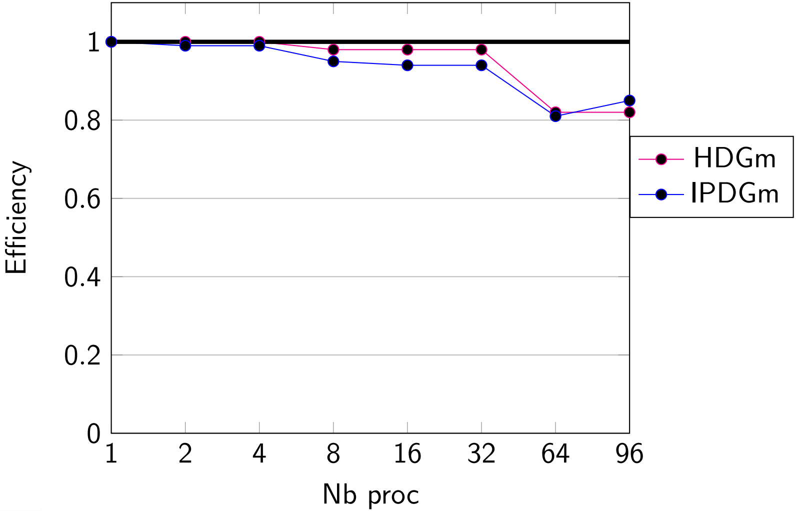
$$\text{Efficiency} = \frac{t\_{ref}}{nb\_cores \times t\_{one\ core}}$$

where :

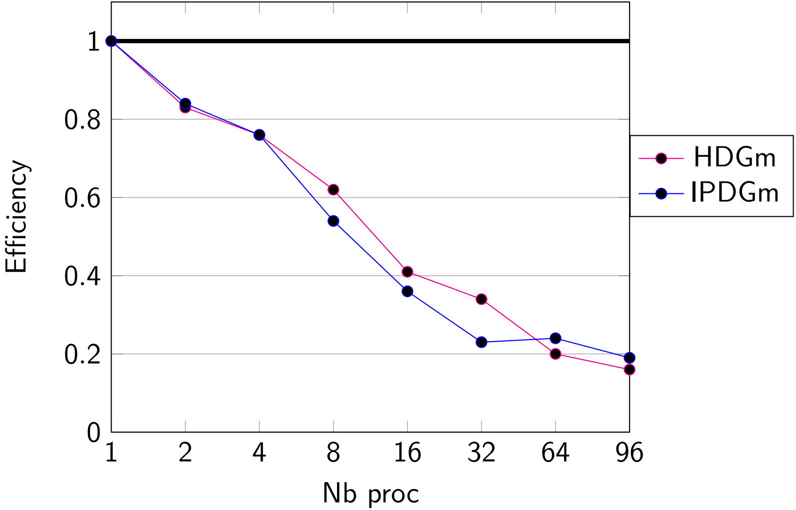
$$t\_{ref} = t\_{seq} = \text{sequential computational time}$$

$$t\_{one\ core} = \text{computational time with one core}$$

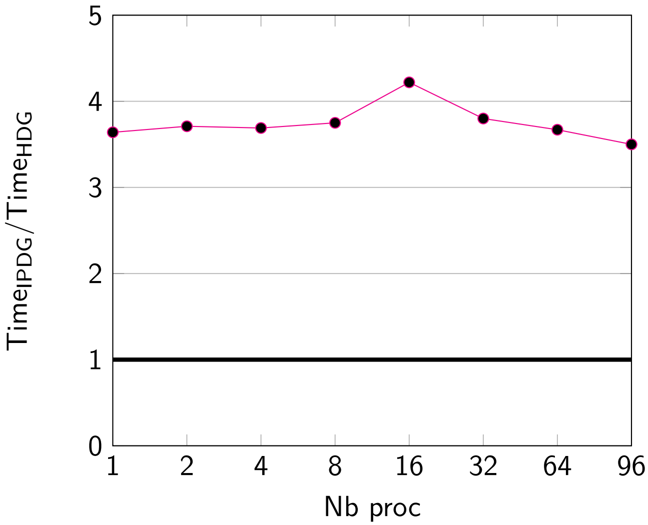
# Efficiency of the parallelism of the construction of the global matrix



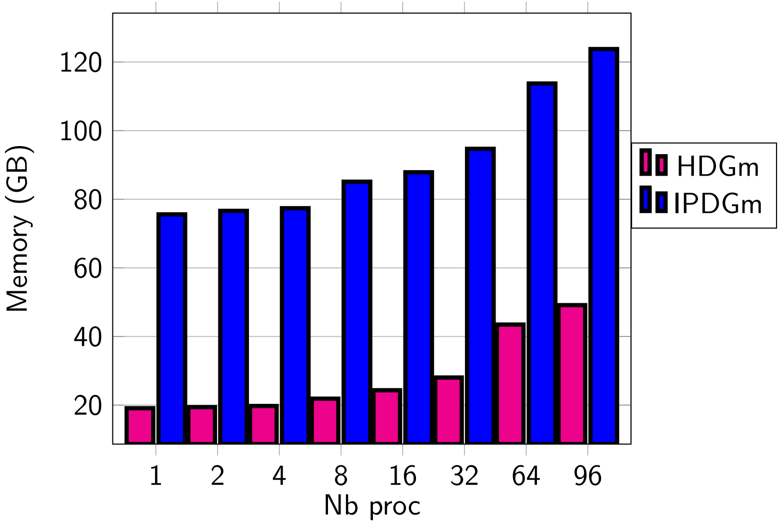
# Efficiency of the parallelism of the whole algorithm



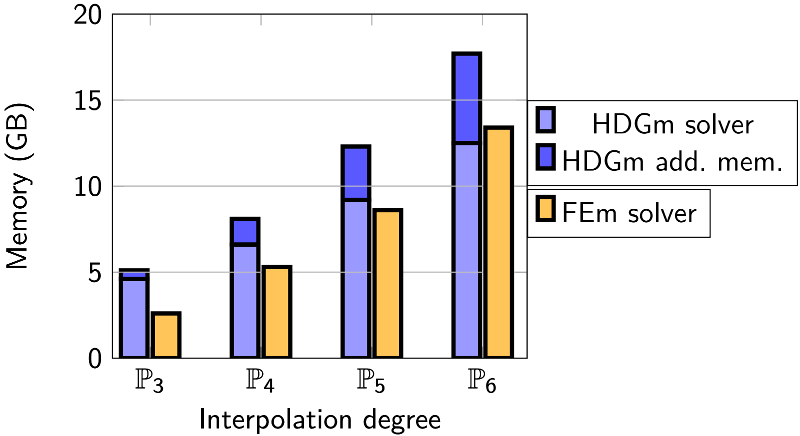
# Speed up for the global matrix construction



# Memory required (GB) for the simulation

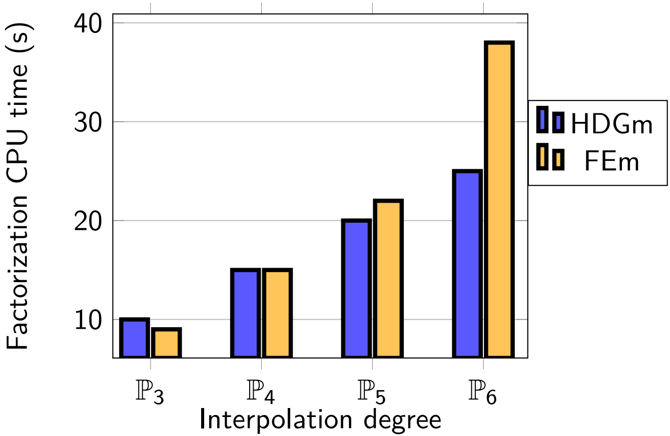


# Comparison with Finite Elements method



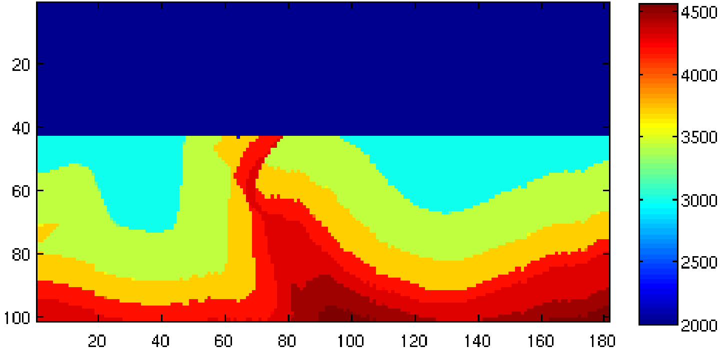

$$\mathbf{FE}_{\mathbf{memory}} \simeq \mathbf{0.7} \times \mathbf{HDG}_{\mathbf{memory}}$$

# Comparison with Finite Elements method



# Contents

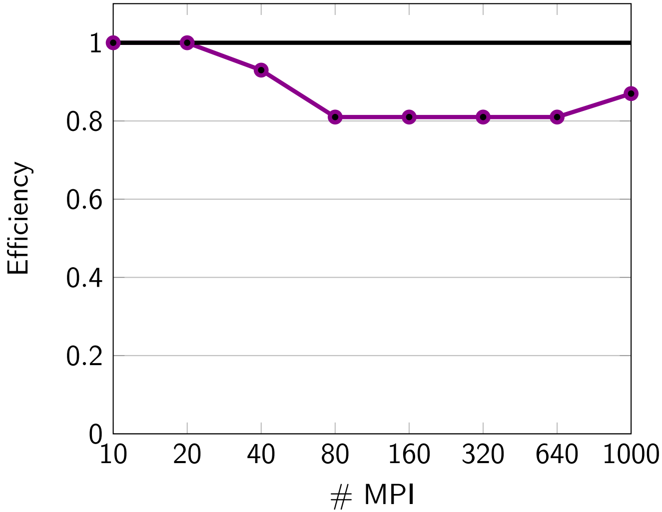
# Epati test-case



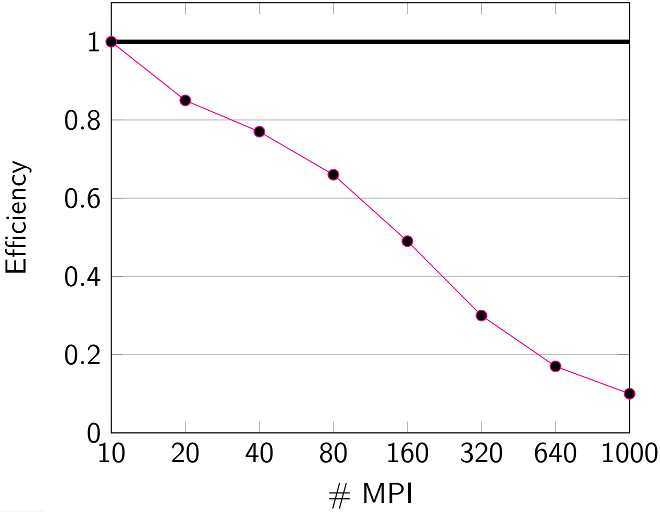

$V_p$-velocity model (m.s$^{-1}$), vertical section at $y = 700$ m

Mesh composed of 25 000 tetrahedrons

# Epati test-case : Scalability, efficiency of the HDG global system construction



# Efficiency of the parallelism of the whole algorithm (construction and resolution)



# Resolution of the linear system $\mathbb{M}\lambda = \mathbf{S}$

- Direct solvers
- Iterative solvers
- Hybrid solvers , combination between direct and iterative solvers
- Direct solvers combined with $H$-matrices
- ...

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- **Direct solvers**
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- **Hybrid solvers**, combination between direct and iterative solvers
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- ...

# Resolution of the linear system $\mathbb{M}\lambda = \mathbf{S}$

- **direct solver : MUMPS** (MUltifrontal Massively Parallel sparse direct Solver) :
  - Direct factorization $A = LU$ or $A = LDL^T$
  - Multiple RHS

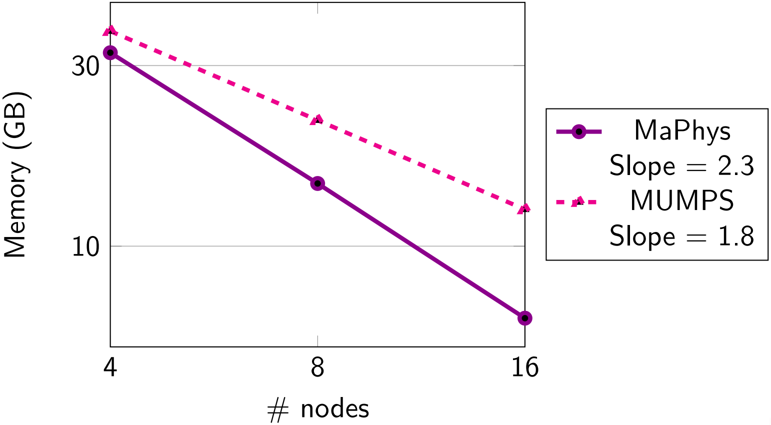
P.R. Amestoy, I.S. Duff and J.-Y. L'Excellent. Multifrontal parallel distributed symmetric and unsymmetric solvers. *Computational Methods in Applied Mechanics and Engineering,* Vol. 184 :501-520, 2000.

- **hybrid solver : MaPhys** (Massively Parallel Hybrid Solver) :
  - combination of direct and iterative methods
  - non-overlapping algebraic domain decomposition method (Schur complement method)
  - resolution of each local problem thanks to direct solver such as MUMPS

https ://gitlab.inria.fr/solverstack/maphys

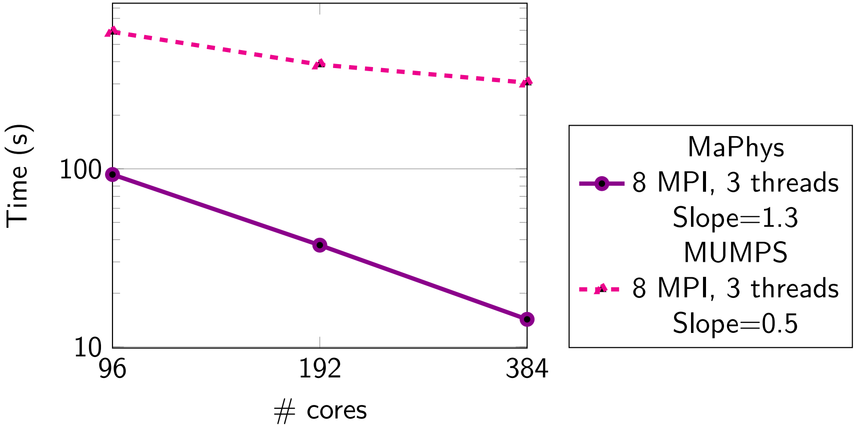
# Epati test-case : Memory consumption

Average memory for one node (8 MPI by node and 3 threads by MPI)

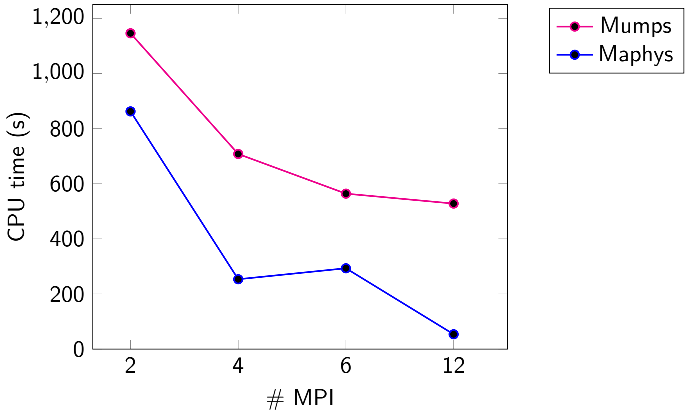


# Epati test-case : Execution time

Execution time for the resolution of the HDG-$\mathbb{P}_3$ system



# Epati test-case : Cumulative time of the two solvers using 96 cores



# Conclusion-Perspectives

## Conclusions

- HDG method more efficient than classical DG methods for a same accuracy (memory and CPU time)
- In 2D, HDG method has equivalent computational performances than FE methods
- Maphys good alternative to MUMPS for the resolution of the HDG system with one RHS
  ✗ Multiple RHS not yet implemented in Maphys

# Conclusion-Perspectives

## Perspectives

- more detailed analysis of the comparison between solvers
  - larger meshes
  - more powerful clusters
- extension to $H$-matrices
- 3D comparison between FE and HDG methods
- extension to elasto-acoustic case
- application to inverse problems

Thank you !