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On wave breaking for Boussinesq-type models

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Abstract

We consider the issue of wave breaking closure for Boussinesq type models, and attempt at providing some more understanding of the sensitivity of some closure approaches to the numerical set-up, and in particular to mesh size. For relatively classical choices of weakly dispersive propagation models, we compare two closure strategies. The first is the hybrid method consisting in suppressing the dispersive terms in breaking regions, as initially suggested by Tonelli and Petti in 2009. The second is an eddy viscosity approach based on the solution of a turbulent kinetic energy. The formulation follows early work by O. Nwogu in the 90's, and some more recent developments by Zhang and co-workers (Ocean Mod. 2014), adapting it to be consistent with the wave breaking detection used here. We perform a study of the behavior of the two closures for different mesh sizes, with attention to the possibility of obtaining grid independent results. Based on a classical shallow water theory, we also suggest some monitors to quantify the different contributions to the dissipation mechanism, differentiating those associated to the scheme from those of the partial differential equation. These quantities are used to analyze the dynamics of dissipation in some classical benchmarks, and its dependence on the mesh size. Our main results show that numerical dissipation contributes very little to the results obtained when using eddy viscosity method. This closure shows little sensitivity to the grid, and may lend itself to the development and use of non-dissipative/energy conserving numerical methods. The opposite is observed for the hybrid approach, for which numerical dissipation plays a key role, and unfortunately is sensitive to the size of the mesh. In particular, when working, the two approaches investigated provide results which are in the same ball range and which agree with what is usually reported in literature. With the hybrid method, however, the inception of instabilities is observed at mesh sizes which vary from case to case, and depend on the propagation model. These results are comforted by numerical computations on a large number of classical benchmarks.

1 Introduction

The last decades have seen the development of several numerical models allowing the simulation of wave propagation from intermediate depths to shallow water by means of some set of depth averaged Boussinesq-type (BT) equations. Many implementations of these are quite well known in the coastal engineering community, to which they are often freely available. We can for example mention the codes BOUSS-2D [1–4], Funwave [5, 6], Coulawave [7, 8] BOSZ [9], MIKE21 [10], TUCWave [11, 12], and many others. These models allow to accurately simulate the dispersive propagation and shoaling of free surface waves, within some asymptotic error w.r.t. nonlinearity and dispersion parameters depending on wave amplitude, wavelength and depth. The reader may refer to the reviews [13, 14] for a broad discussion, and the book [15] for the fundamental aspects concerning the derivation of the underlying partial differential equations. These equations are obtained under the hypotheses of ideal, and most often irrotational flow, and cannot account for the transformation processes taking place in breaking regions. To cope with this limitation, some closure model needs to be introduced.

38 At large scales, the main consequence of wave breaking is a strong energy dissipation. So the first at-
39 tempt to simulate wave breaking was proposed by Zelt [16] introducing a dissipation term in the momentum
40 equation. This term controls the dissipation of energy produced by the wave breaking and it is governed
41 by the value of an eddy viscosity coefficient which must be calibrated with experimental data. Of course,
42 different calibration is needed for different sets of equations. Moreover, to initiate and/or terminate the
43 breaking process some breaking detection criterion needs to be used to activate this eddy viscosity term.
44 The same approach has been followed by many researchers, see for example [17–22]. One of the criticisms
45 to this approach is that, while simple, no direct physical meaning can be attributed to the scaling coeffi-
46 cients involved in the definition of the viscosity [23]. A more relevant physical definition of the effects
47 of breaking on the large scale flow has been attempted using the so-called roller models (see e.g. [23–26]).
48 While based on a better physical background, these models still require some ad-hoc definition of a mo-
49 mentum dissipation, and require some calibration. A more advanced version of these roller models has been
50 proposed in [27], and more recently extended in [28]. These models attempt at accounting for variations
51 along the depth of some of the physical quantities (eddy viscosity, horizontal velocity), thus going beyond
52 the irrotational hypothesis when computing the vorticity and/or dissipation generated in breaking regions.
53 While promising in principle, these models are more complex to implement, require an additional vertical
54 discretization, and have so far been applied only to simple configurations. We also mention the related work
55 presented in [29, 30] in which BT models with vorticity effects are discussed. Beside the requirement of a
56 proper calibration of the model, one of the questions we think is not clearly answered in literature is how
57 much the numerical method interacts with the above modelling approaches, and in particular what is the
58 balance between the numerical and model dissipation. This is also related to the fact that almost system-
59 atically wave breaking benchmarks are presented without any grid convergence analysis. Another issue is
60 the ability of these approaches to describe properly some special cases as, e.g. stationary hydraulic jumps.
61 To the authors’ knowledge there is very little evidence in literature that this type of breaker can be easily
62 modelled with the eddy viscosity approach. Results embedding this type of features, such as e.g. the 2D
63 reef computations presented in [18], again computed on a single mesh. This makes unclear whether the
64 major effect observed is that of the model or that of the limiter/numerical dissipation.

65 Nevertheless, the eddy viscosity method is a very successful one, which is why we consider its use in
66 this paper. Previous work from one of the authors [12, 31] has shown that the classical definition of the
67 eddy viscosity by [17] has trouble detecting stationary hydraulic jumps, and that even modifying the incep-
68 tion mechanisms, the amount of viscosity obtained is not enough for this type of breakers. For this reason
69 we have looked at a more involved approach involving partial differential equations for the main physi-
70 cal quantities: turbulent kinetic energy, energy dissipation, eddy viscosity, and so on. To our knowledge
71 so far only [1, 3, 4, 32–35] have adopted this path, with only [1–4, 32, 33] actually focusing on complex cases.

72
73 As an alternative to the use of eddy viscosity the last ten years have seen the development of a hybrid
74 approach based on a local coupling of the dispersive propagation model with the shallow water equations.
75 It is a simple method in which one first detects breaking regions, and in these the dispersive terms are sup-
76 pressed. In these breaking regions thus one solves the non-linear shallow water (NSW) equations which
77 allow to model a breaker as a shock. Through this discontinuity mass and momentum are conserved, while
78 total energy is dissipated, thus modelling the energy dissipation due to breaking. Due to its relative simplic-
79 ity and effectiveness, this approach has gained substantial attention in the coastal engineering community,
80 see for example [6, 9, 12, 36–38]. The idea was first introduced in [36] in order to exploit the Finite Vol-
81 ume (FV) technique as to simulate discontinuous phenomena such as wave breaking and run-up. In the
82 same work, an indicator criterion for breaking has been extracted based on the similarity between spilling
83 breakers and bores. This criterion has been proven inadequate in some cases [12, 39] since its use leads

84 to less energy dissipation than needed. Several more sophisticated criteria have been developed based on
 85 physical or numerical arguments [12, 40–42]. As pointed out in [37], this approach has a major limitation
 86 in the stability of the coupling which introduces spurious oscillations at the interface between the breaking
 87 and no-breaking region. This phenomenon has been observed by many [43–45], but is unfortunately poorly
 88 documented in literature. One of the issues not fully addressed is the role of numerical dissipation in curing
 89 this flaw. The use of more robust limiting procedures, is advocated by some as e.g. [43, 46] as a means
 90 of stabilizing the numerical solution. For example, for of a fully non-linear weakly dispersive propagation
 91 model in [43] it is suggested that degrading the accuracy of the numerical discretization in correspondence
 92 of the shallow water-Boussinesq interface, thus somehow increasing numerical dissipation, allows to re-
 93 move numerical perturbations on relatively coarse meshes. Unfortunately, the sensitivity to the grid size
 94 for this closure remains unclear. To demonstrate this point we consider the following motivational example
 95 involving the shoaling and breaking of a solitary wave on a slope. This example is part of a set of very
 96 classical benchmarks by C. Synolakis [47] which we will study in more detail in section §7.2. Here, we
 97 present results for a breaking case (wave nonlinearity equal to 0.28), in which we set manually the transi-
 98 tion region according to its known position from the experiments. Compared to actual simulations with the
 99 hybrid method, note that this eliminates one of the causes of instability: the intermittency of the detection.
 100 We then perform simulations degrading the numerical scheme at the interface between the Boussinesq and
 101 shallow water regions, and in all the shallow water region. In these areas we use the most non-oscillatory
 102 and robust possible choice: the first order upwind finite volume method. We consider in figure 1 results in
 103 two very close time instants, on three meshes. The red line shows the detection flag separating the Boussi-
 104 nesq and shallow water regions (one indicates the breaking region, and zero the Boussinesq region). The
 105 figures show that: no instabilities whatsoever are observed in the largest gradient region (which will become
 106 the bore). An oscillation is instead triggered at the interface, and its blow up is almost instantaneous on the
 107 finest mesh, despite the fact that the most dissipative approach available has been used.

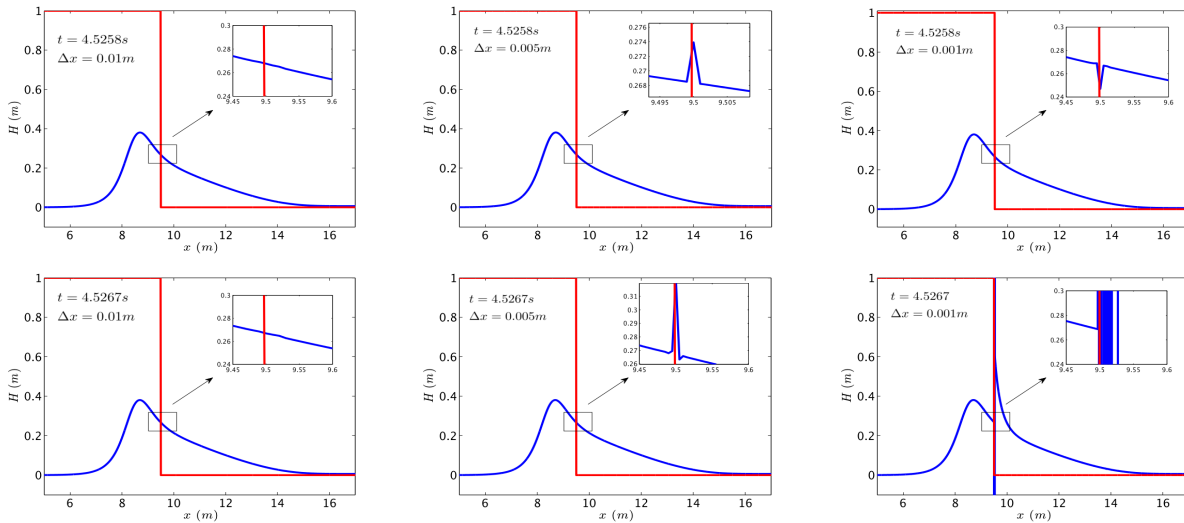


Figure 1: Solitary wave breaking on a slope: hybrid treatment with order reduction at the coupling interface. Wave height at times $t = 4.5258s, 4.5267s$ (top and bottom rows), on mesh sizes (from left to right) $\Delta x = 0.01m, 0.005m, 0.001m$.

108 Grid convergence analysis for breaking cases are quite rare in literature. The only exception we are
 109 aware of is the single computation shown in [6] in which the authors observe convergence in time aver-

110 aged quantities, but report the appearance of increasing oscillations in the pointwise values of the solution
111 without further notice. Clarifying these aspects is of paramount importance. Indeed if one cannot be sure
112 that the mesh size allows the numerical dissipation to be sufficiently large, the initiation of the numerical
113 instabilities may be confused with physical/dispersive effects. A control of this flaw is of course also needed
114 if one wishes for example to use mesh adaptation in breaking regions.

115
116 The aim of this work is to investigate the above issues for choices of propagation models and wave
117 breaking closures representative of classical and well known models such as BOUSS-2D [3, 4], Funwave
118 [5, 6], Coulwave [7, 8], BOSZ [9], MIKE21 [10], TUCWave [11, 12], and others. We thus use two enhanced
119 weakly dispersive Boussinesq models: the weakly nonlinear model of Nwogu (used e.g. in BOUSS-2D,
120 BOSZ, and TUCWave), and a frequency enhanced variant of the fully nonlinear Green-Naghdi equations
121 (with similar properties to those used in Funwave and Coulwave). We compare the hybrid approach to an
122 eddy viscosity model. **Note that with the eddy viscosity closure the breaking wave fronts are smoother than
123 those obtained with the hybrid method which relies on the approximation of these fronts as shocks. For
124 this reason when using the hybrid approach one has to carefully choose the conservative form used to solve
125 the model, which is essential to recover the right jump conditions, and some form of limiting to avoid the
126 creation of additional spurious numerical oscillations in correspondance of the shock [11, 12, 48, 60].**

127 As mentioned before, the original definition of an eddy viscosity model [17] has been previously shown
128 to have difficulties in handling steady hydraulic jumps [12, 31]. For this reason we have chose to use an
129 approach based on the solution of a partial differential equation for the turbulent kinetic energy, similar
130 to the one studied with BOUSS-2D (see [1, 3, 4, 33]). Note that other closures, such as roller models as
131 proposed in [27, 28], or other definitions of the eddy viscosity are certainly as valid a choice as the one
132 made here. A thorough comparison of differences in these approaches ins not in our scopes, and is left for
133 future work. Our main objectives are the following:

- 134 • to perform a systematic study of the behaviour of the two closures for different mesh sizes, with
135 attention to the possibility of obtaining grid independent results ;
- 136 • to gain an insight into the mechanism actually responsible for wave breaking by providing a quan-
137 titative description of the different contributions to the dissipation mechanism, differentiating those
138 associated to the numerical scheme from those introduced at the PDE level;
- 139 • to provide some understanding of the sensitivity of the above mentioned dissipation to the mesh size;
- 140 • to prove the equivalent capabilities of the approaches studied in reproducing simple as well as com-
141 plex wave transformation, while showing the substantial difference in the underlying dissipation
142 mechanisms.

143 The paper is organised as follows. Section two presents the two Boussinesq approximations used in
144 this work. Section 3 discusses the numerical approximation of the models, as well as of the wave breaking
145 closure. The comparison of the two approaches on a wide selection of benchmarks is discussed in section
146 4. The paper is ended by a summary and a sketch of the future and ongoing developments of this work.

147 2 Wave propagation models

148 2.1 The weakly nonlinear-weakly dispersive model of Nwogu

149 With the notation sketched in figure 2, we consider the Boussinesq equations proposed by Nwogu [49]
150 based on a weakly-dispersive and weakly-nonlinear asymptotic approximation in terms of the velocity u^a

151 at an arbitrary distance from a still water level z^a . Denoting partial derivatives with respect to space and
 152 time with the subscripts x and t , Nwogu's equations can be cast in a balance law form as follows

$$\mathbf{U}_t + \mathbf{F}(\mathbf{U}^*)_x = \mathbf{S}_b - \mathbf{S}_d + \mathbf{S}_f + \mathbf{R}_{wb}, \quad (1)$$

153 where \mathbf{U} is the vector of the new variables, \mathbf{U}^* is the vector of the conserved variables, and \mathbf{F} is the flux
 154 vector

$$\mathbf{U} = \begin{bmatrix} H \\ P^* \end{bmatrix}, \quad \mathbf{F}(\mathbf{U}) = \begin{bmatrix} Hu^a \\ H(u^a)^2 + \frac{1}{2}gH^2 \end{bmatrix}.$$

155 The P^* variable is a pseudo-mass flux accounting for the vertical (weakly-dispersive and weakly-nonlinear)
 156 expansion of the velocity profile:

$$P^* = Hu^a + Hz^a \left(\frac{z^a}{2} u_{xx}^a + (du^a)_{xx} \right) \quad (2)$$

157 In the above equations d denotes the still water depth, $H(x, t) = d(x) + \eta(x, t)$ the total water depth, $\eta(x, t)$
 158 the free surface elevation, b the bathymetry height, g is the gravitational acceleration. As done usually, the
 159 value of z^a is chosen to optimize the linear dispersion properties of the model, namely $z^a = -0.531d$.

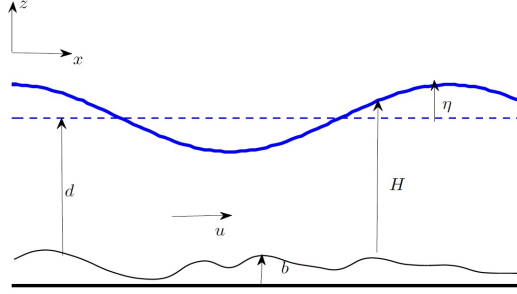


Figure 2: Sketch of the domain.

notation

160 The three source terms on the right hand side of (1) can be expressed as $\mathbf{S}_b = [0 \quad -gHb_x]^T$, accounting
 161 for the effects of the shape of the topography, $\mathbf{S}_f = [0 \quad -gHS_f u^a]$ with $S_f = \frac{r_m^2 ||u||}{H^{4/3}}$, accounting for the
 162 friction on the bottom, and $\mathbf{S}_d = [\psi_C \quad u^a \psi_C - \psi_M]$ introduces additional dispersive terms which do not
 163 contain time derivatives, and in particular

$$\psi_M = -((Hu^a)_x + \psi_C)z^a \left(\frac{z^a}{2} u_{xx}^a + (du^a)_{xx} \right), \quad \psi_C = \left[\left(\frac{(z^a)^2}{2} - \frac{d^2}{6} \right) du_{xx}^a + \left(z^a + \frac{d}{2} \right) d(du^a)_{xx} \right]_x. \quad (3)$$

164 The last term on the right hand side is the turbulent wave breaking term, which is only present when this
 165 approach is activated. Following [1, 32, 33] this term has the form

$$\mathbf{R}_{wb} = \begin{bmatrix} 0 \\ r_{wb} \end{bmatrix}_x, \quad r_{wb} = \nu_t H u_x^a$$

166 with the eddy viscosity ν_t computed from the discretization of the turbulence model, discussed in §4.2.

167 2.2 Fully nonlinear-weakly dispersive Green-Naghdi equations

168 To account for fully nonlinear effects, we also consider the Green-Naghdi (GN) partial differential equations
 169 [50]. In particular, we cast the system in the form suggested in [37] (see also [15, 51, 52] and references
 170 therein) :

$$H_t + (Hu)_x = 0 \quad (4)$$

$$(Hu)_t + (Hu^2)_x + gH\eta_x = H\psi \quad (5)$$

$$H\psi + \alpha HT(\psi) = HT(w) - HQ + gHS_f u + (r_{wb})_x \quad (6)$$

171 where now u denotes the depth averaged velocity, with $w = g\eta_x$, and \mathcal{T} a linear elliptic operator with the
 172 self-adjoint form [51]

$$\mathcal{T}(\cdot) = S_1^* (HS_1(\cdot)) + S_2^* (HS_2(\cdot)) , \quad (7) \quad \boxed{\text{T1}}$$

173 where in one space dimension

$$S_1 = \frac{H}{\sqrt{3}}(\cdot)_x - \frac{\sqrt{3}}{2}b_x(\cdot), \quad S_2 = \frac{1}{2}b_x(\cdot). \quad (8) \quad \boxed{\text{T2}}$$

174 The quantity ψ in (5) and (6) is essentially the gradient of the non-hydrostatic pressure. The right hand side
 175 last in (6) also introduces the nonlinear forcing \mathcal{Q} defined as

$$\mathcal{Q} = 2HH_x(u_x)^2 + \frac{4H^2}{3}u_x u_{xx} + Hb_x(u_x)^2 + Hb_{xx}uu_x + \left[b_{xx}H_x + \frac{1}{2}Hb_{xxx} + b_x b_{xx} \right] u^2 \quad (9)$$

176 Following [38] the value $\alpha = 1.159$ is chosen to optimise the linear dispersion relation of the system. In
 177 absence of friction and of turbulent dissipation, the above system can be solved in two independent steps:
 178 the first to invert the elliptic operator $I + \alpha\mathcal{T}$, the second to evolve physical quantities by solving the shallow
 179 water equations with the algebraic dispersive correction $H\psi$.

180 3 Numerical discretization of the Boussinesq models

181 The numerical treatment of both systems introduced above is done using an implicit treatment of the dis-
 182 sipative components (friction and/or turbulent dissipation). In particular, the kernel of both models is the
 183 hyperbolic component which rules the evolution of the water level and flux variables. Consider then non-
 184 overlapping temporal slabs $[t^n, t^{n+1}]$, with $\Delta t^{n+1} = t^{n+1} - t^n$. The hyperbolic evolution is performed
 185 with the two-stages Adams Bashforth-Adams Moulton predictor-corrector method which, for the ODE
 186 $U' = \mathcal{L}(U)$ reads:

187 1. Predictor stage (Adams-Basforth method)

$$\mathbf{U}^p = \mathbf{U}^n + \Delta t \mathcal{L}^p, \quad \mathcal{L}^p = \frac{23}{12}\mathcal{L}(\mathbf{U}^n) - \frac{16}{12}\mathcal{L}(\mathbf{U}^{n-1}) + \frac{5}{12}\mathcal{L}(\mathbf{U}^{n-2}) \quad (10)$$

188 2. Corrector stage (Adams-Moulton method)

$$\mathbf{U}^{n+1} = \mathbf{U}^n + \Delta t \mathcal{L}^C \quad \mathcal{L}^C = \frac{9}{24}\mathcal{L}(\mathbf{U}^p) + \frac{19}{24}\mathcal{L}(\mathbf{U}^n) - \frac{5}{24}\mathcal{L}(\mathbf{U}^{n-1}) + \frac{1}{24}\mathcal{L}(\mathbf{U}^{n-2}) \quad (11)$$

189 with the time step is computed by means of the standard condition $\Delta t^{n+1} = \text{CFL}\Delta x / \max_i(|u_i^n| + \sqrt{gh_i^n})$.
 190 Within both stages, the evolution operator \mathcal{L} accounts for all the effects except those of friction and turbulent
 191 dissipation (if present). In particular, the shallow water terms are approximated by means of a third order
 192 MUSCL finite volume approximation [53, 54], with Roe-type numerical fluxes [55]. It is useful for some of
 193 the analysis that will follow to report the form of these fluxes reading :

$$\begin{aligned} \mathcal{L}_i^{SW} &= -\frac{1}{\Delta x}(\mathbf{F}_{i+1/2} - \mathbf{F}_{i-1/2}) + \Delta \mathbf{S}_{\mathbf{b}i}^{i+1/2} + \Delta \mathbf{S}_{\mathbf{b}i}^{i-1/2} \\ \mathbf{F}_{i+1/2} &= \mathbf{F}_{i+1/2}^C - \frac{1}{2}|\mathbf{A}|_{i+1/2}\Delta_{i+1/2}\mathbf{U} \end{aligned} \quad (12) \quad \boxed{\text{roeflux}}$$

194 where \mathbf{F}^C is the centered flux, and $\Delta(\cdot)_{i+1/2} = (\cdot)_{i+1/2}^R - (\cdot)_{i+1/2}^L$, and $|\mathbf{A}|_{i+1/2}$ is the usual absolute value
 195 of the shallow water flux Jacobian, computed via eigenvalue decomposition, and modified with an entropy
 196 fix [56, 57]. The source term contributions $\Delta \mathbf{S}_{\mathbf{b}i}^{i\pm 1/2}$ are well balanced, and involve both a centered and an
 197 upwind approximation of the gradient of the bathymetry. We omit details concerning this (quite classical)
 198 aspect, for which the interested reader can consult [58–62] and references therein.

199 Concerning the dispersive terms, the \mathbf{S}_a contribution in (I) is discretized using finite differences. While
 200 for the hyperbolic component the minimization of the dispersion error requires at least a third order approx-
 201 imation, this is not the case for the higher derivatives in the dispersive terms (see [5, 37] for more details on
 202 this issue). Here, following [60], the second and third order derivatives in (I) are treated by means of
 203 second order central differencing. Similarly, the dispersive correction ψ in the GN system (4)-(6) is eval-
 204 uated by means of a second order P^1 continuous finite element approximation of the operator $H + \alpha H\mathcal{T}$
 205 and of the nonlinear forcing term \mathcal{Q} . In absence of friction and turbulent dissipation, equation (6) can be
 206 simplified by dividing through by H , and the self adjoint character of \mathcal{T} (equations (7) and (8)) can be used
 207 to deduce a simple variational form reading

$$\int_{\Omega} (v\psi + S_1(v)HS_1(\psi) + S_2(v)HS_2(\psi)) = \int_{\Omega} (S_1(v)HS_1(w) + S_2(v)HS_2(w)) + Q(v)$$

208 The last expression immediately allows to recover the three diagonal system for the unknown ψ which is
 209 symmetric and positive semi-definite. The term Q on the right hand side is the variational form of the forc-
 210 ing term (9) for which we refer to the full expressions given in [37, 63].

211
 212 The effects of friction and turbulent dissipation (if present) are now embedded in an implicit manner, by
 213 appropriately correcting the velocity values. In particular, for Nwogu's equations, the stage iterations (10)
 214 and (11) are modified as follows:

$$\mathbf{U}^{new} - \Delta t \mathbf{S}_f^{new} - \Delta t \mathbf{R}_{wb}^{new} = \mathbf{U}^n + \Delta t \mathcal{L}$$

215 Accounting for the definitions of the source terms, and of the pseudo mass-flux P^* in (2), we obtain the
 216 following operator defining the new velocity values (the superscript a is dropped for simplicity)

$$\begin{aligned} H^{new}(u^{new} + z^a \left(\frac{z^a}{2} u_{xx}^{new} + (d u^{new})_{xx} \right)) + \Delta t g S_f(H^{new}, u^*) u^{new} - \Delta t (v_t^* H^{new} u_x^{new})_x \\ = H^n u^n + H^n z^a \left(\frac{z^a}{2} u_{xx}^n + (d u^n)_{xx} \right) + \Delta t \mathcal{L} H u \end{aligned} \quad (13) \quad \boxed{\text{eq:vel_nwogu}}$$

217 where \mathcal{L}_{Hu} is the second component of \mathcal{L} , u^* and ν_t^* are the last available values of the velocity and
 218 turbulent viscosity, and H^{new} is independently computed from the first discretized equation. As before,
 219 the derivatives present in (13) are discretized using second order central finite differences, yielding a tri-
 220 diagonal system for the new value of the velocity at each stage.

221 The implementation has been slightly modified for the GN equations. In this case we have added after
 222 each of the iterations (10) and (11) a split (in time) implicit discretization of $\mathbf{U}_t = \mathbf{S}_f^{new} + \mathbf{R}_{wb}^{new}$. Denoting
 223 by u^* the last known value of the velocity (after (10) and/or (11)), we thus obtain the expression

$$H^{new} \frac{u^{new} - u^*}{\Delta t} + gH^{new} S_f(H^{new}, u^*) u^{new} = (\nu_t^* H^{new} u_x^{new})_x.$$

224 The derivatives in the above expression are then approximated by second order finite differences, leading
 225 to a tri-diagonal system again. As in this case the evaluation of the dispersive correction ψ already requires
 226 the inversion of a linear system, we have opted here for a simplified implementation involving a few explicit
 227 Jacobi relaxation iterations which read

$$J_i^m ((u^{new})^{m+1} - (u^{new})^m)_i = - \left(H^{new} \frac{u^{new} - u^*}{\Delta t} + gH^{new} S_f(H^{new}, u^*) u^{new} - (\nu_t^* H^{new} u_x^{new})_x \right)_i^m$$

$$J_i^m = \frac{H_i^{new}}{\Delta t} + gH_i^{new} S_f(H_i^{new}, u_i^*) + \frac{(\nu_t H^{new})_{i+1/2} + (\nu_t H^{new})_{i-1/2}}{\Delta x^2}$$

228 with $H_{i\pm 1/2}^{new}$ arithmetic averaged values, and with $(u^{new})^0 = u^*$. Unless otherwise stated, the number of
 229 relaxation iterations in the results discussed later has been set to 5.

230
 231 Other aspects of the discretizations are related to the modifications of the mass fluxes, velocities, and
 232 bathymetry source terms near wet/dry interfaces. Firstly, as in [64], two cut-off values for H are introduced,
 233 one to identify dry cells (or nodes), the other to mark as troubled cells (or nodes) in which the division by
 234 H may lead to unphysical values of the velocity. To preserve well balancedness in cells containing a dry
 235 node, adverse bathymetry gradients are limited as suggested in [59] (see also [64]). In troubled cells (or
 236 nodes) instead, the mass flux is set to zero, as well as the velocities, and the dispersive corrections \mathbf{S}_d in (11),
 237 the second order terms in (2) and (13), and ψ in (5). The van-Albada slope limiter is used only in breaking
 238 regions, and only if the hybrid approach is chosen.

239 4 Wave breaking closure

240 Boussinesq equations are unable to describe both the overturning of waves, and the dissipation of kinetic
 241 energy originated during wave breaking. A physical closure is necessary. Generally, this closure consists
 242 of two main steps. The first one is a trigger mechanism allowing to localize in space and time the initiation
 243 and the termination of breaking. The second one is a mechanism introducing a dissipation of total energy in
 244 the model. This paper focuses on two techniques to define the second element, which are discussed in some
 245 detail in the following sections. In both cases, the triggering of wave breaking is done following the criteria
 246 proposed [12, 37] which have been found simple and robust. The idea is to introduce a flagging strategy
 247 based on the following conditions:

- 248 • the surface variation criterion: a cell is flagged if $|\eta_t| \geq \gamma \sqrt{gH}$, with $\gamma \in [0.3, 0.65]$ depending on
 249 the type of breaker;

- the local slope angle criterion: a cell is flagged if $\|\nabla\eta\| \geq \tan\varphi_c$, with critical angle $\varphi_c \in [15^\circ, 30^\circ]$ depending on the flow configuration.

The first criterion is usually active in correspondence of moving waves and has the advantage of being completely local. The second criterion acts in a complementary manner, and allows to detect stationary or slow-moving hydraulic jumps [12, 18]. Flagged cells are grouped to form a breaking region. This region is either enlarged to account for the typical roller length, as suggested in [12, 65], or deactivated, depending on the value of the Froude number $Fr^2 = H_{\max}(H_{\max} + H_{\min})/(2H_{\min}^2)$, defined starting from the minimum and maximum wave height in the flagged zone. The interested reader can refer to [12, 41, 65] and references therein for more details regarding the implementation of these detection criteria.

4.1 Hybrid wave breaking model

This closure attempts to exploit the properties of hyperbolic conservation laws embedded with an entropy inequality. For the shallow water equations, in particular, the mathematical entropy coincides with the total energy [66–71]. At the continuous level, while conserved in smooth regions, entropy/total energy is dissipated in discontinuous weak solutions. Provided that the numerical scheme introduces the correct amount of dissipation in shocks [69–72], this lends itself naturally for the wave breaking closure. This approach is in itself neat and simple. It has the limitation that the form of the dissipation is, at best, fixed by that determined by the shallow-water Rankine-Hugoniot jump conditions. This quantity can be analytically computed and it is given by (see e.g. [73] chapter 1.6, and [74])

$$\mathcal{D}_{sw} = g\sqrt{g\frac{H_{\max} + H_{\min}}{2H_{\max}H_{\min}}}\frac{(H_{\max} - H_{\min})^3}{4} \quad (14) \quad \boxed{\text{eq:sw-diss}}$$

This is however a parameter free definition of the dissipation which has been proved to reproduce quite well the large scale decay of the total energy in for several types of breaking waves, and with several different underlying propagation models and relative numerical discretizations [6, 12, 36–39, 41, 75]. The implementation of this closure is somewhat trivial once the wave detection algorithm discussed earlier has been properly set up. For the Nwogu’s equations, it boils down to locally turning off in the whole flagged region the dispersive source \mathbf{S}_d and the second order derivative terms in (13) when evaluating the new nodal velocities. Similarly, for the GN system, the nodal values of ψ in (5) are set to zero in the breaking region.

The most limiting aspect of this approach is the switch between the non-hydrostatic and the hydrostatic equations. What has been reported by many authors in a more or less marked way, is the difficulty of performing this switch in a stable manner. Unless coarse grids are considered, with eventually the addition of local regularization numerical dissipation terms, several authors have reported the appearance of strong oscillations [6, 12, 37, 43]. These artefacts tend to become stronger and stronger as the mesh is refined. To our knowledge, there are no studies in literature reporting fully grid converged solutions with this approach due to this problem. An exception to this is perhaps one result reported in [6] showing some convergence (on only 3 grids) of the time averaged wave heights and setup, even though increasing oscillations in the local profiles are reported for the same test. This behaviour clearly poses a limitation in terms of potential for local automatic adaptation of the mesh, and its investigation is one of the objectives of this article.

4.2 Eddy viscosity closure via a PDE based TKE model

The use of an eddy viscosity model to provide the dissipation required for the breaking closure is one of the earliest approaches [16]. The definition of this artificial viscosity is the key of this approach, as well as the way in which it enters the Boussinesq equations. One of the most common approaches, due to Kennedy

289 and collaborators [17] (see also [12, 18, 22, 31] and references therein), involves a definition of the eddy
 290 viscosity based essentially on the variation in time of the free surface elevation. This term is then embedded
 291 in a viscous flux, as e.g. in (1) and (5). There exist improved variants of this idea, allowing to embed a
 292 richer physical description of the vertical kinematics and of the effects of turbulence (e.g. the so-called roller
 293 models). Some approaches explicitly embed the effects of the dynamics of vorticity (roller-models) [27, 28],
 294 others include partial differential equations for an average turbulent kinetic energy [1, 33], and other intro-
 295 duce a multi-layer description embedding PDEs for a turbulent layer flowing on top and interacting with the
 296 bulk of the wave, well representative of spilling flows [34, 35, 76–78]. Simpler methods have attempted at
 297 improving the behaviour of the total energy dissipation by also including a water elevation viscosity [23].
 298 In this work, we have chosen to adopt a model of intermediate complexity based on the solution of an
 299 additional PDE, weakly coupled to the main Boussinesq system of equations. In particular consider the
 300 approach initially proposed by Nwogu [1] who used a standard TKE (turbulent kinetic energy) equation
 301 coupled to the fully non-linear equations of Wei et al. [79]. A highly non-linear Boussinesq model with
 302 the same turbulence wave breaking model of Nwogu has been used by Elnaggar and Watanabe [32]. More
 303 recent work on the same model is discussed in [33] where the TKE equation is manipulated to obtain a PDE
 304 for the eddy viscosity which is coupled to a fully nonlinear fully dispersive Green-Naghdi model. Here we
 305 propose a variant of the model proposed by Nwogu modified according to some of the definitions proposed
 306 in [33], as well as some definition which improve the consistency of the model with the wave breaking
 307 detection criteria we adopt.

308
 309 Following [33, 80], the eddy viscosity is determined from the amount of the turbulent kinetic energy k ,
 310 produced by the wave breaking, and a turbulent length scale ℓ_t :

$$v_t = C_\nu \sqrt{k} \ell_t \quad (15) \quad \text{eq:nu}_t$$

311 In $k - L$ turbulence models [81, 82] (see also [33]), the constant C_ν is usually set to $C_\nu = (0.09)^{1/4} \approx 0.55$
 312 which is the value used here. We now need a model for the computation of k and ℓ_t . Differently from the
 313 models discussed in [81, 82], here we adopt a one equation approach in which only one PDE is solved for k ,
 314 while the for ℓ_t , inspired by the definition used in [33], we use a vertical average mixing length defined as

$$\ell_t = \kappa H$$

315 where κ is a constant controlling the width and intensity of the breaking. **The length ℓ_t is expected to be of**
 316 **the order of the wave height [1], so κ is a case dependent constant.** Concerning turbulent kinetic energy, it
 317 can be shown that in three space dimensions the following transport equations holds [80]

$$k_t + \mathbf{u} \cdot \nabla k = \mathcal{D} + \mathcal{P} - \mathcal{E} \quad (16) \quad \text{kinturb0}$$

318 with \mathcal{D} , \mathcal{P} , and \mathcal{E} , diffusion, production and dissipation (or destruction) terms respectively. Definitions and
 319 possible expressions of these quantities in terms of mean flow quantities can be found e.g. in the book [80].
 320 When coupling (16) with a depth averaged Boussinesq model, several approximations are possible. Here
 321 we will combine some of the elements suggested in [1] and in [33] in order to obtain a model simple to
 322 implement, to be compared to the hybrid approach. First of all, we will assume that both k (and hence
 323 ν_t) and its transport dynamics are constant along the depth, so that (16) can be replaced by a zero-th order
 324 approximation involving only depth averaged quantities, namely

$$(Hk)_t + (Huk)_x = H\overline{\mathcal{D}} + H\overline{\mathcal{P}} + H\overline{\mathcal{E}}. \quad (17) \quad \text{kintur_con}$$

325 For the definition of the terms on the right hand side of (17) we have followed [1]. In particular, we have
 326 for the diffusion and destruction terms

$$H\bar{D} = H\sigma\nu_t k_{xx}, \quad H\bar{E} = -HC_D \frac{k^{3/2}}{\ell_t} \quad (18) \quad \boxed{\text{DE}}$$

327 where, following [1, 33], we have set $C_D = C_\nu^3$. The constant σ allows to control the smoothness of the
 328 TKE, and hence of the breaking viscosity, in the breaking region. Concerning the production term, the
 329 model used is again the one suggested in [1] assuming this quantity to depend on the vertical gradient of the
 330 velocity at the free surface. Following the notation of (16), and denoting the velocity at the free surface by
 331 $\mathbf{u}^s = \mathbf{u}(t, x, y, z = \eta)$, we have

$$H\bar{P} = HB(t, x) \mu_{\mathcal{P}} \mathbf{u}_z^s \cdot \mathbf{u}_z^s$$

332 As in [1], the turbulent viscosity $\mu_{\mathcal{P}}$ appearing in the production term is defined based on a mixing length
 333 hypothesis assuming a balance between production and dissipation, namely

$$\mu_{\mathcal{P}} = \frac{\ell_t^2}{\sqrt{C_D}} \sqrt{\mathbf{u}_z^s \cdot \mathbf{u}_z^s}$$

334 so that we end with

$$H\bar{P} = HB(t, x) \frac{\ell_t^2}{\sqrt{C_D}} (\mathbf{u}_z^s \cdot \mathbf{u}_z^s)^{3/2}. \quad (19) \quad \boxed{\text{prod}}$$

335 In [1] the parameter B is equal to 0 or 1 depending on a wave breaking criterion. In the reference the
 336 criterion used is based on the ratio between the free surface velocity and the wave celerity being larger
 337 than one. Here, for simplicity B is set to one in the breaking regions detected exactly as discussed in the
 338 beginning of section §4. This also allows to detect wave breaking in the same way for the TKE and hybrid
 339 approach. Having fixed the values of C_ν and C_D , the only “tunable” parameters are κ and σ .

340 Lastly, we need to be able to evaluate the depth averaged and free surface velocities for both Boussinesq
 341 models, as well as the value of the vertical gradient of the velocity at the free surface. For this we use the
 342 vertical asymptotic development underlying the two models. In the weakly nonlinear case, this development
 343 can be used to write the following relations [15, 49]

$$u(z) = u^a - \left(\frac{z^2}{2} - \frac{d^2}{6}\right) u_{xx}^a - \left(z + \frac{d}{2}\right) (du^a)_{xx}$$

344 giving the free surface vertical gradient

$$u_z^s = -\eta u_{xx}^a - (du^a)_{xx}. \quad (20) \quad \boxed{\text{graduz_nowg}}$$

345 and the depth averaged (within the asymptotic accuracy) velocity required for the transport term in (17)

$$u = u^a + \left(\frac{(z^a)^2}{2} - \frac{d^2}{6}\right) u_{xx}^a + \left(z^a + \frac{d}{2}\right) (du^a)_{xx} \quad (21) \quad \boxed{\text{ubar_ua}}$$

346 The GN equations directly provide a value of the depth averaged speed, while the fully nonlinear asymp-
 347 totic development allows to write

$$u(z) = u - \left[\frac{z^2}{2} - \left(\frac{H^2}{6} - \frac{H(H-d)}{2}\right)\right] u_{xx} - \left[z - \left(\frac{H}{2} - d\right)\right] (du)_{xx}$$

348 which yields a similar expression for the vertical gradient of the free surface velocity, this time in function
 349 of the depth averaged velocity u :

$$u_z^s = -\eta u_{xx} - (du)_{xx}. \quad (22)$$

350 The fully discrete distribution of the nodal values of the TKE is obtained by integrating equation (17)
 351 with a semi implicit approach. Before the predictor step (10) is applied to the Boussinesq models, the
 352 nodal TKEs are evolved by first applying an explicit Euler update involving a third order MUSCL upwind
 353 discretisation of the transport operator $(Huk)_x$, essentially the same presented in section §3 for the shallow
 354 water equations. To avoid spurious negative values in this phase, the min-mod limiter is applied [83]. The
 355 predicted values k_i^* are then corrected by means of diagonally semi-implicit relaxation iterations similar to
 356 those used for the breaking dissipation and reading

$$\begin{aligned} \left(\frac{\Delta x}{\Delta t} + \frac{2\sigma v_{t,i}^n}{\Delta x} \right) (k_i^{m+1} - k_i^m) &= \Delta x \frac{k_i^m - k_i^*}{\Delta t} + \sigma v_{t,i}^n \frac{k_{i+1}^m - 2k_i^m + k_{i-1}^m}{\Delta x} \\ &+ \left(\frac{B\ell_{t,i}^2}{\sqrt{C_D}} (u_z^s)_i^{3/2} \right)^n - C_D \left(\frac{k_i^{3/2}}{\ell_{t,i}} \right)^n \end{aligned}$$

357 with an initial condition, $k^0 = k^*$. For the benchmarks discussed in the paper, 4 or 5 relaxation iterations
 358 are used unless otherwise stated. Where necessary, depth average velocity (for the Nwogu model) and
 359 velocity gradient at the free surface (for both Boussinesq models) are obtained by a second order central
 360 finite difference approximation of (21), (20), and (22).

361 As a final note, we will keep in the following the notation TKE when referring to this closure, as this
 362 eddy viscosity method clearly relies on the solution of the PDE for the turbulent kinetic energy.

363 5 A note on the dissipation mechanisms at work

364 One of the key aspects concerning the numerical modelling of wave breaking is the notion of dissipation.
 365 As discussed in the introduction, the mechanisms related to the transformation of potential energy into
 366 mechanical energy, and its subsequent dissipation, are not embedded in Boussinesq models that we study
 367 which stem from a potential description of the flow. The role of the closure model is thus to mimic these
 368 mechanisms. Clearly the main interest in the closure is to be able to predict correctly the dynamics of wave
 369 heights and (in the multidimensional case) currents. It is however interesting to understand what is the un-
 370 derlying dissipation mechanism active during the numerical breaking process. The main question we want
 371 to contribute to answer to in this paper is how much the numerical method is involved in this process, and
 372 if it is at all. We provide here a short discussion of this aspect, and suggest quantities which we will use in
 373 the numerical applications to quantify the contributions to the breaking process of the numerics, as well as
 374 of the PDE model itself.

375
 376 A proper formulation of this analysis requires a formal definition of what is the energy to be dissipated
 377 for the propagation models under consideration. This has to be done at the continuous level, but of course we
 378 must be able to provide an appropriate discrete translation of this energy conservation/dissipation statement.
 379 We recall that the PDE systems used in this paper have been chosen as representatives of models/codes well
 380 known the community such as BOSZ [9], MIKE21 [10], BOUSS-2D [3, 4], TUCWave [11, 12], Funwave
 381 [5, 6], Coulwave [7, 8]. Unfortunately, while the GN equations do have a total energy which one may
 382 choose to use for this purpose, this is not the case for Nowgu's model. For the latter one can only derive

383 conservation statements valid within the limits of the asymptotic accuracy of the model [15, 84]. These
 384 approximate conservation laws, however, are not verified by the solutions of the PDE.

385 When looking at the discrete models, even for the GN equations it is still a matter of research how to
 386 devise a numerical method with a clear associated discrete energy conservation statement. For complete-
 387 ness, we recall that such a construction requires generally an appropriate characterisation of the symplectic
 388 form of the PDE system and, more importantly, an appropriate semi linear form allowing to relate di-
 389 rectly the differential of the total energy to the differentials of the physical quantities which the numerical
 390 scheme solves for. For hyperbolic systems, including the shallow water equations, there is a clear and well
 391 established theory now allowing to construct methods which are exactly entropy-conservative or entropy
 392 stable [56, 67, 68, 85]. For the shallow water equations, in particular, the mathematical entropy coincides
 393 with the total energy. This link between the entropy/total energy, and the physical quantities solved for by
 394 the scheme is played in this case by a so-called entropy (or energy) potential, which is nothing else than the
 395 Legendre transform associated to the entropy, and conservative variables. The interested reader can refer
 396 e.g. to [68–72] for the construction of schemes which are either exactly energy conservative, or energy
 397 stable. Unfortunately, the construction of exactly energy preserving schemes for dispersive equations is still
 398 a subject of research, and the interested reader may refer to [86–89] for some recent results.

399 A consequence of this discussion is that an exact evaluation of the dissipative mechanisms for the type
 of models used here is not within our grasp. So, in order to be able to provide some quantitative information
 on the sources of dissipation, we had to make some choices, and some hypotheses. We start by recasting
 our PDE models as the shallow water system plus a dispersive source

$$\partial_t \mathbf{U} + \partial_x \mathbf{F}(\mathbf{U}) - \mathbf{S}_b - \mathbf{S}_f - \mathbf{R}_{wb} = \mathcal{D}$$

This is a form similar to (I),^{nwoqu} except that in the above equation the left hand side only contains the shallow
 water terms and the eddy viscosity model, if present. All the dispersive terms are included in \mathcal{D} . We
 then look at the contributions to the balance of the shallow water total energy, whose time variation can be
 expressed as (see e.g. [67–69])

$$\partial_t E = \mathbf{V}^t \partial_t \mathbf{U} = -\mathbf{V}^t (\partial_x \mathbf{F}(\mathbf{U}) - \mathbf{S}_b - \mathbf{S}_f - \mathbf{R}_{wb} - \mathcal{D})$$

with $E = H(gH + u^2)/2 + gHb$, and having denoted by \mathbf{V}^t the transpose of the array of the so-called
 energy (or entropy, or symmetrizing) variables $\mathbf{V}^t = [g\eta - u^2/2, u]$. For both numerical models tested in
 the paper, we can easily provide a nodal discrete analog of the last expression which, using the notation of
 (I2),^{coefficient} reads

$$\partial_t E_i = \mathbf{V}_i^t \partial_t \mathbf{U}_i = -\mathbf{V}_i^t (-\mathcal{L}_i^{SW} - \mathbf{S}_{fi} - \mathbf{R}_{wbi} - \mathcal{D}_i)$$

Neglecting the boundary conditions (or assuming periodic or null the boundary fluxes), the total variation
 of the shallow water energy can be deduced using the explicit form \mathcal{L}_i^{SW} and of the central and upwind
 contributions in the bathymetry terms [58–62]. The final result can be recast as

$$\sum_{i \geq 1} \Delta x \partial_t E_i = \sum_{i \geq 1} \Delta x \mathbf{V}_i^t \partial_t \mathbf{U}_i = \sum_{i \geq 1} \Delta \mathcal{F}_i^E - D_{upwind} - D_{friction} - D_{vis} + \Delta E_{\mathcal{D}}.$$

We can now try to say more on the terms on the right hand side. It seems quite reasonable to assume that
 wave breaking is not associated to the dispersive contributions. This means that we will leave out of the
 analysis the contributions of the dispersive source $\Delta E_{\mathcal{D}} = \sum_i \mathbf{V}_i^t \mathcal{D}_i$. Another term which in principle

one would expect not to contribute to the analysis, is the centered part of the flux which enters the above expression via the terms

$$\Delta \mathcal{F}_i^E = -\mathbf{V}_i^t \left(\mathbf{F}_{i+1/2}^C - \mathbf{F}_{i+1/2}^C + \frac{gH_{i+1/2}}{2} \Delta_{i+1/2}[0, b]^t + \frac{gH_{i-1/2}}{2} \Delta_{i-1/2}[0, b]^t \right).$$

400 This is where the analysis provided in e.g. [69, 70] is most useful. Without going into much detail, the
 401 references provide a very simple rule to define the centered flux for which one can show that $\Delta \mathcal{F}_i^E =$
 402 $\mathcal{H}_{i+1/2} - \mathcal{H}_{i-1/2}$, with $\mathcal{H}_{i\pm 1/2}$ consistent numerical approximations of the total energy flux. This algebraic
 403 relation leads to the conclusion $\sum_{i \geq 1} \Delta \mathcal{F}_i^E = 0$ exactly, whether the solution is continuous or not. This
 404 means that, even if slightly different implementations of the central flux are used, this quantity is in principle
 405 not relevant for our analysis.

406

This leaves three quantities to be monitored, associated to the numerical (upwind) dissipation

$$D_{upwind} = \sum_{i \geq 1} \left\{ \Delta \mathbf{V}_{i+1/2}^t \left(\frac{1}{2} |\mathbf{A}|_{i+1/2} \Delta_{i+1/2} \mathbf{U} + \frac{gH_{i+1/2}}{2} \text{sign}(\mathbf{A})_{i+1/2} \Delta_{i+1/2}[0, b]^t \right) \right\},$$

and to the friction and wave breaking (eddy viscosity) model

$$D_{friction} = \sum_{i \geq 1} gH_i S_{fi} u_i^2 \quad \text{and} \quad D_{vis} = \sum_{i \geq 1} \nu_{t,i+1/2} H_{i+1/2} (\Delta_{i+1/2} u)^2.$$

407 Note that with the spatial discretization choices made both $D_{friction}$ and D_{vis} are clearly positive defi-
 408 nite. The same cannot be said a-priori about the upwind dissipation D_{upwind} . To be sure of the positivity
 409 of this term, indeed one should have implemented the dissipation in terms of variations of the entropy
 410 variables [69–71], instead of using in the numerical flux variations at cell interfaces of the conservative
 411 variables, as done in standard implementations of the upwind flux. Furthermore, this term involves both
 412 the reconstruction and the limiter, the latter only in the shallow water regions associated to wave breaking
 413 when using the hybrid approach of section §4.1. This makes it a perfect candidate to monitor the impact
 414 of the numerical choices and their contribution to the wave breaking process, and when possible compare
 415 these contributions to those of the eddy viscosity and friction terms. This analysis has been performed for
 416 three of the benchmarks proposed, involving both periodic and non periodic waves, dry areas, as well as
 417 pure propagating bores. Note that in practice the above expressions have to be evaluated in post-processing,
 418 by saving the different terms evaluated during the computations. The time stepping of course also plays
 419 a role in this analysis. The interested reader can refer to [72] for a discussion on this. To minimize these
 420 effects, while keeping as much as possible of the actual terms computed in the code and used to obtain the
 421 numerical solutions, in all the cases presented we have used $t^{n+1/2}$ half time-step evaluations of these terms
 422 by averaging values at t^n and t^{n+1} .

423 6 Boundary condition and the internal source function

424 In this work we use two types of boundary conditions : solid (reflective) wall and absorbing boundary
 425 conditions. For the wall boundary conditions ghost cells are used with mirrored states for the velocities, as
 426 discussed in [60]. Absorbing boundaries are used for outgoing waves. In this case, an adsorbing layer is
 427 introduced within which surface elevation and the momentum are damped by multiplying their values by a
 428 coefficient $m(x)$ defined as [12]

$$m(x) = \sqrt{1 - \left(\frac{d(x)}{L_s} \right)^2} \quad (23)$$

429 where L_s is the sponge layer width, and $d(x)$ is the distance from the end of the absorbing boundary. As
 430 prescribed in [12], the width L_s should depend on the wavelength of the outgoing wave. For a given wave-
 431 length L , the sponge layer width should be $L \leq L_s \leq 1.5L$.

432
 433 Concerning wave generation, we follow the approach of Wei et al. [90]. To obtain a desired oscillation
 434 signal in the wave generating area, a source function $S(\mathbf{x}, t)$ is added into the mass conservation equation
 435 at each time step, which is expressed as

$$S(\mathbf{x}, t) = D^* \exp(\gamma(x - x_s)^2) \sin(-\omega t) \quad (24)$$

436 in which

$$\gamma = \frac{5}{(\delta L/4)^2} = \frac{80}{\delta^2 L^2} \quad (25)$$

437 where L is the wave length, ω the wave frequency, θ the wave incident angle, x_s is the location of the center
 438 of the wave-making area, δ is a parameter that influences the width $W = \delta L/2$ of the wave generator area
 439 and D^* is the source function's amplitude. For a monochromatic wave, D^* is defined as

$$D^* = \frac{2\sqrt{\gamma}A_0(\omega^2 - \alpha_1 g k^4 h^3)}{\omega k \sqrt{\pi} \exp(-l^2/4\gamma) [1 - \alpha(kh)^2]} \quad (26)$$

440 where h is the still water level at the wave generation region, A_0 the wave amplitude, $l(= k_x)$ the wave
 441 number in the x -direction, $\alpha = -0.390$ and $\alpha_1 = \alpha + 1/3$.

442 7 Numerical results

443 7.1 Wave breaking over a bar

444 This test case of Beji and Battjes [91] examines the sinusoidal wave propagation over a submerged bar. The
 445 scope of this test case is to investigate the frequency dispersion characteristic and non-linear interaction of
 446 complex wave propagation phenomena. A sketch of the problem is provided in figure 5. The computational
 447 domain is $x \in [0, 35m]$, with sponge layers placed at both ends. Periodic waves were generated at $x =$
 448 $10m$ over a mean water depth of $0.4m$. Wave height and period are set to $a = 0.054m$, and $T = 2.5s$,
 449 corresponding to a dispersion parameter $kh \approx 0.52$. Waves propagate over submerged trapezoidal bar with
 450 a toe at $x = 15m$, a front slope of 1 : 20, a 2m long plateau of 0.3m height, and a lee slope of 1 : 10. More
 451 informations on the experiment can be found in [91] and in the references using this test case for model
 452 validation [12, 37, 65, 92].

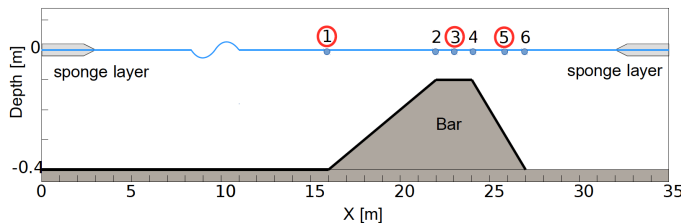


Figure 3: Wave breaking over a bar: problem sketch, and position of the gauges

bar_sketch

Concerning the model parameters, for this highly unsteady problem the surface variation detection parameter γ (cf. section §4) is the one more sensitive to the onset of breaking. For the computations performed here we have set $\gamma = 0.3$. The parameters used for the TKE are not the same for the two Boussinesq propagation models. In particular, we have set $\kappa_{GN} = 2.8$ and $\sigma_{GN} = 1.2$ for the GN equations, while $\kappa_N = 3.2$ and $\sigma_N = 1.2$ for the Nwogu system.

Experimental data are available in several wave gauges placed before, on top, and after the bar. Here we focus on three gauges (cf. figure [3](#)) placed before the toe of the bar, gauge 1 at $x = 16m$, on top of the plateau, gauge 3 at $x = 23m$, and on the lee slope, gauge 5 at $x = 26m$. We will discuss numerical results obtained on three different meshes of size $4cm$, $2cm$, and $1cm$. For the Nwogu model, we could not run the hybrid breaking simulations on the last mesh due to instabilities at the Boussinesq-shallow water interface. Similarly, when using the hybrid approach we could not go below $\Delta x = 1cm$ when using the GN model for propagation. Note also that the results discussed here are those obtained after a transient of 36 seconds, differently from what is done e.g. in [12, 37], where the four first waves are analyzed. The results are presented in figures [4](#), [6](#) and [7](#), for gauges 1, 3, and 5, respectively.

Figure [4](#) allows to visualize the behaviour of the models at the toe of the bar, right at the end of the wave propagation region. This gauge allows to highlight the initial asymmetry of the waves, essentially due to the interaction with the submerged bar. Some preliminary observations can be made. Firstly, the fully nonlinear model (left column) seems to capture better the shape of the waves, the weakly nonlinear one providing a signal which is slightly too peaky. Secondly, we see already at this stage that while the TKE model (blue curves) shows little sensitivity to the mesh size, the signals obtained with the hybrid approach (green curves) depend strongly on this parameter. We can clearly see on the intermediate and fine mesh (in the GN case) higher frequency components absent in the TKE results. These components are generated in correspondence of the boundary of the wave breaking region, as it can be clearly seen in the snapshots of figure [5](#). These instabilities become stronger as the mesh is refined, and may ultimately lead to the blow up of the solution, as it is the case for the Nwogu model on the fines mesh, and of the GN model on finer meshes.

Figures [6](#) and [7](#) confirm the preliminary observations made for the first gauge. In particular we can clearly see the strong dependence of the results of the hybrid model on the mesh size. For this approach we can also see how the breaking waves are represented as very sharp fronts. For the GN model, on the coarse mesh breaking stops early enough for the signal in these two gauges to be smooth. This however leads to a noticeable phase lag. As the mesh is refined, the waves break more strongly. This leads clearly to an improvement on the phase. This behaviour curiously is not observed for the Nwogu model which shows strong and sharp breaking fronts already on the coarsest mesh level, with a correct phase. This allows to highlight the need of tailoring the choice of the breaking detection criterion to the propagation model. Here the same parameters have been used for both. Nevertheless, both set of results allow to visually see the appearance of spurious higher frequencies in the signal. These are the result of the coupling between the dispersive and non-dispersive regions. For the weakly non-linear model (right column) we can see the inception of the instability already on the medium resolution used here in figure [7](#). This is less evident for the GN model, which still provides numerical solutions on the finer level used. We where however unable to refine once more the mesh without solution blow up.

The TKE approach is clearly less sensitive, at least for this test, to both the choice of the model parametrization, and the mesh size. This is summarized in figure [8](#), showing a grid convergence for the gauge 3. We also would like to remark that, for Nwogu's equations and for plunging breakers, Demirblik and Nwogu in [3] resorted to a more complex TKE closure with a PDE for the B coefficient in the production term ([19](#)). We found out that the simplified formulation adopted here, combined with the physical

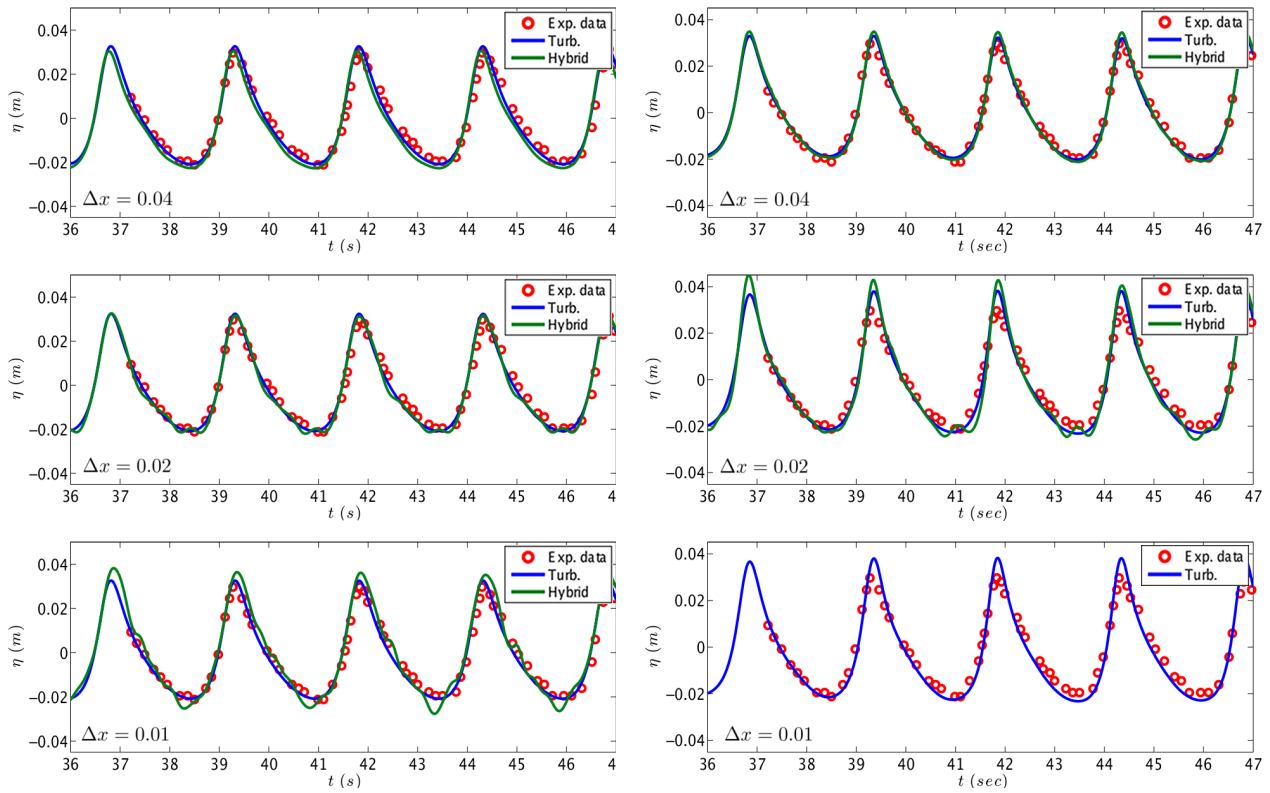


Figure 4: Time series of surface elevation at wave gauge 1 for the GN (left) and Nwogu (right) models using the TKE (blue) and Hybrid (green) wave breaking closure. Mesh size is 0.04, 0.02, 0.01m from top to bottom.

bar_g1

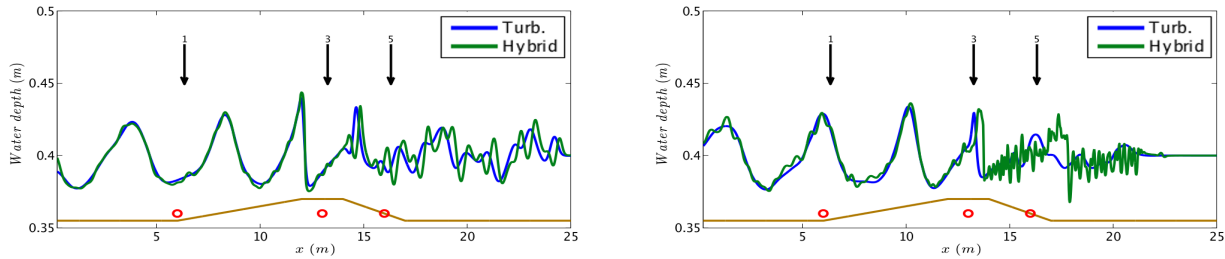


Figure 5: Snapshots of the flow for the GN (left) and Nwogu (right) models using the TKE (blue) and Hybrid (green) wave breaking closure. Mesh size is 0.02m

bar_sn

499 criteria for the initiation and termination of the process discussed in the beginning of section §4 can simulate
 500 reasonably well plunging wave breakers.

501 7.1.1 Dissipation mechanisms

502 We report in figures 9 and 10 the time evolution of the dissipation terms active for this test (cf. section
 503 §5): D_{upwind} (in blue) and D_{vis} (in green). The flow is periodic so we focused on 5 periods from time 12s
 504 to time 14s. The results show the dissipation *flashing* when the tallest wave approaches the bar, and then

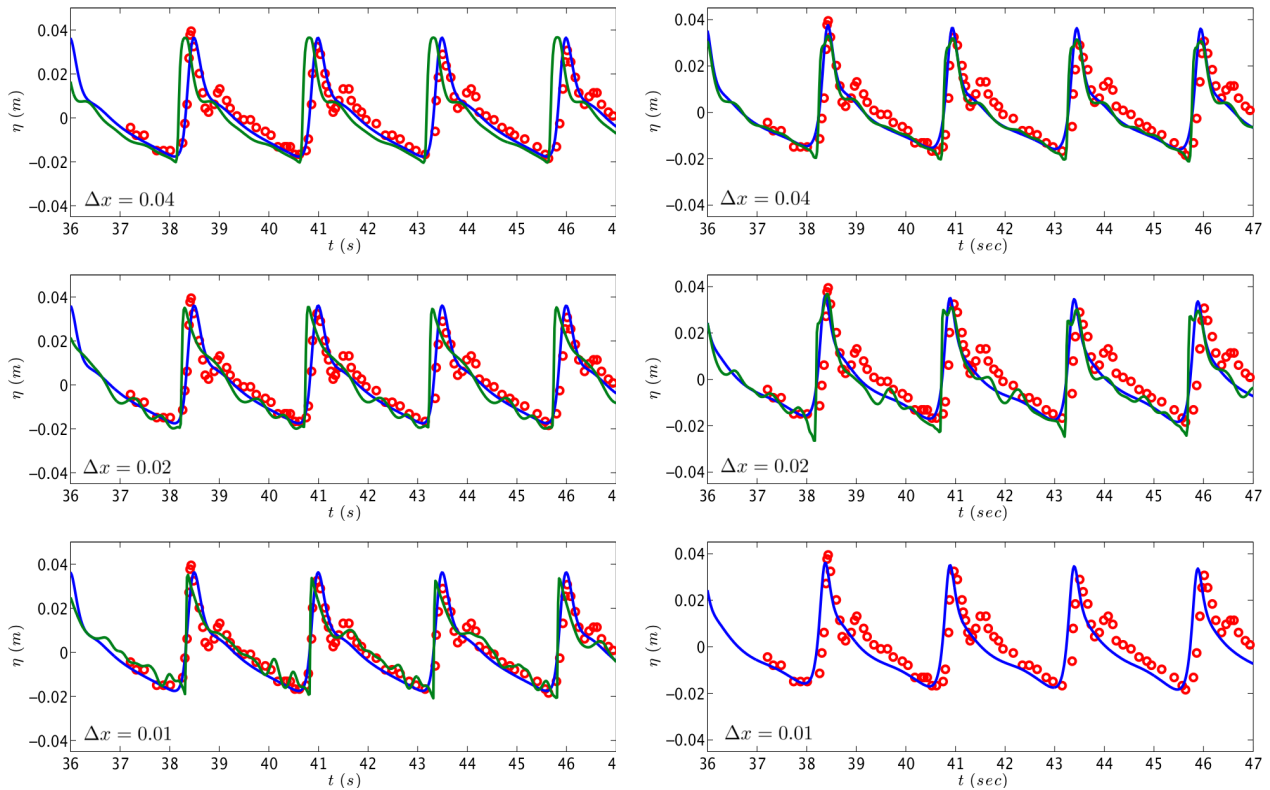


Figure 6: Time series of surface elevation at wave gauge 3 for the GN (left) and Nwogu (right) models using the TKE (blue) and Hybrid (green) wave breaking closure. Mesh size is 0.04, 0.02, 0.01m from top to bottom.

bar_g3

505 reducing as the breaking process continues on top of the bar. Also, the inception of breaking for the Nwogu
 506 model has a phase advance of about one second which can be explained by the over-shoaling characteristics
 507 of this model [93, 94]. The results for the GN model, figure 9, allow the following remarks. The role of
 508 numerical dissipation D_{upwind} when using the eddy viscosity closure (left column) is extremely small. This
 509 term definitely does not contribute at all to the breaking process. On the second mesh, its values approach
 510 machine zero. On the contrary, in the case of the hybrid closure, D_{upwind} is doing all the job. We can also
 511 see that the on the coarser mesh the area under the dissipation bells is larger, which means that the overall
 512 contribution in time to the energy dissipation is more important. When using the model of Nwogu, figures
 513 10, we can see again, from the left column, that the numerical dissipation plays no role in the breaking
 514 process, and it quickly reaches very low values. The right column allows to visualize the inception of the
 515 numerical instabilities (top figure) and their blow up (bottom figure). Note that for the finer meshes used
 516 in figure 8 the behaviour observed for the GN and in general for the TKE closure are the same. Also, we
 517 stress once more that further halvening the mesh size was not possible for the GN. The finest computation
 518 we could perform until the final time is for $\Delta x = 0.008$.

519 The behaviour observed allows to clearly demonstrate that the numerical dissipation has no impact on
 520 the computations performed with the TKE closure. This means that with this closure one could (or should)
 521 in principle use a non-dissipative numerical method to discretize the PDEs. The results, at least those for
 522 the GN equations, also show that the overall numerical dissipation when using the hybrid approach is larger
 523 on coarser meshes.

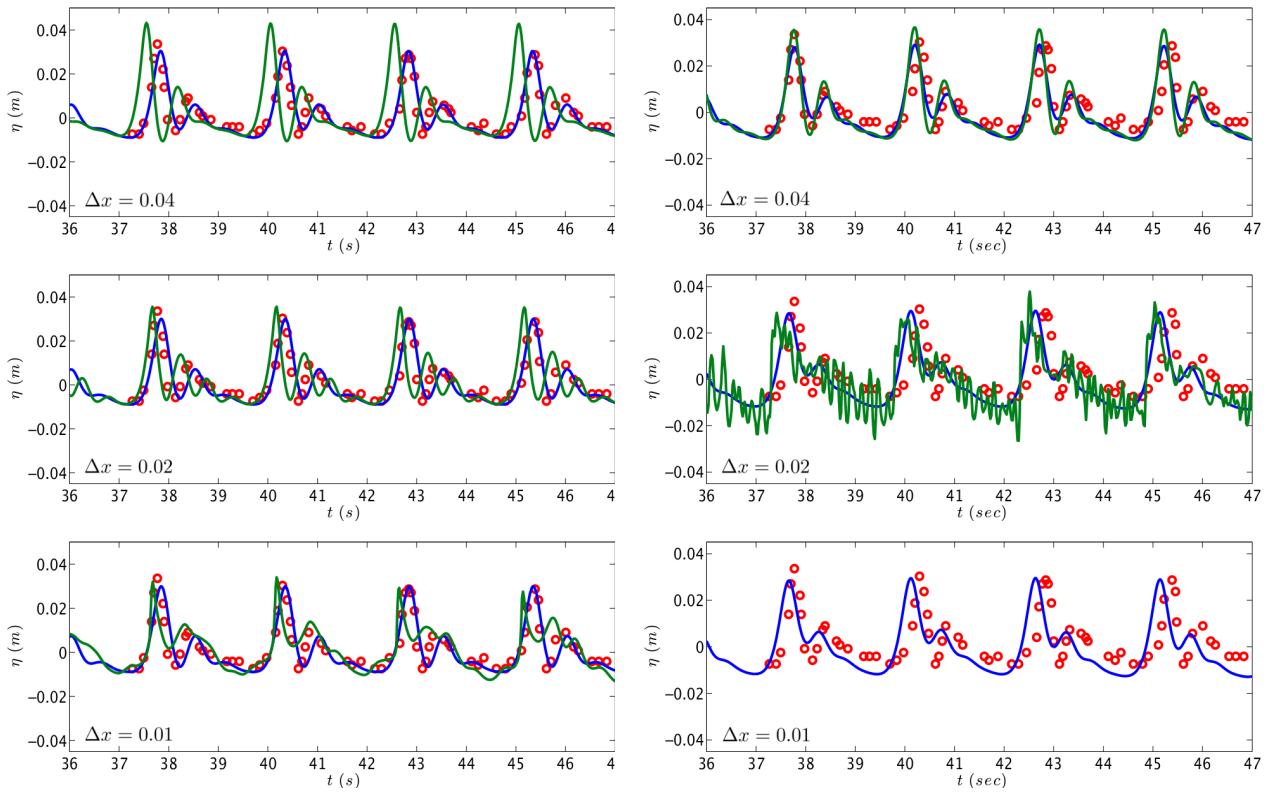


Figure 7: Time series of surface elevation at wave gauge 5 for the GN (left) and Nwogu (right) models using the TKE (blue) and Hybrid (green) wave breaking closure. Mesh size is 0.04, 0.02, 0.01m from top to bottom.

bar_g5

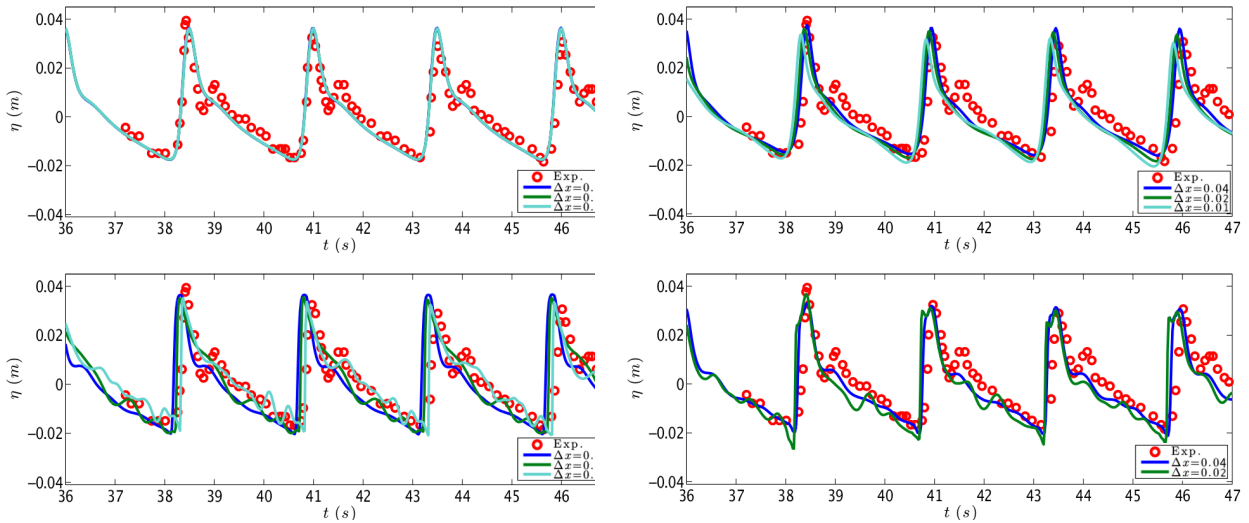


Figure 8: Time series of surface elevation at wave gauge 3: grid convergence for the GN (left) and Nwogu (right) models using the TKE (up) and Hybrid (down) wave breaking closure. Mesh size: 0.04m (blue), 0.02m (green), 0.01m (cyan).

bar_g3_dx

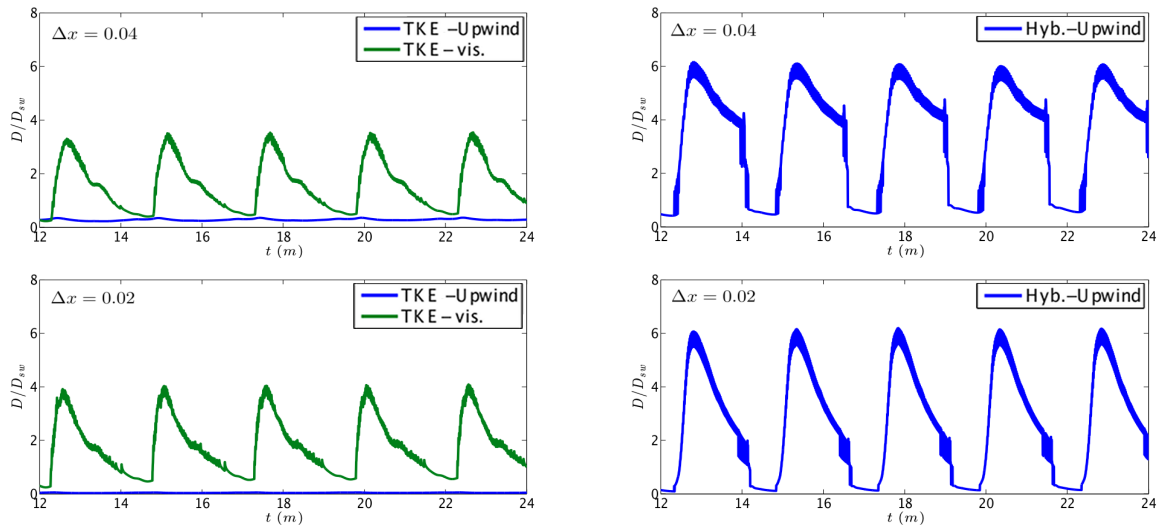


Figure 9: Energy dissipation profile for GN model using the TKE (left) and the Hybrid (right) closures for $\Delta x = 0.04$ (top) and $\Delta x = 0.02$ (bottom)

energy_bar_GN

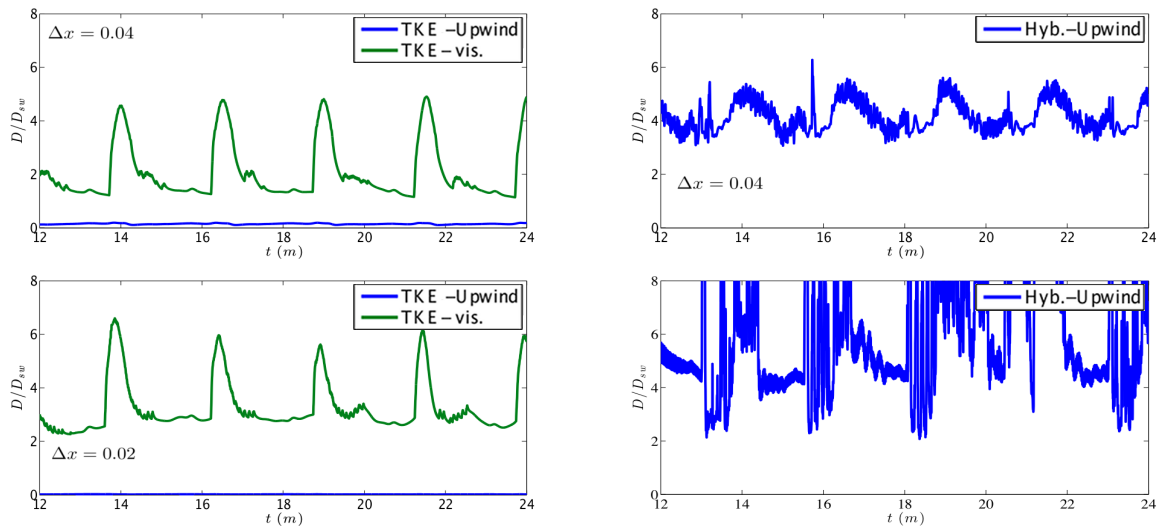


Figure 10: Energy dissipation profile for Nwogu model using the TKE (left) and the Hybrid (right) closures for $\Delta x = 0.04$ (top) and $\Delta x = 0.02$ (bottom)

energy_bar_Nw

7.2 Solitary waves breaking on a slope

One of the most intensively studied problems in long wave modelling is the solitary wave run-up on a plane beach, see for example [12, 16, 18, 47, 75, 95, 96] among others. In this test case we want to study propagation, breaking and run-up of a solitary wave over a planar beach with a slope 1 : 19.85. With this famous test case we assess the ability of our model to describe shoreline motions and wave breaking when it occurs. The incident wave height considered in this case is $A/d = 0.28$ with $d = 1$, so according to Synolakis [47] the wave breaks strongly both in the run-up and in the rundown phase of the motion. The GN and Nwogu's equations are tested and compared, using for each one the turbulent kinetic energy wave

532 breaking model and the hybrid wave breaking model. The same holds for all the test cases that follows.

533 The computation domain is of $120m$, where $x \in [-20, 100]$. The CFL used is 0.3 and sponge layer
534 was applied off-shore with length $L_s = 5m$. A Manning coefficient of $n_m = 0.01$ was used to define the
535 glass surface roughness used in the experiments. As before, computations have been run on three different
536 meshes with size $\Delta x = [0.025, 0.0125, 0.0063m]$. The parameters of the wave breaking criteria used in
537 this test case are $\gamma = 0.6$ and $\phi_c = 30^\circ$ for both models. To properly capture the hydraulic jump generated
538 at during backwash, the TKE parameters depend here both on the propagation model and on the type of
539 breaking criterion satisfied. In particular, for unsteady waves the surface time variation criterion is the one
540 activated. In this case we use $\kappa_{GN} = 0.75$, $\sigma_{GN} = 0.9$ for the GN model and $\kappa_N = 0.8$, $\sigma_N = 1.5$ for
541 Nwogu's model. If the slope criterion is activated, we use instead higher values, namely we set $\kappa_{GN} = 1.5$,
542 $\sigma_{GN} = 1.5$ and $\kappa_N = 1.5$, $\sigma_N = 1.5$.

543 Figure [11](#) compares the numerical surface profiles for the GN equations and the experimental measure-
544 ments. The same is plotted for Nwogu's equations in figure [12](#). The numerical solution was obtained using
545 $\Delta x = 0.05m$. As expected, both mathematical models produced similar behaviour. Until time $t\sqrt{g/h} = 10$
546 the solitary propagates to the shore and the two wave breaking models produce, as expected, identical results
547 since wave breaking hasn't started yet. As expected the Nwogu's model gives a wave which overshoots and
548 breaks slightly earlier compared to the one produced by the GN equations. The experimental wave breaks
549 around $t\sqrt{g/h} = 20$. The numerical solution for the hybrid model is represented like a bore storing the
550 water spilled from the breaking wave behind the front. At time $t\sqrt{g/h} = 20$ the turbulence model rep-
551 represents the solution as a triangular bore considerably closer to the experimental data than the hybrid one.
552 Similar behaviour has been observed by other researchers that used eddy viscosity models [[16-18](#)]. At time
553 $t\sqrt{g/h} = 25$ the bore collapses at the shore, and both approaches show good qualitative agreement with
554 the data. After that the wave starts to run-up, with a maximum run-up occurring at $t\sqrt{g/h} = 45$. As the
555 water recedes, a breaking wave is created near the still water level. The numerical solution is approximated
556 as a hydraulic jump for both numerical models. It is fully resolved using both breaking models, since the
557 breaking criterion recognises the hydraulic jump and the NSW equations are used for the hybrid model
558 while the proper amount of viscosity is added by the turbulent kinetic energy model.

559 Figures [13](#) and [14](#) show the numerical results for both breaking phases (at time $t\sqrt{g/h} = 20$ and
560 $t\sqrt{g/h} = 60$ respectively) while refining the mesh. Up to the authors knowledge it is the first time that
561 such a study is performed for a (quasi-)steady hydraulic jump for an eddy viscosity type model. The first
562 set of figures depict the breaking of the wave which travels on-shore for both GN (left column) and Nwogu's
563 equations (right column). We can clearly see the oscillatory nature of the hybrid wave breaking mechanism.
564 The profiles obtained indicate some sort of convergence of the mean. However this is completely spoiled
565 by the oscillations produced due to the switching between the two sets of equations. On the contrary the
566 turbulent kinetic energy wave breaking mechanism remains stable and gives a convergent solution for both
567 sets of equations. The second set of figures plot the same for the hydraulic jump formed at backwash. The
568 difference between the two approaches is more accentuated here. It is quite hard to see a convergence for
569 the hybrid results, while this is clearly the case for the TKE ones. We must mention that the GN equations
570 combined with the hybrid model is blowing up after $t\sqrt{g/h} = 60$ for $\Delta x = 0.0063$, while Nwogu's
571 equations are more sensitive to the hybrid formulation since numerical solution is obtained only for the first
572 two meshes.

573 We have repeated this test for a more non-linear initial wave with $\epsilon = 0.5$, on the mesh with $\Delta x =$
574 $0.025m$. The results obtained at incipient breaking before the runup and during backwash are reported on
575 figure [15](#). As before the hybrid mechanism produces oscillations, in both breakers, and it is very unstable
576 for Nwogu model. Oscillations are clearly visible for the GN results with the hybrid breaking. Smooth
577 capturing of the breakers is obtained also in this case with the TKE model. Figures [16](#) and [17](#) show again,

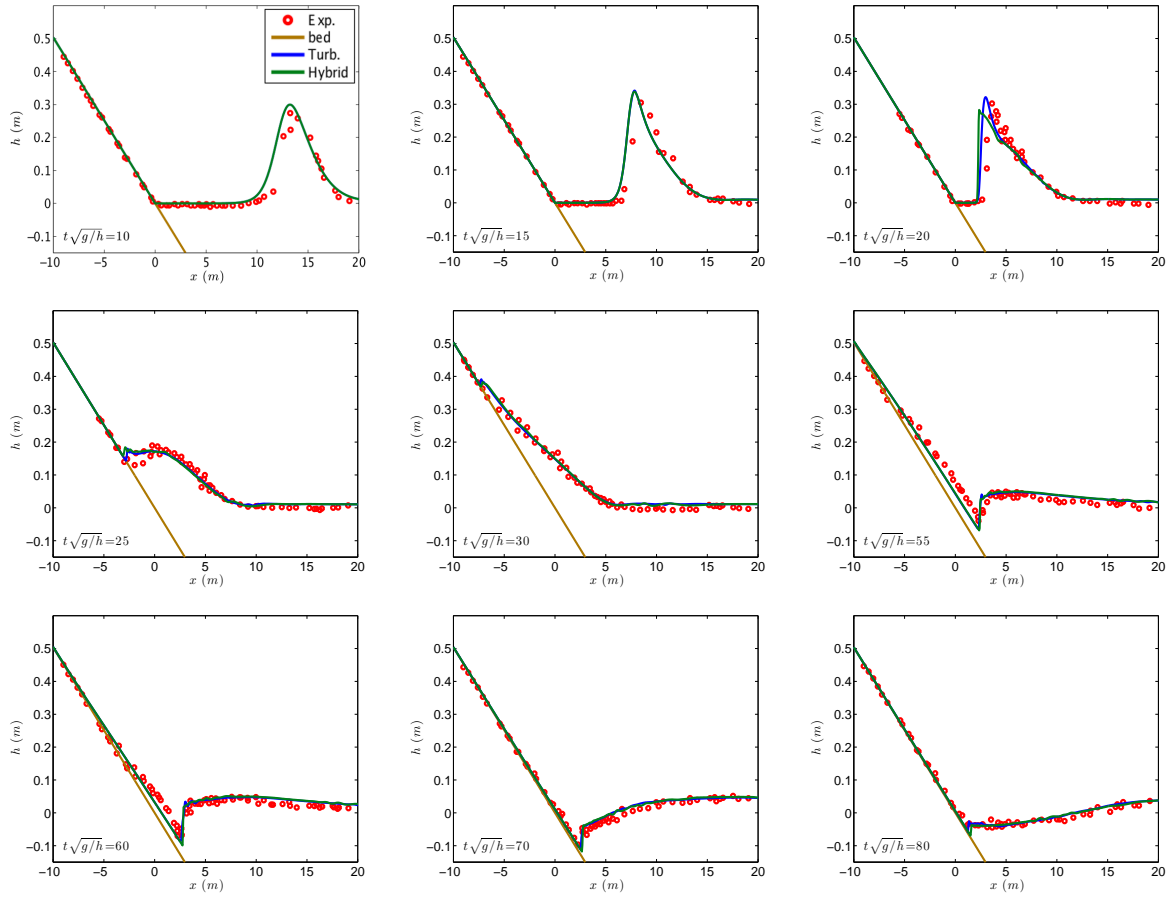


Figure 11: Free surface elevation of solitary wave run-up on a plane beach for the GN model.

syn_gn_1

578 the numerical results for both breaking phases while refining the meshes for the turbulent kinetic energy
 579 mechanism. The Hybrid closure is not converging since the oscillatory nature of the mechanism is more
 580 pronounced in this case.

581 7.2.1 Dissipation mechanisms

582 For both cases we now look at the contributions to the dissipation of energy. In this case, all three sources of
 583 dissipation are active (cf. section ^{note on dis}§5): numerical dissipation D_{upwind} , dissipation due to friction $D_{friction}$,
 584 and the dissipation due to the eddy viscosity D_{vis} when using the TKE closure. Let us first focus on the
 585 results for a nonlinearity of 0.28 reported in figures ^{elementary}18-19 for the GN and Nwogu models. The results with
 586 the Nwogu model are on coarser meshes to allow some comparison on the behaviour of the hybrid closure
 587 on different meshes. The figures allow to see the dynamics of dissipation associated to the different phases
 588 of the flow. The first breaking of the incoming wave is seen in all figures around time 5s, with the Nwogu
 589 model again showing earlier breaking certainly due to its over-shoaling characteristics. As the wave reaches
 590 higher bathymetries and the runup process starts, the friction takes over and dominates the flow, with no or
 591 very little contributions from the other terms. Dissipation is reduced to zero at the end of the runup, and if
 592 increases again during backwash, with again the friction dominating, and the other terms providing again

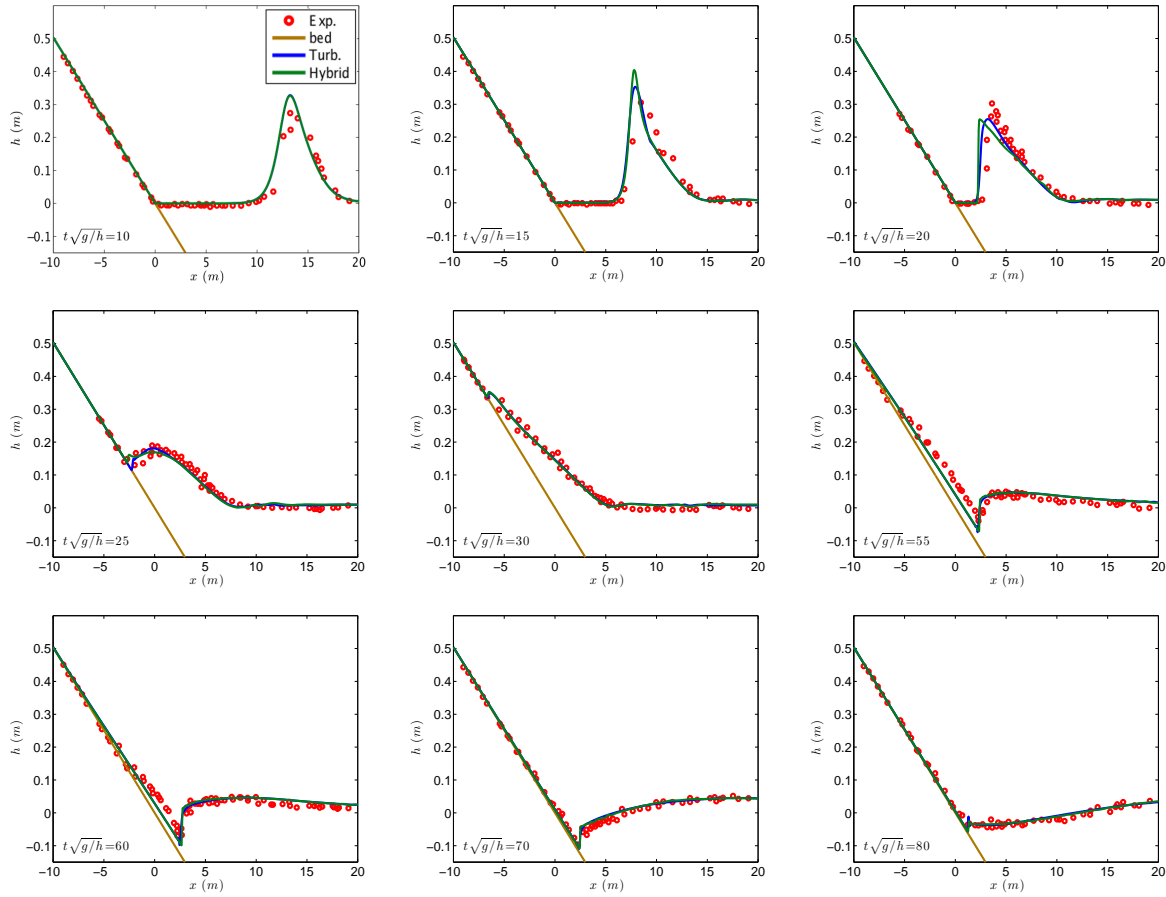


Figure 12: Free surface elevation of solitary wave run-up on a plane beach for Nwogu’s model.

syn_n_1

593 non negligible contributions around time 17s – 20s when the hydraulic jump is formed. Note that these
 594 contributions arise from integrals in space. So the plot may lead to confusion as to which mechanism allows
 595 to capture the hydraulic jump. Indeed, the friction contributions are localized in the region very close to the
 596 wet/dry interface, and they would not allow to capture the hydraulic jump.

597 Looking at the behaviour of the different terms on the meshes considered, we can remark again that
 598 when using the TKE closure the numerical dissipation D_{upwind} is not contributing, or providing very small
 599 contributions, throughout the flow. In the case of the hybrid closure, we can again see that it is indeed
 600 D_{upwind} that provides dissipation during breaking. We can also see from figure 18 that this contribution is
 601 slightly larger on the coarser mesh, even though it is less clear that in the previous case. The oscillations
 602 observed during backwash in both the viscous contribution and numerical dissipation are associated to the
 603 intermittency of the breaking detection criterion, which is certainly something to be improved in the future.
 604 Finally, we remark that the behaviour for finer meshes is exactly the same, and that the finest meshes on
 605 which we managed to run this case until the final time with the hybrid closure are those mentioned earlier,
 606 namely $\Delta x = 0.0063m$ for the GN model, and $\Delta x = 0.025$ for the Nwogu model.

607
 608 We perform the same analysis for the case with a nonlinearity of 0.5. The results are reported on figures

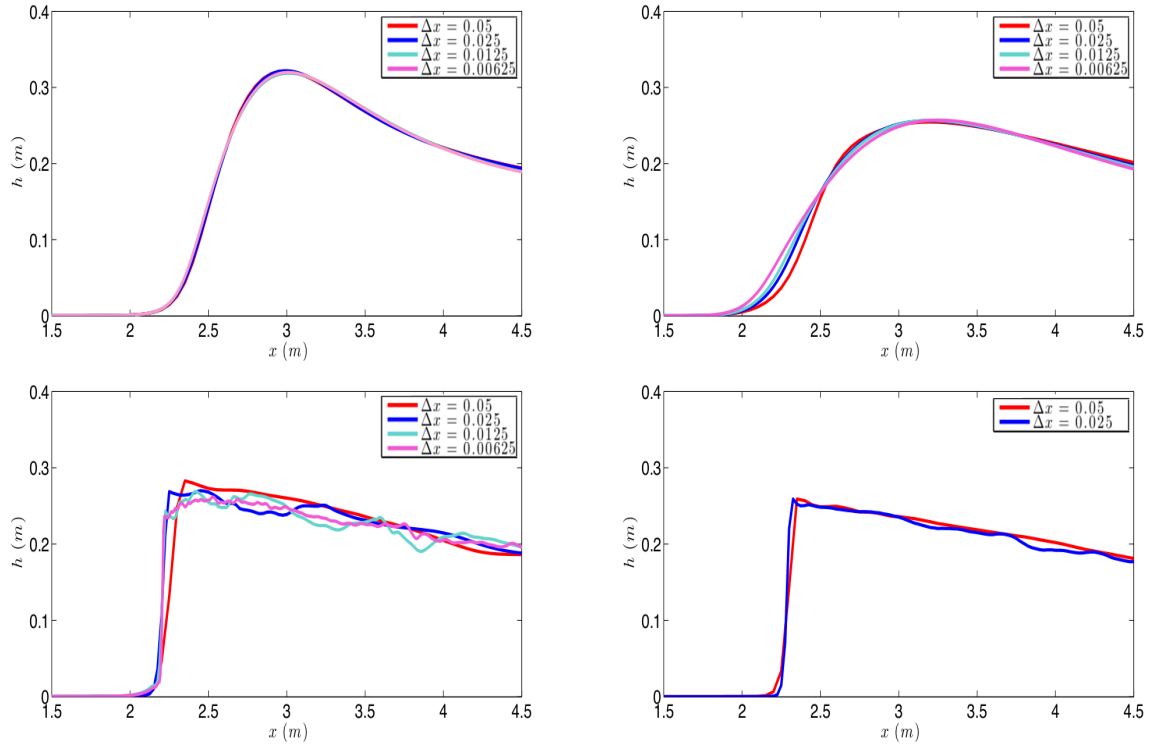


Figure 13: Breaking bore on different meshes for the GN (left) and Nwogu's (right) models, using the TKE (up) and the hybrid (down) wave breaking closure.

syn_con

609 [Nwogu](#) [20-21](#). As mentioned already, in this case we could only run the Nwogu model with hybrid closure on the
610 coarsest resolution of $\Delta x = 0.1m$. The dynamics observed in the figures are very similar to those of the
611 previous case. Of course in this case the first breaking occurs earlier (around $2s$) with the Nwogu model
612 again providing an earlier breaking. We can again see the friction dissipation taking over during the
613 runup process, then decreasing, and increasing again during backwash. As before, breaking is re-activated
614 to capture the hydraulic jump forming during backwash. We can again remark that when using the TKE
615 closure the numerical dissipation is not contributing to the process, which is dominated by the terms em-
616 bedded in the PDE. On the contrary, it is the numerical dissipation term that rules the dynamics of breaking.
617 We can also see quite clearly that a considerable reduction of this contribution is obtained with mesh refine-
618 ment. Again, the contributions of D_{vis} and of D_{upwind} are quite oscillatory during the backwash, and this
619 is related to the intermittency of the detection mechanism. The meshes shown here are the finest we could
620 run this case on until the final time with the hybrid closure.

621
622 As for the previous case, this analysis shows that when using the eddy viscosity closure the numerical
623 dissipation plays very little or no role. This is motivation to look for non-dissipative/energy conserving
624 schemes in this context. The mesh size seems to have an impact on the magnitude of the overall dissipation
625 introduced during breaking. Finer meshes providing overall less dissipation. The TKE closure is very little
626 sensitive to the mesh. This analysis also shows a very interesting interplay between the breaking and friction
627 dynamics.

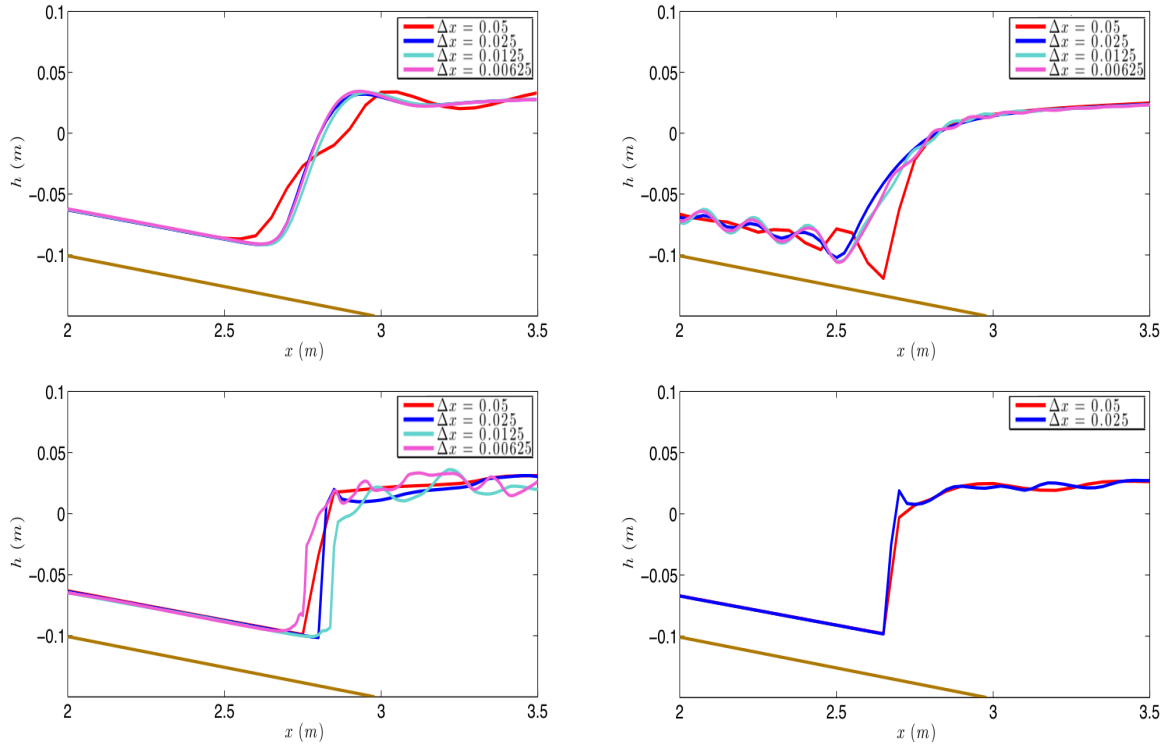


Figure 14: Hydraulic jump on different meshes for the GN (left) and Nwogu's (right) models, using the TKE (up) and the hybrid (down) wave breaking closure.

syn_con_hj

7.3 Bore propagation and dissipation in function of the Froude number

We consider in this test case the propagation of a breaking bore over a flat bottom. We have chosen this benchmark as its simple setting allows to perform some quantitative comparison between the discrete energy dissipation terms analyzed in the paper, and the exact theoretical shallow water dissipation, equation (14), for different values of the Froude number. The test is defined by an initial step which transforms to a bore. The initial solution is defined by

$$\begin{cases} h(x, 0) = \frac{1}{2}(d_b - d_a)(1 - \tanh \frac{x}{a}) + d_a \\ u(x, 0) = \frac{1}{2}(u_b - u_a)(1 - \tanh \frac{x}{a}) + u_a, \end{cases} \quad (27)$$

where d_a and d_b are the water depth in front and behind the bore, u_a and u_b the corresponding depth-averaged velocities. In our case $u_a = 0$, $d_a = 1m$ and $a = 2m$. For each Froude number (F_r), u_b and d_b are computed, solving the mass and momentum conservation conditions across the bore. For $F_r > 1.4$ the initial step evolves into a breaking bore. More informations on the test case can be found in [97] and references therein.

The computational domain used is $x \in [-150, 150]$, $CFL = 0.2$ and $\Delta x = 0.1$. For this type of wave the parameter most sensitive to the onset of breaking is the time derivative of the elevation γ , which we have set here to $\gamma = 0.4$. For the turbulence model we have used $\kappa_{GN} = \kappa_N = 1.5$, $\sigma_{GN} = \sigma_N = 0.8$ for the GN and Nwogu equations respectively. Figure 22 shows the propagated bore at $t = 0, 1, 15s$ for the

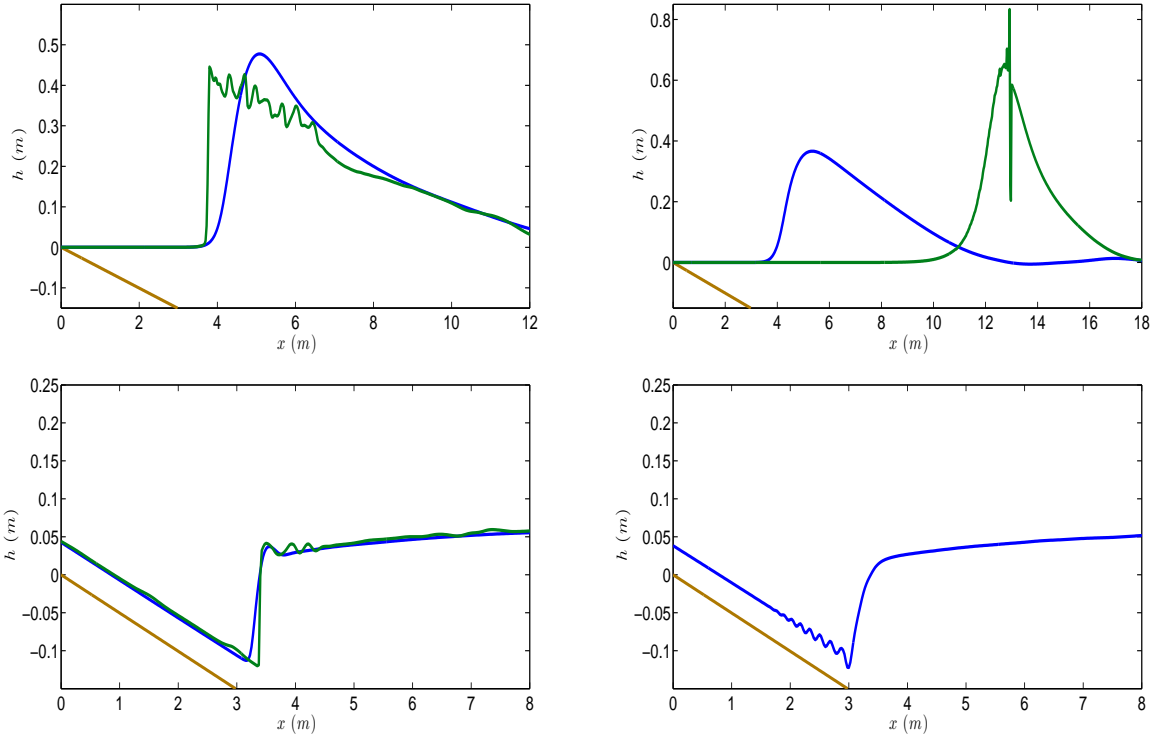


Figure 15: Breaking in the run-up (up) and the run-down (down) phase for GN (left) and Nwogu's equations (right) for $\epsilon = 0.5$, using the TKE (blue) and the hybrid (green) wave breaking closure. Mesh size is 0.025m.

syn_e0P5

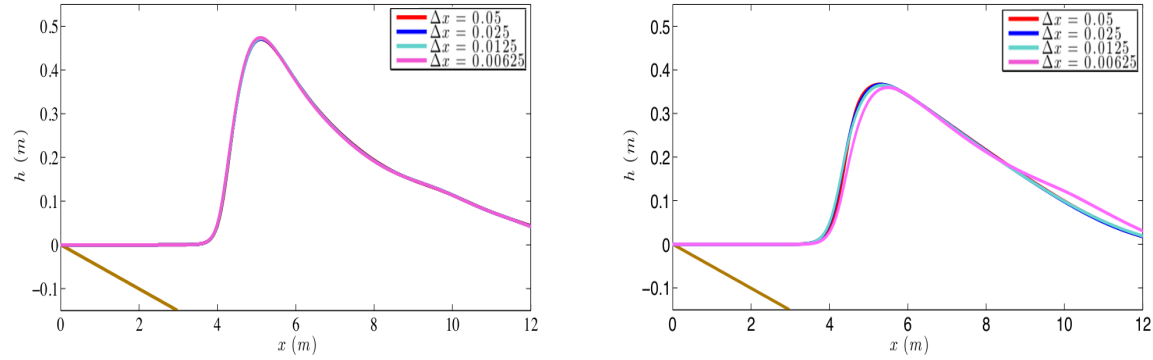


Figure 16: Breaking bore on different meshes for the GN (left) and Nwogu's (right) models for $\epsilon = 0.5$, using the TKE wave breaking closure. Mesh size is 0.025m.

syn_con_0P5

644 two models for a Froude number $F_r = 2$. The bore is breaking as it propagates through the channel, and a
 645 slightly different behaviour is observed for the two breaking closures. Hybrid breaking provides a travelling
 646 shock, for both propagation models, while the turbulent closure presents a more diffusive behaviour, with a
 647 small overshoot before the bore for the GN model.

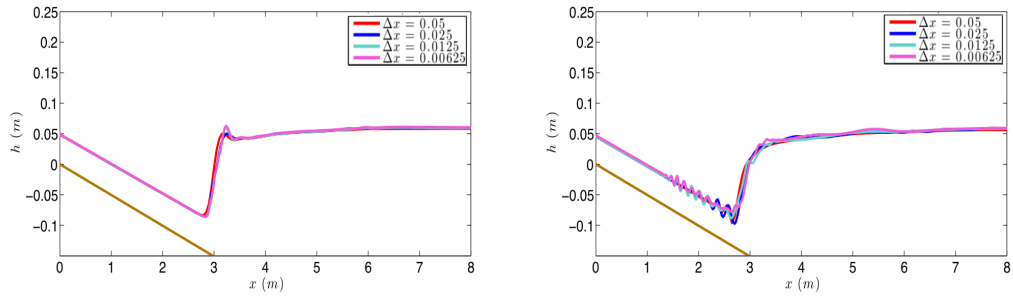


Figure 17: Hydraulic jump on different meshes for the GN (left) and Nwogu's (right) models for $\epsilon = 0.5$, using the TKE wave breaking closure. Mesh size is 0.025m.

syn_con_0P5_h

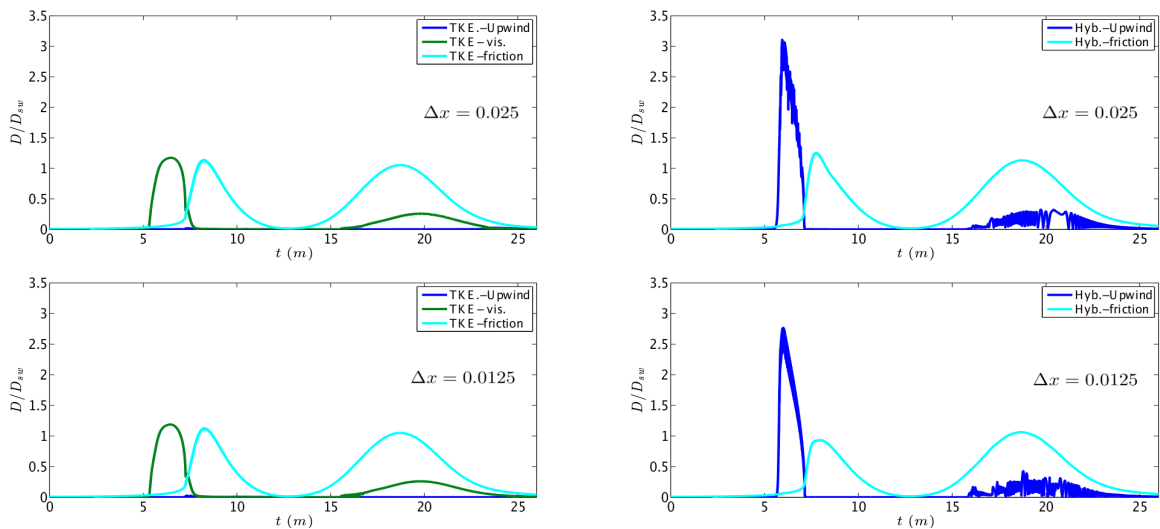


Figure 18: Energy dissipation profile for GN model using the TKE (left) and the Hybrid (right) closures on three different meshes: $\Delta x = 0.025, 0.0125m$ from top to bottom. The nonlinearity of the wave is 0.28.

energy_syn0P2

648
649 The terms related to the upwind dissipation and to eddy viscosity evolve during the transformation of
650 the solution into a bore, as shown on figure 22, and quickly converge to a steady (in time) value, which is
651 plotted in figures 23 and 24 against the shallow water dissipation (14), for different Froude numbers and
652 on different meshes. Note that in this case, the wave breaking interface is located in correspondence of a
653 constant solution region. This makes this case easier compared to the previous ones. This also reduces a lot
654 the impact of mesh size on the final value of the dissipation, essentially dictated by the jump in water height.
655 Nevertheless, exactly as the previous cases with the hybrid approach the initial development of the solution
656 shows instabilities, for meshes finer than those reported in the figures, solution blow up. For the GN model,
657 and for the range of Froude numbers tested, the TKE dissipation is within 10-15% of the value predicted
658 by (14), while the upwind terms basically provide a negligible contribution. Conversely, these terms are,
659 when using the hybrid approach, within 6% of (14). As in the previous tests, this allows to demonstrate
660 that the numerical dissipation does not contribute to the dynamics of wave breaking when using the TKE
661 eddy viscosity closure. It also shows that the particular choice of eddy viscosity we performed allows to

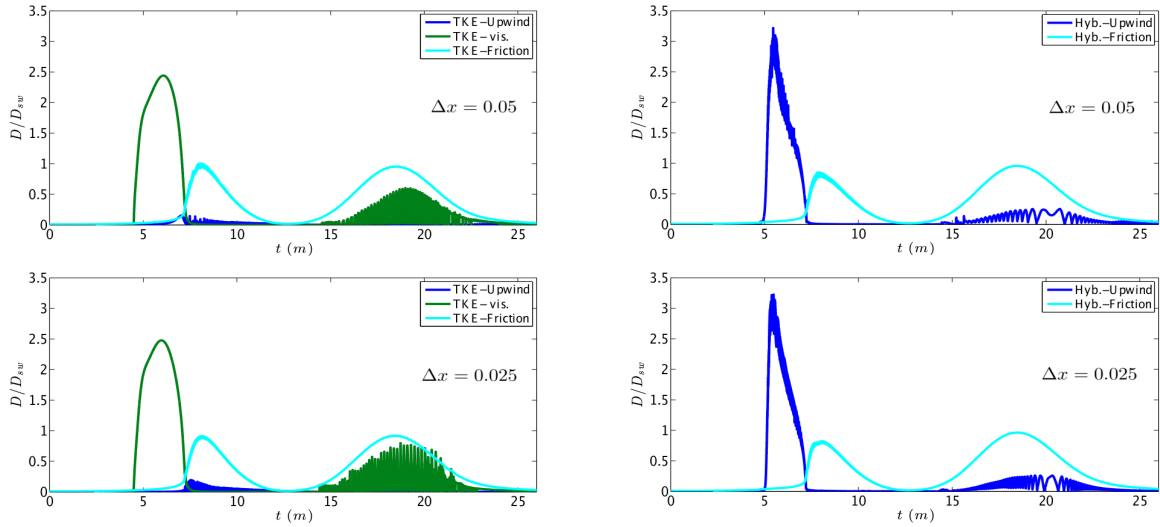


Figure 19: Energy dissipation profile for Nwogu model using the TKE (left) and the Hybrid (right) closures on three different meshes: $\Delta x = 0.05, 0.025m$ from top to bottom. The nonlinearity of the wave is 0.28.

energy_syn0P2

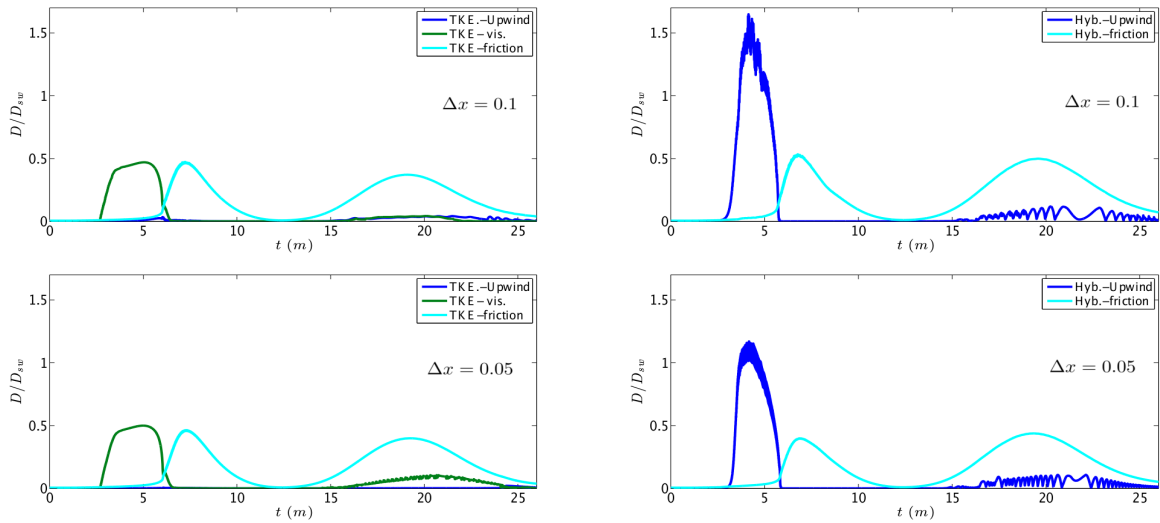


Figure 20: Energy dissipation profile for GN model using the TKE (left) and the Hybrid (right) closures on two different meshes: $\Delta x = 0.1, 0.05m$ from top to bottom. The nonlinearity of the wave is 0.5.

energy_syn0P3

662 reproduce with some accuracy the behaviour with Froude number predicted by the classical formula ^{eq:sw-diss} (14).
 663 Similar conclusions can be drawn for the Nwogu model by looking at figure ^{Nwogu_energy_1} 24.

664 7.4 Wave height and setup prediction

665 The analysis of [74] shows that wave setup is very sensitive to the dissipation mechanism in wave breaking.
 666 So this is an interesting parameter to study for our purposes. To investigate this aspect we consider two
 667 of the experiments performed by Hansen and Svendsen [98]. These experimental studies consider several
 668 different regular waves shoaling and breaking on a sloping beach. Many authors have used these tests to

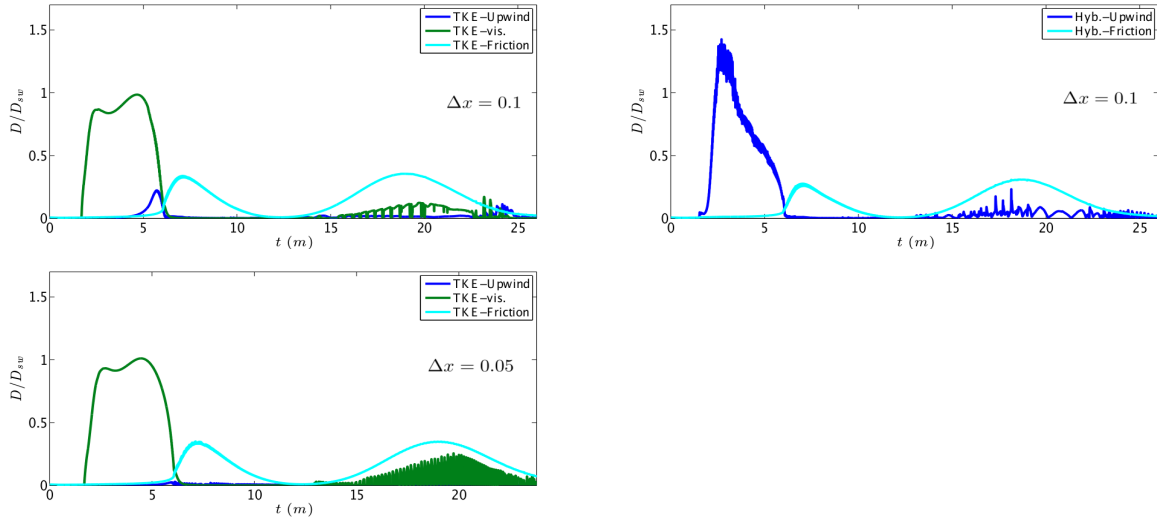


Figure 21: Energy dissipation profile for Nwogu model using the TKE (left) and the Hybrid (right) closures on three different meshes: $\Delta x = 0.1, 0.05m$ from top to bottom. The nonlinearity of the wave is 0.5.

energy_syn0P5

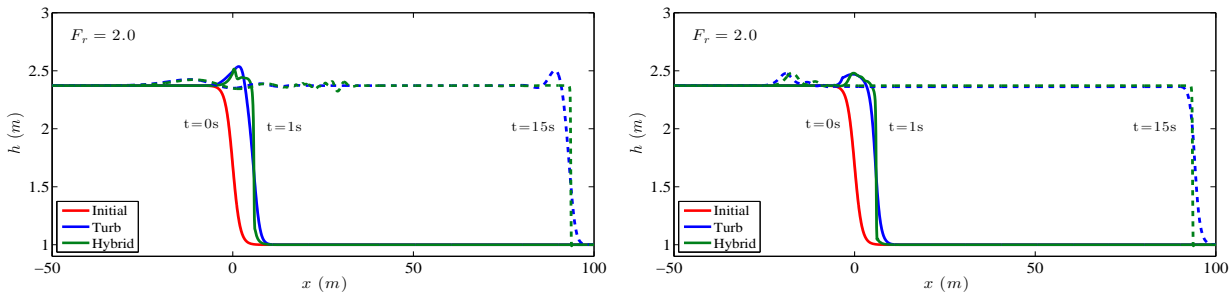


Figure 22: Free surface profiles at $t = 0, 1, 15s$ of hydraulic bores with $Fr = 2.0$. Left: GN model. Right: Nwogu model.

bore_prop

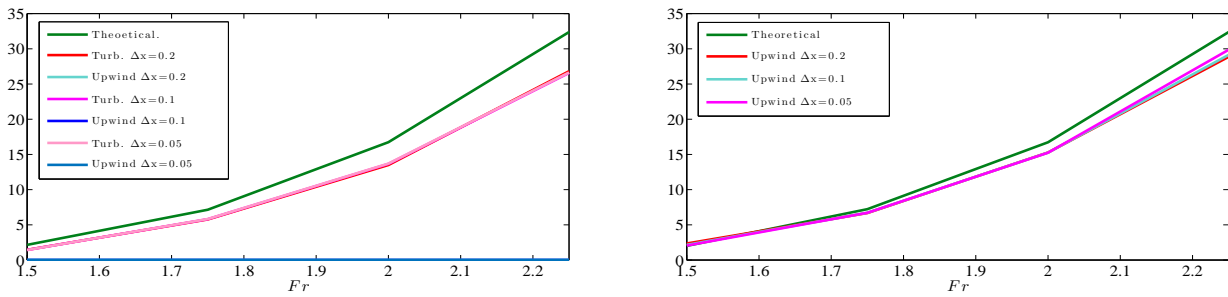


Figure 23: Energy dissipation profile for GN model using turbulent closure (left) and hybrid closure (right).

Gn_energy_1

669 validate their models and the associated breaking closures [6, 12, 17, 23, 75].

670 We consider here two cases, one involving a spilling breaker, the second involving a plunging breaker.
 671 Regular waves are generated over a $0.36m$ horizontal bottom, propagated shoaled and broke over a slope
 672 of $1 : 32.26$. In the spilling breaking case (test number 05041) the regular wave's period T is $2.0s$, and

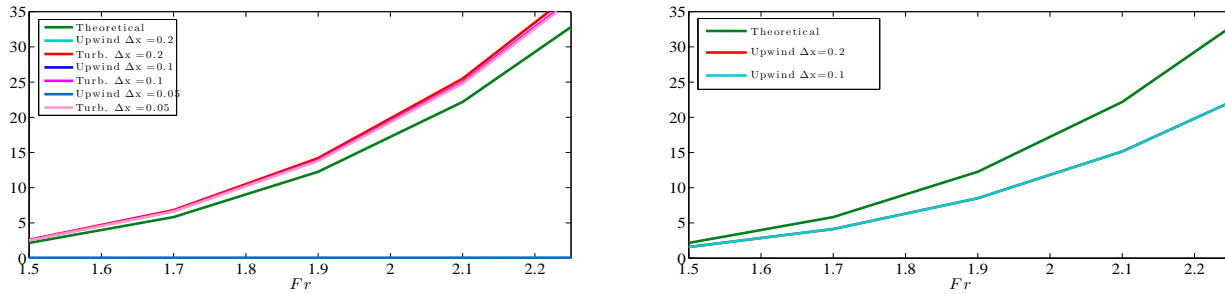


Figure 24: Energy dissipation profile for Nwogu model using turbulent closure (left) and hybrid closure (right).

Nwogu_energy_

673 the wave's height H is $0.036m$. The second test case (test number 03041) is a strong plunging breaking
 674 case with $T = 3.33s$ and $H = 0.043m$. The tests have been run on a $52m$ long domain $x \in [-26 \ 26]$,
 675 discretised with cells of $\Delta x = 0.02m$, and with $CFL = 0.3$. A sponge layer is applied offshore with
 676 length $L_s = 5m$. The wave making internal source was placed $14.78m$ offshore from the toe of the beach,
 677 and bottom friction is neglected. The free surface elevation, recorded at wave gauges which placed every
 678 $0.1m$., is analysed to compute the mean wave height, and the position of the mean water level (MWL). The
 679 value of γ in the surface variation criterion equals to 0.5 for both models. Concerning the wave breaking
 680 closures, we have set $\kappa_{GN} = 0.8$, $\sigma_{GN} = 0.05$ and $\kappa_N = 0.8$, $\sigma_N = 0.4$, for the two GN and Nwogu
 681 models respectively.

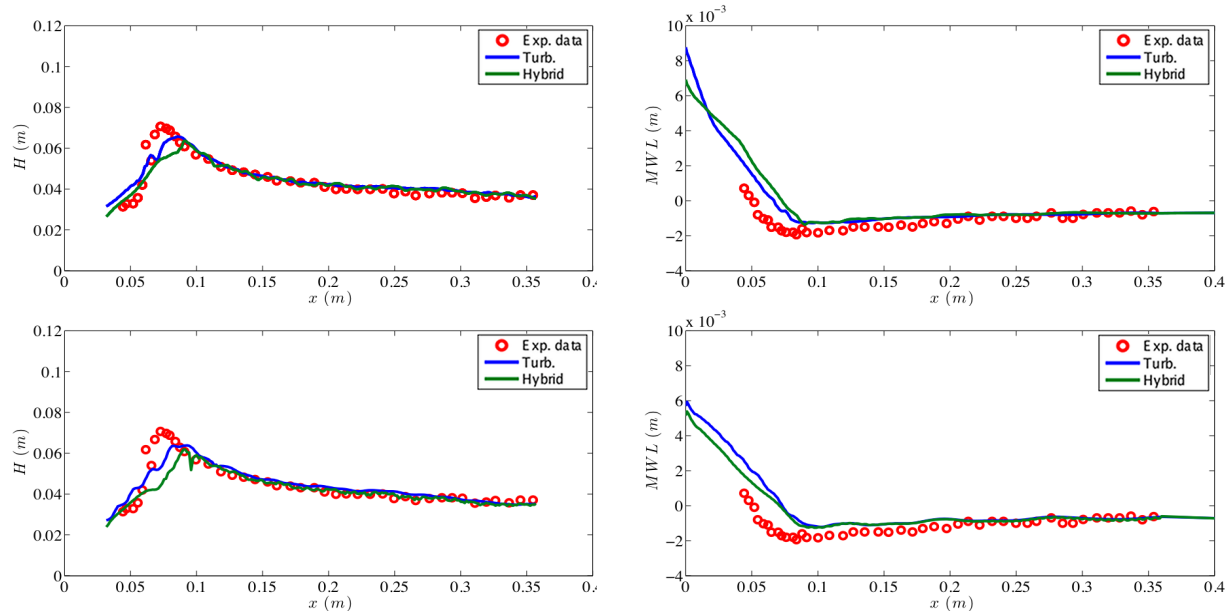


Figure 25: Computed and measured wave heights (left) and set-up (right) using equations. Test number 05041 (spilling breaking). Top: GN equations. Bottom: Nwogu equations. Blue line- TKE closure, green line- Hybrid closure.

case05041

682 The numerical results obtained for the two cases considered are reported on figures [25](#) and [26](#), in terms
 683 of wave height (left) and mean water level (right). As before, the blue lines in the figures refer to the TKE
 684 results, while the green ones to the hybrid wave breaking, and the top row report the computations of the

685 GN model, while the bottom ones the results of the Nwogu equations.

686 For the spilling case, figure [25](#) seems to indicate that in all cases the detection criterion provides an
 687 early breaking. This of course alters the strength of the numerical breaking, which is less intense. This
 688 translates in a wave height decrease slower than the experimental one. Even so, the computations compare
 689 reasonably well with the experiments, especially when compared with results in the published literature
 690 [6, 12, 17, 23, 75]. This is confirmed by the mean water level plots. Although we can clearly observe
 691 the early start of setup, due to the early breaking, the slopes of the numerical signals are quite close to
 692 those of the experimental ones. According to the analysis of [74] this shows that the amount of dissipation
 693 introduced is correct. We stress that the differences between the TKE and hybrid approach are minor, even
 694 though we tend to consider the results obtained with the turbulence model slightly better in terms of both
 695 wave height and slope of the setup.

696

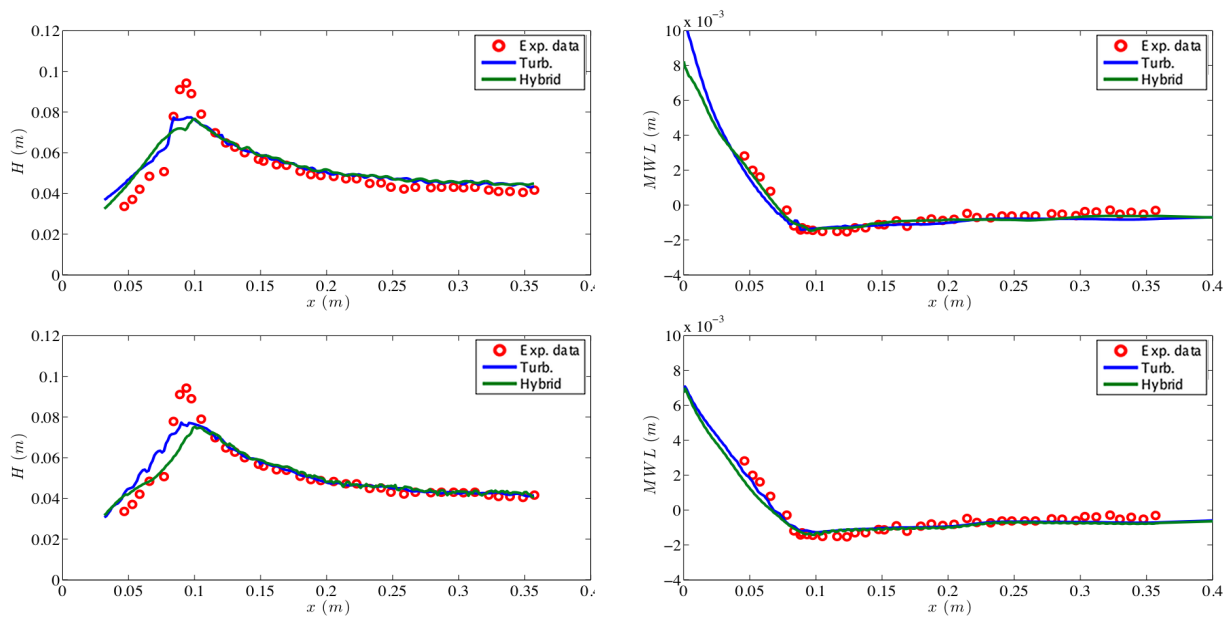


Figure 26: Computed and measured wave heights (left) and set-up (right) using equations. Test number 03041 (plunging breaking). Top: GN equations. Bottom: Nwogu equations. Blue line- TKE closure, green line- Hybrid closure.

case03041

697 For the plunging case, figure [26](#), the agreement with the experimental values is even better. We can
 698 see that the breaking location is detected correctly in this case, even though both the GN and the Nwogu
 699 model provide an underestimation of the shoaling with both breaking closures. The wave height decrease
 700 is predicted with a slightly smaller slope, but the agreement with the data is quite satisfactory. The setup
 701 prediction is very good, with both location of the breaker and slope reproduced correctly by all models.

702 Some conclusions can be drawn from the implementation of this numerical test case. The first one
 703 is that both wave breaking closures allow to detect and handle both spilling and plunging breaking of
 704 regular waves. We stress that the parametrisation used for TKE closure is the same for the two cases
 705 considered. This shows the potential of this type of approach to provide a robust accurate energy dissipation
 706 rate, independently on the number of nodal points per wavelength, and on the nonlinearity of the problem.

707 **7.5 Application: propagation, breaking, and overtopping of a 2D reef**

708 This next test case is reported as a complex application in order to show the potential of the modelling
 709 choices evaluated here to handle the interaction of the whole range of phenomena: dispersive propagation,
 710 shoaling, breaking, overtopping, reflection. The benchmark considered was initially proposed in [18, 99],
 711 and later used by several authors for validation [18, 37, 100, 101]. The problem involves a bathymetry
 712 consisting of a reef with a fore slope of 1/12 and a crest of 0.2m reef crest and an offshore water depth of
 713 2.5m. The reef crest is exposed by 0.06m and hides on the lee side a flat with a depth of 0.14m. Water height
 714 distributions at several time instants and water height time series in 14 wave gauges have been measured
 715 in the flume experiments at Oregon State University within the PhD work of V. Roeber [99] (see also [18]). A
 716 sketch of the reef geometry, showing the positioning of the wave gauges, is reported in figure 27. The initial
 717 state consists of a solitary wave of amplitude $a = 0.75m$ which propagates onshore, shoals and breaks in
 718 front of the reef crest. Walls are present at both ends of the domain. We refer to [18, 99] for a more detailed
 719 description of the experimental and computational setup. Our results have been computed on a mesh with
 720 size $\Delta x = 0.05$, and setting $CFL = 0.3$. Manning friction has been used, with a Manning coefficient
 721 $n_m = 0.012$. Both wave breaking detection criteria are used with $\gamma = 0.6$ and $\phi_c = 30^\circ$. Concerning the
 722 TKE closure $\kappa_{GN} = 0.75$, $\sigma_{GN} = 0.8$ and $\kappa_N = 1.2$, $\sigma_N = 1.5$ but when a hydraulic jump is detected the
 723 values are set to $\kappa_{GN} = 1.5$, $\sigma_{GN} = 15.5$ and $\kappa_N = 3.5$, $\sigma_N = 16$.

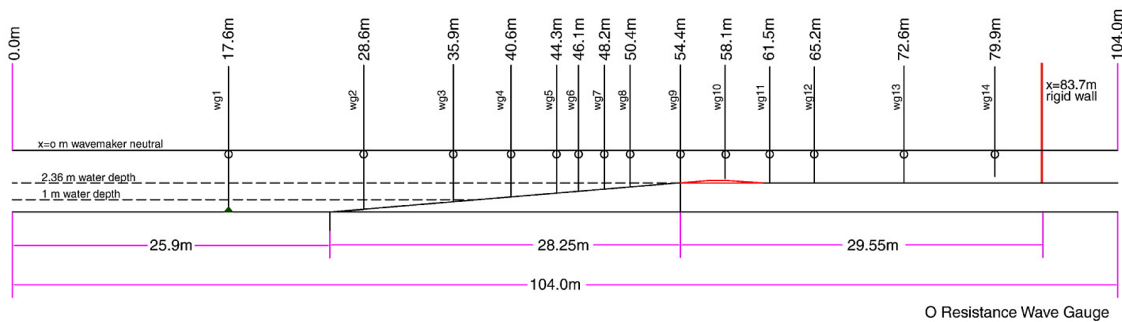


Figure 27: 2D reef geometry and wave gauge locations. Adapted from [18].

reefld_geo

724 To visualise the results we group snapshots of the free surface in three phases : propagation and shoaling
 725 of the initial soliton (figure 28); overtopping and formation, propagation and reflection of a bores on the lee
 726 side of the reef (figure 29); secondary overtopping, with formation of a quasi-steady hydraulic jump and
 727 of an undular bore (figure 30). In all the figures, the top rows report the results obtained with the GN
 728 model, the bottom rows refer to the results of the Nwogu model, the blue lines are those obtained with the
 729 TKE breaking model, and the green lines are those of the hybrid breaking treatment. Symbols refer to the
 730 experimental values provided in [99].

731 The figures show that all models allow, on this mesh resolution, a quite satisfactory prediction of the
 732 water height. The differences between different choices appear to be minor. We can mention that, at least
 733 in our implementation both the fully and the weakly nonlinear models tend to predict the moving bores on
 734 the lee side with some phase advance. This, at least in our implementation, is more pronounced for the fully
 735 nonlinear GN mode, as we can see e.g. on figure 29 (central and right column). This behaviour is independ-
 736 ent on the breaking closure adopted. We can also remark that when using the hybrid wave breaking with
 737 the Nwogu equations some over-prediction of the amplitude of the undulating bores is observed.

738
 739 To have some more insight in the capabilities of the models, we analyze the water height time series

740 in gauges WG5, WG9, WG10, and WG12. The plots are reported on figures [31](#) and [32](#). The dispersive
 741 propagation of the waves is visible in WG5 and, at for the fore side undulating bores, in WG9. We can see
 742 that all the models capture correctly the shoaling of the initial solitary, and that despite a visible phase lag,
 743 provide a quite reasonable amplitude and frequency of the undulating bores on the fore side, as it can be
 744 seen e.g. in the WG5 series on figure [31](#), for times larger than 70s, and in WG9 after 80s. In WG5 we
 745 can see again the over-amplification of the amplitude of the undular bores for the Nwogu model with hybrid
 746 wave breaking.

747 Concerning breaking, we can see the first breaker approximation very well reproduced from the WG9
 748 series at time around 34.5s. The hydraulic jump forming at 55s is also well reproduced in amplitude, albeit
 749 with a phase advance. Similar observations can be made when looking at figure [32](#). The WG12 results,
 750 in particular, show an excellent agreement for the first four bores. All the models give an under-prediction
 751 of the water level behind the slowly moving hydraulic jump which forms behind the main right-going bore
 752 (time 38s). The first reflected bore at time roughly 50s, as well as the second hydraulic jump forming
 753 after the second overtopping (time 60s) are also very well captured by the models. The later reflections
 754 present instead a visible phase error, albeit correct in amplitude. Lastly, the WG10 results in the same figure
 755 show a nice capturing of the first two overtopping phases, although an over-prediction of the water height
 756 is also observed. The later overtoppings are affected by a phase advance already mentioned for the bores
 757 responsible for them.

758
 759 Overall we consider the results quite good for all the models. Some of the differences w.r.t. the experi-
 760 mental water heights we are convinced that are also due to the definition of this quantity in presence of air
 761 entrainment at the free surface, as it was the case for the experimental breakers. We stress very strongly that
 762 with the current implementation the simple TKE breaking closure can handle without any problem simul-
 763 taneous breakers of different types, and of different intensities. For this test, as for all the others analyzed
 764 in the paper, the fully nonlinear GN model with TKE closure provides the most robust combination.

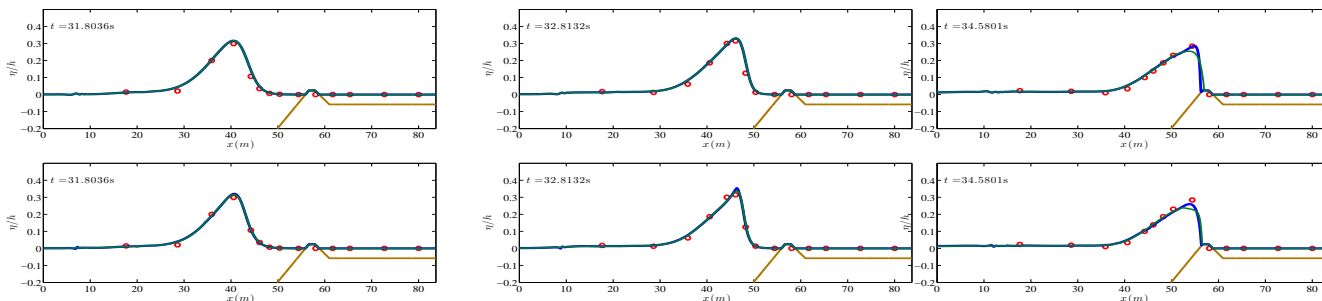


Figure 28: Overtopping of a 2D reef. Propagation, shoaling, and overtopping phases. Top row: GN model. Bottom row: Nwogu model. Blue lines: TKE breaking closure. Green Lines: hybrid wave breaking closure. Left: $t = 31.8036s$. Middle: $t = 32.8132s$. Right: $t = 34.5801s$.

reef_wpl

765 8 Conclusions

766 We have considered the issue of wave breaking closure when using weakly dispersive Boussinesq propaga-
 767 tion models. We studied weakly and fully nonlinear models representative of classical and well known mod-
 768 els/codes such as BOUSS-2D [3, 4], Funwave [5, 6], Coulwave [7, 8], BOSZ [9], MIKE21 [10], TUCWave
 769 [11, 12], and others. We have in particular focused on the enhanced equations of Nwogu [49], and on a

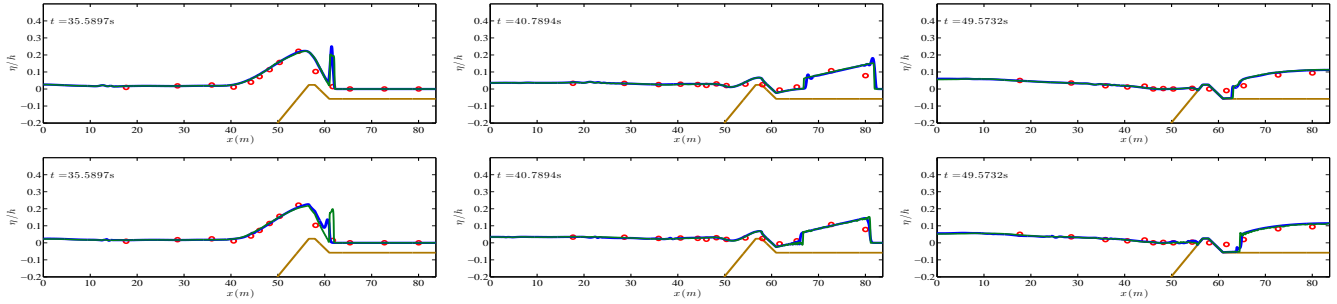


Figure 29: Overtopping of a 2D reef. Bore formation and propagation behind the reef. Top row: GN model. Bottom row: Nwogu model. Blue lines: TKE breaking closure. Green Lines: hybrid wave breaking closure. Left: $t = 35.5897s$. Middle: $t = 40.7894s$. Right: $t = 49.5732s$.

reef_wp2

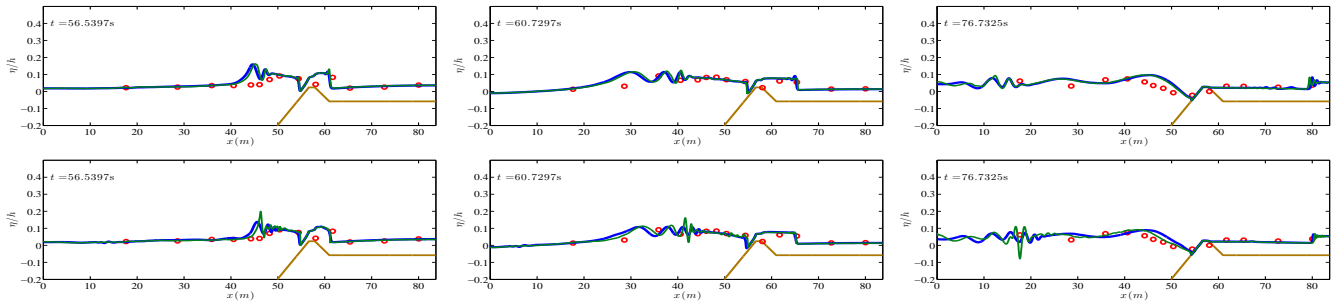


Figure 30: Overtopping of a 2D reef. Second overtopping and undular bore formation. Top row: GN model. Bottom row: Nwogu model. Blue lines: TKE breaking closure. Green Lines: hybrid wave breaking closure. Left: $t = 56.5397s$. Middle: $t = 60.7297s$. Right: $t = 76.7325s$.

reef_wp3

770 frequency enhanced version of the Green-Naghdi system in the form proposed in [37, 52]. We have com-
 771 pared the now popular hybrid closure initially proposed in [36], with an eddy viscosity closure based on an
 772 adaptation of the turbulent kinetic energy closure model of [1], modified to be consistent with the detection
 773 mechanisms proposed of [12, 37], and also used here. The study performed has involved: a systematic
 774 analysis of the behaviour of the two closures for different mesh sizes; the use of dissipation monitors, con-
 775 sistent with the available theory of entropy dissipation for conservation laws [69, 72], to study the dynamics
 776 of breaking for several cases; thorough evidence of the equivalent capabilities of the two approaches to
 777 provide satisfactory results.

778 Our results indicate that indeed, at least with the (rather standard) implementation proposed here, both
 779 closure approaches allow to describe correctly wave transformation and breaking at large scales. We have
 780 shown that when using the TKE eddy viscosity closure the numerical dissipation plays a negligible role,
 781 which motivates to look for non-dissipative/energy conserving numerical methods in the future. Also, the
 782 results clearly show the reduced sensitivity to the mesh of this approach compared to the hybrid one. The
 783 analysis of the wave breaking of solitary waves on a slope also has allowed to quantitatively study the
 784 interplay of the dissipation introduced by friction, eddy viscosity, and numerical dissipation.

785 Of course, one has to keep in mind that the computational cost required by the TKE closure is higher
 786 than the one of the hybrid closure. We judge this overhead justified by the increased robustness.

787
 788 This preliminary study would benefit from further investigation using both improved numerics (e.g. en-

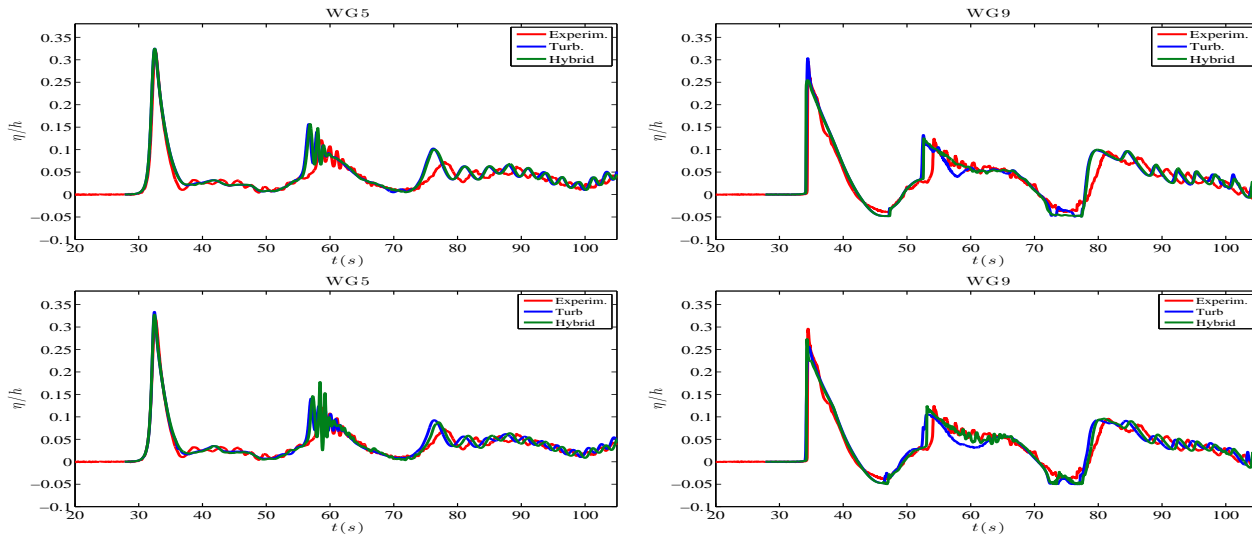


Figure 31: Overtopping of a 2D reef. Free surface time series in wave gauges WG5 (left) and WG9 (right). Top row: GN model. Bottom row: Nwogu model. Blue lines: TKE breaking closure. Green Lines: hybrid wave breaking closure.

reef_wg1

789 ergy preserving approximations in the propagation region), as well as improved models for both the propa-
 790 gation and for the breaking. The models considered at the moment present a dependence on the parameters
 791 of the detection criteria, as well as on the coefficients of the TKE equation. Improved models, including the
 792 effects of vertical variations of the flow in both the propagation and breaking may be considered in future
 793 studies (see e.g. [27,28,35]). The multi-dimensional case will also have to be studied with attention. In this
 794 case more complex effects may come into the picture, related to the interaction with transversal variations
 795 of the bathymetry (see e.g. [102]). These effects, and their interaction with the breaking closure will have
 796 to be assessed systematically.

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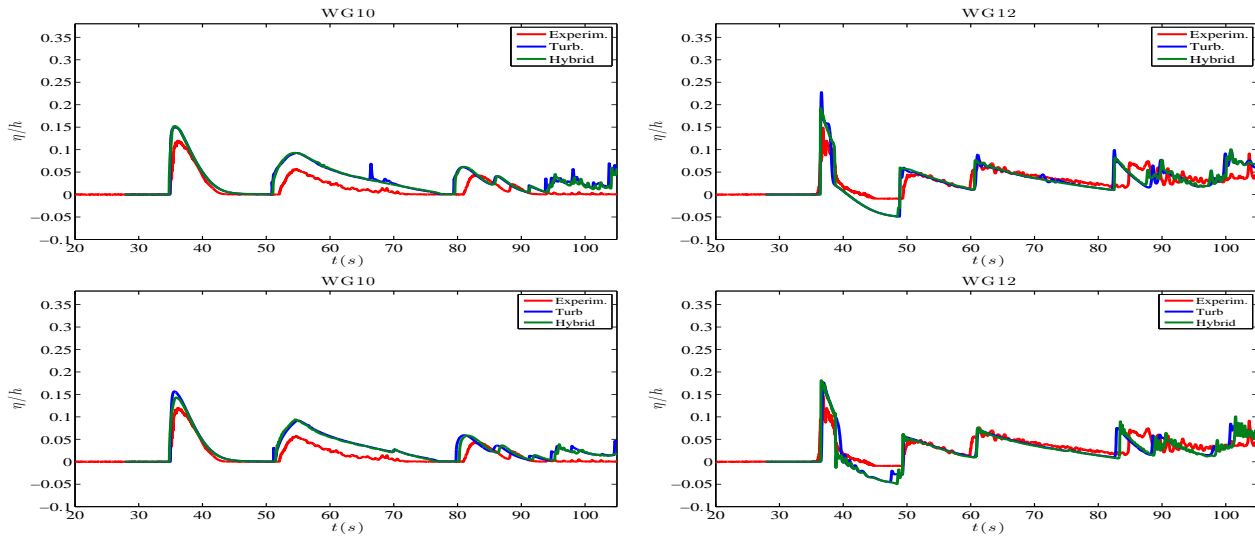


Figure 32: Overtopping of a 2D reef. Free surface time series in wave gauges WG10 (left) and WG12 (right). Top row: GN model. Bottom row: Nwogu model. Blue lines: TKE breaking closure. Green Lines: hybrid wave breaking closure.

reef_wg2

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