THE LARGEST BOND IN 3-CONNECTED GRAPHS

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Abstract

A graph G is connected if given any two vertices, there is a path between them. A bond B is a minimal edge set in G such that $G - B$ has more components than G. We say that a connected graph is dual Hamiltonian if its largest bond has size $|E(G)|-|V(G)|+2$. In this thesis we verify the conjecture that any simple 3-connected graph G has a largest bond with size at least $\Omega(n^{log_3 2})$ (Ding, Dziobiak, Wu, 2015 [3]) for a variety of graph classes including planar graphs, complete graphs, ladders, Möbius ladders and circular ladders, complete bipartite graphs, some unique $(3, g)$-cages, the generalized Petersen graph, and some small hypercubes. We will also go further to prove that a variety of these graph classes not only satisfy the conjecture, but are also dual Hamiltonian.

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Chapter 1

Introduction and Some Background on Graph Theory

1.1 Introduction

Graph Theory is a relatively young area of study in mathematics. The first documented graph theory problem was the Königsberg Bridge problem proposed by Leonhard Euler in 1736, long before the term "Graph Theory" was coined. This problem involved seven bridges connecting two islands to the main city of Königsberg in Prussia, and the question was to devise a path which crossed each bridge only once. Ultimately, it was proven impossible to obtain such a path. It wasn't until 1936 that the first textbook on graph theory was published.

There are many practical applications for graph theory, typically involving relational modeling in areas such as biology, business and computer science. Graphs have many real-world applications such as modeling computer network systems and creating mappings for airline routes.

In this thesis, we will specifically study bond sizes in graphs.

A bond is a minimal edge-cut (a minimal set of edges whose deletion disconnects a connected graph). We know an upper bound for bond size in a graph is $|E(G)| - |V(G)| + 2$, where $|E(G)|$ is the number of edges in the graph and $|V(G)|$ is the number of vertices in the graph. Additionally, we know that a 3-connected graph will have a bond of size at least $\frac{2}{17}\sqrt{logn}$ [3]; however, in 2015, Ding, Dziobiak, and Wu [3] raised the following conjecture:

Conjecture (Ding, Dziobiak, Wu, 2015 [3])**.** *Any simple 3-connected graph G will have a largest bond with size at least $\Omega(n^{log_3 2})$ where $n = |V(G)|$.*

We say that $f(n) = \Omega(g(n))$ if and only if there exists some constant $M > 0$ and some $N \in \mathbb{N}$, such that $f(n) \geq Mg(n)$ for all $n \geq N$.

In this thesis, we will verify this conjecture for a variety of graph classes including planar graphs, complete graphs, ladders, Möbius ladders and circular ladders, complete bipartite graphs, the generalized Petersen graph, a few unique $(3, g)$-cages and some small hypercubes.

Furthermore, we will prove that a variety of these graph classes possess bonds which meet the maximum bound for bond size. These graphs are called *dual Hamiltonian*. We will show that graphs that meet this upper bound for bond size will verify our conjecture on the lower bound.

1.2 What is a graph?

A *simple graph* G with n vertices and m edges consists of a *vertex set* $V(G) = \{v_1, v_2, ..., v_n\}$ and an *edge set* $E(G) = \{e_1, e_2, ..., e_m\}$, where each edge is an unordered pair of vertices from $V(G)$. We say that $G = (V(G), E(G))$. A simple graph has no multiple edges or loops. A graph in which edges are allowed to repeat is called a *multigraph*. Unless otherwise stated, the graphs in this thesis are all simple graphs.

The cardinality of a set S is the number of elements in the set, and is denoted $|S|$. Subsequently, the number of vertices in a graph (also called the *order* of the graph) can be denoted $|V(G)|$ and the number of edges, $|E(G)|$. Most (but not all) graphs have many visual representations, often called *embeddings*.

To illustrate the above definitions, let's use the following graph:

$$G = \{\{A, B, C, D, E\}, \{\{A, B\}, \{B, C\}, \{C, D\}, \{C, E\}, \{D, E\}, \{D, A\}\}\}$$

The edge set of G, $E(G)$ is $\{\{A, B\}, \{B, C\}, \{C, D\}, \{C, E\}, \{D, E\}, \{D, A\}\}$.
The vertex set of G, $V(G)$ is $\{A, B, C, D, E\}$.
$|V(G)| = 5$, $|E(G)| = 6$
Two possible embeddings of G are illustrated in Figure 1:

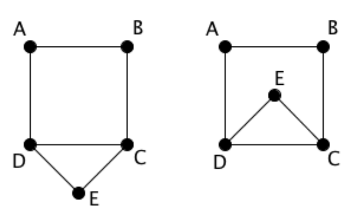

Figure 1: Example of a Graph

We may abbreviate the edge $\{a, b\}$ as ab. We say that a and b are called *neighbors* when they are joined by an edge. Any two neighbors are said to be *adjacent*. When a and b are neighbors, we say that a and b are each *incident* to the edge ab. Subsequently, the *neighborhood* of a vertex v, denoted $N(v)$ is the set of all neighbors of v. Using the example from Figure 1, we can see that $N(A) = \{B, D\}$. The size of the neighborhood of a vertex, $|N(v)|$, is called the *degree* of v, frequently denoted $d(v)$. A graph in which all vertices have degree r is called r-regular. Many of the graphs we will discuss are *cubic*, meaning they are 3-regular.

The number of edges in a graph can be determined from the number of vertices and their degree by the following formula:

The Handshake Lemma. $2|E(G)| = \sum_{v \in V(G)} d(v)$.

1.3 Cycles and Subgraphs

A *walk* is an alternating sequence of vertices and incident edges. In other words, a walk is a route that can be traveled within a graph from vertex to vertex along edges. A *trail* is a walk in which no edges are repeated, and a *path* is a walk in which no vertices are repeated. A walk is said to be *closed* if it begins and ends with the same vertex. A closed walk that repeats no vertices or edges is called a *cycle*.

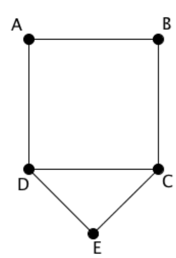

Figure 2: A Graph Containing Cycles

In Figure 2, $\{A, B, C, D, A\}$ is an example of a cycle. It follows a path from vertex A to B, C, D and back to A to complete the cycle. Another example of a cycle is $\{D, C, E, D\}$. We call the size of the largest cycle in a graph the *circumference* of the graph, denoted $c(G)$, and the size of the smallest cycle in a graph the *girth*.

A cycle on a graph G is called *Hamiltonian* if it contains every vertex from $V(G)$.

A *subgraph* of a graph G is a graph whose vertex set is a subset of $V(G)$ and whose edge set is a subset of $E(G)$. We call a subgraph *induced* if it contains all edges which have both ends in the vertex set of the subgraph. The subgraph induced by the set S is denoted $G[S]$.

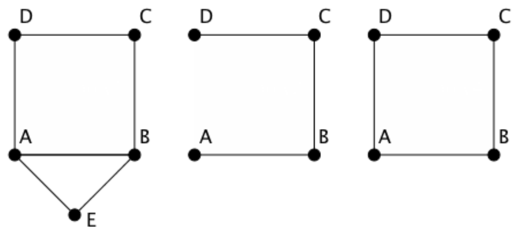

Figure 3: Subgraphs

In Figure 3, both the graphs in the center and on the right are subgraphs of the leftmost graph. However, only the subgraph on the right is induced, as the edge AD has both ends in the vertex set of the subgraphs and thus, is required in the central subgraph in order for it to be induced.

1.4 Graph Connectivity

A graph is called *connected* if given any two vertices, there exists a path between them.

The *edge connectivity* of a graph G, denoted $\kappa'(G)$, is the minimum size of an edge set F, such that $G - F$ is disconnected, where $G - F$ is the graph G with all edges included in F removed. We call F an *edge-cut*. An example is illustrated below.

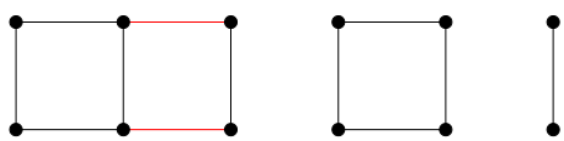

Figure 4: An Example of an Edge-Cut

In Figure 4, let the edges in red represent an edge-cut F. The graph G on the left has only one component, but $G - F$ on the right has two. One component is a 4-cycle and the other is a path consisting of two vertices. Components can also be isolated vertices. We call an edge e a *cut edge* if $G - e$ has more components than G.

Similarly, a vertex v is called a *cut vertex* if $G - v$ has more components than G. When we remove an edge from a graph, we simply remove it and leave behind any incident vertices. However, when removing a vertex from a graph, all incident edges must be removed along with it.

The *vertex connectivity* of a connected graph G, denoted $\kappa(G)$, is the minimum size of a vertex set S such that $G - S$ is disconnected or has only one vertex. We call S

a *cut set*. We say that G is k-connected where $\kappa(G) \geq k$. Thus, a k-connected graph is also t-connected for $t \leq k$. We can also specify certain cuts as (u, v)-cuts, which is to say that such a cut removes any possible paths between vertices u and v in the graph. An important property of graph connectivity is given by *Menger's Theorem*.

Theorem 1 (Menger, 1927 [4])**.** *The minimum size of a (u, v)-cut is equal to the maximum number of pairwise disjoint paths from u to v.*

A *bond* is a non-empty minimal edge-cut. Figure 5 is a connected graph with an edge-cut F illustrated in red. F is also a bond as if we remove any edge from F, $G - F$ is still connected. If we include the edge e, we create the edge-cut $F \cup e$; however, e is not required to disconnect the graph, as F does this on its own. Thus, $F \cup e$ is not a *minimal* edge-cut, and subsequently is not an example of a bond.

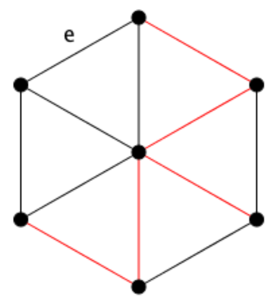

Figure 5: An Example of a Bond

1.5 Trees

A *tree* is an acyclic, connected graph. Any nontrivial tree will contain at least two vertices of degree 1, called *leaves*. A tree with only two leaves is a path. Trees are minimally connected, as they include only one path between any two vertices in the graph. A tree with n vertices has $n - 1$ edges.

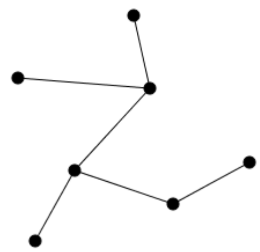

Figure 6: An Example of a Tree

1.6 Planar Graphs

One special class of graphs is the planar graph. A graph is *planar* if it can be embedded in the 2-dimensional plane in such a way that no edges cross. A few examples are illustrated in Figure 7:

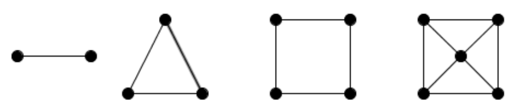

Figure 7: A Few Planar Graphs

The following are a few examples of non-planar graphs, or graphs for which all embeddings contain edge crossings:

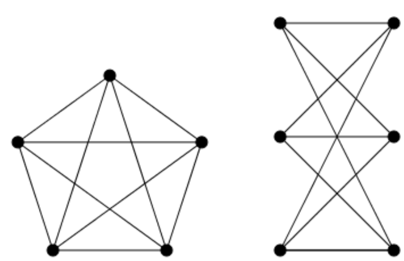

Figure 8: Non-Planar Graphs

Planar graphs can be drawn in a way where edges do cross. In this case, we call the graph drawn without edge crossings a *plane graph*. Such is the case in the Figure 9. Both illustrations represent the same planar graph, but one drawing is a plane graph, while the other is not.

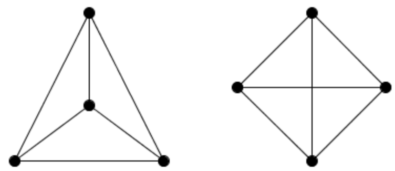

Figure 9: Two Possible Drawings of a Planar Graph

An important formula for planar graphs is *Euler's Formula*.

Euler's Formula. *If G is a plane graph, then $n(G) - e(G) + f(G) = 2$.*

From this point forward, we will assume that $n(G) = |V(G)|$, $e(G) = |E(G)|$ and $f(G)$ is the number of faces in G. A *face* is a maximal region of the plane that contains no point used in the embedding of the graph. Every plane graph contains one unbounded face.

1.6.1 The Planar Dual Graph

An important property of plane graphs is that their *dual graph* can be drawn in the 2-dimensional plane. The dual graph of a plane graph G, denoted G^*, is obtained by placing a vertex in each face of G, and placing an edge anywhere two faces of G are separated by an edge of G. A simple example is illustrated below. The graph on the right has the dual graph illustrated in red.

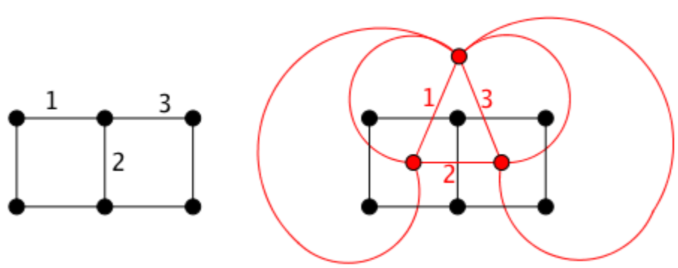

Figure 10: A Graph and its Dual Graph

A bond in G corresponds to a cycle in G^*, as illustrated by the edges labeled $1, 2, 3$. On the left, we see the bond $\{1, 2, 3\}$ in the original graph. The corresponding edges in the dual graph create the cycle $\{1, 2, 3\}$. Remember that we call the largest cycle in a graph its circumference, $c(G)$. A largest cycle in the dual graph $c(G^*)$ will be equal in size to the largest bond in the original graph, denoted $c^*(G)$.

If G is a plane graph then $n(G^*) = f(G)$, $e(G^*) = e(G)$, $f(G^*) = n(G)$.

The original graph in Figure 10 is not 3-connected, as the two vertices incident to edge 2 create a cut set of size 2. As a result, the dual graph is a multigraph. However, the dual graph of a simple 3-connected plane graph will also be a simple 3-connected plane graph.

1.7 Dual Hamiltonian Graphs

In Figure 10, the original graph G has bonds of size 2 and 3. We know that $\{1, 2, 3\}$ is a bond of size 3. One example of a bond of size two is the two edges incident to the top right vertex. The largest bond has size 3. Exactly how large can a bond be?

Proposition 1. *Let G be a connected graph. Then every bond in G has size $\leq |E(G)| - |V(G)| + 2$.*

Proof. Let G be a connected graph which contains a bond B. The removal of B will separate G into two components, H_1 and H_2. B will have size $|E(G)| - |E(H_1)| - |E(H_2)|$. Let n_1 be the number of vertices in H_1 and n_2 be the number of vertices in H_2. For H_1 and H_2 to be connected, they must contain at least $n_1 - 1$ and $n_2 - 1$ edges, respectively. Thus $|B| \leq |E(G)| - (n_1 - 1) - (n_2 - 1) = |E(G)| - (n_1 + n_2) + 2$. Since all of the vertices in G are contained in either H_1 or H_2, $n_1 + n_2 = |V(G)|$. Then $|B| \leq |E(G)| - |V(G)| + 2$. □

A graph G is called *dual Hamiltonian* if it contains a bond of size $|E(G)| - |V(G)| + 2$.

As previously discussed, a bond in a plane graph G corresponds to a cycle in G^*. In our previous example from Figure 10, the bond $\{1, 2, 3\}$ had size $|E(G)| - |V(G)| + 2$. The corresponding cycle in the dual graph is a Hamiltonian cycle. Thus, it makes sense that a graph whose largest bond attains this bound would be called dual Hamiltonian.

Chapter 2

The Main Problem

2.1 The Conjecture

There are many known results about a multitude of aspects of graph connectivity; however, there are very few results about the largest bond size in a general graph. We know that an upper bound on bond size for a connected graph is $|E(G)|-|V(G)|+2$. In 2015, Ding, Dziobiak, and Wu [3] raised the following conjecture about the lower bound on the largest bond size in a 3-connected graph:

Conjecture 1 (Ding, Dziobiak, Wu, 2015 [3]). *Any simple 3-connected graph G will have a largest bond with size at least $\Omega(n^{log_3 2})$ where $n = |V(G)|$.*

In this thesis, we will verify this conjecture for a variety of graph classes including planar graphs, complete graphs, ladders, Möbius ladders and circular ladders, complete bipartite graphs, the generalized Petersen graph, some unique $(3, g)$-cages and some small hypercubes.

2.2 Some Known Results

2.2.1 The Lower Bound on Bond Size in a 3-connected Graph

In 2015, Ding, Dziobiak and Wu [3] proved the following lower bound for 3-connected graphs:

Theorem 2 (Ding, Dziobiak, Wu, 2015 [3]). *Any simple 3-connected graph G contains a bond of size at least $\frac{2}{17}\sqrt{logn}$.*

2.2.2 The Largest Cycle in a Planar Graph

In a previous section, we discussed planar graphs and their duals. Recall that a the dual graph of a simple 3-connected plane graph is also a simple 3-connected plane graph, and that a cycle in the dual graph will correspond to a bond in the original graph. While the largest bond size of a planar graph has not been proven, we have some useful results about the largest cycle.

Theorem 3 (Chen, Yu 2002 [2])**.** *Let G be a simple 3-connected planar graph on n vertices. Then $c(G) \geq n^{log_3 2}$.*

2.2.3 Results About Dual Hamiltonian Graphs

As previously stated, a dual Hamiltonian graph will possess a bond B which satisfies the condition $|B| = |E(G)| - |V(G)| + 2$.

Proposition 2. *A connected graph is dual Hamiltonian if and only if its vertex set can be partitioned into two subsets whose induced subgraphs are trees.*

Proof. ($\Rightarrow$) Let G be a dual Hamiltonian graph. Then G has a maximum bond B with size $|E(G)|-|V(G)|+2$. Then $G-B$ has exactly two connected components G_1 and G_2 with n_1 and n_2 vertices, respectively. Thus, $|E(G_1)| \geq n_1 - 1$ and $|E(G_2)| \geq n_2 - 1$. $|E(G)| - |V(G)| + 2 = |B| = |E(G)| - |E(G_1)| - |E(G_2)| \geq |E(G)| - (n_1 - 1) - (n_2 - 1) = |E(G)| - (n_1 + n_2) + 2 = |E(G)| - |V(G)| + 2$. Then $|E(G_1)| = n_1 - 1$ and $|E(G_2)| = n_2 - 1$. Thus both G_1 and G_2 are indeed trees.
($\Leftarrow$) Conversely, if both G_1 and G_2 are trees, it is easily seen from the above proof that the corresponding bond has size $|E(G)| - |V(G)| + 2$. $\square$

In addition, Aldred et al. observed the following:

Proposition 3 (Aldred, Van Dyck, Brinkmann, Fack, McKay 2008 [1])**.** *If a dual Hamiltonian graph is regular with degree ≥ 3, the partition of the vertex set from the largest bond will result in two sets of equal size.*

Chapter 3

Main Results

3.1 Introduction

In this section we will verify Conjecture 1 for a number of graph classes. The graphs discussed will include planar graphs, complete graphs, ladders, Möbius ladders and circular ladders, complete bipartite graphs, the generalized Petersen graph, a few unique $(3, g)$-cages, and some small hypercubes. We will also prove that many of these graphs are dual Hamiltonian.

3.2 Planar Graphs

As previously discussed, a planar graph has an embedding in the plane with no edge crossings.

Recall Theorem 3 from Section 2.2.2. We will use this theorem to verify Conjecture 1 for planar graphs as follows:

Theorem 4. *Let G be a 3-connected planar graph. Then $c^*(G) = \Omega(n^{log_3 2})$.*

Proof. We may assume that G is a 3-connected plane graph. Let G^* be the dual graph of G. We know that $n(G^*) = f(G)$ and, by Euler's Formula, $n(G) - e(G) + f(G) = 2$. Note that $2|E(G)| = \sum_{v \in V(G)} d(v) \geq 3[n(G)]$ as G is 3-connected. So $e(G) \geq \frac{3}{2}n(G)$. Then $f(G) = e(G) - n(G) + 2 \geq \frac{3}{2}n(G) - n(G) + 2 > \frac{1}{2}n(G)$. Hence, by Theorem 3, $c^*(G) = c(G^*) \geq [n(G^*)]^{log_3 2} = [f(G)]^{log_3 2} > [\frac{1}{2}n(G)]^{log_3 2} = (\frac{1}{2})^{log_3 2}[n(G)]^{log_3 2}$. Thus $c^*(G) = \Omega(n^{log_3 2})$. $\square$

3.2.1 Complete Graphs

A complete graph is one where each vertex is adjacent to every other vertex in the graph. A complete graph on n vertices is denoted K_n. Subsequently, complete graphs on n vertices are $(n-1)$-regular. A few examples are illustrated below.

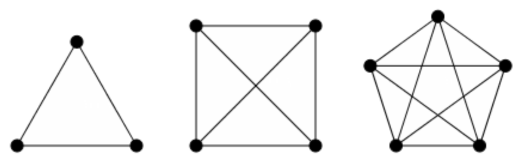

Figure 11: K_3, K_4, K_5

Theorem 5. *The largest bond in K_n has size $\lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil$*

Proof. Deleting any bond from a graph leaves two connected components. Since each vertex in a complete graph is adjacent to every other vertex, the size of a bond will be the product of the sizes of the two components it creates. The largest possible product results from partitioning the vertex set into $\{X, Y\}$ with $|X| - |Y|$ being minimum. These sets will have size $\lfloor \frac{n}{2} \rfloor$ and $\lceil \frac{n}{2} \rceil$. $\square$

Clearly, this bond size confirms the conjecture.

3.3 Dual Hamiltonian Graphs

As previously discussed, a dual Hamiltonian graph is one whose largest bond meets the upper bound on bond size, $|E(G)| - |V(G)| + 2$. The existence of such a bond can be proven by showing that the vertices of a graph can be partitioned into two subsets, each of which induces a tree. A 3-connected graph which is dual Hamiltonian clearly confirms the conjecture that the largest bond size is at least $\Omega(n^{log_3 2})$ as $|E(G)| - |V(G)| + 2 \geq \frac{3}{2}|V(G)| - |V(G)| + 2 = \frac{1}{2}|V(G)| + 2 > \frac{1}{2}|V(G)|$. We will explore a few different graph classes for which dual Hamiltonicity can be proven.

3.3.1 Ladders, Möbius Ladders, and Circular Ladders

A *ladder* is a graph obtained from two paths of length n, and edges joining corresponding vertices in the paths. These edges are called *rungs*.

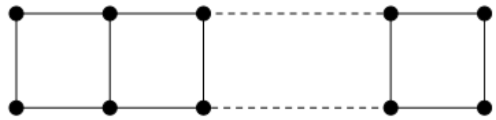

Figure 12: A Ladder Graph

Theorem 6. *Any ladder graph is dual Hamiltonian.*

Proof. Let G be a ladder graph with $2n$ vertices. G will have $3n-2$ edges. We can partition the vertices of G into two sets, each of which induces a path of length n. Then the rungs of G will create a bond of size $n = (3n-2) - 2n + 2 = |E(G)| - |V(G)| + 2$. Thus, the ladder is dual Hamiltonian. □

Figure 13 illustrates a maximum bond on the ladder graph, shown in blue.

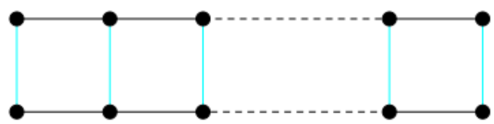

Figure 13: A Maximum Bond in a Ladder Graph

A *Möbius ladder* is a graph obtained from an n-cycle where n is even, and each opposite vertex is joined a rung.

The Möbius ladder can also be illustrated with a ladder like structure, where the opposite vertices on each end of the ladder are adjacent. Figure 14 illustrates this embedding.

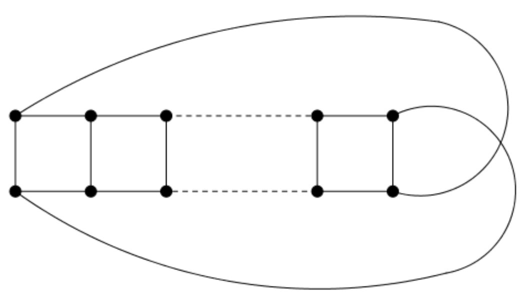

Figure 14: The Möbius Ladder

Theorem 7. *The Möbius ladder is dual Hamiltonian.*

Proof. Let G be a Möbius ladder with $2n$ vertices. We know that the maximum possible bond size for any graph is $|E(G)|-|V(G)|+2$. Since the Möbius Ladder is cubic, it will have $\frac{3}{2}(2n) = 3n$ edges. Thus the largest possible bond in a Möbius Ladder will have size $3n - 2n + 2 = n + 2$. In any Möbius Ladder, a bond can be formed by removing each of the rungs (of which there are n) and then removing any two additional non-incident edges, such as those connecting the vertices at each end of the ladder. Since this always provides a bond of size $|E(G)| - |V(G)| + 2 = 3n - 2n + 2 = n + 2$, the Möbius ladder is dual Hamiltonian. □

Such a bond is illustrated in blue in Figure 15.

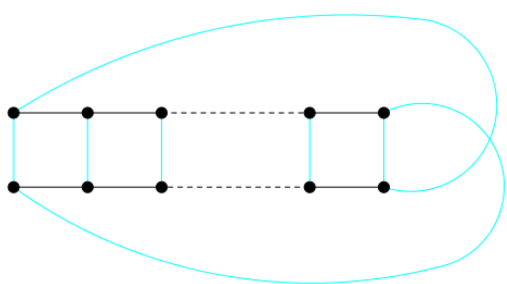

Figure 15: A Maximum Bond in the Möbius Ladder

The *circular ladder graph* is a graph obtained from two n-cycles where corresponding vertices from each cycle are made adjacent by a rung. Such a graph is sometimes referred to as a *prism graph*.

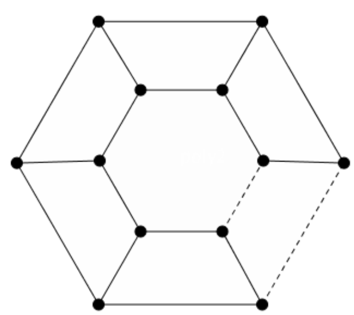

Figure 16: A Circular Ladder

Theorem 8. *The circular ladder is dual Hamiltonian.*

Proof. Let G be a circular ladder graph with $2n$ vertices where $n \geq 3$. G will have $3n$ edges. We can partition of the vertices of G into two sets, each of which induces a cycle of length n. We can then ensure that the induced subgraphs of this partition are trees by swapping the first vertex of one cycle with the last vertex of the other. The resulting bond will include 2 edges from each of the original cycles and all but 2 rungs. This bond will have size $4+(n-2) = n+2 = 3n-2n+2 = |E(G)|-|V(G)|+2$. Thus, the circular ladder is dual Hamiltonian. □

Figure 17 illustrates the proof of Theorem 8. The induced graphs of the new vertex sets are shown in red and black and the bond is shown in blue.

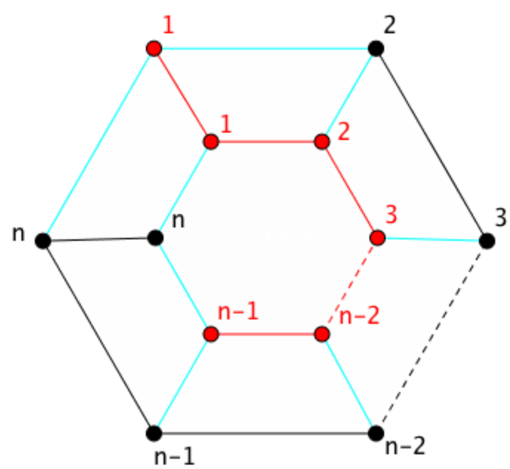

Figure 17: A Maximum Bond in a Circular Ladder

3.3.2 The Complete Bipartite Graph

A *bipartite graph* is a graph whose vertices can be divided into two disjoint sets (S, T) with s and t vertices, respectively, such that no vertex is adjacent to another vertex within the same set. The *complete bipartite graph*, denoted $K_{s,t}$, is a bipartite graph in which each vertex from the first set is adjacent to every vertex in the second set. In Figure 18, the graph on the left is bipartite, while the graph on the right is a complete bipartite graph.

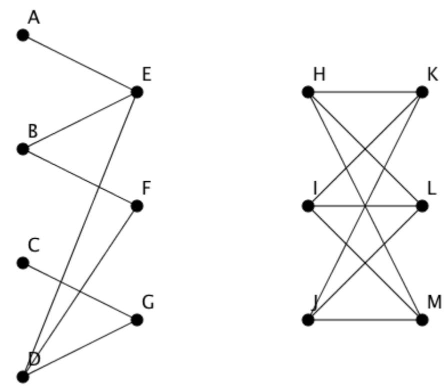

Figure 18: Examples of Bipartite Graphs

Theorem 9. *Any Complete Bipartite Graph is Dual Hamiltonian.*

Proof. Let G be a complete bipartite graph $K_{s,t}$ with vertex partitions S and T. Let $|S| = s$ and $|T| = t$ so that $s + t = |V|$. Select one vertex a from S. The vertex a will form an induced tree when joined with all but one vertex b in T. Moreover, b will then form an induced tree when joined with every vertex in S, except a. The first tree contains $t - 1$ edges, and the second contains $s - 1$. There are then $|E(G)| - (s - 1) - (t - 1) = |E(G)| - (s + t) + 2 = |E(G)| - |V(G)| + 2$ edges in the resulting bond. Thus $K_{s,t}$ is dual Hamiltonian. □

The proof of Theorem 9 is illustrated in Figure 19. The two trees are shown in red and black and the bond in blue.

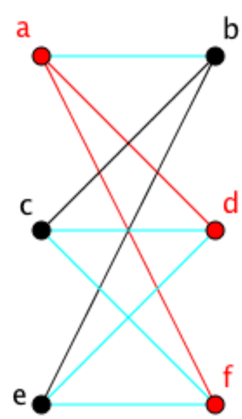

Figure 19: A Maximum Bond in $K_{3,3}$

3.3.3 The Generalized Petersen Graph

The graph below is *the Petersen graph*.

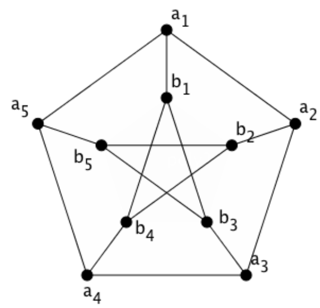

Figure 20: The Petersen Graph

The generalized Petersen graphs are a class of graphs which follow a similar design. The generalized Petersen graph, denoted $P(n,k)$ for $n \geq 3$ and $1 \leq k < \frac{n}{2}$, is defined as follows:

- The vertex set is $\{a_1, a_2, ..., a_n, b_1, b_2, ..., b_n\}$.
- The edge set is composed of $\{a_i a_{i+1}, a_i b_i, b_i b_{i+k}\colon\ i = 1, ..., n\}$ where subscripts are read modulo n.

The generalized Petersen graphs satisfy the following conditions:

- The graph has $2n$ total vertices.
- The graph has $3n$ total edges.
- The graph is 3-regular.

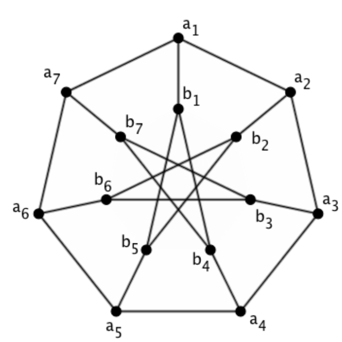

Figure 21: $P(7,3)$

As previously stated, a graph is dual Hamiltonian if its vertex set can be partitioned into two sets whose induced subgraphs are trees. In addition, if the graph is regular with degree $r \geq 3$, then these two parts must be equal in size.

Since the generalized Petersen graph is 3-regular, it will be dual Hamiltonian if its vertex set can be separated into two parts of equal size whose induced subgraphs are trees. Since $P(n, k)$ has $2n$ total vertices, each of these parts must contain n vertices.

Theorem 10. *$P(n, k)$ is dual Hamiltonian and has at least n dual Hamiltonian cycles.*

Proof. Let $V(P(n, k)) = \{a_1, a_2, ..., a_n\} \cup \{b_1, b_2, ..., b_n\}$ where $a_i \sim b_i$, $a_i \sim a_{i+1}$, $b_i \sim b_{i+k}$ where index is read mod $n, 1 \leq k < \frac{n}{2}$.
Let $L = \{a_1, a_2, ...a_{n-k}, b_1, b_2, ..., b_k\}$, $M = \{a_{n-k+1}, ..., a_n, b_{k+1}, ..., b_n\}$, $\overline{B} = \{b_{k+1}, b_{k+2}, ..., b_n\}$, $B = \{b_1, b_2, ..., b_n\}$. Clearly, $G[L]$ is a tree. We need only show that $G[M]$ is also a tree.
In $P(n, k)$, we show that

(i) $G[M]$ is connected, and

(ii) $G[M]$ is acyclic.

First, we show that $G[M]$ is connected. We need only show that for any $i, k + 1 \leq i \leq n - k$, there is a path between b_i to $\overline{B_1} = \{b_{n-k+1}, b_{n-k+2}, ..., b_n\}$. $\overline{B_1} \subseteq \overline{B}$ as $k < \frac{n}{2}$. By the definition of $P(n, k)$, $b_i \sim b_{i+k}, b_{i+k} \sim b_{i+2k}$, etc. We assume that $i + tk$ is the greatest positive integer $\leq n - k$ for $t \geq 0$. Then, as $|\overline{B_1}| = k$, we conclude that $n - k + 1 \leq i + (t + 1)k \leq n$. Therefore, there is a path from b_i to $b_{i+(t+1)k}$ which is in $\overline{B_1}$. Thus, we conclude that $G[M]$ is connected.

Next we show that $G[M]$ has no cycles.

(a) We show that $G[B]$ consists of the union of some vertex disjoint cycles and each such cycle meets $\{b_1, b_2, ..., b_k\}$. In fact, as each vertex in $G[B]$ clearly has degree two, $G[B]$ is a vertex disjoint union of cycles. Now we show that each cycle meets $\{b_1, b_2, ..., b_k\}$. Let C be such a cycle and $b_t \in V(C)$. We may assume that $t > k$. Then by the proof of (i), b_t is connected by a path to some vertex b_i where $n - k + 1 \leq i \leq n$. Clearly, b_i is adjacent to some vertex in $\{b_1, b_2, ..., b_k\}$. Thus, (a) holds.

(b) If both b_i, b_j are in a cycle C of $G[B]$ where $n - k + 1 \leq i \neq j \leq n$, then any path in C connecting b_i and b_j contains some b_s where $1 \leq s \leq k$. We may assume that $i < j$. There are two paths in C connecting b_i and b_j. In the b_i, b_j-path, the neighbor of b_i must be in $\{b_1, b_2, ..., b_k\}$ as $n - k + 1 \leq i \leq n$; while in the b_j, b_i-path, the neighbor of b_j must be in $\{b_1, b_2, ..., b_k\}$ as $n - k + 1 \leq j \leq n$. Thus (b) is true.

By (a) and (b), $G[\overline{B}]$ consists of vertex disjoint paths as all of $\{b_1, b_2, ..., b_k\}$ are removed from B. Moreover, as each of b_i $(n - k + 1 \leq i \leq n)$ is adjacent to some vertex in $\{b_1, b_2, ..., b_k\}$, b_i has degree one in $G[\overline{B}]$ and there is no b_i, b_j-path between any b_i and b_j for $n-k+1 \leq i \neq j \leq n$ by (b). We conclude that $G[M]$ is acyclic, and thus is a tree by (i).

Therefore, $P(n, k)$ is dual Hamiltonian. By symmetry, $P(n, k)$ has at least n such dual Hamiltonian cycles. □

The proof above is demonstrated in Figure 22 on the Petersen graph ($P(5, 2)$), and in Figure 23 on $P(7, 3)$. L is shown in red and M is shown in black. The bond is shown in blue.

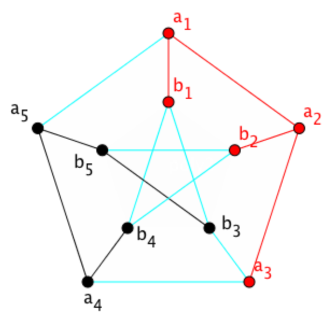

Figure 22: A Maximum Bond in the Petersen Graph ($P(5, 2)$)

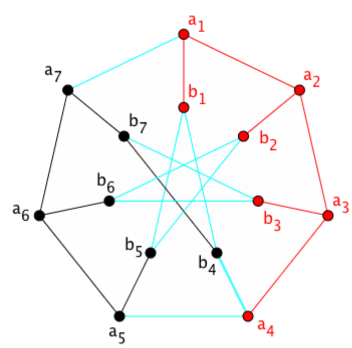

Figure 23: A Maximum Bond in $P(7, 3)$

3.3.4 Unique $(3, g)$-Cages

Given fixed values of r and g, a cage graph is an r-regular graph with the minimum possible number of vertices to obtain a graph with girth g. Such a graph is denoted as an (r, g)-cage. While some cages are known and can be satisfied by many possible graphs, others are known and unique, and many cages are still unknown. The $(r, 3)$-cages are complete graphs on $r+1$ vertices and $(r, 4)$-cages are complete bipartite graphs $K_{r,r}$. Complete graphs and complete bipartite graphs have already been discussed in this thesis. In this section, we will discuss results on a few of the known, unique $(3, g)$-cages. The (3,3)-cage is the complete graph K_4, the (3,4)-cage is the complete bipartite graph $K_{3,3}$. The unique (3,5)-cage is the Petersen graph. As such, we will begin with the (3,6)-cage, also known as the Heawood graph.

Proposition 4. *The (3,6)-cage (Heawood graph), (3,7)-cage (McGee graph), (3,8)-cage (Levi graph or Tutte-Coxeter graph), and the (3,11)-cage (Balaban 11-cage) are all dual Hamiltonian.*

The following figures demonstrate partitions of the vertex sets of these cages whose induced subgraphs are trees, shown in red and black, which give a maximum possible bond (illustrated in blue). The (3,9)-cages and (3,10)-cages were not studied, as they are not unique.

The (3,6)-cage, or the Heawood graph, with a maximum possible bond of size $|E| - |V| + 2 = 21 - 14 + 2 = 9$:

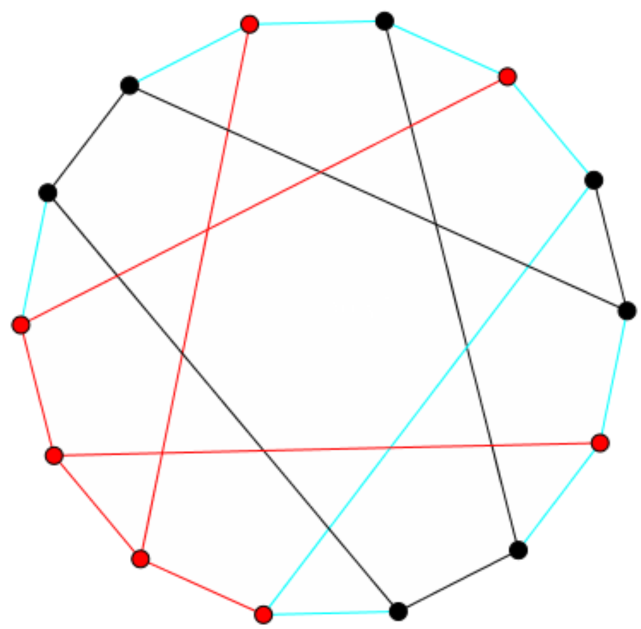

Figure 24: A Maximum Bond on the (3,6)-Cage

The (3,7)-cage, or the McGee graph, with a maximum possible bond of size $|E| - |V| + 2 = 36 - 24 + 2 = 14$:

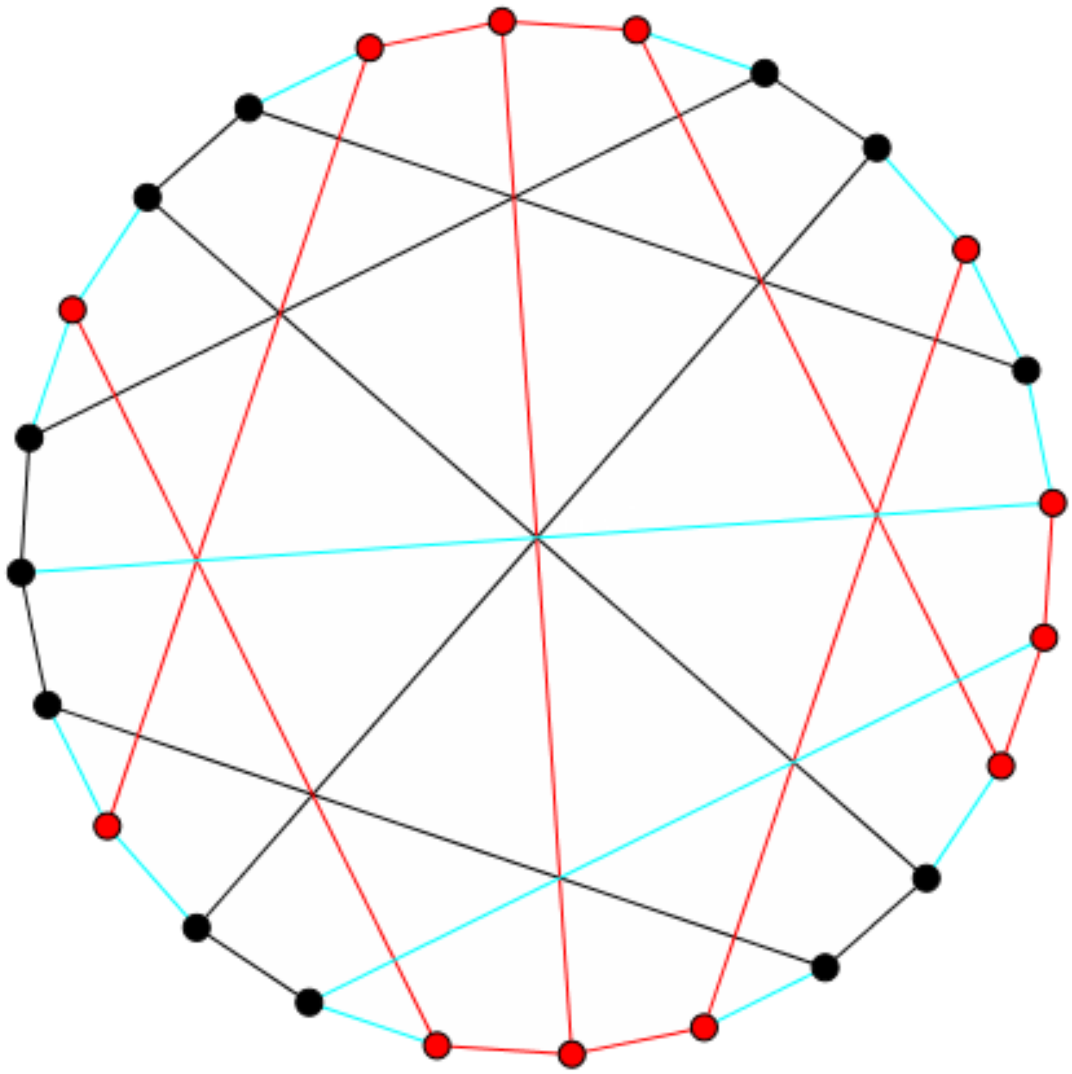

Figure 25: A Maximum Bond on the (3,7)-Cage

The (3,8)-cage, also known as the Tutte-Coxeter graph or the Levi graph, with a maximum possible bond of size $|E| - |V| + 2 = 45 - 30 + 2 = 17$:

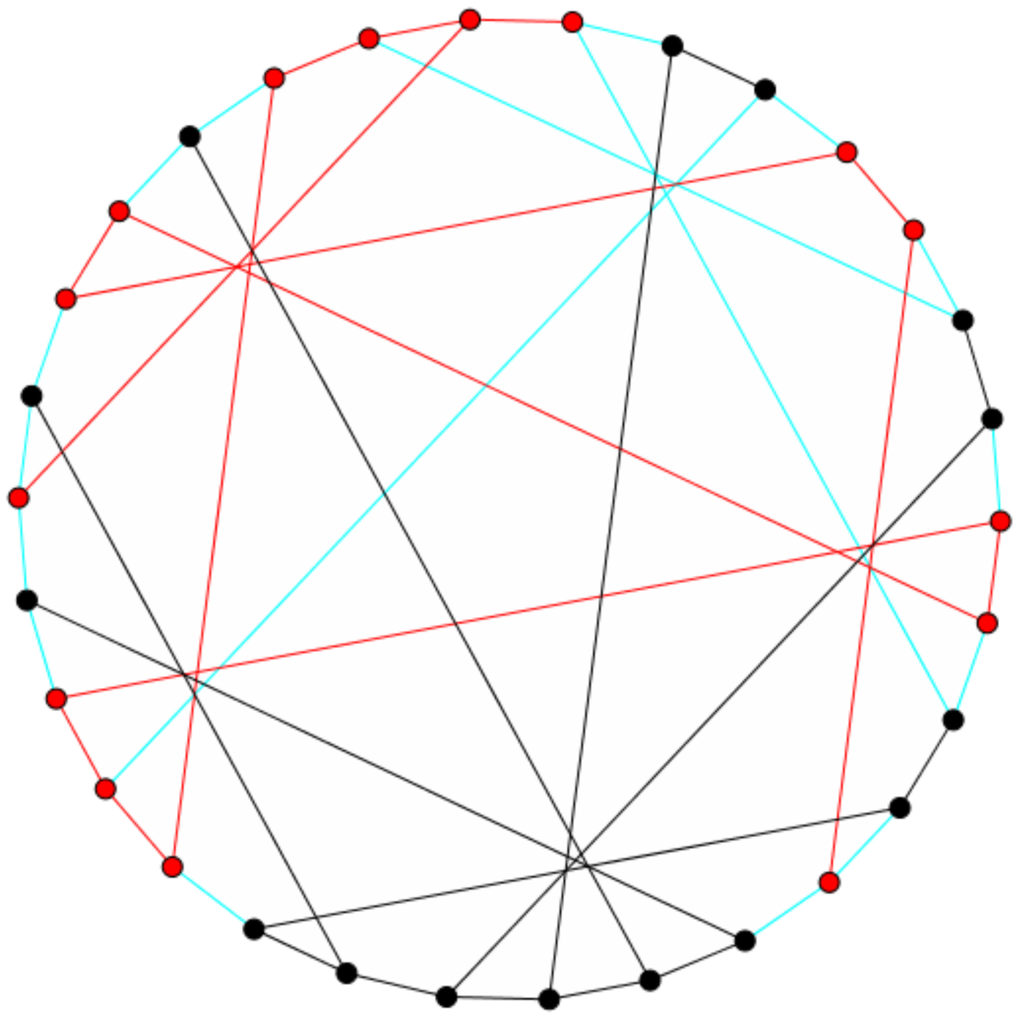

Figure 26: A Maximum Bond on the (3,8)-Cage

The (3,11)-cage, or the Balaban 11-cage, with a maximum possible bond of size $|E| - |V| + 2 = 168 - 112 + 2 = 58$:

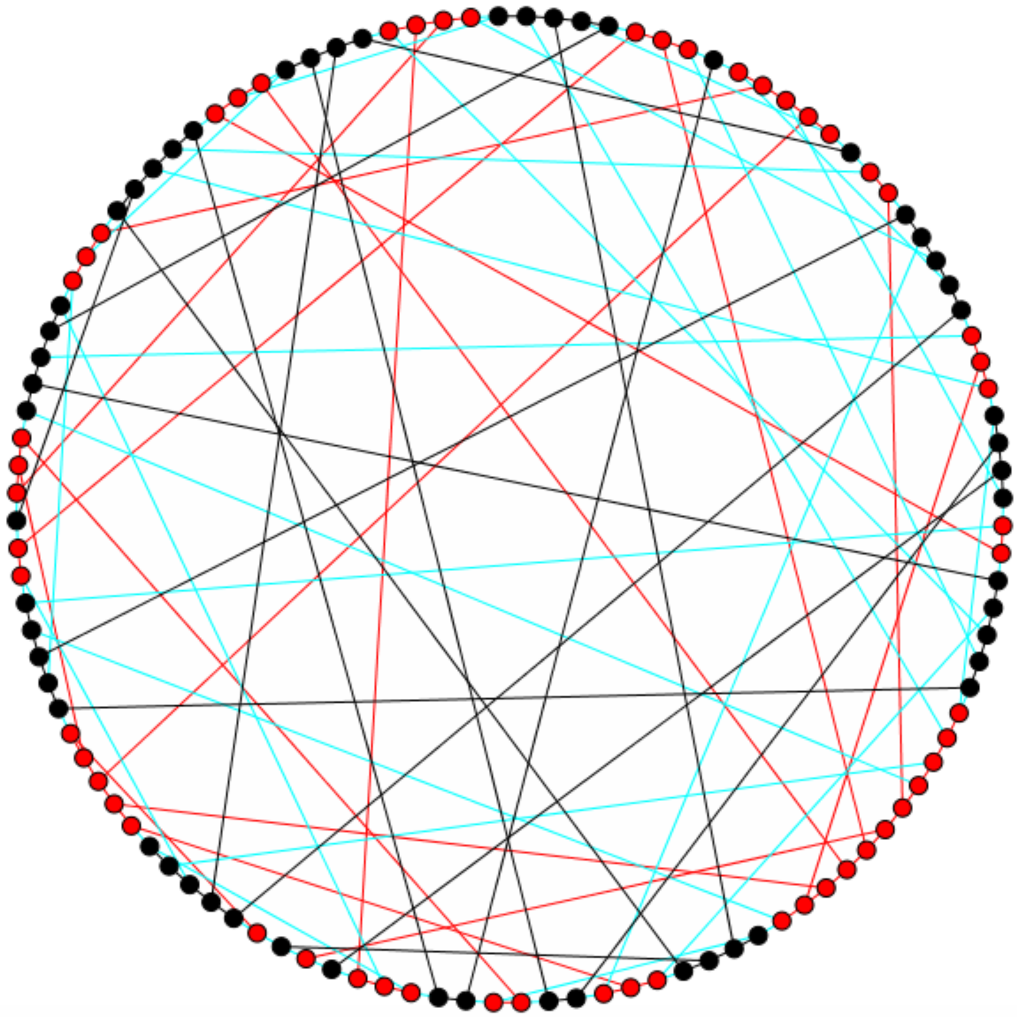

Figure 27: A Maximum Bond on the (3,11)-Cage

3.4 A Conjecture on Hypercubes

A *hypercube*, typically written as an n-cube for $n \geq 2$, is a cube in n dimensions. To illustrate this concept, we will begin with a 2-cube, a cube in two dimensions. Each vertex in the 2-cube is labeled as follows:

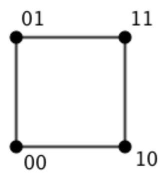

Figure 28: The 2-cube

n-cubes can be created iteratively. To build a 3-cube, we will take two copies of a 2-cube. In the first copy, a "0" will be added to the end of the number sequence label on each vertex. In the second copy, each vertex label will gain a "1". Then corresponding vertices in the first and second copy will be made adjacent. Thus, each vertex is adjacent to any other vertex whose number sequence label differs from its own by only one digit.

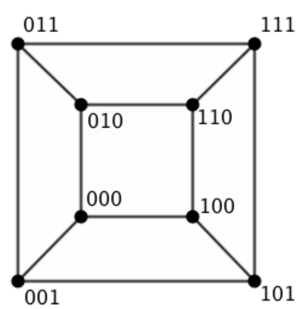

Figure 29: The 3-cube

This same process can be repeated by using two copies of the 3-cube to obtain the 4-cube, two copies of the 4-cube to obtain the 5-cube, and so on such that two copies of the $(n-1)$-cube are used to obtain the n-cube. As dimensions increase, visual representations of n-cubes become increasingly difficult.

Hypercubes are often used in studying computer networks.

As we know from previous sections, a graph is dual Hamiltonian if its vertex set can be partitioned into two sets, each of which induces a tree. If the graph is regular (as is the hypercube) these sets will be equal in size. Using this property, we prove the following results for the n-cube in dimensions 2 through 5.

Proposition 5. *The 2-cube is Dual Hamiltonian.*

Proof. There are a few possible bonds which will satisfy that the 2-cube is Dual Hamiltonian; however, only one bond is necessary to prove this to be true. In the figure below, the bond is illustrated in blue, while the induced subgraphs on the vertex subsets S and T are illustrated in black and red.

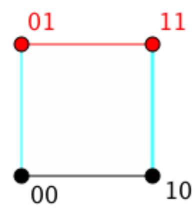

Figure 30: A Maximum Bond in the 2-Cube

$S = \{01, 11\}$
$T = \{00, 10\}$ □

Proposition 6. *The 3-cube is Dual Hamiltonian.*

Proof. Like the 2-cube, there are many possible bonds to satisfy that the 3-cube is Dual Hamiltonian. We will show one example of a working bond of maximum size that satisfies partitioning the vertices of the 3-cube into two subsets S and T whose induced subgraphs are trees, illustrated in black and red. The bond, once more, is shown in blue.

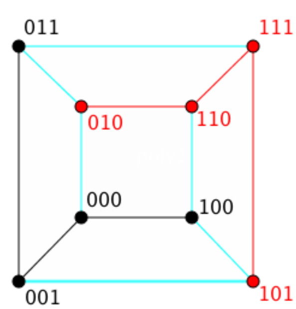

Figure 31: A Maximum Bond in the 3-Cube

$S = \{011, 001, 111, 100\}$
$T = \{010, 110, 111, 101\}$ □

Proposition 7. *The 4-cube is Dual Hamiltonain.*

Proof. Since visualization becomes difficult when working with cubes in higher dimensions, we will provide the two subsets of the vertices of the 4-cube S and T whose induced subgraphs are trees. Based on previous theorems, such a partition is satisfactory to prove dual Hamiltonicity. Although there are many possible partitions, the following example satisfies the necessary conditions.

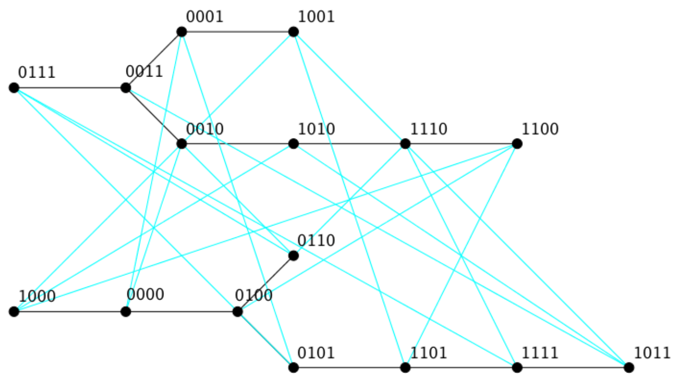

Figure 32: A Maximum Bond in the 4-Cube

$$S = \{0111, 0011, 0001, 0010, 1010, 1110, 1100, 1001\}$$
$$T = \{1000, 0000, 0100, 0110, 0101, 1101, 1111, 1011\}$$

□

Proposition 8. *The 5-cube is Dual Hamiltonain.*

Proof. Similarly to the 4-cube, we will provide an example of a partition of the vertices of the 5-cube into sets S and T, each of which induces a tree. Thus, by previous theorems, the 5-cube is dual Hamiltonian.

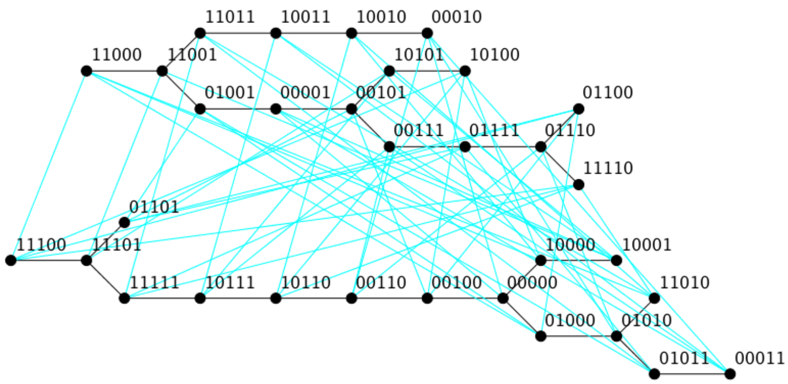

Figure 33: A Maximum Bond in the 5-Cube

$$S = \{11000, 11001, 11011, 10011, 10010, 00010, 01001, 00001, 00101, 10101, 10100, 00111, 01111, 01110, 01100, 11110\}$$

$$T = \{11100, 11101, 01101, 11111, 10111, 10110, 00110, 00100, 00000, 10000, 10001, 01000, 01010, 01011, 00011, 11010\}$$

□

Although it has proven difficult to provide a proof of dual Hamiltonicity for the general case of the hypercube, the prior results have led us to raise the following conjecture:

Conjecture 2. *All hypercubes are dual Hamiltonian.*

Chapter 4

Further Study

In summary, our results have made strides towards verifying the conjecture raised by Ding, Dziobiak, and Wu that the largest bond size for any simple 3-connected graph is at least $\Omega(n^{log_3 2})$ [3] and have also proven the existence of even larger bonds for a variety of classes. We have shown that the conjecture is verified for planar graphs, complete graphs, ladders, Möbius ladders and circular ladders, complete bipartite graphs, the generalized Petersen graph, a few known unique $(3, g)$-cages, and some small hypercubes. Furthermore, we have proven that the complete bipartite graph, ladders, Möbius ladders and circular ladders, the generalized Petersen graph, a few known unique $(3, g)$-cages, and some small hypercubes are dual Hamiltonian, and have raised a conjecture that states that all hypercubes are dual Hamiltonian.

For further study, we would like to work to verify the conjecture about maximum bond size for more classes of graphs, potentially including triangle-free graphs, product graphs and grids, and eventually verify the conjecture for the general case.

We would also like to further study the hypercube to continue looking for a pattern or algorithm that may prove dual Hamiltonicity for the general case.

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