Approximate controllability of coupled 1-d wave equations on star-shaped graphs

Contrôlabilité approchée des équations d'ondes 1d couplées sur graphes étoiles

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A B S T R A C T

In this Note, we study the approximate controllability of a cascade system of two 1-d wave equations defined on a star-shaped planar graph. Only the first equation is controlled, with controls applied at the simple vertices. The controls act on the second equation through the coupling at the multiple vertex. We give a necessary and sufficient condition for the approximate controllability of the system: all the ratios of the lengths of any two different edges of the graph should be irrational numbers. The result implies that, under these conditions, the approximate desensitizing controllability of a star-shaped network of strings holds, when the desensitized functional is the L^2-norm of the displacement of the multiple node.

R É S U M É

Dans cette Note, on étudie la contrôlabilité approchée d'un système en cascade de deux équations d'ondes 1d définies sur un graphe planaire en forme d'étoile. Seulement la première équation est contrôlée, avec des contrôles qui agissent sur les sommets simples. Les contrôles agissent sur la deuxième équation à travers le couplage au sommet multiple. On donne une condition nécessaire et suffisante pour la contrôlabilité approchée du système : tous les rapports des longueurs des arêtes doivent être des nombres irrationnels. Le résultat implique que, dans ces conditions, la contrôlabilité désensibilisante approchée d'un réseau a lieu lorsque la fonctionnelle désensibilisée est la norme L^2 du déplacement du nœud multiple.

1. Introduction

Let $n \geq 2$ be a natural number and $\ell_1, \ldots, \ell_n$ be positive numbers. Consider the following system of $2n$ coupled 1-d wave equations controlled by means of n control functions:

$$\begin{cases} y_{i,tt} - y_{i,xx} = 0 & \text{in } \mathbb{R}\times(0,\ell_i),\ i=1,\ldots,n \\ y_i(\cdot,\ell_i) = \mathbf{v}_i & \text{in } \mathbb{R},\ i=1,\ldots,n \\ y_i(\cdot,0) = y_j(\cdot,0) & \text{in } \mathbb{R},\ i,j=1,\ldots,n \\ y_{1,x}(\cdot,0)+\cdots+y_{n,x}(\cdot,0)=0 & \text{in } \mathbb{R} \\ y_i(0,\cdot)=y_i^0,\ \ y_{i,t}(0,\cdot)=y_i^1 & \text{in } (0,\ell_i),\ i=1,\ldots,n, \end{cases} \tag{1}$$

$$\begin{cases} q_{i,tt} - q_{i,xx} = 0 & \text{in } \mathbb{R}\times(0,\ell_i)\ i=1,\ldots,n \\ q(\cdot,\ell_i) = 0 & \text{in } \mathbb{R},\ i=1,\ldots,n \\ q_i(\cdot,0) = q_j(\cdot,0) & \text{in } \mathbb{R},\ i,j=1,\ldots,n \\ q_{1,x}(\cdot,0)+\cdots+q_{n,x}(\cdot,0)=y_i(\cdot,0) & \text{in } \mathbb{R} \\ q_i(0,\cdot)=q_i^0\ \ q_{i,t}(0,\cdot)=q_i^1 & \text{in } (0,\ell_i),\ i=1,\ldots,n. \end{cases} \tag{2}$$

The systems (1) and (2) model the motion of two networks of n homogeneous vibrating strings connected at one point in a star-shaped configuration. The scalar real functions $y_i(t,x)$, $q_i(t,x)$, defined in $\mathbb{R}\times(0,\ell_i)$, denote the transversal displacement of the i-th string of length ℓ_i of each network, respectively. For every string the values 0 and ℓ_i of the variable x correspond to its ends: 0 to the end where the strings are joined (the multiple node) and ℓ_i to the other one (the simple node).

The network given by (1) is a controlled one. At every simple node, a control $\mathbf{v}_i$ is applied that regulates its displacement. At the multiple nodes, the third and fourth equations in (1) ensure the continuity of the network and the balance of the tensions.

The network (2) is controlled in an indirect way: the sum of the tensions is equal to $y_i(\cdot,0)$, the displacement of the multiple node. Through that coupling, the system receives the action of the control mechanism applied to the first network.

To simplify the notation we denote $\bar{y} := (y_1,\ldots,y_n)$, $\bar{q} := (q_1,\ldots,q_n)$, $\bar{y}^0 := (y_1^0,\ldots,y_n^0)$, $\bar{y}^1 := (y_1^1,\ldots,y_n^1)$, $\bar{q}^0 := (q_1^0,\ldots,q_n^0)$, $\bar{q}^1 := (q_1^1,\ldots,q_n^1)$, $\bar{\mathbf{v}} := (\mathbf{v}_1,\ldots,\mathbf{v}_n)$.

Our goal is to investigate the controllability properties of (1)–(2): given $T>0$ and initial states $\bar{y}^0,\bar{y}^1,\bar{q}^0,\bar{q}^1$ find controls $\bar{\mathbf{v}} \in \left(L^2(0,T)\right)^n$ such that both networks are at rest in time T, i.e., $\bar{y}(T)=\bar{y}_t(T)=\bar{q}(T)=\bar{q}_t(T)=\bar{0}$.

To be precise we introduce the Hilbert spaces

$$V := \left\{\bar{\psi} \in \prod_{i=1}^{n} H^1(0,\ell_i),\ \psi_i(\ell_i)=0,\ \psi_i(0)=\psi_j(0),\ i,j=1,\ldots,n\right\},\quad H := \prod_{i=1}^{n} L^2(0,\ell_i),$$

endowed with the natural product structures.

It is known (see, e.g., [6, p. 23]), that system (1)–(2) is well posed for initial data $\left(\bar{y}^0,\bar{y}^1\right) \in H\times V'$, $\left(\bar{q}^0,\bar{q}^1\right)\in V\times H$ and $(\bar{q},\bar{y}) \in C([0,T],V\times H)\cap C^1([0,T],H\times V')$ for any $T>0$.

The controlled systems with the type of coupling defined between (1) and (2) are called controlled cascade systems. Their study is motivated by the desensitizing controllability, a concept introduced by Lions in [8].

Indeed, consider the following desensitizing controllability problem for the network defined by (1): given the initial data $\left(\bar{y}^0,\bar{y}^1\right)\in H\times V'$, find controls $\bar{\mathbf{v}}\in\left(L^2(0,T)\right)^n$, such that the functional

$$\Phi(\bar{h}^0,\bar{h}^1) := \int_0^T (y_i(t,0))^2\,\mathrm{d}t, \tag{3}$$

defined for the solutions to (1) with perturbed initial data $\bar{y}^0+\bar{h}^0, \bar{y}^1+\bar{h}^1$, is locally linearly independent of the small, unknown perturbation $\bar{h}^0\in H, \bar{h}^1\in V'$, that is, $D\Phi(0,0)=0$. Then, the controls $\bar{\mathbf{v}}$ have that property if and only if, the solution to (1)–(2) with those controls and initial data $\bar{y}^0,\bar{y}^1$, $\bar{q}^0=\bar{q}^1=\bar{0}$ satisfies $\bar{q}(T)=\bar{q}_t(T)=\bar{0}$ (see, [2]). The latter property is clearly implied by the controllability of (1)–(2).

Additionally, the weaker property that for fixed $\left(\bar{y}^0,\bar{y}^1\right)\in H\times V'$ and every $\varepsilon>0$ it holds that $\|D\Phi(0,0)\|_V<\varepsilon$ for some controls $\bar{\mathbf{v}}_\varepsilon$ (known as approximate desensitizing controllability, see [1]) coincides with the fact that the set of all final states $(\bar{q}(T),\bar{q}_t(T))$ of solutions to (1)–(2) with $\bar{q}^0=\bar{q}^1=\bar{0}$ as $\bar{\mathbf{v}}$ ranges over $\left(L^2(0,T)\right)^n$ is dense in $V\times H$, that is, with a weakened version of the approximate controllability of (1)–(2).

The main result of this Note related to the approximate controllability of (4)–(5) is given in Theorem 1. Besides, in Theorem 5, a weighted observability inequality is obtained that allows us to provide in Corollary 6 an alternative more precise description of the approximate desensitizing controllability for the functional (3).

2. Approximate controllability results

We should mention that, as it was proved in [9], the control mechanism in (1) is strong enough to guarantee the controllability of all initial states in $H \times V'$, even if one of the controls is kept switched off (e.g., $\mathbf{v}_1 = 0$). However, the coupling mechanism between (1) and (2) is very weak. Indeed, a star-shaped network of strings controlled from the multiple node fails to be controllable in the natural energy space $H \times V'$ and even the approximate controllability may fail in some circumstances, namely, when the ratio of the length of two of the strings is a rational number (see [4]). As we shall see, that is also the case in the present problem.

We will say that the positive numbers $\ell_1, \ldots, \ell_n$ *satisfy the D-condition* if all the ratios ℓ_i/ℓ_j for $i, j = 1, \ldots, n$, $i \neq j$, are irrational numbers. As we shall see below, the D-condition allows us to characterize the networks that are approximately controllable.

Besides, we denote $\ell_* := \max\{\ell_i : i = 1, \ldots, n\}$ and $T_0 := 2(\ell_1 + \cdots + \ell_n) + 2\ell_*$.

The following is the main result of this Note:

Theorem 1. *System (1)–(2) is approximately controllable in time $T \geq T_0$ if $\ell_1, \ldots, \ell_n$ satisfy the D-condition. In addition, when the D-condition is not satisfied, the approximate controllability of (1)–(2) fails for every $T > 0$.*

To prove this theorem we consider the auxiliary adjoint system

$$\begin{cases} p_{i,tt} - p_{i,xx} = 0 & \text{in } \mathbb{R}\times(0,\ell_i),\ i = 1, \ldots, n \\ p_i(\cdot, \ell_i) = 0 & \text{in } \mathbb{R},\ i = 1, \ldots, n \\ p_i(\cdot, 0) = p_j(\cdot, 0) & \text{in } \mathbb{R},\ i, j = 1, \ldots, n \\ p_{1,x}(\cdot, 0) + \cdots + p_{n,x}(\cdot, 0) = 0 & \text{in } \mathbb{R} \\ p_i(T, \cdot) = p_i^0,\ \ p_{i,t}(T, \cdot) = p_i^1 & \text{in } (0, \ell_i),\ i = 1, \ldots, n, \end{cases} \tag{4}$$

$$\begin{cases} z_{i,tt} - z_{i,xx} = 0 & \text{in } \mathbb{R}\times(0,\ell_i)\ i = 1, \ldots, n \\ z(\cdot, \ell_i) = 0 & \text{in } \mathbb{R},\ i = 1, \ldots, n \\ z_i(\cdot, 0) = z_j(\cdot, 0) & \text{in } \mathbb{R},\ i, j = 1, \ldots, n \\ z_{1,x}(\cdot, 0) + \cdots + z_{n,x}(\cdot, 0) = p_i(\cdot, 0) & \text{in } \mathbb{R} \\ z_i(T, \cdot) = z_i^0\ \ z_{i,t}(T, \cdot) = z_i^1 & \text{in } (0, \ell_i),\ i = 1, \ldots, n. \end{cases} \tag{5}$$

System (4)–(5) is well posed for $(\bar{p}^0, \bar{p}^1) \in H \times V'$, $(\bar{z}^0, \bar{z}^1) \in V \times H$ and $(\bar{z}, \bar{p}) \in C([0,T], V \times H) \cap C^1([0,T], H \times V')$ for any $T > 0$.

It is known (see [6, p. 142] or [5, p. 15]) that the eigenvalue problem associated with (4)

$$-w_{i,xx} = \mu w_i,\ \ w_i \in H^2(0, \ell_i),\ \ w_i(0) = w_j(0),\ \ w_{1,x} + \cdots + w_{n,x}(0) = 0,\ \ w_i(\ell_i) = 0,\ \ i, j = 1, \ldots, n,$$

has a positive unbounded increasing sequence of eigenvalues $(\mu_k)_{k\in\mathbb{N}}$ and that the corresponding eigenfunctions $\bar{w}_k := (w_{1,k}, \ldots, w_{n,k})$, $k \in \mathbb{N}$ may be chosen so that the sequence $(\bar{w}_k)_{k\in\mathbb{N}}$ is an orthonormal basis of H. Further, denote $\lambda_k := \sqrt{\mu_k}$.

In the space V_* of all the finite linear combinations of the eigenfunctions $(\bar{w}_k)_{k\in\mathbb{N}}$, we define the norms $\|\cdot\|_\alpha$ for $\alpha \in \mathbb{R}$ as

$$\|\bar{\psi}\|_\alpha := \left(\sum_{k\in\mathbb{N}} |\lambda_k^\alpha \psi_k|^2\right)^{\frac{1}{2}},$$

for $\bar{\psi} \in V_*$ given by $\bar{\psi} = \sum \psi_k \bar{w}_k$, and V^α as the completion of V_* with respect to $\|\cdot\|_\alpha$. The spaces V, H and V' can be identified with V^1, V^0 and V^{-1}, respectively.

Besides, the solution to (4) with initial (final) data

$$\bar{p}^0 = \sum_{k\in\mathbb{N}} p^{0,k} \bar{w}_k,\ \ \bar{p}^1 = \sum_{k\in\mathbb{N}} p^{1,k} \bar{w}_k$$

at $t = t_0$ may be written as

$$\bar{p}(t) = \sum_{k\in\mathbb{N}} \left(p^{0,k} \cos\lambda_k (t - t_0) + \frac{p^{1,k}}{\lambda_k} \sin\lambda_k (t - t_0)\right) \bar{w}_k$$

and its α-energy

$$\mathbf{E}^{\alpha}_{\bar p}(t) := \|\bar p(t)\|^2_{\alpha+1} + \|\bar p_t(t)\|^2_{\alpha} = \sum_{k\in\mathbb{N}} \left(\left(\lambda_k p^{0,k}\right)^2 + \left(p^{1,k}\right)^2\right)(\lambda_k)^{2\alpha}$$

is conserved.

For $(\theta_1,\theta_2) \in H \times V'$ and $(\vartheta_1,\vartheta_2) \in V \times H$ we denote $\langle(\theta_1,\theta_2),(\vartheta_1,\vartheta_2)\rangle := \langle\theta_1,\vartheta_2\rangle_H + \langle\theta_2,\vartheta_1\rangle_{V',V}$.

Proposition 2. *For any $T > 0$ and any controls $\bar{\mathbf{v}} \in \left(L^2(0,T)\right)^n$ the following duality identity holds for the solutions to (1)–(2) and (4)–(5)*

$$\sum_{i=1}^{n} \int_0^T \mathbf{v}_i(t) z_{i,x}(t,\ell_i) = \left\langle(\bar y(T), -\bar y_t(T)), \left(\bar z^0, \bar z^1\right)\right\rangle + \left\langle\left(\bar y^0, \bar y^1\right), (\bar z(T), -\bar z_t(T))\right\rangle + \tag{6}$$

$$+ \left\langle\left(\bar p^0, \bar p^1\right), (\bar q(T), -\bar q_t(T))\right\rangle + \left\langle(\bar p(T), -\bar p_t(T)), \left(\bar q^0, \bar q^1\right)\right\rangle.$$

From identity (6), we can prove

Proposition 3. *For every $\left(\bar y^0, \bar y^1\right) \in H \times V'$, $\left(\bar q^0, \bar q^1\right) \in V \times H$ the sets*

$$\left\{(\bar q(T), \bar q_t(T)) : \bar{\mathbf{v}} \in \left(L^2(0,T)\right)^n\right\}, \quad \left\{(\bar y(T), \bar y_t(T)) : \bar{\mathbf{v}} \in \left(L^2(0,T)\right)^n\right\}$$

are dense in $V \times H$ and $H \times V'$ respectively, if and only if, the following unique continuation property holds for all solutions to (4)–(5) with $\left(\bar p^0, \bar p^1\right) \in H \times V'$, $\left(\bar z^0, \bar z^1\right) \in V \times H$:

$$z_{i,x}(t,\ell_i) = 0, \ i = 1,\dots,n, \ \text{a.e. in } (0,T) \text{ implies } \left(\bar p^0, \bar p^1\right) = \left(\bar z^0, \bar z^1\right) = (\bar 0, \bar 0). \tag{7}$$

From Proposition 3, it follows that the approximate controllability of (1)–(2) for $\left(\bar y^0, \bar y^1\right) \in H \times V'$, $\left(\bar q^0, \bar q^1\right) \in V \times H$ is equivalent to the unique continuation property (7).

Proposition 4. *The following holds:*

1) the unique continuation property (7) holds with $T \geq T_0$ if and only if the lengths $\ell_1,\dots,\ell_n$ of the strings satisfy the D-condition;
2) if the lengths $\ell_1,\dots,\ell_n$ of the strings do not satisfy the D-condition the unique continuation property (7) fails for any $T > 0$.

Proof. 1) Recall the following fact derived from the D'Alembert formula (see [5, p. 26]): if $u(t,x)$ solves the wave equation $u_{tt} - u_{xx} = 0$ in $\mathbb{R} \times (0,\ell)$ and satisfies the Dirichlet boundary condition $u(t,\ell) = 0$ for $t \in (t_1,t_2)$ with $t_2 - t_1 \geq 2\ell$ then, for every $t \in (t_1 + \ell, t_2 - \ell)$,

$$u_x(t,0) = \frac{1}{2}\left(u_x(t+\ell,\ell) + u_x(t-\ell,\ell)\right). \tag{8}$$

Now assume that $z_{i,x}(t,\ell_i) = 0$ for every $i = 1,\dots,n$, a.e. in $(0,T)$. Applying (8) in each string, we conclude that $\bar z(t) = \bar 0$ in $[\ell_*, T_0 - \ell_*]$. Consequently, $p_i(t,0) = 0$ for every $i = 1,\dots,n$, in $[\ell_*, T_0 - \ell_*]$ and system (4) becomes

$$\begin{cases} p_{i,tt} - p_{i,xx} = 0 & \text{in } [\ell_*, T_0 - \ell_*] \times (0,\ell_i), \ i = 1,\dots,n \\ p_i(\cdot,0) = p_i(\cdot,\ell_i) = 0 & \text{in } [\ell_*, T_0 - \ell_*], \ i = 1,\dots,n \\ p_{1,x}(\cdot,0) + \dots + p_{n,x}(\cdot,0) = 0 & \text{in } [\ell_*, T_0 - \ell_*]. \end{cases}$$

If the lengths $\ell_1,\dots,\ell_n$ satisfy the D-condition, we can apply the unique continuation property associated with the simultaneous control of n strings proved in [4] to conclude that $\bar p(t) = \bar 0$ in $[\ell_*, T_0 - \ell_*]$. Then, as the (-1)-energy of $\bar p$ is conserved, $\left(\bar p^0, \bar p^1\right) = (\bar 0, \bar 0)$. Plugging this information into (5), the 0-energy of $\bar z$ is also conserved. Thus, $\bar z(t) = \bar 0$ in $[\ell_*, T_0 - \ell_*]$ implies $\left(\bar z^0, \bar z^1\right) = (\bar 0, \bar 0)$ and the unique continuation property (7) is true.

2) Assume the D-condition is not verified and take $\bar p = \bar 0$ and $z_i = 0$ for $i \neq s,r$ with $\ell_s/\ell_r \in \mathbb{Q}$. Choose $m,n \in \mathbb{N}$ such that $\ell_s/\ell_r = n/m$ and define

$$z_s(t,x) = \ell_s m \cos\left(\frac{n\pi(T-t)}{\ell_s}\right)\sin\left(\frac{n\pi x}{\ell_s}\right), \quad z_r(t,x) = -\ell_r n \cos\left(\frac{m\pi(T-t)}{\ell_r}\right)\sin\left(\frac{m\pi x}{\ell_r}\right).$$

Clearly,

$$z_{s,x}(t,0) + z_{r,x}(t,0) = mn\pi\left(\cos\left(\frac{n\pi(T-t)}{\ell_s}\right) - \cos\left(\frac{m\pi(T-t)}{\ell_r}\right)\right) = 0$$

and then, $z_{1,x}(t,0)+\cdots+z_{n,x}(t,0)=0$ for any $t\in\mathbb{R}$. Besides, since $z_r(T,x)=\sin(\frac{n\pi x}{\ell_r})$, $\bar{z}^0\neq\bar{0}$. Consequently, $\bar{p}$, $\bar{z}$ are solutions to (4)–(5) for which the unique continuation property (7) is false with any value $T>0$.

Now, Theorem 1 is a consequence of the Propositions 3 and 4.

3. Desensitizing controllability

As mentioned before, the approximate controllability of system (1)–(2) implies the approximate desensitizing controllability of the functional Φ defined by (3) along the solutions to (1), thus, in view of Theorem 1, it holds for $T\geq T_0$ if the lengths of the strings satisfy the D-condition. However, the desensitizing controllability is implied by an apparently weaker version of the unique continuation property (7). It is enough that, for every solution to (4)–(5) with final data $\bar{p}^0,\bar{p}^1\in V_*$, $\bar{z}^0,=\bar{z}^1=\bar{0}$,

$$z_{i,x}(t,\ell_i)=0,\ i=1,\ldots,n,\ \textit{a.e. in } (0,T)\ \textit{imply}\ \left(\bar{p}^0,\bar{p}^1\right)=(\bar{0},\bar{0}),\tag{9}$$

to guarantee the desensitizing controllability.

Using the technique developed in [5, Chapter 5], the latter unique continuation property may be quantified by means of the following weighted observability inequality.

Theorem 5. *Let $T\geq T_0$. Then, there exists a positive constant C such that, for any final data $\bar{p}^0=\sum p^{0,k}\bar{w}_k$, $\bar{p}^1=\sum p^{1,k}\bar{w}_k$, $\bar{z}^0$, $\bar{z}^1$, that belong to V_*, the corresponding solution to (4)–(5) satisfies*

$$C\sum_{i=1}^{n}\int_0^T\left(z_{i,x}(t,\ell_i)\right)^2dt\geq\sum_{k\in\mathbb{N}}\left(\left(p^{0,k}\right)^2+\left(\lambda_k^{-1}p^{1,k}\right)^2\right)\omega_k^2,\quad \textit{where}\quad \omega_k=\prod_{i=1}^{n}\sin(\lambda_k\ell_i).\tag{10}$$

Besides, all the numbers ω_k are different from zero whenever the lengths of the strings satisfy the D-condition. Therefore, Theorem 5 implies the unique continuation property (9). Unfortunately, for any values of the lengths $\ell_1,\ldots,\ell_n$, the sequence $(|\omega_k|)_{k\in\mathbb{N}}$ is not uniformly bounded from below with a positive bound, so (10) leads to the observation of norms weaker than $\|\cdot\|_{H\times V'}$.

Additionally, based on the Hilbert Uniqueness Method (J.-L. Lions [7]), Theorem 5 allows us to give an alternative, more precise characterization of the approximate desensitizing controllability of system (1):

Corollary 6. *If $\ell_1,\ldots,\ell_n$ satisfy the D-condition, the functional Φ defined by (3) can be desensitized along any solution to (1) with initial data $\bar{y}^0$, $\bar{y}^1$ that are finite linear combinations of the eigenfunctions $(\bar{w}_k)_{k\in\mathbb{N}}$.*

Remark 7. All the results in this Note remain true when $n=1$. We have excluded this value of n to guarantee that the terms multiple and simple nodes make sense.

Remark 8. As far as we know, desensitizing controllability problems on networks of strings have not been studied before. A similar approach, based on the technique developed in [3] and in chapters 4 and 5 of [5], should work as well when a smaller number of controllers is used or when the network is supported on a tree-shaped graph.

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