A NOTE ON SOME INVERSE PROBLEMS ARISING IN LUBRICATION THEORY

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Abstract. It is well-known that the pressure of a lubricating fluid filling the gap between two solid surfaces satisfies the Reynolds equation involving the distance function, h, between both planes, as a crucial coefficient. Nevertheless, in most of the applications the function h is not known a priori. Here we consider the simple case in which the surfaces are two parallel planes and assume prescribed the total force applied upon one of the surfaces. We give some sufficient conditions on the total force in order to solve this inverse problem. We show that in the incompressible case, such a condition is also necessary.

1. Introduction

Since the pioneering work by O. Reynolds, in 1886, it has been well-known that the pressure of a lubricating fluid filling the gap between two solid surfaces satisfies the so-called Reynolds equation involving the distance function, h, between both planes, as a crucial coefficient. Nevertheless, in most of the applications the function h is not known a priori. The hard disc of computers or the compact disc player are two examples of the many real situations where this kind of problem appears.

Although several works have been devoted to the study of this problem when some extra information is added to the formulation (see, e.g., the articles Bayada [1], and Bayada and El Alaoui Talibi [2], in which the total load supported by the surfaces is prescribed), the necessity of imposing suitable conditions on the additional information in order to get a well-posed formulation seems not well observed.

Here we consider the simple case in which the surfaces are two parallel planes, and so the unknown distance between both planes is merely a time function $h = h(t)$, for $t \in (0, T)$ with $T > 0$ given. So, the unknowns are the functions $(h(t), P(x, y, t))$, where P denotes the pressure and $(x, y) \in \Omega$, an open and bounded set of $\mathbb{R}^2$. We assume given the initial distance between the planes

$$h(0) = h_0, \tag{1.1}$$

the external pressure P_a (a positive constant), the initial pressure distribution $P(x, 0) = P_0(x)$ (only for the case of a compressible fluid), and the relative velocity (U, V) of the superior plane (in fact here assumed to be a constant vector). In this note we also assume to be known the total force applied upon the superior plane and that it has only a nonzero component, $F(t)$, in the z-direction (orthogonal to the planes).

The main goal of the note is to give some sufficient conditions on $F(t)$ in order to solve this inverse problem. Moreover, in the incompressible case, we shall show that our sufficient condition on $F(t)$ is also necessary for the existence of a solution (h, P). We recall that in the case of an incompressible fluid, under the above conditions, the Reynolds equation deals with the linear elliptic inverse problem: assuming $F(t)$ is known, find (h, P) such that

$$\left\{ \begin{array}{cl} -div(h(t)^3 \nabla P) = -h'(t), & \text{in } \Omega \times (0, T), \\ P = P_a, & \text{on } \partial\Omega \times (0, T) \\ F(t) = \int_\Omega (P(x, y, t) - P_a) dx\, dy & \text{for } t \in (0, T). \end{array} \right. \tag{1.2}$$

In spite of the simplicity of the above formulation, it seems that the study of necessary and sufficient conditions on $F(t)$ was not clearly indicated before in the literature.

The compressible case is more delicate since the associated problem becomes parabolic and of quasilinear type. To simplify the formulation, we consider the simpler case in which the spatial domain is reduced to a one-dimensional interval $I = (0, L)$ (so, there is no dependence of data and unknowns with respect to the y variable). Then, under some conditions on the degree of compressibility of the gas (see, e.g., Friedman and Tello [4]) we arrive at the following inverse problem for the Reynolds equation: assuming $F(t)$ is known, find (h, P) such that

$$\begin{cases} \dfrac{\partial (P h(t))}{\partial t} + U h(t) \dfrac{\partial P}{\partial x} - \epsilon \dfrac{\partial}{\partial x}((\alpha h(t)^2 + \beta h(t)^3 P) \dfrac{\partial P}{\partial x}) = 0, & \text{in } I \times (0, T), \\ P(x, 0) = P_0(x), & \text{in } I, \\ P(0, t) = P(L, t) = P_a, & \\ F(t) = \displaystyle\int_\Omega (P(x, y, t) - P_a) dx\, dy, & \text{for } t \in (0, T), \end{cases} \tag{1.3}$$

where P_a, α, β, ϵ, and U are known positive constants and T is small enough.

2. Incompressible case

Problem (1.1) and (1.2) can be solved by using the auxiliary problem

$$\begin{cases} -\Delta w = 1, & \text{in } \Omega, \\ w = 0, & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

Consider

$$K(\Omega) = \int_\Omega w(x,y)dx\,dy,$$

and assume that

$$F(t) > -\frac{P_a K(\Omega)}{\max_{(x,y)\in\Omega}\{w(x,y)\}}, \quad t \in (0,\infty). \tag{2.2}$$

Theorem 1.

a) *If*

$$\int_0^t F(s)ds > -\frac{K(\Omega)}{2h_0^2} \quad \forall\ t \in (0,\infty), \tag{2.3}$$

then there exists a unique solution $(h(t), P(x,y,t))$ *of the problem* (1.1), (1.2) *such that*

1.

$$\frac{h'(t)}{h^3(t)} = -\frac{F(t)}{K(\Omega)} \ (\textit{and therefore, } sign(h') = -sign(F(t))). \tag{2.4}$$

In particular,

$$h(t) = [\frac{1}{h_0^2} + \frac{2}{K(\Omega)}\int_0^t F(s)ds]^{-\frac{1}{2}}.$$

2.

$$P(x,y,t) = \frac{F(t)}{K(\Omega)}w(x,y) + P_a. \tag{2.5}$$

b) *If there exists* $t_0 > 0$ *such that*

$$\int_0^{t_0} F(s)ds = -\frac{K(\Omega)}{2h_0^2} \ \textit{and} \ \int_0^t F(s)ds < -\frac{K(\Omega)}{2h_0^2}\ \forall t \in (0,t_0), \tag{2.6}$$

then $h(t) \to \infty$ *when* $t \nearrow t_0$ *and* $P(x,y,t_0) = \frac{F(t_0)}{K(\Omega)}w(x,y) + P_a$.

Proof. We construct $P = \frac{F(t)}{K(\Omega)}w + P_a$, where w is the solution of the problem (2.1), and h is defined as the unique solution of the problem

$$\frac{h'(t)}{h^3(t)} = -\frac{F(t)}{K(\Omega)}, \qquad \forall t \in (0, \infty);$$

that is,

$$h(t) = \Big[\frac{1}{h_0^2} + \frac{2}{K(\Omega)} \int_0^t F(s)ds\Big]^{-\frac{1}{2}}. \tag{2.7}$$

By construction of P we know that

$$\min_{(x,y)\in\Omega} \{P(x,y,t)\} \geq \frac{\min\{F(t), 0\}}{K(\Omega)} \max_{(x,y)\in\Omega} \{w(x,y)\} + P_a.$$

From (2.2) we obtain that $P > 0$; furthermore,

$$\int_\Omega (P(x,y,t) - P_a) dx\, dy = \int_\Omega \frac{F(t)}{K(\Omega)} w(x,y) dx\, dy = F(t).$$

By uniqueness of the problem (2.1), the solution $P(x,y,t)$ of (1.1) is

$$P(x,y,t) = -\frac{h'(t)}{h^3(t)} w(x,y) + P_0.$$

Substituting in (1.1) we obtain that (P, h) is the solution of the problem (1.1), (1.3). If $F(t)$ satisfies (2.3), we obtain (2.4), (2.5), and h exists there for all $t \in (0, \infty)$.

If $F(t)$ satisfies (2.6), then

$$\frac{h'(t)}{h^3(t)} = -\frac{F(t)}{K(\Omega)}, \forall t \in (0, t_0),$$

and solving the O.D.E. for $h(t)$ we obtain that

$$h(t) = [\frac{1}{h_0^2} + \frac{2}{K(\Omega)} \int_0^t F(s)ds]^{-\frac{1}{2}} \ \forall t \in (0, t_0),$$

and therefore

$$P(x,y,t) = \frac{F(t)}{K(\Omega)} w + P_a, \forall t \in (0, t_0).$$

By taking limits when $t \nearrow t_0$, we obtain that $h(t_0) = +\infty$, which proves that the sufficient condition (2.6) is also a necessary assumption. Notice that, even if $h(t_0) = +\infty$, the function P is well defined at this time t_0 since $P(x,y,t_0) = \frac{F(t_0)}{K(\Omega)} w + P_a$.

We recall that the derivation of the Reynolds equation requires knowing that the function gap h is small enough. So, the above estimates allow us to get some quantitative conditions in terms of $F(t)$ to justify such a derivation. We also notice that the uniqueness of the couple $(h(t), P(x, y, t))$ is a consequence of the uniqueness of the equation $h'(t) = -\frac{F(t)}{K(\Omega)} h^3(t)$.

Remark 2.1. By (2.1) we know $\|\Delta w\|_{L^2(\Omega)} \leq |\Omega|^{\frac{1}{2}}$. Taking w as a test function in (2.1), we obtain $\|w\|_{H_0^1(\Omega)} = (K(\Omega))^{\frac{1}{2}}$, and deduce that

$$\|w\|_{H^2(\Omega)\cap H_0^1(\Omega)} \leq |\Omega|^{\frac{1}{2}} + (K(\Omega))^{\frac{1}{2}}.$$

Since $H^2(\Omega) \hookrightarrow L^\infty(\Omega)$ is a continuous inclusion, it results that $\|w\|_{L^\infty(\Omega)} \leq C(\Omega)$, and therefore

$$P \geq \min\{0, F(t)\} \frac{C(\Omega)}{K(\Omega)} + P_a.$$

If $\min\{F(t)\} > -P_a K(\Omega)/C(\Omega)$, we obtain that $P > 0$.

3. Compressible case

In this section we study the one-dimensional compressible problem (1.3) and (1.1) for small time $T(P_a, P_0, h_0, F, F')$ (see (3.13)). The following existence result requires again a total force $F(t)$ small enough.

Theorem 2. *Let $P_0 \geq 0$ such that $P_0 - P_a \in W_0^{1,\infty}(0, L)$, and let $F \in C^1(0, T)$, satisfying*

$$h_0 = \frac{\int_0^L P_0(s)ds}{F(0) + P_a L}, \qquad 0 < F_0 \leq F(t) + P_a L \leq F_1, \qquad P_0 \geq 0, \tag{3.1}$$

for some positives constants $F_0 \leq F_1$. Then, there exists $\epsilon^* > 0$ such that if $0 \leq \epsilon < \epsilon^*$ problem (1.3) and (1.1) has, at least, one solution $(h(t), P(x, y, t))$.

Proof. Limit case: $\epsilon = 0$. In the special case $\epsilon = 0$, the system (1.3), (1.1) reduces to

$$\left\{ \begin{array}{ll} \frac{\partial(Ph)}{\partial t} + U\frac{\partial(Ph)}{\partial x} = 0, \quad 0 < x < L, & 0 < t < T, \\ P(x, 0) = P_0(x), & 0 < x < L, \\ P(0, t) = P_a, & 0 < t < T. \end{array} \right. \tag{3.2}$$

The solution "Ph" of (3.2) is given by $Ph = \varphi = \varphi(x - Ut)$, where

$$\varphi(s) = \left\{ \begin{array}{ll} h_0 P_0(s), & \text{if } 0 < s < L, \\ h(-\frac{s}{U}) P_a, & \text{if } -L < s < 0. \end{array} \right.$$

Integrating, we obtain

$$F(t)=\int_0^L (P(x,t)-P_a)dx=\int_0^L (\frac{\varphi(x-Ut)}{h(t)}-P_a)dx,$$

and then

$$h(t)=\frac{1}{F(t)+P_aL}\int_0^L \varphi(x-Ut)dx$$
$$=\frac{1}{F(t)+P_aL}(P_a\int_0^{Ut} h(t-\frac{1}{U}x)dx+h_0\int_{Ut}^L P_0(x-Ut)dx).$$

Introducing the change of variables $s=t-\frac{1}{U}x$, $r=x-Ut$, we obtain

$$h(t)=\frac{1}{F(t)+P_aL}(P_aU\int_0^t h(s)ds+h_0\int_0^{L-Ut} P_0(r)dr). \qquad (3.3)$$

By (3.1) we obtain that the unique solution h of (3.3) is positive, and by defining

$$P(x,t)=\frac{\varphi(x-Ut)}{h(t)},$$

it results that $(h(t),P(x,t))$ is the unique solution of problem (3.2).

Remark 3.1. Notice that $h\in C^1(0,\frac{L}{U})$.

Case $\epsilon>0$**.** Let $\xi:=hP$. Then

$$\frac{\partial\xi}{\partial t}+U\frac{\partial\xi}{\partial x}-\epsilon\frac{\partial}{\partial x}(\alpha h\frac{\partial\xi}{\partial x}+\beta h\xi\frac{\partial\xi}{\partial x})=0, \qquad 0<x<L, \qquad 0<t<T. \quad (3.4)$$

Through this section we use the constants

$$\alpha_0:=\frac{1}{4}h_0>0, \qquad \alpha_1:=2h_0+1<\infty, \qquad (3.5)$$

$$M:=(h_0+1)(P_a+\max_{x\in(0,L)}\{P_0(x)\}), \quad m:=\frac{h_0}{4}\min\{P_a,\min_{x\in(0,L)}\{P_0(x)\}\}, \quad (3.6)$$

$$F_2:=\max_{t\in[0,\frac{L}{U}]}|F'(t)|, \qquad \alpha_2:=\frac{2MLF_2}{F_0^2}+\frac{1}{P_a}<\infty. \qquad (3.7)$$

We consider the closed, convex set $G\subset C^0(0,T)$, defined by

$$G=\{h\in W^{1,\infty}(0,T),\ h(0)=h_0,\ \ \alpha_0\le h(t)\le\alpha_1<\infty,\ \ |h'(t)|\le\alpha_2\}.$$

We introduce the truncature function ϕ defined by

$$\phi(s)=\begin{cases} s, & \text{if } \alpha_0\le s\le\alpha_1,\\ \alpha_0, & \text{if } s\le\alpha_0,\\ \alpha_1, & \text{if } \alpha_1\le s.\end{cases}$$

It is useful to start considering the truncated problem

$$\frac{\partial \xi}{\partial t} + U\frac{\partial \xi}{\partial x} - \epsilon\frac{\partial}{\partial x}(\alpha\widehat{h}\frac{\partial \xi}{\partial x} + \beta\widehat{h}\xi\frac{\partial \xi}{\partial x}) = 0, \qquad 0 < x < L, \quad 0 < t < T, \tag{3.8}$$

$$\xi(0,t) = \xi(L,t) = P_a\widehat{h}(t), \qquad 0 < t < T, \tag{3.9}$$

$$\xi(x,0) = h_0P_0(x) \geq 0, \qquad 0 < x < L, \tag{3.10}$$

$$\phi(\frac{\int_0^L \xi(x,t)dx}{F(t) + P_aL}) = \widehat{h}. \tag{3.11}$$

Proposition 3.1. *Given $\widehat{h} \in G$ and $\xi_0 \geq 0$ the problem* (3.8)–(3.10) *has a unique weak solution satisfying $m \leq \xi \leq M$.*

Proof. We consider the problem

$$\begin{cases} \dfrac{\partial \xi}{\partial t} + U\dfrac{\partial \xi}{\partial x} - \epsilon\dfrac{\partial}{\partial x}(\widehat{h}\dfrac{\partial}{\partial x}(\alpha\xi + \dfrac{\beta}{2}|\xi|\xi)) = 0, & 0 < x < L, \quad 0 < t < T \\ \xi(0,t) = \xi(L,t) = P_a\widehat{h}(t), & 0 < t < T, \\ \xi(x,0) = h_0(P_0(x) - P_a) \geq 0, & 0 < x < L. \end{cases}$$

Since $(\alpha\xi + \frac{\beta}{2}|\xi|\xi)$ is an increasing function of ξ and $\widehat{h} \geq \alpha_0 > 0$, it results that the operator

$$A(u) := U\frac{\partial u}{\partial x} - \epsilon\frac{\partial}{\partial x}(\widehat{h}\frac{\partial}{\partial x}(\alpha u \; + \frac{\beta}{2}|u + p_a|(u + p_a)))$$

is a maximal monotone operator in $H^1(0,L)$, and then there exists a unique weak solution $\xi \in L^2(0,T : H^1(0,L)) \cap C([0,T] : L^2(0,L))$ with $\frac{\partial \xi}{\partial t} \in L^2(0,T : (H^1(0,L))')$. Taking $-(\xi - m)^-$ and $(\xi - M)^+$ as test function in (3.8) we obtain

$$\frac{d}{dt}\Big(\int_0^L [(\xi - m)^-]^2dx + \int_0^L [(\xi - M)^+]^2dx\Big) \leq 0.$$

Since $m \leq \xi(x,0) \leq M$, we deduce $m \leq \xi(x,t) \leq M$; then $|\xi| = \xi$, and we get that ξ satisfies (3.8). □

Lemma 3.1. *Let $\widehat{h} \in G$ and ξ be the solution of the problem* (3.8)–(3.11), *where*

$$0 < \epsilon < \min\{\frac{MLF_2U}{F_0(\alpha\alpha_1U + 2\beta\alpha_1M(P_a\alpha_2 + 1))}, \frac{U^2}{\alpha_1\beta(P_a\alpha_2 + 1)}\} = \epsilon^*. \tag{3.12}$$

Then

$$|\frac{\partial \xi}{\partial x}| < \frac{P_a\alpha_2 + 1}{U}, \qquad \text{at the points } \; x = 0, L.$$

Proof. Let $\overline{\xi} = P_a\widehat{h} + \bar{\lambda}x$, where $\bar{\lambda} = \frac{P_a\alpha_2+1}{U}$. We obtain

$$\frac{\partial\overline{\xi}}{\partial t} + U\frac{\partial\overline{\xi}}{\partial x} - \epsilon\frac{\partial}{\partial x}(\alpha\widehat{h}\frac{\partial\overline{\xi}}{\partial x} + \beta\widehat{h}\overline{\xi}\frac{\partial\overline{\xi}}{\partial x}) = P_a\widehat{h}' + U\bar{\lambda} - \epsilon\widehat{h}\beta\bar{\lambda}^2.$$

By the choice of $\bar{\lambda}$ and since $\epsilon \le \epsilon^*$, it results that $P_a\widehat{h}' + U\bar{\lambda} - \epsilon\widehat{h}\beta\bar{\lambda}^2 > 0$. Since $\overline{\xi} \ge \xi$ in $x = 0, L$ it results that $\overline{\xi}$ is a supersolution of the problem (3.8)–(3.11). In the same way we prove that $\overline{\overline{\xi}} = P_a\widehat{h}(t) + \bar{\lambda}(1-x)$ is a supersolution in a neighborhood of L. Taking

$$\underline{\xi} = P_a\widehat{h} - \frac{\alpha_2}{U}x, \qquad \text{in } x \in (0, \frac{P_a h(t)U}{\alpha_2}), \quad t \in (0, \frac{L}{U}),$$

we obtain that $\underline{\xi}$ is a subsolution and $\underline{\underline{\xi}} = \widehat{h} + \frac{\alpha_2}{U}(x-1)$ is also a subsolution in a neighborhood of L. By comparison it results that

$$-\frac{\alpha_2}{U} = \frac{\partial\underline{\xi}}{\partial x} \le \frac{\partial\xi}{\partial x} \le \frac{\partial\overline{\xi}}{\partial x} = \bar{\lambda}, \qquad \text{on } x = 0,$$

and we also deduce that

$$-\frac{\alpha_2}{U} = \frac{\partial\underline{\underline{\xi}}}{\partial x} \ge \frac{\partial\xi}{\partial x} \ge \frac{\partial\overline{\overline{\xi}}}{\partial x} = \bar{\lambda}, \qquad \text{on } x = L,$$

which proves the lemma. □

We consider the function $\Upsilon : G \longrightarrow C^0(0, \frac{L}{U})$ defined by

$$\Upsilon(\widehat{h}) = \phi(\frac{\int_0^L \xi(x,t)dx}{F(t) + P_a L}) = h,$$

where ξ is the solution of the problem (3.8)–(3.10); then we have

Lemma 3.2. Υ *has a fixed point in* G.

Proof. By construction of ϕ we know $\alpha_0 \le h(t) \le \alpha_1$, and furthermore, integrating by parts in (3.8) we get

$$\int_0^L \xi_t dx = \epsilon(\alpha\widehat{h} + \beta\widehat{h}\xi)\frac{\partial\xi}{\partial x}\Big|_0^L.$$

As a consequence of Lemma 3.1, we obtain that

$$\Big|\int_0^L \xi_t dx\Big| \le \epsilon(\alpha\alpha_1 + 2\beta\alpha_1 M\frac{P_a\alpha_2 + 1}{U}).$$

From (3.11) it results that

$$\frac{dh}{dt} = \phi'\Big(\frac{\int_0^L \xi(x,t)dx}{F(t) + P_a L}\Big)\Big[\frac{\int_0^L \frac{\partial\xi(x,t)}{\partial t}dx}{F(t) + P_a L} - \frac{\int_0^L \xi(x,t)dxF'(t)}{(F(t) + P_a L)^2}\Big],$$

and we deduce that

$$|\frac{dh}{dt}| \leq \frac{\epsilon}{F_0}(\alpha\alpha_1 + 2\beta\alpha_1 M\frac{P_a\alpha_2+1}{U}) + \frac{MLF_2}{F_0^2}.$$

By (3.12) and (3.7) it results that $|\frac{dh}{dt}| \leq \alpha_2$, and we deduce that $Im\Upsilon \subset G$. Since the inclusion $W^{1,\infty}(0,\frac{U}{L}) \subset C^0(0,\frac{U}{L})$ is continuous and compact, $Im\Upsilon$ is a compact subset of C^0. Applying Schauder's fixed-point theorem we obtain the wished-for result. $\square$

End of the proof of Theorem 2. It is clear that each fixed point of Υ is a solution of the problem (3.8)–(3.11). We denote by $(\xi_\epsilon, h_\epsilon)$ the solution of the problem (3.8)–(3.11). Since $|h'_\epsilon| \leq \alpha_2$, for all $0 \leq \epsilon \leq \epsilon^*$, we obtain that $h_0 - \alpha_2 T \leq h_\epsilon(t) \leq h_0 + \alpha_2 T$. Then, by taking

$$T < \min\Big\{\frac{3h_0}{4\alpha_2}, \frac{L}{U}\Big\}, \tag{3.13}$$

we get that $\alpha_0 < h_\epsilon(t) < \alpha_1$, for all $0 \leq \epsilon \leq \epsilon^*$, and therefore

$$h_\epsilon = \phi\Big(\frac{\int_0^L \xi_\epsilon(x,t)dx}{F(t)+P_aL}\Big) = \frac{\int_0^L \xi_\epsilon(x,t)dx}{F(t)+P_aL}.$$

Then $(\xi_\epsilon, h_\epsilon)$ is a solution of the problem (3.8)–(3.11), and $(P_\epsilon = \frac{\xi_\epsilon}{h_\epsilon}, h_\epsilon)$ satisfies (1.2), (1.1) which conclude the proof. $\square$

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