Stability of steady states of the Cauchy problem for the exponential reaction–diffusion equation

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Abstract

We consider the Cauchy problem

$$\begin{cases} u_t = \Delta u + e^u, & x \in \mathbb{R}^N,\ t \in (0,T), \\ u(x,0) = u_0, & x \in \mathbb{R}^N, \end{cases}$$

where $u_0 \in C(\mathbb{R}^N)$ and $T > 0$. We first study the radial steady states of the equation and the number of intersections distinguishing four different cases: $N = 1$, $N = 2$, $3 \leqslant N \leqslant 9$ and $N \geqslant 10$, writing explicitly every steady state for $N = 1$ and $N = 2$. Then we study the large time behavior of solutions of the parabolic problem.

Keywords: Stability; Blow up; Exponential reaction–diffusion equation

1. Introduction and main results

In this paper we consider the following Cauchy problem:

$$\begin{cases} u_t = \Delta u + e^u, & x \in \mathbb{R}^N,\ t \in (0,T), \\ u(x,0) = u_0(x), & x \in \mathbb{R}^N, \end{cases} \tag{1.1}$$

where $u = u(x,t)$, Δ is the Laplace operator in x and u_0 belongs to the space of continuous functions $C(\mathbb{R}^N)$. It is well known that for any initial data u_0, satisfying $-ke^{c|x|^2} \leqslant u_0(x) \leqslant c$ for $c, k > 0$ there exists $T = T(u_0) > 0$ such that (1.1) has a unique classical solution $u(x,t;u_0)$ in $C^{2,1}(\mathbb{R}^N \times (0,T)) \cap C(\mathbb{R}^N \times [0,T))$.

In Section 2 we study the stationary states of (1.1) under the assumption of radial symmetry, i.e.

$$u := \tilde{u}(|x|).$$

For simplicity we drop the tilde and denote $|x| = r$, then u satisfies the equation

$$u_{rr} + \frac{N-1}{r} u_r + e^u = 0, \qquad r > 0. \tag{1.2}$$

In that section we also study the number of intersections between the steady states. This result is enclosed in the following theorem:

Theorem 1.1.

(i) *For $N = 1$, every solution to* (1.2) *is of the form*

$$u_\alpha(r) := \alpha - 2\log\left(\cosh\left(\frac{\sqrt{2}}{2} e^{\alpha/2} r\right)\right),$$

for $\alpha \in \mathbb{R}$, and every two solutions intersect each other once.

(ii) *For $N = 2$, every solution to* (1.2) *is of the form*

$$u_{\alpha\beta}(r) := \alpha + (\beta - 2)\log r - 2\log\left(1 + \frac{e^\alpha}{2\beta^2} r^\beta\right),$$

for $\alpha \in \mathbb{R}$ and $\beta > 0$, and

(a) $u_{\alpha_1\beta_1}$ *intersects* $u_{\alpha_2\beta_2}$ *twice if* $\beta_1 \neq \beta_2$;

(b) $u_{\alpha_1\beta_1}$ *intersects* $u_{\alpha_2\beta_2}$ *once if* $\beta_1 = \beta_2$, $\alpha_1 \neq \alpha_2$.

(iii) *For $3 \leqslant N \leqslant 9$ there exists one singular solution and every regular solution intersects the singular one infinitely many times. Every two regular solutions intersect each other infinitely many times.*

(iv) *For $N \geqslant 10$, there exists one singular solution to* (1.2) *and solutions do not intersect each other.*

The singular solution to (1.2) for $N > 2$ is given by

$$\Phi^*(r) := -2\log r + \log(2N - 4). \tag{1.3}$$

For $N \geqslant 2$, we denote by u_α the regular solutions to

$$u_{rr} + \frac{N-1}{r} u_r + e^u = 0, \qquad u(0) = \alpha, \qquad u_r(0) = 0. \tag{1.4}$$

In Section 3 we study the parabolic problem and the *blow up*, of solutions in the sense:

– the solution "blows up to $-\infty$" if

$$\lim_{t \to T} \max_{x \in \mathbb{R}^N} u(x,t;u_0) = -\infty, \qquad \text{for } T < \infty;$$

– the solution "blows up to ∞" if

$$\lim_{t\to T}\max_{x\in\mathbb{R}^N} u(x,t;u_0)=\infty, \qquad \text{for } T<\infty;$$

and prove the following results:

– under the assumption

$$-c_1' e^{c_2'|x|^2}+c_3' \leqslant u_0(x) \leqslant -c_1 e^{c_2|x|^2}+c_3, \qquad \text{for } c_i'>0\ (i=1,2) \text{ and } c_3, c_3' \in \mathbb{R},$$

the solution blows up to $-\infty$;

– if u_0 satisfies

$$u_0(x) \geqslant -c_1 e^{c_2|x|^{2-\epsilon}}, \qquad \text{for } c_1, c_2>0 \text{ and } \epsilon\in(0,1),$$

the solution remains bounded below in every compact sub-set of $\mathbb{R}^N\times[0,\infty)$;

– **Theorem 1.2.** *Suppose that $N=1$ or $2\leqslant N<10$, and consider $u_0(x)\geqslant -c_1e^{c_2|x|^{2-\epsilon}}$ for $\epsilon>0$ and some positive constants c_1 and c_2. Then the following conclusions hold.*
  (i) *If $u_0\leqslant u_\alpha$ and $u_0\neq u_\alpha$, for $\alpha\in\mathbb{R}$, the solution is global and*

$$\max_{x\in\mathbb{R}^N} u(x,t)\to-\infty \quad \textit{as } t\to\infty.$$

  (ii) *If $u_0\geqslant u_\alpha$ and $u_0\neq u_\alpha$, then u blows up at finite time to ∞.*
  (iii) *For $N=2$, if $u_0\leqslant u_{\alpha\beta}$ for $\beta\in(0,2)$, the solution is global and*

$$\max_{x\in\mathbb{R}^N} u(x,t)\to-\infty \quad \textit{as } t\to\infty.$$

In Section 4 we study the stability of the radial steady states for $N\geqslant 10$ with respect to the norms $|||\cdot|||$ and $\|\cdot\|_s$ defined by

$$|||\psi|||=\sup_{x\in\mathbb{R}^N}\frac{(1+|x|)^4}{\log(2+|x|)}\big|\psi(x)\big| \quad \text{and} \quad \|\psi\|_s=\sup_{x\in\mathbb{R}^N}\big(1+|x|\big)^{-s}\big|\psi(x)\big|,$$

for $s\in\mathbb{R}$. These results presented in Theorems 1.3–1.5 concern stability and weak stability. For readers convenience we give definitions of stability and weak asymptotic stability.

Definition 1.1. A stationary solution u_α is stable with respect the norm $\|\cdot\|_{\lambda_+}$, if for $\epsilon>0$ there exists $\delta>0$ such that $\|u_0-u_\alpha\|_{\lambda_+}<\delta$ then $\|u(t,\cdot,u_0)-u_\alpha\|_{\lambda_+}<\epsilon$ for $t>0$.

Definition 1.2. A stationary solution u_α is weak asymptotically stable with respect the norm $\|\cdot\|_{\lambda_+}$, if there exists $\delta>0$ such that for $\|u_0-u_\alpha\|_{\lambda_+}<\delta$ then $\|u(t,\cdot,u_0)-u_\alpha\|_{\lambda'}\to 0$ as $t\to\infty$ for $\lambda'>\lambda_+$.

Theorem 1.3. *Let λ_+ and λ_- be defined by*

$$\lambda_\pm=\frac{1}{2}\big(-N+2\pm\sqrt{N-2}\sqrt{N-10}\big),$$

then:

(i) *If $N=10$, any radial steady state u_α is stable with respect to the norm $|||\cdot|||$ and is weakly asymptotically stable with respect to the norm $\|\cdot\|_\lambda$, for $\lambda=-4$.*

(ii) *If* $N > 10$, *any radial steady state* u_α *is stable with respect to the norm* $\|\cdot\|_{\lambda_+}$ *and is weakly asymptotically stable with respect to the norm* $\|\cdot\|_{\lambda_-}$.

Theorem 1.4. *Let* $\alpha_0, \alpha_1, \alpha_2 \in \mathbb{R}$ *such that* $\alpha_0 < \alpha_1$ *and* $\alpha_2 \leqslant \alpha_0$. *Assume* v_0, $u_0 \in C(\mathbb{R}^N)$ *for* $N \geqslant 10$ *satisfying*

$$u_{\alpha_0} \leqslant u_0 \leqslant u_{\alpha_1}, \tag{1.5}$$

$$u_{\alpha_2} \leqslant u_0 + v_0 \leqslant \Phi^*, \tag{1.6}$$

$$\lim_{|x|\to\infty} |x|^s \big|v_0(x)\big| = 0, \quad \text{where } s = 4 \text{ for } N = 10 \text{ and } s = -\lambda_+ \text{ for } N > 10, \tag{1.7}$$

then $u(x, t; u_0 + v_0)$ *is a global solution and satisfies*

$$\big\|u(\cdot, t; u_0 + v_0) - u(\cdot, t; u_0)\big\|_{L^\infty(\mathbb{R}^N)} \to 0 \quad \text{as } t \to \infty. \tag{1.8}$$

Theorem 1.5. *Let* $N \geqslant 10$ *and* $u_0 \in C(\mathbb{R}^N)$ *satisfy*

$$-c_1 e^{c_2|x|^{2-\epsilon}} \leqslant u_0(x) \leqslant \Phi^*\big(|x|\big), \quad x \in \mathbb{R}^N, \tag{1.9}$$

for c_1, $c_2 > 0$ *and* $\epsilon \in (0, 1)$,

$$\lim_{|x|\to\infty} \phi(x)\big|\Phi^*\big(|x|\big) - u_0(x)\big| = 0, \quad \text{for } \phi(x) = \begin{cases} \frac{|x|^4}{\log|x|}, & \text{if } N = 10, \\ |x|^{-\lambda_+}, & \text{if } N > 10, \end{cases} \tag{1.10}$$

then the solution $u(x, t; u_0)$ *is global and satisfies*

$$\lim_{t\to\infty} u(\cdot, t; u_0) = \Phi^*. \tag{1.11}$$

2. Steady states: Proof of Theorem 1.1

In this section we give the proof of Theorem 1.1 concerning the steady states and the number of intersections between them.

The proof for the cases $N \geqslant 10$ and $3 \leqslant N \leqslant 9$ follows the ideas of Joseph and Lundgren [4] developed to study the Dirichlet problem. See also [2,3,9].

Proof of Theorem 1.1.

Case $(N = 1)$. Explicit solutions of the problem are well known (see [1,7]):

$$u(r) = \alpha - 2\log\left(\cosh\left(\frac{\sqrt{2}}{2} e^{\alpha/2} r\right)\right).$$

To prove the non-existence of singular solutions, we multiply (1.4) by u' and integrate over $(\epsilon, 1)$ to obtain

$$\frac{1}{2}\big[u'(\epsilon)\big]^2 + e^{u(\epsilon)} = \frac{1}{2}\big[u'(1)\big]^2 + e^{u(1)}.$$

Then

$$\big|u'(\epsilon)\big| \leqslant \big[\big(u'(1)\big)^2 + 2e^{u(1)}\big]^{1/2},$$

which proves the boundedness of u in every bounded sub-set of R and the non-existence of singular solutions.

To see that every two solutions intersect each other once, we consider u_i, satisfying $u_i(0) = \alpha_i$ (for $i = 1, 2$ and $\alpha_1 > \alpha_2$), and define f by

$$f(r) := u_1(r) - u_2(r).$$

Notice that

$$f(0) = \alpha_1 - \alpha_2 > 0, \qquad \lim_{r\to\infty} f(r) = -\infty \quad \text{and} \quad f'(r) < 0.$$

By continuity and monotonicity of f we obtain (i).

Case *($N = 2$).* There exists a two parameters family of solutions defined by

$$u_{\alpha\beta}(r) := \alpha + (\beta - 2)\log r - 2\log\left(1 + \frac{e^{\alpha}}{2\beta^2} r^{\beta}\right) \tag{2.1}$$

for $\alpha \in \mathbb{R}$ and $\beta \in (0, \infty)$. The explicit solutions were already known for the unit ball with Dirichlet boundary conditions, see [5]. Notice that, if $\beta = 2$, $u_{\alpha\beta}$ is a regular solution and if $\beta \neq 2$, $u_{\alpha\beta}$ is singular and satisfies

$$\lim_{r\to 0} u_{\alpha\beta} = \begin{cases} \infty, & 0 < \beta < 2, \\ -\infty, & \beta > 2. \end{cases}$$

For simplicity we will denote by u_α the solution $u_{\alpha\beta}$ for $\beta = 2$.

To prove that every solution is of the form $u_{\alpha\beta}$, we consider the Cauchy problem

$$\begin{cases} p' = q, & r > r_0 > 0, \\ q' = \frac{q}{r} - e^{p}, & r > r_0 > 0, \\ p(r_0) = a, & q(r_0) = b, \end{cases} \tag{2.2}$$

for $r_0 = 1$. For every $a, b \in \mathbb{R}$, $u_{\alpha\beta}$ is a solution to (2.2) for α and β defined by

$$\alpha = a + 2\log 2, \quad \beta = \left(2e^{\alpha}\right)^{-1/2}, \qquad \text{if } b = -2,$$

$$\alpha = a + 2\log(1+k), \quad \beta = \frac{(b+2)(1+k)}{1-k}, \qquad \text{if } b \neq -2,$$

where k satisfies

$$g(k) := \frac{e^{a}(1+k)^2}{2(b+2)^2}\left(1 - \frac{2k}{1+k}\right)^2 - k = 0.$$

Notice that $g(1) = -1$ and

$$g(0) > 0, \quad \text{if } b > -2, \qquad \text{and} \qquad \lim_{k\to\infty} g(k) > 0, \quad \text{if } b > -2,$$

which guarantees the existence of $k > 0$. By uniqueness of (2.2) we obtain that every solution is given by (2.1).

To study the intersections between two solutions $u_{\alpha_1\beta_1}$ and $u_{\alpha_2\beta_2}$ we distinguish two cases:

(a) $\beta_1 > \beta_2$. Let h be defined by

$$h(r,\alpha_1,\alpha_2,\beta_1,\beta_2) := u_{\alpha_1\beta_1}(r) - u_{\alpha_2\beta_2}(r)$$
$$= \alpha_1 - \alpha_2 + (\beta_1 - \beta_2)\log r + 2\log\left[\frac{1+\frac{e^{\alpha_2}}{2\beta_2^2}r^{\beta_2}}{1+\frac{e^{\alpha_1}}{2\beta_1^2}r^{\beta_1}}\right].$$

Notice that

$$\lim_{r\to 0} h(r,\alpha_1,\alpha_2,\beta_1,\beta_2) = \lim_{r\to\infty} h(r,\alpha_1,\alpha_2,\beta_1,\beta_2) = -\infty, \quad \alpha_1,\alpha_2 \in \mathbb{R}. \tag{2.3}$$

We consider first the case $\alpha_i = \log(2\beta_i^2)$. Since $h(1,\alpha_1,\alpha_2,\beta_1,\beta_2) > 0$,

$$\frac{\partial h}{\partial r} = f(r)\big[(\beta_2-\beta_1)r^{\beta_1} + 2\beta_2 - 2\beta_1 r^{\beta_1-\beta_2}\big],$$

where

$$f(r) := \frac{r^{\beta_2-1}}{(1+r^{\beta_2})(1+r^{\beta_1})},$$

and there exists a unique $r_0 \geqslant 0$ such that $\partial h(r_0)/\partial r = 0$, we obtain that h has two zeros (for $\alpha_i = \log(2\beta_i^2)$).
By using the previous case and continuous dependence, we prove by contradiction the general case. We assume there exist $\tilde\alpha_1, \tilde\alpha_2 \in \mathbb{R}$ such that the number of zeros of h is different from two. Let I_i be the compact interval $[\log(2\beta_i^2), \tilde\alpha_i]$ if $\tilde\alpha_i \geqslant \log(2\beta_i^2)$ and $[\tilde\alpha_i, \log(2\beta_i^2)]$ if $\tilde\alpha_i < \log(2\beta_i^2)$, for $i = 1, 2$. Since $|\partial h/\partial\alpha_i| \leqslant 1$ and by (2.3), there exist r_0, r_1, satisfying $0 < r_0 < 1 < r_1 < \infty$, such that

$$h(r,\alpha_1,\alpha_2,\beta_1,\beta_2) < 0, \quad \text{for } r < r_0 \ (\alpha_1,\alpha_2) \in I_1 \times I_2,$$
$$h(r,\alpha_1,\alpha_2,\beta_1,\beta_2) < 0, \quad \text{for } r > r_1 \ (\alpha_1,\alpha_2) \in I_1 \times I_2.$$

Then, there exist $\alpha_1^* \in I_1$, $\alpha_2^* \in I_2$ and $r^* \in (r_0, r_1)$ such that $h(r^*, \alpha_1^*, \alpha_2^*, \beta_1, \beta_2) = 0$ and $\partial h(r^*)/\partial r = 0$. Then

$$u_{\alpha_1^*\beta_1}(r_*) = u_{\alpha_2^*\beta_2}(r_*), \qquad u'_{\alpha_1^*\beta_1}(r_*) = u'_{\alpha_2^*\beta_2}(r_*),$$

by uniqueness of (2.2), $\beta_1 = \beta_2$, which contradicts $\beta_1 > \beta_2$ and proves (a).

(b) $\beta_1 = \beta_2 = \beta$. We consider the function v defined by

$$v(r,\alpha_1,\alpha_2,\beta) := u_{\alpha_1\beta}(r) - u_{\alpha_2\beta}(r) = \alpha_1 - \alpha_2 + 2\log\left[\frac{1+\frac{e^{\alpha_2}}{2\beta^2}r^{\beta}}{1+\frac{e^{\alpha_1}}{2\beta^2}r^{\beta}}\right].$$

Since

$$\frac{dv}{dr} = \frac{\left(\frac{e^{\alpha_2}}{\beta} - \frac{e^{\alpha_1}}{\beta}\right)r^{\beta-1}}{\left(1+\frac{e^{\alpha_2}}{2\beta^2}r^{\beta}\right)\left(1+\frac{e^{\alpha_1}}{2\beta^2}r^{\beta}\right)} < 0,$$

$$v(0,\alpha_1,\alpha_2,\beta) = \alpha_1 - \alpha_2 \quad \text{and} \quad \lim_{r\to\infty} v(r,\alpha_1,\alpha_2,\beta) = \alpha_2 - \alpha_1,$$

we obtain (b), for $\alpha_1 \neq \alpha_2$.

Case *(3 ⩽ N ⩽ 9)*. In order to prove (iii), we introduce s and w defined by

$$s := \log r \quad \text{and} \quad w := u_\alpha - \Phi^*. \tag{2.4}$$

Then, Eq. (1.2) is converted into the following one:

$$w_{ss} + (N-2)w_s + 2(N-2)\big(e^w - 1\big) = 0, \quad -\infty < s < \infty. \tag{2.5}$$

We now rewrite the above equation as a system of ODEs:

$$w' = q, \qquad q' = -2(N-2)\big(e^w - 1\big) - (N-2)q. \tag{2.6}$$

In the w–q plane, $(0,0)$ is the unique steady state. Since the general solution of the linearized system

$$W' = Q, \qquad Q' = -2(N-2)W - (N-2)Q, \tag{2.7}$$

is given by

$$W(s) = k_1 e^{\lambda_+ s} + k_2 e^{\lambda_- s},$$

for

$$\lambda_\pm = \frac{1}{2}\big(-N + 2 \pm \sqrt{N-2}\sqrt{N-10}\big), \tag{2.8}$$

we obtain that $(0,0)$ is a stable focus.

Multiplying (2.6) by $(2(N-2)e^w - 1, q)$ and adding both equations we get

$$2(N-2)\big(e^w - w\big)' + \frac{1}{2}\big(q^2\big)' = -(N-2)q^2. \tag{2.9}$$

Integrate (2.9) over $(0,s)$ to obtain

$$2(N-2)\big(e^w - w\big) + \frac{1}{2}q^2 < k,$$

which proves the boundedness of the solution.

Notice that, from (2.9), we also deduce that there is no periodic solutions (apart of $(0,0)$). Then, $(w,q) \to (0,0)$ as $s \to \infty$. Since $(0,0)$ is a focus we get that w has infinitely many zeros.

To see that there exist infinitely many intersections, we argue by contradiction. We consider two solutions $\overline{w}$ and $\underline{w}$ which satisfy:

$$\underline{w}_{ss} + (N-2)\underline{w}_s + 2(N-2)\big(e^{\underline{w}} - 1\big) = 0, \quad -\infty < s < \infty, \tag{2.10}$$

$$\overline{w}_{ss} + (N-2)\overline{w}_s + 2(N-2)\big(e^{\overline{w}} - 1\big) = 0, \quad -\infty < s < \infty, \tag{2.11}$$

$$|\overline{w}| < 1 \quad \text{and} \quad |\underline{w}| < 1, \quad S_0 < s < \infty. \tag{2.12}$$

We assume that $\overline{w} > \underline{w}$ for $s > S_1 > S_0$ (i.e. there is no intersection after S_1). Since $\overline{w}$ is oscillatory, there exist $s_0, s_1 > S_1$ such that $\overline{w}(s_0) = \overline{w}(s_1) = 0$ and $\overline{w} > 0$ at (s_0, s_1). Multiplying (2.10) by $\overline{w}$ and (2.11) by $\underline{w}$, and subtracting both expressions, it results

$$(\overline{w}_{ss}\underline{w} - \underline{w}_{ss}\overline{w}) + (N-2)(\overline{w}_s\underline{w} - \overline{w}\underline{w}_s) + 2(N-2)\big(e^{\overline{w}}\underline{w} - e^{\underline{w}}\overline{w}\big) = 0,$$

since $((e^{\overline{w}} - 1)\underline{w} - (e^{\underline{w}} - 1)\overline{w}) < 0$ for $s > S_0$, multiplying by $e^{(N-2)s}$ we have

$$\big(e^{(N-2)s}(\overline{w}_s\underline{w} - \overline{w}\underline{w}_s)\big)_s > 0.$$

Integrating over (s_0, s_1) we obtain

$$-e^{(N-2)s_1}\overline{w}_s(s_1)\underline{w}(s_1) + e^{(N-2)s_0}\overline{w}_s(s_0)\underline{w}(s_0) > 0, \tag{2.13}$$

since $\overline{w}_s(s_1) < 0$ and $\overline{w}_s(s_0) > 0$, from (2.13) we have that $\underline{w}(s_0) > 0$ or $\underline{w}(s_1) > 0$, which contradicts $\overline{w} > \underline{w}$ for $s > S_1$, and (iii) is proven.

Case *($N \geqslant 10$)*. We first consider the solution to (2.6), which satisfies

$$\lim_{s\to-\infty} w(s) = -\infty, \qquad \lim_{s\to-\infty} q(s) = 2.$$

In the w–q plane, we have:

(1) there is no steady state (apart of $(0,0)$);
(2) there is no periodic solution (see case $3 \leqslant N \leqslant 9$);
(3) at $q = 0$, $w < 0$, we have $w' = 0$, $q' > 0$;
(4) at the half-line $q = -\frac{N-2}{2}w$, $w < 0$, it results

$$\frac{q'}{w'} = -(N-2) + 4\frac{e^w - 1}{w},$$

since $(e^w - 1)/w < 1$ for $w < 0$ we have

$$\frac{q'}{w'} < -N + 6 \leqslant -\frac{N-2}{2}.$$

Consequently, the region $q < 0$, $w < 0$ and $q < -\frac{N-2}{2}w$ is invariant, $w \to 0$ as $s \to \infty$ and the case $N \geqslant 10$ is proven.

To see that there is no intersection, we consider two solutions, (w_1, q_1) and (w_2, q_2), satisfying

$$\lim_{s\to-\infty} w_1 = \alpha_1, \qquad \lim_{s\to-\infty} w_2 = \alpha_2 \quad \text{for } \alpha_1 < \alpha_2.$$

We define $w := w_1 - w_2$, $q := q_1 - q_2$ which satisfies

$$w' = q, \qquad q' = -(N-2)q - 2(N-2)e^{\tilde{w}}w, \tag{2.14}$$

for some $\tilde{w} \in (w_1, w_2)$ if $w_1 < w_2$ and $\tilde{w} \in (w_2, w_1)$ if $w_1 > w_2$ (notice that in both cases $\tilde{w} < 0$).

In the w–q plane, we have, as before:

- $(0,0)$ is the unique stationary state;
- at $q = 0$, $w < 0$ we have $w' = 0$, $q' > 0$;
- at the half-line $q = -\frac{N-2}{2}w$, $w < 0$, we have

$$\frac{q'}{w'} = -(N-2) + 4e^{\tilde{w}} < -N + 6 \leqslant -\frac{N-2}{2}.$$

As before, we deduce that the region $q < 0$, $w < 0$ and $q < -\frac{N-2}{2}w$ is invariant and w remains negative for $s < \infty$ which ends the proof of the theorem. □

Lemma 2.1. *Let u be a solution to (1.2) for $N \geqslant 10$, then:*

(i) *If $N = 10$,*

$$u(r) = -2\log r + \log(16) + r^{-4}(a\log r + b) + r^{-8}\big(c(\log r)^2 + d\log r\big) + O\big(r^{-8}\big),$$

as $r \to \infty$, for some constants a, b, c and d.

(ii) *If* $N > 10$,

$$u(r) = -2\log r + \log(2N-4) + ar^{\lambda_+} + br^{\lambda_-} + cr^{2\lambda_+} + dr^{\lambda_+ + \lambda_-} + O\big(r^{2\lambda_-}\big),$$

for

$$\lambda_\pm = \frac{1}{2}\big(-N + 2 \pm \sqrt{N-2}\sqrt{N-10}\big),$$

as $r \to \infty$, *and some constants* a, b, c *and* d.

Proof. We start with the proof of (ii). We consider w, the solution to (2.5) and

$$W(s) = k_1 e^{\lambda_+ s} + k_2 e^{\lambda_- s},$$

the solution to the linearized equation.

By standard arguments, as in [8], we get

$$w(s) = ae^{\lambda_+ s} + be^{\lambda_- s} - \frac{2(N-2)}{\lambda_+ - \lambda_-} \int_s^\infty \big(e^{\lambda_+(s-s')} - e^{\lambda_-(s-s')}\big)\big(e^{w(s')} - 1 - w(s')\big)\, ds'. \tag{2.15}$$

Since

$$e^w - 1 - w = \sum_{n=2}^{n=\infty} \frac{1}{n!} w^n$$

and $w(s) = ae^{\lambda_+ s} + O(e^{\lambda_- s})$, we have $e^{w(s)} - 1 - w(s) = \frac{a^2}{2} e^{2\lambda_+ s} + O(e^{(\lambda_+ + \lambda_-)s})$. Then, by (2.15) we have

$$w(s) = ae^{\lambda_+ s} + be^{\lambda_- s} + c' e^{2\lambda_+ s} + O\big(e^{(\lambda_+ + \lambda_-)s}\big). \tag{2.16}$$

Substituting again in (2.15) we have

$$w(s) = ae^{\lambda_+ s} + be^{\lambda_- s} + ce^{2\lambda_+ s} + de^{(\lambda_+ + \lambda_-)s} + O\big(e^{2\lambda_- s}\big). \tag{2.17}$$

Introducing the variables, r and u, we obtain

$$u(r) = -2\log r + \log(2N-4) + ar^{\lambda_+} + br^{\lambda_-} + cr^{2\lambda_+} + dr^{\lambda_+ + \lambda_-} + O\big(r^{2\lambda_-}\big).$$

In the same fashion we prove (i). □

3. The parabolic problem

In this section we study the blow up of solutions under suitable assumptions. We present first a necessary and sufficient condition in order to obtain blow up in the sense

$$\lim_{t \to T} \max_{x \in \mathbb{R}^N} u(x,t) = -\infty,$$

for $T < \infty$. The proof of Theorem 1.2, which follows the ideas of Gui, Ni and Wang [8], is also enclosed in this section. See also [6,11].

Lemma 3.1. *If* $u_\alpha \geqslant u_0 \geqslant -c_1 e^{c_2|x|^{2-\epsilon}}$, *for* $\epsilon \in (0,1)$, $c_1 > 0$, $c_2 > 0$ *and* u_α *is a regular steady state, the solution to* (1.1) *remains bounded in every compact sub-set of* $\mathbb{R}^N \times [0, \infty)$.

Proof. Since $u_\alpha \geqslant u_0$, we obtain, by maximum principle, that $u \leqslant u_\alpha$. In order to prove the lemma, we argue by contradiction and assume the solution blows up at $t = T < \infty$. Let us consider the function

$$\underline{u} = -c_1 \exp\left(\frac{c_2}{1 - t/(T+1)}\left(|x|^2 + c_3\right)^{1-\epsilon/2}\right),$$
$$\text{for} \quad c_3 = \left((T+1)\left(c_2\left(1 - \frac{\epsilon}{2}\right)^2 + 2N\left(1 - \frac{\epsilon}{2}\right)\right)\right)^{2/\epsilon} > 1.$$

After routine computations, by selection of c_3, we have that $\underline{u}$ is a sub-solution to the problem. Since $\underline{u}$ remains bounded in every compact sub-set of $\mathbb{R}^N \times [0, T+1)$, we obtain a contradiction which proves the lemma. □

Proposition 3.1. *Suppose*

$$-c_1' e^{c_2'|x|^2} + c_3' \leqslant u_0(x) \leqslant -c_1 e^{c_2|x|^2} + c_3, \quad \textit{for } c_i' > 0 \ (i = 1, 2) \textit{ and } c_3, c_3' \in \mathbb{R}.$$

Then, the solution u blows up at finite time $T \leqslant 1/(4c)$, for $c := \min\{e^{-c_3-1}/(4c_1), c_2\}$, in the sense

$$\lim_{t\to T} \max_{x\in\mathbb{R}^N} u(x,t) = -\infty.$$

Proof. We consider the function

$$\bar{u}(x,t) = -c_1 \exp\left(\frac{cx^2}{1-4ct} - \frac{t}{4c}\right) + c_3, \quad \text{for } t \leqslant \frac{1}{4c}.$$

After routine computations, we have

$$\bar{u}_t - \Delta\bar{u} - e^{\bar{u}} > 0,$$

which implies that $\bar{u}$ is a super-solution to the problem and $u \leqslant \bar{u}$. Since

$$\lim_{t\to T} \max_{x\in\mathbb{R}^N} \bar{u}(x,t) = -\infty,$$

at finite time $T = 1/(4c)$ we get the desired result. □

Proof of Theorem 1.2. In order to prove (i) we assume, without loss of generality, that

$$u_0(x) < u_\alpha\big(|x|\big), \quad x \in \mathbb{R}^N. \tag{3.1}$$

If $u_0(x) = u_\alpha(|x|)$ at some point, we consider $u_0(x) = u(x, t_0)$ for $t_0 > 0$ that, by the strong maximum principle, satisfies (3.1).

We consider u_β, the solution to (1.2), and $r_0(\beta)$ defined as the first intersection point of u_β and u_α. Since $u_\beta \to u_\alpha$ as $\beta \to \alpha$ uniformly in any compact sub-set $[0,k]$ and $u_0 < u_\alpha$, there exists $\beta_0 < \alpha$ such that the function ψ_1 defined by

$$\psi_1(x) = \begin{cases} u_{\beta_0}(|x|), & \text{if } |x| \leqslant r_0(\beta_0), \\ u_\alpha(|x|), & \text{if } |x| \geqslant r_0(\beta_0), \end{cases}$$

satisfies $u_0 \leqslant \psi_1$. Since $u(x,\cdot;\psi_1)$ is monotone decreasing and every steady state u_α intersects ψ_1 for $\alpha < \beta_0$ (see Theorem 1.1 for $N < 10$), we have

$$\lim_{t\to\infty} \max_{x\in\mathbb{R}^N} u(x,t;\psi_1) \to -\infty.$$

By standard comparison arguments we have that $u(x,t;\psi_1) \geqslant u(x,t;u_0)$. Now, (i) follows from Lemma 3.1.

In order to prove (ii), we define the function

$$\psi_2(x) = \begin{cases} u_{\beta_1}(|x|), & \text{if } |x| \leqslant r_0, \\ u_\alpha(|x|), & \text{if } |x| \geqslant r_0, \end{cases}$$

where r_0 is the first intersection point between u_α and u_{β_1}, for $\beta_1 > \alpha$ sufficiently close to α. We may assume that $u_0 > \psi_2$ and ψ_2 is a sub-solution to the problem. Since $u(x,\cdot;\psi_2)$ is a monotone increasing function and every radial steady state u_β intersects ψ_2 for $\beta > \beta_1$ (see Theorem 1.1 for $N < 10$), we have

$$\lim_{t\to T} \max_{x\in\mathbb{R}^N} u(x,t;\psi_2) \to \infty,$$

for some $T \leqslant \infty$. To prove $T < \infty$, we consider first the case in which the blow up set contains a neighborhood "B" of the origin and use a standard Kaplan's argument. Let λ_1 be the first eigenvalue of $-\Delta$ in $H_0^1(B)$ and w_1 the positive and normalized in $L^1(\Omega)$ associated eigenfunction. We define

$$U := \int_B u w_1 \, dx,$$

then

$$\dot{U} + \lambda_1 U \geqslant e^U, \qquad U(t_0) = k,$$

for t_0 large, such that k is bigger than the larger root of the equation $e^s - \lambda_1 s = 0$, we obtain finite time blow up for U, which proves $T < \infty$. If the blow-up set of $u(x,t;\psi_2)$ is a single point, the radially symmetric function v defined by

$$v(|x|) := \sup_{t\in[0,\infty)} u(x,t;\psi_2)$$

is a singular steady state satisfying $v > \psi_2 \geqslant u_\alpha$, which contradicts Theorem 1.1.

In order to prove (iii) we argue as in (i) and assume, without loss of generality, that $u_0 < u_{\alpha\beta}$. Consider now $u_{\alpha\beta^*}$ for $\beta^* > \beta$, and $r_1 > 0$ such that $u_{\alpha\beta^*}(r+\epsilon) > u_0(r)$ for $r \in (0,r_1)$ and $u_{\alpha\beta}(r_1+\epsilon) = u_{\alpha\beta^*}(r_1)$ for ϵ small enough, such that, the function

$$\psi_3(x) = \begin{cases} u_{\alpha\beta^*}(|x|+\epsilon), & \text{if } |x| \leqslant r_1, \\ u_{\alpha\beta}(|x|), & \text{if } |x| > r_1, \end{cases}$$

satisfies $u_0 \leqslant \phi_3$. As in parts (i) and (ii), we can see that $u(x,\cdot;\psi_3)$ is monotone decreasing and

$$\lim_{t\to\infty} \max_{x\in\mathbb{R}^N} u(x,t;\psi_3) \to -\infty.$$

Since $u(x,t;u_0) \leqslant u(x,t;\psi_3)$ and thanks to Lemma 3.1 we obtain (iii). □

4. Stability of solutions for $N \geqslant 10$

In this section we study the stability of solutions for $N \geqslant 10$. Theorem 1.3 concerns the stability of steady states and the weak asymptotic stability, the proof follows the ideas of Gui, Ni and Wang [8] developed to study

$$u_t = \Delta u + u^p, \quad x \in \mathbb{R}^N,\ t \in (0,T). \tag{4.1}$$

Theorems 1.4 and 1.5 concerning asymptotic stability are proven in a similar fashion as Theorems 4.2 and 6.1 in Poláčik and Yanagida [10] concerning (4.1).

Proof of Theorem 1.3. First we show that u_α is stable with respect to the norm $\|\cdot\|_{\lambda_+}$ for $N > 10$ and with respect to the norm $|||\cdot|||$ for $N = 10$. We divide the interval $(0,\infty)$ in two parts $[0, R_{\alpha\epsilon}]$ and $(R_{\alpha\epsilon},\infty)$, where $R_{\alpha\epsilon}$ will be defined later. By continuous dependence on α we have that

$$\lim_{\beta\to\alpha}\sup_{r\in[0,R_{\alpha\epsilon}]}|u_\beta - u_\alpha| = 0. \tag{4.2}$$

Then, if $N > 10$,

$$u_\alpha(r) = -2\log r + \log(2N-4) + A(\alpha)r^{\lambda_+} + O\big(r^{\lambda_-}\big),$$

where $A(\alpha)$ is continuous and increasing in α. We choose $R_{\alpha\epsilon}$ such that

$$\sup_{r\in[R_{\alpha\epsilon},\infty)}\big|\big(u_\alpha(r) - u_\beta(r)\big)r^{-\lambda_+}\big| \leqslant \big|A(\alpha) - A(\beta)\big| + \frac{\epsilon}{3}, \tag{4.3}$$

for $|\beta - \alpha| < \alpha/2$. Taking $\beta_+ > \alpha > \beta_-$, such that $|A(\alpha) - A(\beta_\pm)| \leqslant \epsilon/3$, and

$$\sup_{r\in[0,R_{\alpha\epsilon}]}\big|u_{\beta_\pm}(r) - u_\alpha(r)\big| \leqslant \frac{\epsilon}{3},$$

we get

$$\big\|u_{\beta_\pm}\big(|x|\big) - u_\alpha\big(|x|\big)\big\|_{\lambda_+} \leqslant \epsilon. \tag{4.4}$$

Then, for any $\epsilon > 0$ there exists $\delta > 0$, such that if $\|u_0(x) - u_\alpha(|x|)\|_{\lambda_+} \leqslant \delta$, there exists $\beta_\pm$ such that $u_{\beta_-}(|x|) \leqslant u_0(x) \leqslant u_{\beta_+}(|x|)$ and (4.4) holds. By comparison, we have $u_{\beta_-}(|x|) \leqslant u(x,t;u_0) \leqslant u_{\beta_+}(|x|)$, which proves the stability of u_α.

The case $N = 10$ may be proven in the same way.

To see that u_α is weakly asymptotically stable we introduce the next proposition.

Proposition 4.1. *For each radial solution u_α and for $N \geqslant 10$ there exists a sequence of radial strict super-solutions $\overline{u}_\alpha^k$ and sub-solutions $\underline{u}_\alpha^k$ such that*

$$\overline{u}_\alpha^1 > \overline{u}_\alpha^2 > \cdots > u_\alpha > \cdots > \underline{u}_\alpha^2 > \underline{u}_\alpha^1,$$

and u_α is the unique solution in the interval $\overline{u}_\alpha^k > u_\alpha > \underline{u}_\alpha^k$. Moreover, for $k \in \mathbb{N}$

$$\lim_{|x|\to\infty}\big|\overline{u}_\alpha^k - u_\alpha\big||x|^{-\lambda_-} > 0, \qquad \lim_{|x|\to\infty}\big|\underline{u}_\alpha^k - u_\alpha\big||x|^{-\lambda_-} > 0. \tag{4.5}$$

The proposition may be proven in the same fashion that Theorem 4.1 in [8], by using the auxiliary problem

$$\begin{cases} v'' + \frac{N-1}{r}v' + (1\pm h)e^v = 0, & \text{in } (0,\infty),\\ v(0) = \beta, & v'(0) = 0, \end{cases} \tag{4.6}$$

where h is a non-negative and non-trivial regular function with compact support.

In order to finish the proof of Theorem 1.3 we consider a sub-solution $\underline{u} := \underline{u}_\alpha^k$ and a super-solution $\overline{u} := \overline{u}_\alpha^k$ (which existence is shown in Proposition 4.1). From (4.5) we have that there exists $\delta > 0$ such that

$$\big|\underline{u}(x) - u_\alpha\big(|x|\big)\big|\big(1+|x|\big)^{-\lambda_-} > \delta, \qquad \big|\overline{u}(x) - u_\alpha\big(|x|\big)\big|\big(1+|x|\big)^{-\lambda_-} > \delta,$$

for $N > 10$. Then, if $\|u_0 - u_\alpha\|_{\lambda_-} < \delta$, we have $\underline{u} \leqslant u_0 \leqslant \overline{u}$ and

$$\underline{u}(x) \leqslant u(x,t;\underline{u}) < u(x,t;u_0) < u(x,t;\overline{u}) \leqslant \overline{u}(x).$$

By monotonicity of $u(x,\cdot;\underline{u})$ and $u(x,\cdot;\overline{u})$ and Proposition 4.1, we have

$$\lim_{t\to\infty} u(\cdot,t;\underline{u}) = u_\alpha = \lim_{t\to\infty} u(\cdot,t;\overline{u}).$$

Then, for $\lambda' < \lambda_-$ and $R > 0$, we have

$$\begin{aligned}
&\left|(1+|x|)^{-\lambda'}\big(u(x,t;u_0) - u_\alpha(|x|)\big)\right| \\
&\quad \leqslant \begin{cases} c(1+|x|)^{-\lambda'}|x|^{\lambda_-}, & \text{if } |x| \geqslant R, \\ c(1+R)^{-\lambda'}\|u(x,t;u_0) - u_\alpha(|x|)\|_{L^\infty(0,R)}, & \text{if } |x| < R, \end{cases} \\
&\quad \leqslant \begin{cases} cR^{-\lambda'+\lambda_-}, & \text{if } |x| \geqslant R, \\ c(1+R)^{-\lambda'}\|u(x,t;u_0) - u_\alpha(|x|)\|_{L^\infty(0,R)}, & \text{if } |x| < R, \end{cases}
\end{aligned}$$

and taking limits, we obtain

$$\limsup_{t\to\infty} \left\|u(\cdot,t;u_0) - u_\alpha\right\|_{\lambda'} \leqslant CR^{-\lambda'+\lambda_-}.$$

Since R is arbitrary, we conclude the proof for $N > 10$. The case $N = 10$ can be proven in the same fashion. $\square$

Before giving the proof of Theorem 1.4, we introduce Lemma 4.1 and Proposition 4.2. The proofs are given for reader's convenience, the proof of Lemma 4.1 follows the proof of Lemma 3.3 in Poláčik and Yanagida [10] and the results of Proposition 4.2 are enclosed in Proposition 4.1 and Lemma 4.3 of the same work.

Lemma 4.1. *Let u_α be the solution to* (1.4) *and let v be the solution to*

$$\begin{cases} v_t - \Delta v = e^{u_\alpha} v, & x \in \mathbb{R}^N,\ t > 0, \\ v(x,0) = v_0(x), & x \in \mathbb{R}^N, \end{cases} \tag{4.7}$$

where $v_0 \in C(\mathbb{R}^N)$. Then,

(i) *There exists a constant $C > 0$ independent of v_0 such that*

$$\left\|v(\cdot,t;v_0)\right\| \leqslant C\|v_0\|,$$

where $\|\cdot\|$ is any of the norms $\|\cdot\|_\lambda$, $\lambda \in [\lambda_-, \lambda_+]$ for $N > 10$ and $|||\cdot|||$ for $N = 10$.

(ii) *If v_0 has compact support then*

$$\left\|v(\cdot,t;v_0)\right\|_{L^\infty(\mathbb{R}^N)} \to 0, \quad \textit{as } t \to 0. \tag{4.8}$$

More generally (4.8) *holds if*

$$\lim_{|x|\to\infty} |x|^\lambda \left|v_0(x)\right| = 0. \tag{4.9}$$

Proof. We only consider the case $N > 10$ in the proof. The case $N = 10$ can be treated similarly.

By the linearity of the equation we may restrict ourselves to the case $v_0 \geqslant 0$. Since $v(x,t;v_0) \geqslant 0$, we have that $u_\alpha + v$ is a sub-solution to the problem, i.e.

$$u_\alpha(x) + v(x,t;v_0) \leqslant u(x,t;u_\alpha + v_0). \tag{4.10}$$

Since u_α is stable, there exists $\delta > 0$ such that if $\|v_0\| \leqslant \delta$ then

$$\left\| u_\alpha - u(\cdot, t; u_\alpha + v_0) \right\|_\lambda \leqslant 1.$$

By (4.10) we obtain that

$$\left\| v(\cdot, t; v_0) \right\|_\lambda \leqslant 1 = \frac{1}{\delta} \|v_0\|_\lambda,$$

by linearity, we have proven statement (i) for $C = 1/\delta$.

To prove (ii) we first consider the case where v_0 has compact support. Using the weak asymptotic stability (see Theorem 1.3) and (4.10) we obtain (4.8) for $\|v_0\|_\lambda$ small enough. By linearity of (4.7), (4.8) holds for any $v_0 \in C(\mathbb{R}^N)$ with compact support.

We next assume that (4.9) is satisfied, then, for any $\epsilon > 0$ we can decompose v_0 as

$$v_0 = v_1 + v_2,$$

where v_1 is continuous with compact support and v_2 satisfies $\|v_2\|_\lambda \leqslant \epsilon$. By linearity of the problem and (i) we have that

$$\limsup_{t\to\infty} \left\| v(\cdot, t; v_0) \right\|_\lambda \leqslant \limsup_{t\to\infty} \left\| v(\cdot, t; v_1) \right\|_\lambda + \limsup_{t\to\infty} \left\| v(\cdot, t; v_2) \right\|_\lambda \leqslant C\epsilon.$$

Since ϵ is arbitrarily small we get (4.8). □

Proposition 4.2. *Assume u_0 and v_0 satisfy* (1.5)–(1.7), *then*:

(i) *There exist $C > 0$ and $\delta > 0$ such that for each $v_0 \in C^0(\mathbb{R}^N)$ with $\|v_0\| \leqslant \delta$ we have*

$$\left\| u(\cdot, t; u_0 + v_0) - u(\cdot, t; u_0) \right\| \leqslant C \|v_0\|, \tag{4.11}$$

where $\|\cdot\|$ is any of the norms $\|\cdot\|_\lambda$, $\lambda \in [\lambda_-, \lambda_+]$ for $N > 10$ and $|||\cdot|||$ for $N = 10$.

(ii) *If v_0 has compact support, then*

$$\lim_{|x|\to\infty} |x|^\lambda \left| u(\cdot, \tau; u_0 + v_0) - u(\cdot, \tau; u_0) \right| = 0,$$

for $\tau > 0$ small enough.

Proof.

(i) Since $\|v_0\| \leqslant \delta$ for δ small enough, we have

$$u_{\beta-1} \leqslant u_0 + v_0 \leqslant u_{\alpha+1}.$$

We consider the problems

$$\begin{cases} \underline{v}_t - \Delta \underline{v} = e^{u_{\beta-1}} \underline{v}, & x \in \mathbb{R}^N,\ t > 0, \\ v(x, 0) = (v_0)_-, & x \in \mathbb{R}^N, \end{cases} \tag{4.12}$$

and

$$\begin{cases} \overline{v}_t - \Delta \overline{v} = e^{u_{\alpha+1}} \overline{v}, & x \in \mathbb{R}^N,\ t > 0, \\ v(x, 0) = (v_0)_+, & x \in \mathbb{R}^N, \end{cases} \tag{4.13}$$

where $(\cdot)_+$ is the positive part function (i.e. $(s)_+ = 0$ if $s \leqslant 0$ and $(s)_+ = s$ if $s \geqslant 0$) and $(s)_- = (-s)_+$.

As in Lemma 4.1 we can see that $u(x,t;u_0)+\overline{v}(x,t;(v_0)_+)$ is a super-solution and $u(x,t;u_0)-\underline{v}(x,t;(v_0)_-)$ is a sub-solution. Then

$$\big\|u(\cdot,t;u_0+v_0)-u(\cdot,t;u_0)\big\|_{L^\infty(\mathbb{R}^N)}\leqslant \max\{\overline{v},\underline{v}\}.$$

Applying Lemma 4.1 we obtain (i).

(ii) By the stability of u_0 (see Proposition 4.1) we have that

$$\big\|u(\cdot,t;u_0\pm w_0)-u(\cdot,t;u_0)\big\|_\lambda\leqslant C\|w_0\|_\lambda, \tag{4.14}$$

for w_0 positive continuous function, satisfying $-w_0<v_0<w_0$ and such that $\|w_0\|_\lambda<\infty$. Then, since the $\sup\{v_0\}\subset B_R$, for some $R>0$ we have

$$u(\cdot,t;u_0-w_0)<u(\cdot,t;u_0+v_0)<u(\cdot,t;u_0+w_0),\quad |x|=R,\quad t<\tau,$$

for τ small enough. Then, by maximum principle we obtain

$$u(\cdot,t;u_0-w_0)<u(\cdot,t;u_0+v_0)<u(\cdot,t;u_0+w_0),\quad |x|>R,\quad t<\tau.$$

Then, thanks to (4.14) we obtain (ii). □

Proof of Theorem 1.4. The proof can be done now, following the steps of Theorem 4.2 in [10]. □

Proof of Theorem 1.5. We first prove that the solution is global, i.e. it does not blow up at finite time. Arguing by contradiction, we assume that the solution u blows up at time $T<\infty$. By Lemma 3.1 the solution remains bounded below in every compact sub-set of $\mathbb{R}^N\times[0,\infty)$ and by the maximum principle we obtain that

$$u(x,t;u_0)\leqslant \Phi^*\big(|x|\big),\quad \text{in } \mathbb{R}^N\times[0,T], \tag{4.15}$$

as long as the solution exists. Then, if the solution blows up at finite time, it has to be at $x=0$. By maximum principle u remains strictly less than Φ^* on the compact sub-set $|x|=\delta$, $T'\leqslant t\leqslant T$, for $\delta>0$ and $0<T'<T$.

Since $u_\alpha\to\Phi^*$ as $\alpha\to\infty$ in every compact sub-set of $(0,\infty)$, there exists α_* large enough such that

$$u(x,t;u_0)<u_{\alpha_*}(x),\quad |x|=\delta,\ t\in[T',T],$$
$$u(x,T';u_0)<u_{\alpha_*}(x),\quad |x|<\delta.$$

By maximum principle the solution remains below u_{α_*} at $t=T$ for $|x|\leqslant\delta$, which contradicts the assumption of blow up at finite time and proves the global existence. The proof ends following [10]. □

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