AAS 15-661

COMPACT SOLUTION OF CIRCULAR ORBIT RELATIVE MOTION IN CURVILINEAR COORDINATES

Claudio Bombardelli[*], Juan Luis Gonzalo[†] and Javier Roa[‡]

A compact approximate solution of the highly non-linear relative motion in curvilinear coordinates is provided under the assumption of circular orbit for the chief spacecraft and moderately small inclination and eccentricity for the follower. The rather compact three-dimensional solution, which employs time as independent variable, is obtained by algebraic manipulation of the individual Keplerian motions in curvilinear coordinates and Taylor expansion for small eccentricity of the follower orbit. Numerical test cases are conducted to show that the approximate solution can be effectively employed to extend the classical linear Clohessy-Wiltshire solution to include non-linear relative motion without significant loss of accuracy up to a limit of 0.4-0.5 in eccentricity and a few degrees in inclination.

INTRODUCTION

The most common representation of relative motion in circular orbit comes from the solution of the well-known Clohessy-Wiltshire (CW) equations. These equations, easily solvable in closed analytical form, derive from the linearization of the gravitational acceleration acting on the follower spacecraft and, as a consequence, are quite accurate when its separation distance from the chief is a sufficiently small fraction of the orbital radius. When such condition is not fulfilled non-linear gravitational effects influence the relative dynamics in such a way that the original CW solution fails to accurately reproduce the relative motion.

Due to the intrinsic instability of orbital motion the chief-follower separation distance can grow in a secular way even if the initial conditions of relative position and velocity are small. This happens every time the follower orbit semi-major axis differs from the one of the chief and can represent an important limitation for the use of CW equations for long time propagation. Fortunately, there is a simple solution to this problem: by formally replacing the along-track 'y' Cartesian coordinate with the curvilinear abscissa describing the follower-chief separation along the chief orbit and the radial 'x' coordinate with the radial distance between the follower and chief orbit the CW solution can be "projected" along the curved orbital path providing a quite accurate fully analytical solution even for relatively large time intervals as long as the distance between the follower and chief *orbits* remain small. This fact has been known since decades and widely used especially in the Russian literature.[1] Nevertheless, a mathematical proof of the correspondence between the linearized equations of relative motion in Cartesian and curvilinear coordinates appeared (to the authors' knowledge) only recently.[2]

[*]Research Associate, Space Dynamics Group, Department of Aeronautical Engineering
[†]Graduate Student, Space Dynamics Group, Department of Aeronautical Engineering
[‡]Graduate Student, Space Dynamics Group, Department of Aeronautical Engineering

In the most general, fully non-linear case in which the distance between the *orbits* of the follower and chief is large compared with the chief orbital radius the linearized curvilinear solution also fails and more complicated solution strategies are required.

One straightforward approach, also known in the literature, is to vectorially subtract the individual Keplerian conics therefore obtaining an exact solution to the problem. An elegant vectorial relative motion formulation has been proposed by Condurache and Martinusi.[3]

This approach would however still require the numerical solution of Kepler´s equation for one of the two orbit and would in general be affected by a loss of accuracy for short relative distances due to the subtraction of nearly equal numbers. An interesting algorithm to overcome the short separation distance limitation was proposed in 1970 by Lancaster.[4]

A considerable improvement with respect to the vectorial solution has been proposed by other authors. Berreen and Crisp[5] and Berreen[6] in the 70s propose a coordinate transformation of the known Keplerian orbital motions to curvilinear rotating coordinates. The use of a series expansion in eccentricity and a change of independent variable from time to eccentric anomaly leads to a rather compact and insightful analytical expression of the "cycloidal" relative motion that is valid for small eccentricities of the follower orbit, or, equivalently, for small initial relative velocities. The work by Berreen and Berreen and Crisp is limited by the fact that it only deals with coplanar motion and that it does not provide a time-explicit solution. More recently, Gurfil and Kasdin[7] develop an approximate solution that overcomes these limitations by resorting to a Taylor expansion around a degenerate set of orbital elements composed by the semi-major axis, eccentricity, inclination and mean longitude of the follower orbit. The results is a Cartesian coordinate solution separating periodic and secular terms and offering improved accuracy.

The present work is a step forward with respect to the previous references based on a Taylor expansion with respect to the orbit eccentricity only (retaining the full influence of orbital inclination in the out-of-plane dynamics) and the use of curvilinear coordinates instead of Cartesian, which provides a much more robust and compact solution. In addition, it provides key relations between the relative motion initial state and the resulting orbital elements of the follower in compact analytical form.

The structure of the article is as follows. First we introduce the curvilinear coordinate parametrization of the relative motion describing all relevant reference frames and transformations. We then derive the relations between the relative motion initial state and orbital element. The full equations of relative motion, already obtained implicitly by Alfriend et al.[2] and in explicit form by Geller and Lovell[8] are then derived following a different approach. After reporting the corresponding linearized solution we move on to the derivation of the exact non-linear solution starting from the individual Keplerian orbits. With the purpose of obtaining a good approximation, approximate solution with time as independent variable we perform a Taylor series expansion of the exact solution for small eccentricity of the follower spacecraft obtaining a compact solution in the form of a Fourier series and a secular term. Finally, the accuracy of the quasi-planar solution is tested by comparison with the exact numerical solution.

CURVILINEAR COORDINATES

With reference to Figure 1, let a chief spacecraft be in a circular Keplerian orbit whose radius and inverse mean motion are employed, throughout this article, as unit of distance and time, respectively.

Let us introduce a local vertical local horizontal (LVLH) rotating orbital reference frame $\mathcal{F}$ attached to the chief with orthonormal basis $\{\mathbf{i}', \mathbf{j}', \mathbf{k}'\}$ so that the relative position and velocity of a follower spacecraft relative to $\mathcal{F}$ can be written, respectively, as:

$$\mathbf{r}' = x\mathbf{i}' + y\mathbf{j}' + z\mathbf{k}'. \tag{1}$$

$$\mathbf{v}' = \dot{x}\mathbf{i}' + \dot{y}\mathbf{j}' + \dot{z}\mathbf{k}'. \tag{2}$$

Let us define the curvilinear coordinates ρ and θ of the follower as:

$$\rho = -1 + \sqrt{(x+1)^2 + y^2} \tag{3}$$

$$\theta = \mathrm{atan2}\,(y, 1 + x) \tag{4}$$

such that the in-plane relative Cartesian coordinates of the follower with respect to the chief can always be obtained as:

$$x = -1 + (1 + \rho)\cos\theta \tag{5}$$

$$y = (1 + \rho)\sin\theta \tag{6}$$

and the derivatives:

$$\dot{x} = \dot{\rho}\cos\theta - \dot{\theta}\,(1 + \rho)\sin\theta \tag{7}$$

$$\dot{y} = \dot{\rho}\sin\theta + \dot{\theta}\,(1 + \rho)\cos\theta. \tag{8}$$

It is convenient to introduce the radial and transversal unit vectors $\mathbf{u}_\rho$, $\mathbf{u}_\theta$, defined, respectively, as:

$$\mathbf{u}_\rho = \cos\theta\mathbf{i}' + \sin\theta\mathbf{j}'$$

$$\mathbf{u}_\theta = -\sin\theta\mathbf{i}' + \cos\theta\mathbf{j}',$$

so that the follower position, velocity and acceleration relative to $\mathcal{F}$ can also be written as:

$$\mathbf{r}' = (1 + \rho - \cos\theta)\,\mathbf{u}_\rho + \sin\theta\mathbf{u}_\theta + z\mathbf{k}'. \tag{9}$$

$$\mathbf{v}' = \dot{\rho}\mathbf{u}_\rho + \dot{\theta}\,(1 + \rho)\,\mathbf{u}_\theta + \dot{z}\mathbf{k}'. \tag{10}$$

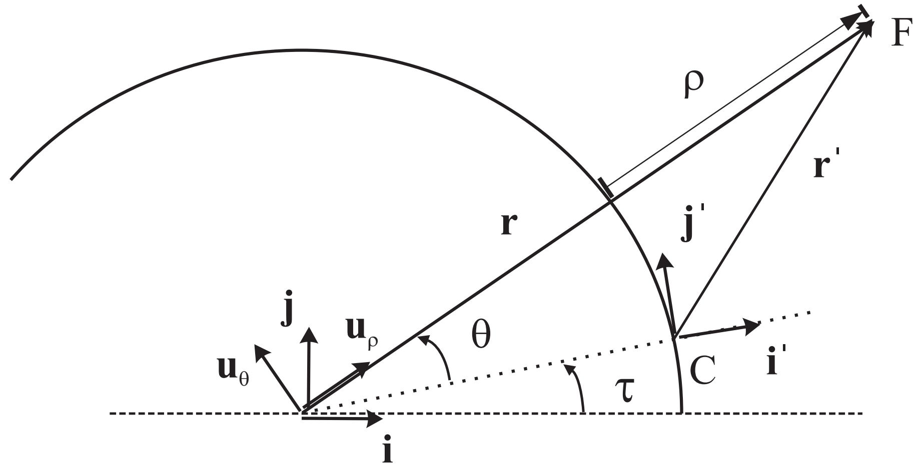

Figure 1. Relative motion geometry

$$\mathbf{a}' = \left[\ddot{\rho} - \dot{\theta}^2 \left(1+\rho\right)\right] \mathbf{u}_\rho + \left[\left(1+\rho\right) \ddot{\theta} + 2\dot{\theta}\dot{\rho}\right] \mathbf{u}_\theta + \ddot{z}\mathbf{k}'. \tag{11}$$

Orbital elements

The inertial position and velocity vectors of the follower using curvilinear variables (ρ, θ) can be obtained from the corresponding position (Eq. 1) and velocity (Eq. 2) with respect to the $\mathcal{F}$ frame as (see also Figure 1):

$$\mathbf{r} = \mathbf{r}_C + \mathbf{r}' \tag{12}$$

$$\mathbf{v} = \mathbf{v}' + \mathbf{v}_C + \boldsymbol{\omega}_C \times \mathbf{r}' \tag{13}$$

where $\mathbf{r}_C$,$\mathbf{v}_C$ and $\boldsymbol{\omega}_C$ are, respectively the inertial position and velocity of the chief and the angular rate of $\mathcal{F}$ and yield:

$$\mathbf{r}_C = \mathbf{i}' = \cos\theta \mathbf{u}_\rho - \sin\theta \mathbf{u}_\theta,$$

$$\mathbf{v}_C = \mathbf{j}' = \sin\theta \mathbf{u}_\rho + \cos\theta \mathbf{u}_\theta,$$

$$\boldsymbol{\omega}_C = \mathbf{k}'. \tag{14}$$

By taking into account the preceding relations and Eqs. (1,2) one readily obtains:

$$\mathbf{r} = r_\rho \mathbf{u}_\rho + r_z \mathbf{k}' \tag{15}$$

$$\mathbf{v} = v_\rho \mathbf{u}_\rho + v_\theta \mathbf{u}_\theta + v_z \mathbf{k}' \tag{16}$$

with:

$$r_\rho = 1 + \rho, \qquad r_z = z,$$

$$v_\rho = \dot{\rho}, \qquad v_\theta = (1+\rho)\left(1+\dot{\theta}\right), \qquad v_z = \dot{z}.$$

The follower angular momentum can now be computed as:

$$\mathbf{h} = \mathbf{r} \times \mathbf{v} = (h_\rho \cos\theta + h_\theta \sin\theta)\,\mathbf{i}' + (h_\rho \sin\theta - h_\theta \cos\theta)\,\mathbf{j}' + h_z \mathbf{k}'$$

with:

$$h_\rho = -z\,(1+\rho)\left(1+\dot{\theta}\right),$$

$$h_\theta = z\dot{\rho} - \dot{z}\,(1+\rho)\,.$$

$$h_z = (1+\rho)^2\left(1+\dot{\theta}\right),$$

from which the right ascension of the ascending node, referred to an inertial frame parallel to the $\mathcal{F}$ frame at $\tau = 0$, can be obtained as:

$$\Omega = \operatorname{atan2}\left(-\frac{\mathbf{h}\cdot\mathbf{i}'}{|\mathbf{h}|}, \frac{\mathbf{h}\cdot\mathbf{j}'}{|\mathbf{h}|}\right) + \tau = \theta + \alpha + \tau,$$

with:

$$\alpha = \operatorname{atan2}\left(h_\rho, -h_\theta\right),$$

while the inclination obeys:

$$\cos i = \frac{h_z}{h},$$

with:

$$h = \sqrt{h_\rho^2 + h_\theta^2 + h_z^2}.$$

The semi-major axis follows from the vis-viva equation as:

$$a = \frac{r}{2 - rv^2},$$

where:

$$r = \sqrt{(1+\rho)^2 + z^2}$$

$$v = \sqrt{(1+\rho)^2 \left(1+\dot{\theta}\right)^2 + \dot{\rho}^2 + \dot{z}^2}.$$

The eccentricity vector can be computed as:

$$\mathbf{e} = \mathbf{v} \times \mathbf{h} - \frac{\mathbf{r}}{r} = e_\rho \mathbf{u}_\rho + e_\theta \mathbf{u}_\theta + e_z \mathbf{k}',$$

with:

$$e_\rho = (1+\rho)^3 \left(1+\dot{\theta}\right)^2 - \dot{z} z \dot{\rho} + (1+\rho)\, \dot{z}^2 - \frac{1+\rho}{r},$$

$$e_\theta = -(1+\rho)\left(1+\dot{\theta}\right)\left[(1+\rho)\,\dot{\rho} + z\dot{z}\right],$$

$$e_z = \dot{\rho}^2 z - \dot{\rho}\dot{z}\,(1+\rho) + z\,(1+\rho)^2 \left(1+\dot{\theta}\right)^2 - \frac{z}{r},$$

so that the eccentricity yields:

$$e^2 = e_\rho^2 + e_\theta^2 + e_z^2.$$

The argument of pericenter ω can now be derived as:

$$\omega = \operatorname{atan2}\left(\frac{(\mathbf{n} \times \mathbf{e}) \cdot \mathbf{h}}{|\mathbf{h}|\,|\mathbf{e}|}, \frac{\mathbf{n} \cdot \mathbf{e}}{|\mathbf{e}|}\right),$$

where $\mathbf{n}$ is the node line unit vector:

$$\mathbf{n} = \cos\left(\mathit{\Omega} - \tau\right)\mathbf{i}' + \sin\left(\mathit{\Omega} - \tau\right)\mathbf{j}' = \cos\alpha \mathbf{u}_\rho + \sin\alpha \mathbf{u}_\theta$$

so that:

$$\omega = \operatorname{atan2}\left(\frac{\sin\alpha\,(e_z h_\rho - h_z e_\rho) - \cos\alpha\,(e_z h_\theta - h_z e_\theta)}{he}, \frac{e_\rho \cos\alpha + e_\theta \sin\alpha}{e}\right).$$

The true anomaly reads:

$$\nu = \operatorname{atan2}\left(\frac{(\mathbf{n} \times \mathbf{r}) \cdot \mathbf{h}}{|\mathbf{h}|\,|\mathbf{r}|}, \frac{\mathbf{n} \cdot \mathbf{r}}{|\mathbf{r}|}\right) - \omega = -\alpha - \omega.$$

From a reference value ν_0 of the true anomaly at epoch one can obtain the corresponding eccentric anomaly as:

$$E_0 = \operatorname{atan2}\left(\sqrt{1-e^2}\sin\nu_0,\; e + \cos\nu_0\right),$$

so that the time evolution of the follower mean anomaly reads:

$$M = M_0 + n\tau = E_0 - e\sin E_0 + \frac{\tau}{a^{3/2}}.$$

Equations of motion

The exact equations of relative motion obey:

$$\mathbf{a}' = -\boldsymbol{\omega}_C \times \left(\boldsymbol{\omega}_C \times \mathbf{r}'\right) - 2\boldsymbol{\omega}_C \times \mathbf{v}' - \mathbf{a}_C - \frac{\mathbf{r}}{r^3}, \tag{17}$$

where:

$$\mathbf{a}_C = -\mathbf{i}' = -\cos\theta \mathbf{u}_\rho + \sin\theta \mathbf{u}_\theta \tag{18}$$

is the chief inertial acceleration.

By substituting Eqs. (9,10,14,15,18) into Eqs. (17) one obtains the exact relative motion equations in curvilinear coordinates (see[8] for an alternative derivation):

$$\left\{\begin{array}{c} \ddot{\rho} - 2\dot{\theta} - 3\rho = a_{i\rho} + a_{g\rho} \\ \ddot{\theta} + 2\dot{\rho} = a_{i\theta} \\ \ddot{z} + z = a_{gz} \end{array}\right. \tag{19}$$

where the four right hand side terms contain the non-linear contributions of the generalized inertial ($a_{i\rho}$,$a_{i\theta}$) and gravitational ($a_{g\rho}$,a_{gz}) accelerations and read:

$$a_{i\rho} = \dot{\theta}^2 \left(1 + \rho\right) + 2\dot{\theta}\rho \tag{20}$$

$$a_{g\rho} = -2\rho + 1 - \frac{(1+\rho)}{\left[(1+\rho)^2 + z^2\right]^{3/2}} \tag{21}$$

$$a_{i\theta} = \frac{2\dot{\rho}\left(\rho - \dot{\theta}\right)}{1+\rho} \tag{22}$$

$$a_{gz} = z - \frac{z}{\left[(1+\rho)^2 + z^2\right]^{3/2}} \tag{23}$$

Linearized solution

As noted by other authors,[8] when $\left(\rho_0, \theta_0, z_0, \dot{\rho}_0, \dot{\theta}_0, \dot{z}_0\right) \ll \mathbf{1}$ all non-linear perturbing terms can be neglected and the equations of relative motion become linear with the same structure of the Clohessy-Wiltshire equations.

$$\left\{\begin{array}{c} \ddot{\rho}_l - 2\dot{\theta}_l - 3\rho_l = 0 \\ \ddot{\theta}_l + 2\dot{\rho}_l = 0 \\ \ddot{z}_l + z_l = 0 \end{array}\right.$$

whose solution is:

$$\begin{bmatrix} \rho_l \\ \theta_l \\ z_l \\ \dot{\rho}_l \\ \dot{\theta}_l \\ \dot{z}_l \end{bmatrix} = \begin{bmatrix} 4-3\mathrm{C}\tau & 0 & 0 & \mathrm{S}\tau & 2-2\mathrm{C}\tau & 0 \\ -6\tau+6\mathrm{S}\tau & 1 & 0 & -2+2\mathrm{C}\tau & -3\tau+4\mathrm{S}\tau & 0 \\ 0 & 0 & \mathrm{C}\tau & 0 & 0 & \mathrm{S}\tau \\ 3\mathrm{S}\tau & 0 & 0 & \mathrm{C}\tau & 2\mathrm{S}\tau & 0 \\ -6+6\mathrm{C}\tau & 0 & 0 & -2\mathrm{S}\tau & -3+4\mathrm{C}\tau & 0 \\ 0 & 0 & -\mathrm{S}\tau & 0 & 0 & \mathrm{C}\tau \end{bmatrix} \begin{bmatrix} \rho_0 \\ \theta_0 \\ z_0 \\ \dot{\rho}_0 \\ \dot{\theta}_0 \\ \dot{z}_0 \end{bmatrix}, \quad (24)$$

where $S\tau = \sin\tau, C\tau = \cos\tau$.

The corresponding Cartesian coordinate solution can then be obtained through Eqs. (5,6) and reduces to the Clohessy-Wiltshire solution for $\theta_l \ll 1$.

EXACT SOLUTION

Eqs. (19) cannot be solved analytically. However, a solution can be found by indirectly looking at the individual Keplerian orbits of the chief and the follower. Although this approach is certainly not new and similar solutions are available in the literature we report here the derivation for completeness.

The relative position of the follower with respect to the chief projected onto the chief-centered orbital frame can be expressed in Cartesian coordinates as:

$$\begin{cases} x = (X_f - X_c)\cos\tau + (Y_f - Y_c)\sin\tau \\ y = -(X_f - X_c)\sin\tau + (Y_f - Y_c)\cos\tau \\ z = Z_f - Z_c \end{cases}, \quad (25)$$

where the planetocentric inertial position of the chief and the follower obey, respectively:

$$\begin{bmatrix} X_c \\ Y_c \\ Z_c \end{bmatrix} = \begin{bmatrix} \cos\tau \\ \sin\tau \\ 0 \end{bmatrix}, \quad (26)$$

$$\begin{bmatrix} X_f \\ Y_f \\ Z_f \end{bmatrix} = a \begin{bmatrix} C\Omega & -S\Omega & 0 \\ S\Omega & C\Omega & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & Ci & -Si \\ 0 & Si & Ci \end{bmatrix} \begin{bmatrix} C\omega & -S\omega & 0 \\ S\omega & C\omega & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos E - e \\ \sqrt{1-e^2}\sin E \\ 0 \end{bmatrix}. \quad (27)$$

From Eqs. (26,27,3,4) one can obtain the exact relative motion in cylindrical coordinates:

$$\begin{cases} \rho = \sqrt{X_f^2 + Y_f^2} - 1, \\ \theta = -\tau + \mathrm{atan2}\left(\mathrm{Y_f}, \mathrm{X_f}\right) \\ z = Z_f \end{cases} \quad (28)$$

Although exact, the solutions (25,28) cannot provide a fully analytical description of relative motion in time. This is because the time evolution of E needs to be obtained numerically through the solution of Kepler´s equation. For this reason and given their complex structure, they do not provide any particular insight into the kinematic structure of the problem as noted by other authors (see for instance[5]).

LOW-ECCENTRICITY, LOW-INCLINATION SOLUTION

When the follower eccentricity is not too large one can find an approximated analytical solution for Kepler´s equation. The expansion of $\cos E$ and $\sin E$ using Bessel function of the first kind J_k obeys (see Battin[9]):

$$\cos E = -\frac{e}{2} + \sum_{k=1}^{\infty} \frac{2}{k^2} \frac{dJ_k\left(ke\right)}{de} \cos kM,$$

$$\sin E = \frac{2}{e} \sum_{k=1}^{\infty} \frac{1}{k} J_k\left(ke\right) \sin kM.$$

After substituting the preceding relations into Eqs.(28) , expanding for small eccentricities and setting $i = 0$ one obtains the compact expressions (here displayed up to the third order in eccentricity):

$$\begin{aligned} \rho \simeq \; & -1 + a\left[1 + \frac{e^2}{2} + \left(-e + \frac{3e^3}{8}\right)\cos M + \right. \\ & \left. -\frac{e^2}{2}\cos 2M - \frac{3e^3}{8}\cos 3M\right] \end{aligned} \tag{29}$$

$$\begin{aligned} \theta \simeq \; & \Omega + \omega + M_0 + \tau\left(n-1\right) + \left(2e - \frac{e^3}{4}\right)\sin M + \\ & + \frac{5}{4}e^2 \sin 2M + \frac{13}{12}e^3 \sin 3M. \end{aligned} \tag{30}$$

where the orbital parameter e ,M, ω and Ω can be obtained from the initial relative motion conditions using the expressions derived in the first section of the article.

An approximate expression for the out-of-plane motion can be obtained in a similar way but without eliminating the inclination:

$$\begin{aligned} z \simeq \; & \frac{a\,\sin i}{8}\left\{\sin\omega\left[-12e + \left(8-3e^2\right)\cos M + \left(4e - \frac{8}{3}e^3\right)\cos 2M + 3e^2\cos 3M\right] + \right. \\ & \left. \cos\omega\left[\left(8-5e^2\right)\sin M + \left(4e - \frac{10}{3}e^3\right)\sin 2M + 3e^2 \sin 3M\right]\right\} \end{aligned} \tag{31}$$

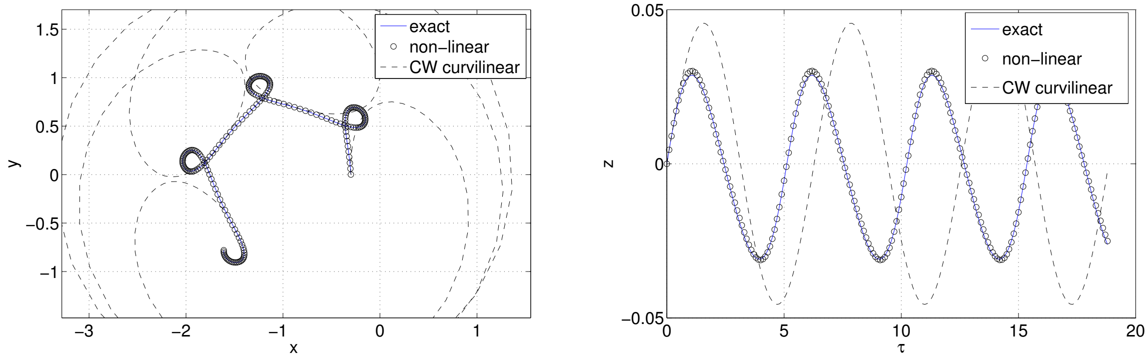

Figure 2. Comparison of different relative motion formulations. The follower initial conditions are set to $\rho_0 = -0.3, \dot{\rho}_0 = 0, \theta_0 = 0, \dot{\theta}_0 = 0.8667, z_0 = 0, \dot{z}_0 = 0.0456$, providing eccentricity and inclination of 0.2 and 2 deg, respectively.

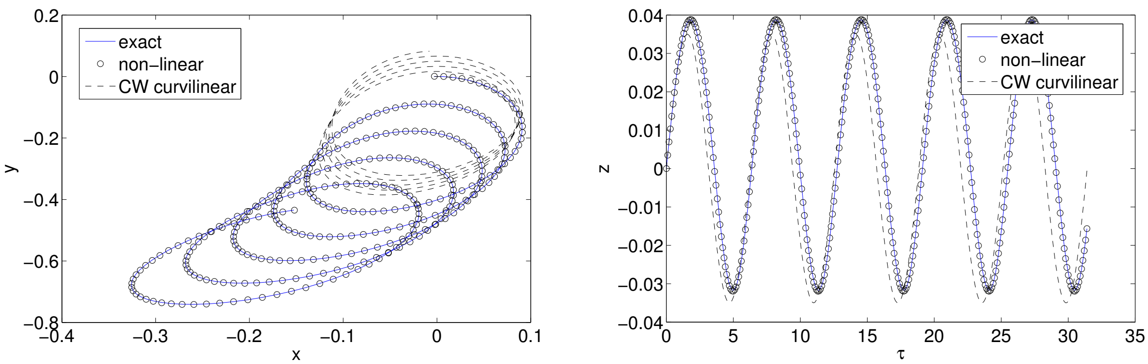

Figure 3. Comparison of different relative motion formulations. The follower initial conditions are set to $\rho_0 = -0.003, \dot{\rho}_0 = 0.1, \theta_0 = 0, \dot{\theta}_0 = 0.0051, z_0 = 0, \dot{z}_0 = 0.035$, providing eccentricity and inclination of 0.1 and 2 deg, respectively.

RESULTS

Figures 2 and 3 display two test cases in which the proposed compact non-linear solution (Eqs.(29-31)) is compared to its linear counterparts (Eqs(24)) and to the exact solution obtained by numerical integration of Eqs.(19) . As it can be seen the linearized solution struggles to reproduce the real motion as the initial conditions grow. The improvement of the proposed non-linear solution is remarkable.

CONCLUSIONS

A new approximate solution for the relative motion with respect to a satellite in circular orbit has been developed based on the use of curvilinear coordinates together with a Taylor expansion for small eccentricity. The solution greatly improves its linearized counterpart and can be employed to study the relative motion in the presence of nonlinearities up to moderately small values of the eccentricity and inclination of the follower orbit. Useful relations between the relative motion curvilinear coordinates and the follower orbital elements have also been provided in compact form.

ACKNOWLEDGMENTS

This work has been supported by the Spanish Ministry of Economy and Competitiveness within the framework of the research project "Dynamical Analysis, Advanced Orbital Propagation, and Simulation of Complex Space Systems" (ESP2013-41634-P). The authors also want to thank the funding received from the European Union Seventh Framework Programme (FP7/2007-2013) under grant agreement N 607457 (LEOSWEEP).

REFERENCES

[1] V. Il'in and G. Kuzmak, "Optimal'nye perelety kosmicheskikh apparatov (Optimal Flights of Spacecraft)," 1976.

[2] K. Alfriend, S. R. Vadali, P. Gurfil, J. How, and L. Breger, *Spacecraft formation flying: Dynamics, control and navigation*, Vol. 2. Butterworth-Heinemann, 2009.

[3] D. Condurache and V. Martinusi, "Quaternionic Exact Solution to the Relative Orbital Motion Problem," *Journal of guidance, control, and dynamics*, Vol. 33, No. 4, 2010, pp. 1035–1047.

[4] E. Lancaster, "Relative motion of two particles in elliptic orbits," *AIAA Journal*, Vol. 8, No. 10, 1970, pp. 1878–1879.

[5] T. Berreen and J. Crisp, "An exact and a new first-order solution for the relative trajectories of a probe ejected from a space station," *Celestial mechanics*, Vol. 13, No. 1, 1976, pp. 75–88.

[6] T. Berreen, "The trajectories of a spaceprobe ejected from a space station in circular orbit," *Celestial mechanics*, Vol. 20, No. 4, 1979, pp. 405–431.

[7] P. Gurfil and N. J. Kasdin, "Nonlinear modelling of spacecraft relative motion in the configuration space," *Journal of Guidance, Control, and Dynamics*, Vol. 27, No. 1, 2004, pp. 154–157.

[8] D. K. Geller and T. A. Lovell, "Relative Orbital Motion and Angles-Only Relative State Observability in Cylindrical Coordinates," *AAS/AIAA Space Flight Mechanics Meeting*, American Astronomical Soc. Washington, DC, 2014, pp. 133–148.

[9] R. Battin, *An introduction to the mathematics and methods of astrodynamics*. Aiaa, 1999.