"Design and Implementation of Maple Packages for Processing Offsets and Conchoids"
Annals of Mathematics and Artificial Intelligence (2016). DOI: 10.1007/s10472-016-9504-z
and it is available at link.springer.com

Design and Implementation of Conchoid and Offset Processing Maple Packages

Juana Sendra[1], David Gómez[2] and Valerio Morán[2]

[1] Dpto. de Matemática Aplicada a la I.T. de Telecomunicación.
Research Center on Software Technologies and Multimedia Systems
for Sustainability (CITSEM)
Universidad Politécnica de Madrid (UPM)
[2] E.U.I.T.Telecomunicación,
Universidad Politécnica de Madrid
E-28031 Madrid, Spain

Abstract. This paper is framed within the problem of analyzing the rationality of the components of two classical geometric constructions, namely the offset and the conchoid to an algebraic plane curve and, in the affirmative case, the actual computation of parametrizations. We recall some of the basic definitions and main properties on offsets (see [18]), and conchoids (see [20]) as well as the algorithms for parametrizing their rational components (see [1] and [21], respectively). Moreover, we implement the basic ideas creating two package in the computer algebra system Maple to analyze the rationality of conchoids and offset curves, as well as the corresponding help pages. In addition, we present a brief atlas where the offset and conchoids of several algebraic plane curves are obtained, its rationality analyzed, and parametrizations are provided using the created packages.

Key Words: Offset Curves, Conchoids Curves, Rational curves, parametrization, symbolic mathematical software, computer-aided geometric design.

Introduction

The main geometric objects that we deal with in this paper are the offsets and the conchoids to an algebraic plane curve. Whereas the offset operation is well known and implemented in most CAD-software systems, the conchoid operation is less known, although already mentioned by the ancient Greeks, and recently studied by some authors. These two operations are algebraic and create new objects from the given input objects. There is a surprisingly simple relation between the offset and the conchoid operation. There exists a rational bijective quadratic map which transforms a given curve F and its offset curve F_d to a curve G and its conchoidal curve G_d, and vice versa (see [15]).

Essentially, the intuitive idea of these geometric constructions are the following. Let $\mathbb{K}$ be an algebraically closed field of characteristic zero (say $\mathbb{K} = \mathbb{C}$), and $\mathcal{C}$ an irreducible curve in $\mathbb{K}^2$. *The offset curve (or parallel curve) to $\mathcal{C}$ at distance d*, denoted by $\mathcal{O}_d(\mathcal{C})$, is *essentially* the envelope of the system of circles centered at the points of $\mathcal{C}$ with fixed radius d (see Fig.1, left and, for a formal definition, see [1]). In particular, if $\mathcal{C}$ is parametrized by $\mathcal{P}(t) \in \mathbb{K}(t)^2$, the offset to $\mathcal{C}$ corresponds to the Zariski closure of the set in $\mathbb{K}^2$ generated by the formula

$$\mathcal{P}(t) \pm d\,\frac{\mathcal{N}(t)}{\|\mathcal{N}(t)\|}$$

Email addresses: `jsendra@euitt.upm.es` (Corresponding author: Juana Sendra), `dgsp@alumnos.upm.es` (David Gómez), `vmoran@alumnos.upm.es` (Valerio Morán)

where $\mathcal{N}(t)$ is the normal vector to $\mathcal{C}$ associated with $\mathcal{P}(t)$.

The term "parallel" was apparently introduced by Leibniz in [8] for the case of plane curves. Also, in elementary texts on differential geometry or in some books on algebraic geometry (see [17]) some elementary aspects of parallel curves are studied. Nevertheless, in the 1980s, CAGD (Computer Aided Geometric Design) community started to be interested on the topic, and they began to address problems related to offsets to curves and surfaces, due to the important role that offsets play in practical applications as tolerance analysis, geometric control, robot path-planning and numerical-control machining problems, etc. [6]. As a consequence of this applicability, many interesting questions directly related to algebraic geometry have been addressed (see, e.g. [1], [2], [4], [5], [9], [16], [18], [19]) and, currently, the study of offsets continue being an active research area.

The conchoid of a plane curve arise in a big variety of practical applications, namely the design of the construction of buildings, in astronomy [7], in electromagnetism research [24], optics [3], physics [23], mechanical engineering and biological engineering [10], [11], in fluid mechanics [22], etc. The conchoid concept is rather intuitive. More precisely, given a plane curve $\mathcal{C}$ (base curve) and a fixed point A (focus), consider the line $\mathcal{L}$ joining A to a point P of $\mathcal{C}$. Consider now the points Q of intersection of $\mathcal{L}$ with a circle of radius d centered at P. The geometric locus of Q as P moves along $\mathcal{C}$ is called *the conchoid of $\mathcal{C}$ from focus A at a distance d* and denoted by $\mathfrak{C}_d^A(\mathcal{C})$ (see Fig.1 right, for the geometric construction of the conchoid and, for a formal definition, see [20]). The *Conchoid of Nicomedes* and the *Limacon of Pascal* are the two classic examples of conchoids, and the best known. They appear when the base curve is a line or a circle, respectively.

In [18], and [20] the authors analyze theoretically offsets and conchoids and the most important properties of them, from the point of view of Algebraic Geometry. For this, incidence diagrams are used to give a formal definition of these concepts of an algebraic curve. Certain aspects as the rationality of the new curve and the possible inheritance of this property from the base curve have been treated by the authors. Furthermore, the study of the conchoids has been extended to surfaces (see [12], [13], [14]), and the relation between offset and conhcoid is an active problem (see [15]).

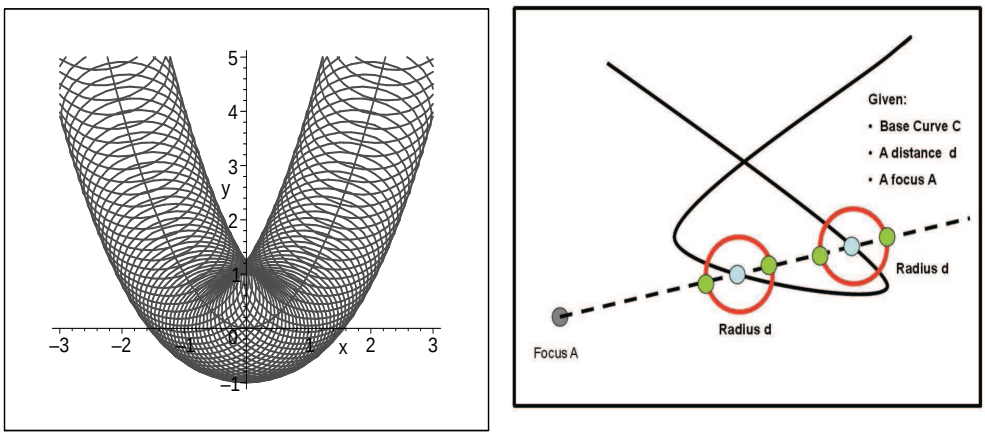

Fig. 1. Left: Construction of the offset to the parabola, Right: Geometric construction of the conchoid.

The paper is structured as follows. In Section 1, we recall some of the basic definitions and main properties on offsets and conchoids of algebraic plane curves (see [18], [20]). We provide algorithms to analyze the rationality of the components of these new objects (see [1], [21]), and in the affirmative case, rational parametrizations are given. In Section 2 we present the creation of two packages in the computer algebra system Maple to analyze the rationality of offset and conchoids curves respectively, whose procedures are based on the above algorithms, as well as the corresponding help pages. In Section 3, we illustrate the package running presenting a brief atlas where the offset and conchoids of several algebraic plane curves are obtained, and its rationality analyzed. Furthermore, in case of genus zero, a rational parametrization is computed.

1 Parametrization Algorithms

In this Section we summarize the results on the rationality of the offsets and conchoids of curves, presented in [1], [21] respectively, by deriving an algorithm for parametrizing them.

The Offset Rationality Problem

The rationality of the components of the offsets is characterized by means of the existence of parametrizations of the curve whose normal vector has rational norm, and alternatively by means of the rationality of the components of an associated curve, that is usually simpler than the offset. As a consequence, one deduces that offsets to rational curves behave as follows: they are either reducible with two rational components (double rationality), or rational, or irreducible and not rational.

For this purpose, we first introduce two concepts: *Rational Pythagorean Hodographs* and *curve of reparametrization of the offset*. Let $\mathcal{P}(t) = (P_1(t), P_2(t)) \in \mathbb{K}(t)^2$ be a rational parametrization of $\mathcal{C}$. Then, $\mathcal{P}(t)$ is RPH (**Rational Pythagorean Hodograph**) if its normal vector $\mathcal{N}(t) = (N_1(t), N_2(t))$ satisfies that

$$N_1(t)^2 + N_2(t)^2 = m(t)^2,$$

with $m(t) \in \mathbb{K}(t)$. For short we will express this fact writing $\|\mathcal{N}(t)\| \in \mathbb{K}(t)$. On the other hand, we define the **reparametrizing curve of $\mathcal{O}_d(\mathcal{C})$ associated with** $\mathcal{P}(t)$ as the curve generated by the primitive part with respect to x_2 of the numerator of

$$x_2^2\, P_1^{'}(x_1) - P_1^{'}(x_1) + 2\, x_2\, P_2^{'}(x_1),$$

where $P_i^{'}$ denotes the derivative of P_i. In the following, we denote by $\mathcal{G}_{\mathcal{P}}^{\mathcal{O}}(\mathcal{C})$ the reparametrizing curve of $\mathcal{O}_d(\mathcal{C})$ associated with $\mathcal{P}(t)$.

Summarizing the results in [1], one can outline the following algorithm for offsets.

<u>Algorithm: offset parametrization</u>

- GIVEN: a proper rational parametrization $\mathcal{P}(t)$ of a plane curve $\mathcal{C}$ in $\mathbb{K}^2$ and $d \in \mathbb{K}$.
- DECIDE: whether the components of $\mathcal{O}_d(\mathcal{C})$ are rational.
- DETERMINE: (in the affirmative case) a rational parametrization of each component of $\mathcal{O}_d(\mathcal{C})$.

1. Compute the normal vector $\mathcal{N}(t)$ of $\mathcal{P}(t)$. IF $||\mathcal{N}(t)|| \in \mathbb{K}(\bar{t})$ THEN RETURN $\mathcal{O}_d(\mathcal{C})$ has two rational components parametrized by $\mathcal{P}(t) \pm \frac{d}{||\mathcal{N}(t)||}\mathcal{N}(t)$.
2. Determine $\mathcal{G}_{\mathcal{P}}^{\mathcal{O}}(\mathcal{C})$, and decide whether $\mathcal{G}_{\mathcal{P}}^{\mathcal{O}}(\mathcal{C})$ is rational.
3. IF $\mathcal{G}_{\mathcal{P}}^{\mathcal{O}}(\mathcal{C})$ is not rational THEN RETURN no component of $\mathcal{O}_d(\mathcal{C})$ is rational.

4. Else compute a proper parametrization $\mathcal{R}(t) = (\tilde{R}(t), R(t))$ of $\mathcal{G}_{\mathcal{P}}^{\mathcal{O}}(\mathcal{C})$ and return that $\mathcal{O}_d(\mathcal{C})$ is rational and that $\mathcal{Q}(t) = \mathcal{P}(\tilde{R}(t)) + \frac{2\,d\,R(t)}{N_2(\tilde{R}(t))(R(t)^2+1)}\mathcal{N}(\tilde{R}(t))$ where $\mathcal{N} = (N_1, N_2)$, parametrizes $\mathcal{O}_d(\mathcal{C})$.

The Conchoid Rationality Problem

In [21], it is proved that conchoids having all their components rational can only be generated by rational curves. Moreover, it is shown that reducible conchoids to rational curves have always their two components rational (double rationality). From these results, one deduces that the rationality of the conchoid component, to a rational curve, does depend on the base curve and on the focus but not on the distance. To approach the problem we use similar ideas to those for offsets introducing the notion of *reparametrization curve* as well as the notion of *rdf parametrization*. The rdf concept allows us to detect the double rationality while the reparametrization curve is a much simpler curve than the conchoid, directly computed from the input rational curve and the focus, and that behaves equivalently as the conchoid in terms of rationality. As a consequence of these theoretical results [21] provides an algorithm to solve the problem. The algorithm analyzes the rationality of all the components of the conchoid and, in the affirmative case, parametrizes them. The problem of detecting the focuses from where the conchoid is rational or with two rational components is, in general, open.
We say that a rational parametrization $\mathcal{P}(t) = (P_1(t), P_2(t)) \in \mathbb{K}(t)^2$ of $\mathcal{C}$ is at *rational distance to the focus* $A = (a, b)$ if

$$(P_1(t) - a)^2 + (P_2(t) - b)^2 = m(t)^2,$$

with $m(t) \in \mathbb{K}(t)$. For short, we express this fact saying that $\mathcal{P}(t)$ is rdf or A-rdf if we need to specify the focus. On the other hand, we define the *reparametrization curve of the conchoid* $\mathfrak{C}_d^A(\mathcal{C})$ *associated to* $\mathcal{P}(t)$, denoted by $\mathcal{G}_{\mathcal{P}}^{\mathfrak{C}}(\mathcal{C})$, as the primitive part with respect to x_2 of the numerator of

$$-2x_2(P_1(x_1) - a) + (x_2^2 - 1)(P_2(x_1) - b).$$

<u>Algorithm: conchoid parametrization</u>

- Given: a proper rational parametrization $\mathcal{P}(t)$ of a plane curve $\mathcal{C}$ in $\mathbb{K}^2$, a focus $A = (a, b)$, and $d \in \mathbb{K}$.
- Decide: whether the components of the concchoid $\mathfrak{C}_d^A(\mathcal{C})$ are rational.
- Determine: (in the affirmative case) a rational parametrization of each component of $\mathfrak{C}_d^A(\mathcal{C})$.

1. Compute the primitive part $g(x_1, x_2)$ w.r.t. x_2 of the numerator of $-2x_2(P_1(x_1) - a) + (x_2^2 - 1)(P_2(x_1) - b)$.
2. If g is reducible return that $\mathfrak{C}_d^A(\mathcal{C})$ is double rational and that $\mathcal{P}(t) + \frac{d}{\pm\|\mathcal{P}(t)-A\|}(\mathcal{P}(t) - A)$ parametrize the two components.
3. Check whether the genus of $\mathcal{G}_{\mathcal{P}}^{\mathfrak{C}}$ is zero. If not return that $\mathfrak{C}_d^A(\mathcal{C})$ is not rational.
4. Compute a proper parametrization $(\phi_1(t), \phi_2(t))$ of $\mathcal{G}_{\mathcal{P}}^{\mathfrak{C}}$ and return that $\mathfrak{C}_d^A(\mathcal{C})$ is rational and that $\mathcal{P}(\phi_1(t)) + \frac{d}{\pm\|\mathcal{P}(\phi_1(t))-A\|}(\mathcal{P}(\phi_1(t)) - A)$ parametrizes $\mathfrak{C}_d^A(\mathcal{C})$.

We can note that the rationality of the both constructions are not equivalent. For instance, if $\mathcal{C}$ is the parabola of equation $y_2 = y_1^2$, that can be parametrized as (t, t^2), the offset at distance d is rational. However, the rationality of the conchoid of the parabola depends on the focus.

2 Implementation of conchoid and offset processing packages and help pages

In this section, we present the creation of two packages in the computer algebra system Maple, that we call **Conchoid** and **Offset**. These packages compute the implicit equation, and analyze the rationality and the reducibility of conchoids and offset curves respectively, providing rational parametrizations in case of genus zero. In addition, it allows us to display plots. These packages consist in several procedures that are based on the above parametrization algorithms.

In the following, we give a brief description of the procedures and we show one of the help pages for one of the maple functions. The procedure codes and packages are available contacting with the corresponding author.

2.1 Procedures of the Conchoid Package

getImplConch

This procedure determines the implicit equation of the conchoid of an algebraic plane curve at a fixed focus and a fixed distance. For this purpose, we use Gröebner Basis to solve the system of equations consisting on the circle centered at generic point of the initial curve $\mathcal{C}$ and radius d, the straight line from the focus A to the generic point of the initial curve $\mathcal{C}$, and the initial curve $\mathcal{C}$.

getParamConch

Firstly this procedure checks whether the conchoid of a rational curve is irreducible or it has two rational components. For this purpose, a proper rational parametrization of the initial curve is RDF is analized. In affirmative case, the procedure outputs a message indicating reducibility (the conchoid has two rational components) and a rational parametrization for each component is displayed. Otherwise, the conchoid is irreducible and the reparametrization curve is computed in order to study its rationality. In the affirmative case, it provides a rational parametrization by means of a rational parametrization of the reparametrizing curve and it outputs a message indicating irreducibility and rationality.

plotImplConch

This procedure computes the conchoids curve using *getImplConch* procedure, and then it plots both the initial curve and its conchoid within the coordinates axes interval $[-a, a] \times [-a, a]$.

2.2 Procedures of the Offset Package

ImplicitOFF

This procedure determines the implicit equation of the offset of an algebraic plane curve at a fixed distance. For this purpose, we use resultants to solve the system of equations consisting on the circle centered at a generic point of the initial curve $\mathcal{C}$ and radius d, and the normal line at each point of $\mathcal{C}$.

OFFparametric

This procedure analyzes the rationality of the offset of a rational plane curve. For this purpose, first it decides whether the offset is irreducible or it has two rational components. In case of reducibility, the procedure outputs a rational parametrization for each component, using the RPH concept. Otherwise, it checks whether the offset is rational or not. In the affirmative case, it provides a rational parametrization by means of a rational parametrization of the reparametrizing curve.

OFFplot

This procedure computes the offset curve at a generic distance, d, and then replaces d with a fixed value, *dist*. Finally, it plots both the initial curve and its offset at a distance *dist* within the coordinates axes interval $[-a, a] \times [-a, a]$.

Once we have implemented the conchoids and offset procedures in Maple, we have created a two packages containing them, called *Conchoid* and *Offset*, respectively. Now, if we want to use the package, first of all we have to specify, with the command *libname*, the directory where the file containing the offset procedures is located in our computer. Finally we have to use the command *with* followed by the name of the package, to charge the package in memory and then it will be ready to be used.

In addition, we have created the help pages associated to the procedures. Figure 2 shows one of the Maple help pages created.

Conchoid[getParamConch] - Analyze the rationality of the conchoid of a given rational plane curve and find a parametrization in case of a rational conchoid.

Calling Sequence

getParamConch(P, a, b, d)

Parameters

P – proper parametrization of the initial plane curve C.
d – fixed distance, d>0.
(a,b) – values of 'y1' and 'y2' axis, in focus A(a,b) of conchoid.

Description

- The getParamConch command analyzes the rationality of inital curve's Conchoid and in case of rationality a rational parametrization is computed.
- Firstly, reducibility of Conchoid is studied: if the Conchoid is reducible then the conchoid has two rational components and the parametric expression of Conchoid will be reported.
- If Conchoid is irreducible then the reparametrization curve will be calculated, and the study of rationality of Conchoid will be done by studing the reparametrization curve.
- One of the following messages will be reported: "The Conchoid is reducible, so it is double-rational. Its components can be parametrized by: (Conchoid's components)"; "The Conchoid is irreducible and not rational."; "The Conchoid is irreducible and rational. It can be parametrized by: (Conchoid's component)"; "A mistake has been obtained in parametrization of Gp."; "Error, 'f' must be a curve."; "The value of 'd' must be bigger than cero.".
- Details of computation will also reported when the Conchoid is rational. More precisely, m(t) will be reported in case of a reducible conchoid, and Qp(t) and Delta(t) will reported in case of an irreducible conchoid.
- Parametric expresion of Conchoid´s component is reported when Conchoid is rational.
- For a description of the algorithm used see: J. Sendra, J. R. Sendra, "Rational parametrization of conchoids to algebraic curves", AAECC (2010) 21:285–308.

Examples

```
> with(Conchoid):
> P := [-2*(-1+t^2)/(1+t^2), 4*t/(1+t^2)];
```

$$P := \left[-\frac{2\left(-1+t^2\right)}{1+t^2}, \frac{4t}{1+t^2}\right] \tag{2.1}$$

```
> getParamConch(P, 1, 0, 0);
```

The Conchoid is reducible, so it is double-rational. Its components can be parametrized by: CONCHmas = $\left[-\frac{3\left(-1+t^2\right)}{1+t^2}, \frac{6t}{1+t^2}\right]$, *and CONCHmenos =* $\left[-\frac{-1+t^2}{1+t^2}, \frac{2t}{1+t^2}\right]$

$$\textit{Details of computation: } m(t) = \frac{2+2t^2}{1+t^2} \tag{2.2}$$

```
> P:=[-8*t/(4+t^2), -(t^2-4)/(4+t^2)];
```

$$P := \left[-\frac{8t}{4+t^2}, -\frac{t^2-4}{4+t^2}\right] \tag{2.3}$$

```
> getParamConch(P, 1, -2, 0);
```

The Conchoid is irreducible and rational. It can be parametrized by: CONCH = $\left[\frac{3t^6-21t^4-47t^2+1}{\left(1+t^2\right)\left(t^4+14t^2+1\right)}, \frac{2\left(-3+5t^4+14t^2\right)t}{\left(1+t^2\right)\left(t^4+14t^2+1\right)}\right]$

$$\textit{Details of computation: } Qp(t) = \left[-\frac{2\left(-4t+t^2-1\right)}{t^2-1+4t}, t\right], \textit{Delta}(t) = \left[\frac{2\left(t^2-1+4t\right)\left(-4t+t^2-1\right)}{t^4+14t^2+1}, \frac{8t\left(-1+t^2\right)}{t^4+14t^2+1}\right] \tag{2.4}$$

```
> getParamConch(P, 1, 0, 2);
```

The Conchoid is irreducible and not rational. (2.6)

```
>
```

See Also

getImplConch, plotImplConch, checkParam

Fig. 2. getParamConch help page.

3 Atlas of Conchoid and Offset Curves

In this section we illustrate the previous results applying the packages *Offset* and *Conchoid*. We analyze the rationality of the offset and the conchoid of several classical rational curves, and in the case of rationality we compute rational parametrizations. We give a table summarizing the main details of the process for each geometric construction, such as the degree of the implicit equation, rational character and rational parametrization in case of genus zero. In case of Offsets, we also give the implicit equation of the reparametrizing curve (the *oracle* curve to study the rationality of the offset curve). In case of Conchoids, the rationality depends on the focus, therefore in the table we study the rationality for different focus position, distinguishing if the focus is on the base curve or not. We don't include the implicit equation of the reparametrizing curve because

of space limitations. The implicit equations, plots and more details of the computation of these atlas are available contacting with the corresponding author.

Offsets Curves

Base Curve	Offset Degree	Rationality Parametrization	$\mathcal{G}_{\mathcal{P}}$
Circle	4	Double Rational $\left(\pm\frac{(d\pm r)2t}{t^2+1},\ \mp\frac{(d\pm r)(t^2-1)}{t^2+1}\right)$	N/A
Parabola	6	Rational $\left(\frac{(t^2-1)(-t^2-1+4\,dat)}{4at(t^2+1)},\ \frac{t^6-t^4-t^2+1+32\,dt^3a}{16at^2(t^2+1)}\right)$	$x_2^2-1+4x_2ax_1$
Hyperbola	8	Irreducible and non rational	$x_2^2a^2x_1-x_1a^2-a^2x_2-b^2x_1^2x_2$
Ellipse	8	Irreducible and non rational	$-x_2^2ax_1+ax_1+x_2b-x_2bx_1^2$
Cardioid	14	Rational $\left(\frac{(-9+t^2)(dt^6-117dt^4+3456t^3-1053dt^2+729d)}{(243t^2+27t^4+t^6+729)(t^2+9)},\ \frac{-18(dt^6-16t^5-21dt^4+864t^3-189dt^2-1296t+729d)t}{(243t^2+27t^4+t^6+729)(t^2+9)}\right)$	$32\,x_2^2\,x_1^3-6\,x_2^2\,x_1-32\,x_1^3+6\,x_1-48\,x_2\,x_1^2+x_2$
Three-leafed Rose	14	Irreducible and non rational	$-5\,x_2^2\,x_1+3\,x_2^2\,x_1^3+5\,x_1-3\,x_1^3-12\,x_2\,x_1^2+3\,x_2\,x_1^4+x_2$
Trisectrix of Maclaurin	10	Irreducible and non rational	$4x_1x_2^2-4x_1+6x_2x_1^2+x_2x_1^4-3x_2$
Folium of Descartes	14	Irreducible and non rational	$x_2^2-2\,x_2^2\,x_1^3-1+2\,x_1^3+4\,x_2\,x_1-2\,x_2\,x_1^4$
Tacnode	20	Irreducible and non rational	$486x_2^2x_1^6+2304x_2^2x_1^5+3882x_2^2x_1^4+3144x_2^2x_1^3+1303x_2^2x_1^2+256x_2^2x_1+17x_2^2-486x_1^6-2304x_1^5-3882x_1^4-3144x_1^3-1303x_1^2-256x_1-17+432x_2x_1^6+3960x_2x_1^5+9360x_2x_1^4+10080x_2x_1^3+5640x_2x_1^2+1604x_2x_1+184x_2$
Epitrochoid	10	Irreducible and non rational	$32\,x_2^2\,x_1^3-256\,x_2^2\,x_1-32\,x_1^3+256\,x_1-5\,x_2\,x_1^4+288\,x_2\,x_1^2-256\,x_2$
Ramphoid Cusp	20	Irreducible and non rational	$9\,x_2^2\,x_1^5-9\,x_1^5+12\,x_2\,x_1^5-16\,x_2\,x_1^4+9\,x_2^2\,x_1^4-9\,x_1^4+4\,x_2\,x_1^3-6\,x_1^3+6\,x_2^2\,x_1^3-4\,x_2\,x_1^2-6\,x_1^2+6\,x_2^2\,x_1^2+x_2^2\,x_1-x_1+4\,x_2+x_2^2-1$
Lemniscata of Bernoulli	16	Irreducible and non rational	$x_2^2-3x_2^2x_1^4+3x_2^2x_1^2-x_1^6x_2^2-1+3x_1^4-3x_1^2+x_1^6+2x_2-6x_2x_1^4-6x_2x_1^2+2x_1^6x_2$

N/A Not available.

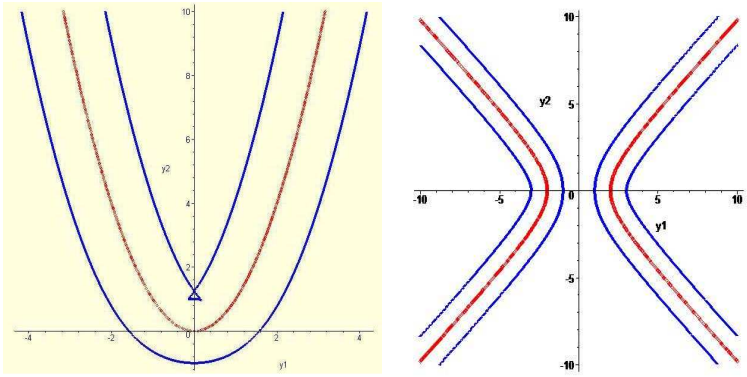

Fig. 3. Parabola (red curve) and the offset at $d = 2$. Hyperbola (red curve) and the offset at $d = 1.5$.

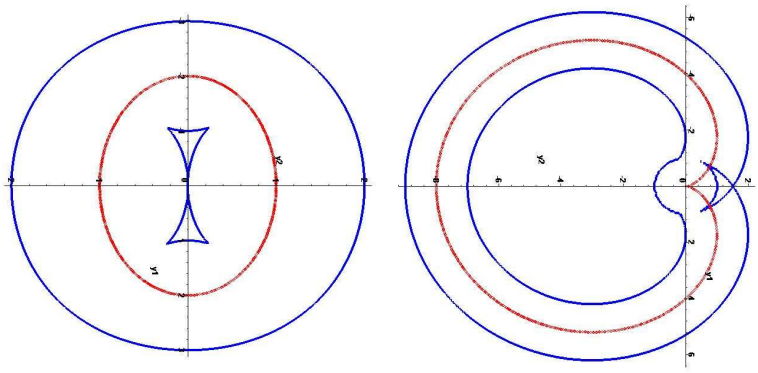

Fig. 4. Left: Ellipse (red curve) and the offset at $d = 1$. Right: Cardioid (red curve) and the offset at $d = 1$.

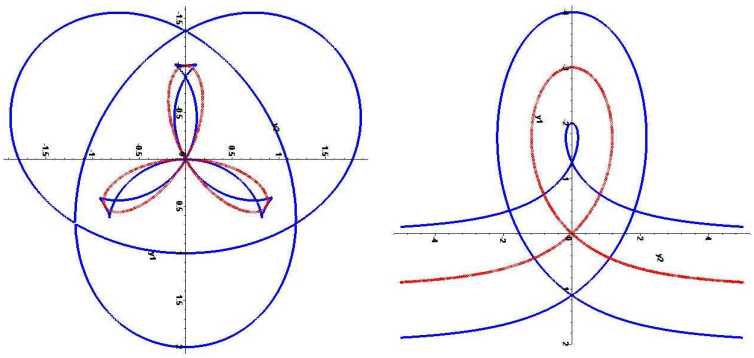

Fig. 5. Three-leafed Rose (red curve) and the offset at $d = 1$. Trisectrix of Maclaurin (red curve) and the offset at $d = 1$.

Conchoids Curves

Base Curve ($\mathcal{C}$)	**Focus of Curve (O) & Focus of Conchoid (A)**	**Rationality**	**Conchoid ($\mathfrak{C}$) Parametrization**
Circle $x_2^2 + x_1^2 - 4$	A=O=(0,0) A $\in \mathcal{C}$, A=(-2,0) A $\neq$ O, A $\notin \mathcal{C}$, A=(-4,0)	Double Rational Irr. rational Irr. not rational	$\left(\frac{-2(-1+t^2)\pm(1-t^2)}{1+t^2}, \frac{4t\pm 2t}{1+t^2}\right)$ $\left(\frac{3t^4-12t^2+1}{1+2t^2+t^4}, \frac{2t(-3+5t^2)}{1+2t^2+t^4}\right)$ N/A
Parabola $x_2 - x_1^2$	A=O=(0,1/4) A $\in \mathcal{C}$, A=(0,0) A $\neq$ O, A $\notin \mathcal{C}$, A=(0,-2)	Double Rational Irr. rational Irr. not rational	$\left(t \pm \frac{4t}{1+4t^2}, t^2 \pm \frac{4t^2-1}{1+4t^2}\right)$ $\left(\frac{2t+2t^3+1-2t^2+t^4}{(-1+t^2)(1+t^2)}, \frac{2t(2t+2t^3+1-2t^2+t^4)}{(-1+t)^2(1+t)^2(1+t^2)}\right)$ N/A
Hyperbola $\frac{x_1^2}{16} - \frac{x_2^2}{9} - 1$	A=O=(5,0) A $\in \mathcal{C}$, A=(-4,0) A $\neq$ O, A $\notin \mathcal{C}$, A=(0,0)	Double Rational Irr. rational Irr. not rational	$\left(\frac{-2(9+t^2)}{3t} \pm \frac{2(-t-6)(2t+3)}{45+24t+5t^2}, \frac{t^2-9}{2t} \pm \frac{3(t^2-9)}{45+24t+5t^2}\right)$ $\left(\frac{(45t^6+129t^4+311t^2+27)}{(1+t^2)(-9+t^2)(9t^2-1)}, \frac{2(-63+81t^4-82t^2)t}{(1+t^2)(-9+t^2)(9t^2-1)}\right)$ N/A
Ellipse $\frac{x_1^2}{25} + \frac{x_2^2}{16} - 1$	A=O=(3,0) A $\in \mathcal{C}$, A=(0,4) A $\neq$ O, A $\notin \mathcal{C}$, A=(0,0)	Double Rational Irr. rational Irr. not rational	$\left(\frac{5(t^2-1)}{t^2+1} \pm \frac{t^2-4}{t^2+4}, \frac{8t}{t^2+1} \pm \frac{4t}{t^2+4}\right)$ $\left(\frac{(1-t^2)(100t+100t^3+4t^4+17t^2+4)}{(4t^4+17t^2+4)(1+t^2)}, \frac{2(-58t^4+8t^6-58t^2+8-4t^5-17t^3-4t)}{(4t^4+17t^2+4)(1+t^2)}\right)$ N/A
Cardioid $(x_1^2 + 4x_2 + x_2^2)^2 - 16(x_1^2 + x_2^2)$	A $\in \mathcal{C}$, A=(0,0) A $\notin \mathcal{C}$, A=(-9,0)	Double Rational Irr. not rational	$\left(\frac{-1024t^3}{(16t^2+1)^2} \pm \frac{-8t}{16t^2+1}, \frac{-128t^2(16t^2-1)}{(16t^2+1)^2} \pm \frac{1-16t^2}{16t^2+1}\right)$ N/A
Three-leafed Rose $(x_1^2 + x_2^2)^2 + x_1(3x_2^2 - x_1^2)$	A $\in \mathcal{C}$, A=(0,0) A $\notin \mathcal{C}$, A=(-2,0)	Irr. rational Irr. not rational	$\left(\frac{2(t^4-6t^2+9)t^2(t-1)(t+1)}{(t^4+2t^2+1)^2}, \frac{4t^3(t^4-6t^2+9)}{(t^4+2t^2+1)^2}\right)$ N/A
Trisectrix of Maclaurin $x_1(x_1^2 + x_2^2) - (x_2^2 - 3x_1^2)$	A $\in \mathcal{C}$, A=(0,0) A $\notin \mathcal{C}$, A=(-4,0)	Irr. rational Irr. not rational	$\left(\frac{-2(-5t^2+2t^4+1)}{(t^4+2t^2+1)}, \frac{-4t(-5t^2+2t^4+1)}{(t^4+2t^2+1)(t^2-1)}\right)$ N/A
Folium of Descartes $x_1^3 + x_2^3 - 3x_1x_2$	A $\in \mathcal{C}$, A=(0,0) A $\notin \mathcal{C}$, A=(-1,-1)	Irr. rational Irr. not rational	$\left(\frac{(-6t+6t^5+t^6-3t^4+3t^2-1+8t^3)(t-1)(t+1)}{(t^2+1)(t^6-3t^4+3t^2-1+8t^3)}, \frac{2(-6t+6t^5+t^6-3t^4+3t^2-1+8t^3)t}{(t^2+1)(t^6-3t^4+3t^2-1+8t^3)}\right)$ N/A
Tacnode $2x_1^4 - 3x_1^2x_2 + x_2^2 - 2x_2^3 + x_2^4$	A $\in \mathcal{C}$, A=(0,0) A $\in \mathcal{C}$, A=(0,1)	Irr. not rational Irr. not rational	N/A N/A
Epitrochoid $x_2^4 + 2x_1^2x_2^2 - 34x_2^2 + x_1^4 - 34x_1^2 + 96x_1 - 63$	A $\in \mathcal{C}$, A=(3,0) A $\notin \mathcal{C}$, A=(0,0)	Double Rational Irr. rational	$\left(\frac{-7t^4+288t^2+256}{(t^2+16)^2} \pm \frac{16-t^2}{t^2+16}, \frac{-16t(5t^2-16)}{(t^2+16)^2} \pm \frac{(-8t)}{t^2+16}\right)$ N/A
Ramphoid Cusp $x_1^4 + x_1^2x_2^2 - 2x_1^2x_2 - x_1x_2^2 + x_2^2$	A $\in \mathcal{C}$, A=(0,0) A $\notin \mathcal{C}$, A=(-1,-1)	Irr. not rational Irr. not rational	N/A N/A
Lemniscata of Bernoulli $(x_1^2 + x_2^2)^2 - 4(x_1^2 - x_2^2)$	A $\in \mathcal{C}$, A=(0,0) A $\in \mathcal{C}$, A=(-2,0)	Irr. not rational Irr. not rational	N/A N/A

N/A Not available.

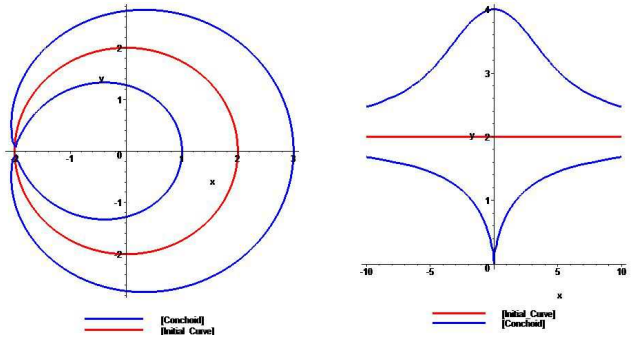

Fig. 6. Left: Circle (red curve) and the conchoid (blue curve) at $A = (-2, 0)$ and $d = 1$ (Limaçon of Pascal). Right: Straight line (red line) and the conchoid (blue curve) at $A = (0, 0)$ and $d = 2$ (Conchoid of Nicomedes).

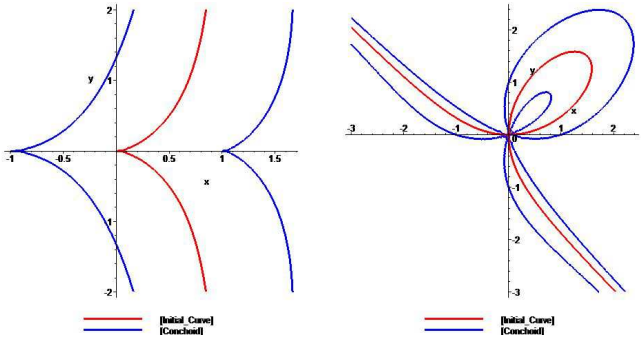

Fig. 7. Left: Conchoid of Sluze (red curve) and the conchoid (blue curve) at $A = (-2, 0)$ and $d = 1$. Right: Folium of Descartes (red curve) and the conchoid (blue curve) at $A = (0, 0)$ and $d = 2$.

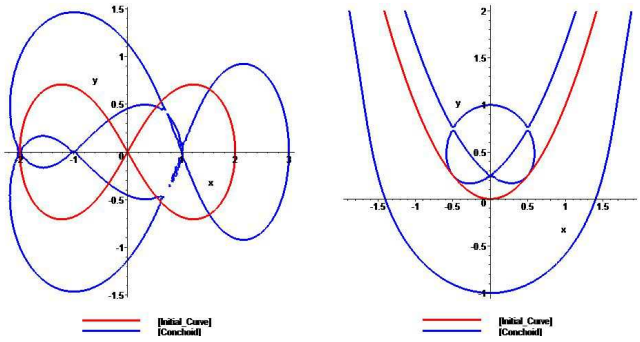

Fig. 8. Left: Lemniscata of Bernoulli (red curve) and the conchoid (blue curve) at $A = (-2, 0)$ and $d = 1$. Right: Parabola (red curve) and the conchoid (blue curve) at $A = (0, 1/4)$ and $d = 1$.

Acknowledgements. This work has been partially funded by the Spanish "Ministerio de Economia y Competitividad" under the project MTM2011-25816-C02-01.

References

1. E. Arrondo, J. Sendra, J.R. Sendra. Parametric Generalized Offsets to Hypersurfaces. *Journal of Symbolic Computation* **23**(2–3), 267–285, 1997.
2. E. Arrondo, J. Sendra, J.R. Sendra. Genus Formula for Generalized Offset Curves. *Journal of Pure and Applied Algebra* **136**(3), 199–209, 1999.
3. Azzam R.M.A. *Limacon of Pascal locus of the complex refractive indices of interfaces with maximally flat reflectance-versus-angle curves for incident unpolarized light.* J. of the Optical Society of America A: Optics, Imagen Science, and Vision, Vol 9. pp. 957–963, 1992.

4. R.T. Farouki, C.A. Neff. Analytic Properties of Plane Offset Curves. *Comput. Aided Geom. Des.* **7**, 83–99, 1990.
5. R.T. Farouki, C.A. Neff. Algebraic Properties of Plane Offset Curves. *Comput. Aided Geom. Des.* **7**, 100–127, 1990.
6. C.M. Hoffmann. *Geometric and Solid Modeling.* Morgan Kaufmann Publishers, Inc. 1993.
7. Kerrick A.H.,*The limacon of Pascal as a basis for computed and graphic methods of determining astronomic positions.* J. of the Institute of Navigation. vol.6, No.5. 1959.
8. G.W. Leibniz. Generalia de natura linearum, anguloque contactus et osculi povocationibis aliisque cognatis et eorum usibus nonnullis. *Acta Eruditorum*, 1692.
9. W. Lü Offset-Rational Parametric Plane Curves. *Comput. Aided Geom. Des.* **12**, 601–617, 1995.
10. Menschik F., *The hip joint as a conchoid shape.* J. of Biomechanics. 1997 Sep; 30(9):971-3 9302622., 1997.
11. MyungJin Kang, *Hip joint center location by fitting conchoid shape to the acetabular rim images*. Engineering in Medicine and Biology Society, 2004. IEMBS'04. 26th Annual International Conference. Vol 2. pp. 4477-4480 vol 6. ISBN:0-7803-8439-3, 2004.
12. Peternell, M. and Gruber, D. *Conchoid surfaces of quadrics*, Journal of Symbolic Computation, 2013. doi:10.1016/j.jsc.2013.07.003.
13. Peternell, M., Gruber, D. and Sendra, J. *Conchoid surfaces of rational ruled surfaces*, Comp. Aided Geom. Design 28, 427–435, 2011.
14. Peternell, M., Gruber, D. and Sendra, J. *Conchoid surfaces of spheres*, Comp. Aided Geom. Design 30, issue 1, pp. 35–44, 2013.
15. Peternell, M., Gotthart, L., Sendra, JR., and Sendra, J. *The Relation Between Offset and Conchoid Constructions*, arXiv:1302.1859v2 [math.AG] 10 Jun 2013.
16. M. Peternell, H. Pottmann. *A Laguerre geometric approach to rational offsets.* Computer Aided Geometric Design, **15**, 223–249, 1998.
17. G. Salmon. *A Treatise an the Higher Plane Curves.* Chelsea, New York ,1960.
18. J. Sendra, J.R. Sendra. Algebraic Analysis of Offsets to Hypersurfaces. *Mathematische Zeitschrift* **234**, 697–719, 2000.
19. J. Sendra, J.R. Sendra. Rationality Analysis and Direct Parametrization of Generalized Offsets to Quadrics. *Applicable algebra in Engineering, Communication and Computing* **11**, 111–139, 2000.
20. Sendra J., Sendra J. R. . *An Algebraic Analysis of Conchoids to Algebraic Curves.* Applicable Algebra in Engineering, Communication and Computing vol. 19, pp. 413–428, 2008.
21. J. Sendra and J.R. Sendra. *Rational parametrization of conchoids to algebraic curves*, Applicable Algebra in Engineering, Communication and Computing, Vol.21, No.4, pp. 413–428, 2010.
22. Sultan A., *The Limaçon of Pascal: Mechanical Generating Fluid Processing.* J. of Mechanical Engineering Science. vol. 219,Number 8/ 2005. pp. 813–822. ISSN.0954-4062, 2005.
23. Szmulowicz F., *Conchoid of Nicomedes from reflections and refractions in a cone.* American Journal of Physics, vol. 64, pp. 467–471, 1996.
24. Weigan L., Yuang E., Luk K.M. *Conchoid of Nicomedes and Limaçon of Pascal as Electrode of Static Field and a Wavwguide of High Frecuency Wave.* Progress In Electromagnetics Research Symposium, PIER 30, 273–284, 2001.