Autonomous use of a Computer Algebra System for Learning Linear Algebra

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A Computer Algebra System (CAS) can be defined as mathematical software with numerical, graphic and symbolic capacities. Many studies have addressed the teaching of mathematics in a CAS environment. This paper describes a teaching experience carried out in a linear algebra course, for engineering students, with methodologies adapted to the European Higher Education Area. Generic competences such as *self-learning*, *team work* or *use of technology* have been enhanced through autonomous work of students, who had worked in small groups (2-3 students) using a CAS for solving proposal exercises. The experience was completed with a competition which was announced and promoted between the students with the best grades. They developed a project related to a real problem in science or engineering, where orthogonal transformations are used for modelling and solving the problem. A brief summary of the winner project is included in the paper. The experiment proves that technology provides the students with material for enhancing the apprenticeship and improving the motivation.

Keywords: Computer Algebra Systems, Competences, Orthogonal Transformations, Linear Algebra.

Introduction

Since their development in the 1970s, CAS has been recognized as a powerful technology for doing mathematics and a tool to support teaching. During the last twenty years, the capabilities of CAS have improved considerably and there has been a significant increase in the use of CAS for learning and teaching mathematics. The literature reflects many experiences gained through their use (see Marshall, Buteau, Jarvis, Lavicza, 2012).

Nowadays, there is a new problem. It is necessary to find the best way of integrating CAS into a learning model based on competences. This problem is a consequence of the new directives for curricula in European universities (European Commission, 2009). Therefore, we have to change the way of teaching and learning mathematics in engineering. The Framework for Mathematics Curricula in Engineering Education (SEFI, 2014) shows the potential of CAS in mathematics competence development. Diaz et al (2011) described how the use of CAS in problem-solving promotes the development of generic competences like *critical thinking*, *communication and writing* or the *use of technology*.

In this paper, we propose a way of using CAS, as an element of constructive learning, in accordance with the generic and specific competences to be acquired by the students. We emphasize our proposal with an experiment carried out in linear algebra topics. This experiment has been designed to enhance the following generic competences: *team-work*, *self-learning*, *critical thinking* and the *use of technology*. It has been developed in two stages: the first one involved autonomous teamwork of students, using a CAS to analyze documents produced by the teachers, for solving linear algebra exercises and problems. The second one was *The Spring Math Competition*, which was promoted between the best grade students. They developed a project related with the applications of linear algebra to real problems in science or engineering.

The paper is organized as follows. We initially address general issues, offering a brief introduction to the role of CAS in the new learning model based on competences, and introducing our didactical proposal. Then, we describe the two stages of the experiment and finally, we offer some conclusions.

A New Scenario for Teaching and Learning Mathematics in Engineering

The Bologna Declaration and the creation of the European Higher Education Area (European Commission, 2009) have promoted a structural change in the Spanish universities. This change, which has been increasingly addressed in recent years, has not been limited to a mere restructuring of the academic curriculum, but has served to change the paradigm of university teaching. In this new scenario, the advance of constructivist methods in teaching practices seems to have become the hegemonic method. Some of the characteristics of such methods that are currently being imposed in general, not only within the field of mathematics, are as follow:

- Student-focused teaching.
- An increase in participation by the students themselves, favoring discussion between students and instructors and departing from the traditional method of teaching through lectures.
- The use of educational technology.

The change is now also affecting traditional methods of assessment, although for the time being there is no new methodological paradigm, such that numerous proposals have been made concerning the mechanisms of evaluation of group work and the weight that new methods should have in the assessment of students' work. There is indeed still much inertia regarding the need for some face-to-face examination using the traditional format.

In this general context, mathematical contents in engineering studies and teaching methods have also changed. Regarding mathematical contents, the new structure has relegated the math subjects to the period of basic training and, in many instances, this period is basically the first academic year of university studies. There has also been a reduction in the math contents since the students accessing university degrees have less mathematical knowledge. As a result, in many engineering degrees, the mathematical content has been reduced to one semester of linear algebra and one semester of calculus, which in general are used in an attempt to unify the students' mathematical knowledge. Accordingly, the constructivist methodology must face up to the task of filling the "gap", which sometimes prevents individual work by the students. This work is crucial in the new teaching paradigm.

As well as the above characteristics, a core part of the teaching process is the contextualization in the case of mathematics, in the scenario of realistic problems, with special emphasis on the use of technology.

On the other hand, teaching has to be aimed at preparing the students for continuous education, enhancing the self-learning methods, because the students have to learn throughout their career.

The new technologies are opening a broad range of methodological possibilities in the teaching-learning process. There is an increasing number of universities that are now completing their offer of face-to-face tuition with modalities based on e-learning or b-learning. The use of remote teaching platforms as support tools to teaching for helping students to use learning resources is now widely available. In this way, students have a tool for efficient communication between themselves and their instructors, and instructors have a good way of monitoring the students' production. Through these tools, it is possible to provide students with documentation, tutorials, etc., and open forums to debate the development of projects.

The application of the new teaching styles involves the introduction of new problems, now subject to different ways of resolution. Again, it is worth mentioning the different opinions regarding the suitability and efficiency of the new methods of assessment, which are patently opposite to the earlier habits both among instructors and students.

Teaching Alternatives: Learning Based on Competences

The concept of competence can be defined (Weinert, 2001) as the ability to carry out tasks or to deal with situations effectively using knowledge, skills and attitudes. According to Mulder et al. (2006) "*knowledge is captured as cognitive competence, skills as functional competence and attitudes as social competence*".

All the subjects in the curriculum must to be designed according to the competences to be acquired: generic and specific. Learning activities will be planned taking into account these competences to define their content and to decide the resources and methodologies to be used.

The Tuning-AHELO conceptual Framework (OECD, 2011) defines engineering as "the profession that deals with the application of technical, scientific, and mathematical knowledge in order to use natural laws and physical resources to help design and implement materials, structures, machines, devices, systems and processes that safely accomplish a desired objective". This framework offers a summary of some of the most influential learning outcomes in the engineering field. Graduates should possess generic skills needed to practice engineering. Among these are: The capacity to analyze and synthesize, apply knowledge to practice, adapt to new situations, ensure quality, manage information, and generate new ideas. More particularly, graduates are expected to have achieved the following generic skills: "the ability to function effectively as an individual and as a member of a team; the ability to communicate effectively with the engineering community and with society at large; the ability to recognize the need for and engage in independent life-long learning; and the ability to demonstrate awareness of the wider multidisciplinary context of engineering".

In the table 1, generic competences for an engineering bachelor degree are described.

Table 1

Basic Generic Competencies for Engineering

Competencies	**Description**
Self-Learning	The ability to engage in independent life-long learning
Critical Thinking	The ability to select, analyze, synthesize and apply relevant information, knowledge, methods and logical and well- motivated argument
Use of Technology	The ability to use modern ICT technology for communication and engineering practice
Problem solving	The ability to apply knowledge of mathematics, science, and engineering for formulating and solving engineering problems
Technical Communication	The ability to communicate effectively, by oral or written form, with the engineering community and with society at large
Teamwork	The ability to function effectively as a member of a multi-disciplinary team

Obviously, mathematical teaching for engineering students has the responsibility to master the mathematical techniques that allow students to solve engineering problems.

Savery (2006) argues that Problem Based Learning (PBL) methodologies are adequate for learning competences, but sometimes are constrained by a strict curriculum. The study of Strobel and Barneveld (2009) shows that PBL is more effective when the development of competences and students satisfaction are evaluated, but traditional learning is more effective for the knowledge assessment category. Perrenet et al. (2000) suggest that PBL is less suitable as an overall strategy for engineering education. However, any model of learning is improved by proposing the student to carry out any project based on a real problem. The motivation for the best students must include tasks for building on acquired knowledge and achieving the highest levels. Projects could be a good opportunity to convince the students of the great potential that mathematical approach can offer for engineering problems.

Using CAS for Improving Competences

A reflection about learning mathematics in a CAS environment is presented in Artigue (2002). Z. Lavizca (2007) analyzed factors influencing the integration of CAS into university level math education. In the field of mathematics learning of engineering students, the CAS has been widely used as a tool for graphic support (Amrhein, Gloor & Maeder, 1997), for learning algorithms (Cariaga & Nualart, 2002) and for problem solving (Díaz, García & de la Villa, 2011). The most frequent way of integrating the CAS is by laboratory sessions in which the students work with the CAS, under teacher supervision. In García, García, Martín, Rodríguez & de la Villa (2014) the use of CAS in all the learning and assessment activities is proposed as an element of constructive learning and a tool for improving autonomous work.

In our didactical framework, students supplement face to face learning with autonomous work by doing learning activities, which include studying and solving direct applications or problems.

As an aid in the learning process, students use technology: digital tools for communicating with the teacher and for self-assessment activities and a CAS for working in mathematics. Projects are used as a formative supplement.

The Experiment

The experiment has been carried out in Pontificia Comillas University of Madrid, in a linear algebra course for engineers whose content is the standard one: vector spaces, linear applications, eigenvalues and eigenvectors, Euclidean space and orthogonal transformations. The textbook is Villa (2010). The face-to-face teaching (4 hours per week) is divided into two hours of formal lectures and two hours of practical sessions. Furthermore, 4 additional sessions of computer laboratory (2 hours per session) have been planned. The required tutorials are provided to the students. Students' personal work includes the solving of exercises and problems posed by the instructor. In addition, we have designed voluntary activities for students' autonomous work, in order to complete their knowledge and skills on orthogonal transformations and appropriate use of mathematical software in problem solving.

Our experiment was designed so that students would be able to develop the following generic competences:

- Self-learning
- Teamwork
- Critical thinking
- Use of technology,
- together with the specific competences inherent to the subject:
- Knowledge, understanding and use of linear algebra (orthogonal transformations, matrix representation and classification)
- Ability to apply the above concepts for solving engineering problems.

We have designed learning activities for autonomous teamwork in a CAS environment. The choice of the CAS is made taking into account many parameters: students' background, software policy of the university (free or commercial software), usability, accessibility in computer rooms and laptops, etc. From the view point of teaching objectives to be achieved, the CAS used is not important. Most CAS are useful for teaching purposes. In short, from our point of view the key point is to develop interesting mathematical work using any CAS. For the experiment, the CAS chosen has been DERIVE, because students are familiar with this software. But similar activities can be performed with different CAS, including free open source CAS, like Maxima (see García, García, Rodríguez & de la Villa, 2011) or SAGE (see Botana, Abánades & Escribano, 2012).

The first stage of the experiment has been designed according to the following steps:

1. Five DERIVE tutorials for orthogonal transformations were provided to the students:
   - rot2, rotations in two dimensions
   - ref2, reflection across a line in the plane
   - rot3, rotations in three dimensions
   - ref13, reflection across a line in three dimensions
   - ref23, reflection through a plane in three dimensional space.

Also, DERIVE files that automatically implement matrices representing reflections respect lines or planes in three dimensions have been designed for the students' work. All tutorials are similar and include DERIVE functions for finding the image of a vector, drawing a vector with its image and finding invariant vectors. An example is shown in figures 1 and 2.

2. Worksheets with exercises, with similar difficulty as the tutorial's examples, have been prepared to be solved by the students organized in teams (2-3 students per team), using DERIVE functions previously defined in the tutorials.
3. The students sent the solutions of the proposed exercises to the instructor and received the feedback with comments and suggestions.

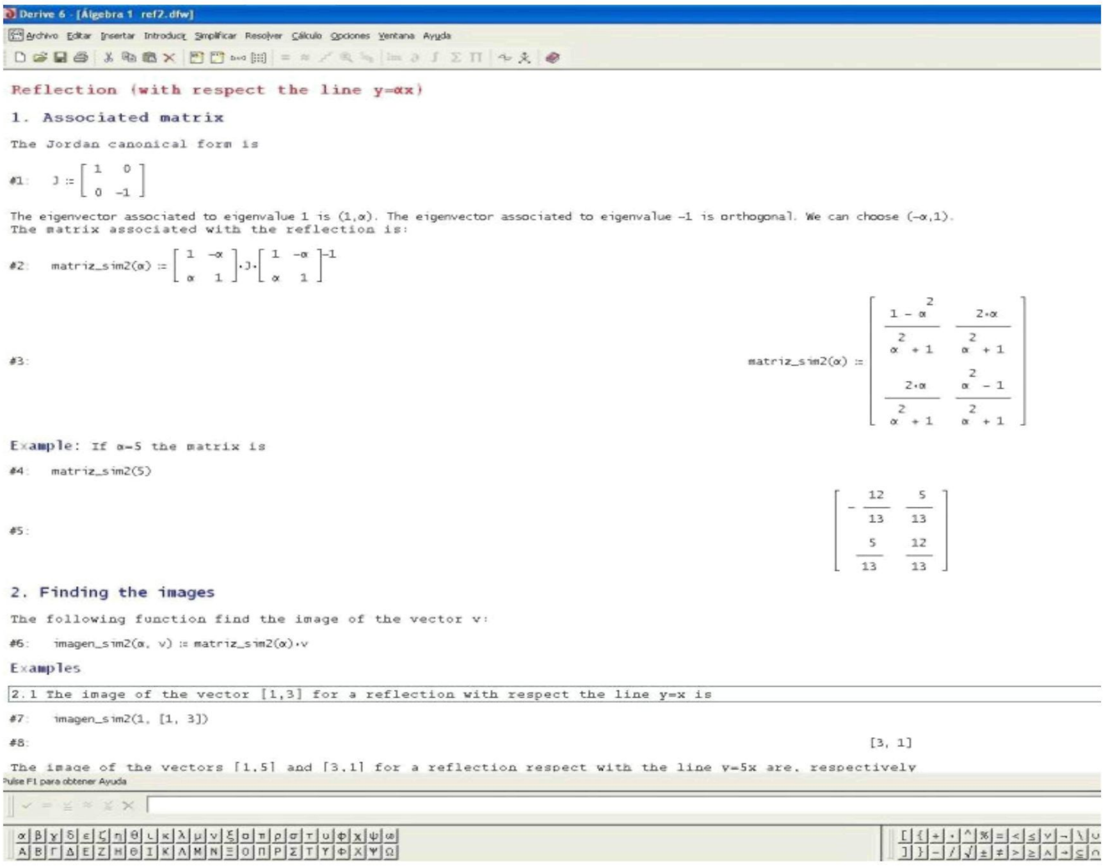

Figure 1. Tutorial file Reflection across a line (ref2)

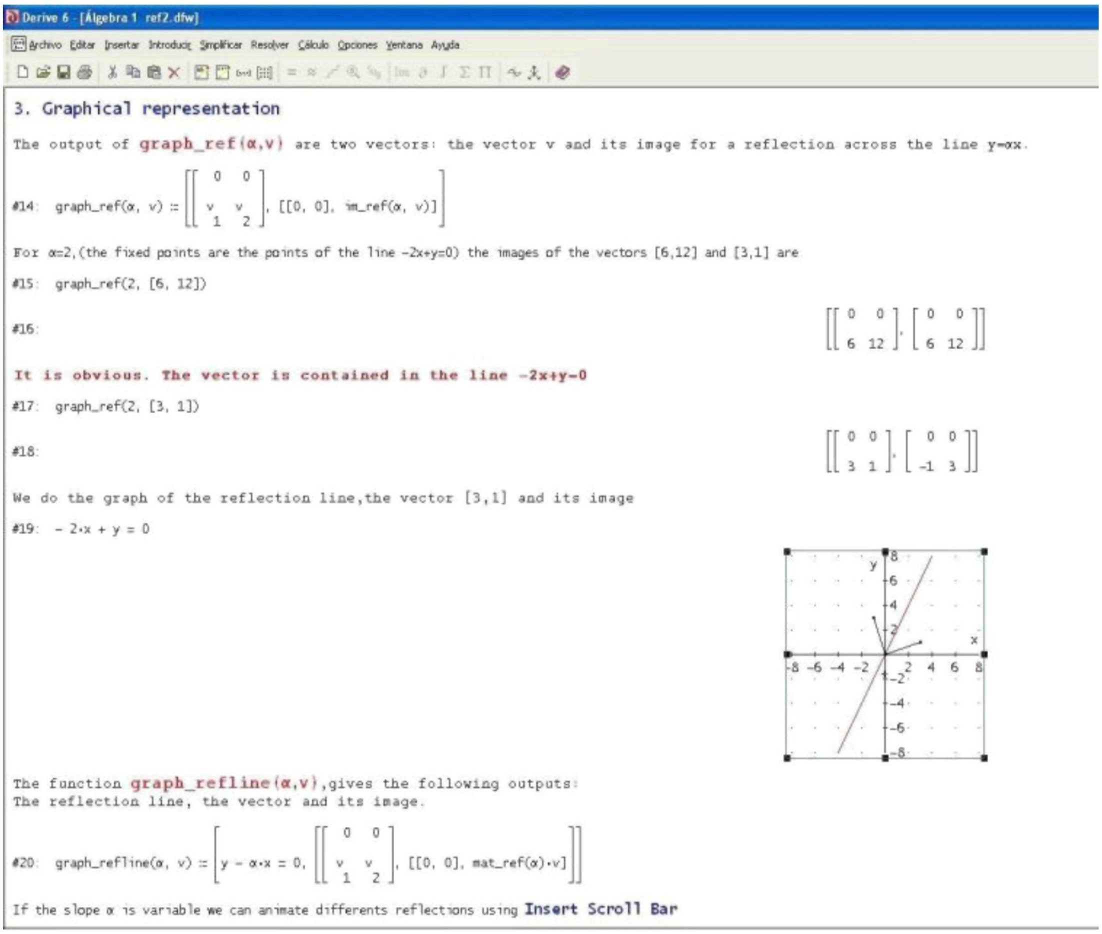

Figure 2. Tutorial file ref2 (Graphical representation)

Results

To evaluate the effectiveness of the experiment, we have used:

- Observation and evaluation of student's work.
- Surveys about mathematical and technical aspects.
- Surveys about personal aspects including the student's perception regarding the development of generic competencies.

Twenty five students distributed in ten teams of 2-3 people participated by using the DERIVE files. Initially, the teams worked autonomously to understand the tutorials and they performed the proposed tutorial tasks. The teams make most of the proposed exercises correctly. They have successfully performed the tasks of independent learning and team-work. They did not have any difficulty with using technology. But, we cannot prove that the work has allowed them to improve their critical thinking competence. The students found the greatest difficulties in the reasoning exercises because they had to conjecture about possible situations with rotations and reflections.

Regarding their satisfaction, the students filled a survey about the following items: ease of use of the files, usefulness of the tutorials for understanding the mathematical topics and the time spent in the task. The survey is easy to answer, since not many items are involved. With a Likert scale 1=strongly disagree, 2=disagree, 3=agree, 4=strongly agree, the mean scores were as depicted in table 2.

After analyzing the answers of the satisfaction survey we can appreciate that the students highly value the utility functions defined in the tutorials and also believe they have greatly improved the self-learning and team-work competence.

Table 2

Mean scores for the satisfaction survey (scale: 1-4)

ITEMS	**Rot2**	**Ref2**	**Rot3**	**Ref13**	**Ref23**
Easy to work with the files	3.7	3.7	3.3	3.5	3.4
Usefulness of defined functions	3.9	3.9	3.7	3.9	3.8
Useful for understanding concepts	3.6	3.5	3.3	3.2	3.4
The exercises are affordable	3.5	3.4	2.8	3	3.2
Useful for solving problems	3.6	3.6	3.2	3.5	3.5
Suitable for exams	3.1	3.2	3.1	3.1	3.2
Improvesself-learning	3.7	3.7	3.3	3.5	3.4
Improvesteam-work	3.6	3.6	3.5	3.5	3.6

The Competition

The experiment was completed with a competition. The *Spring Math Competition* was announced and promoted between the top students of linear algebra (first academic year of engineering degrees). The contest prize was a trip supported by the association of ICAI engineers. The jury was composed by the authors of this paper.

Students participating in the competition developed a project, related to a real problem in science or engineering, where orthogonal transformations are used for modelling and solving the problem. The students looked for the problem and used DERIVE in the solution. Each student had to develop their project individually and autonomously consulting the teachers for advice and help when needed. The students' work in the projects included: the mathematical modelling of a real-world problem; the self-learning of concepts or algorithms; the application of mathematical concepts for solving the problem (using mathematical software for programming appropriate functions) and the answer to the driving question.

All the presented projects were initially analyzed. Not all of them had the same mathematical difficulty but we wanted to stand out the ability of students to find applications of a particular mathematical topic. The students were able to extract appropriate information from very different sources and worked more enthusiastically. The four selected projects, for the final competition, were:

- Computing angles associated with trajectories.
- Orthogonal transformation associated with the Rubik's cube.
- Optimal and automatic process for container unloading.
- Digital image processing.

After the presentation by the corresponding authors of the quoted projects, and after some discussions, the jury declared the winner project: the project entitled *Optimal and automatic process for container unloading*. The author of this project is *Jordi Vila*.

Here we briefly describe the project and the main characteristics: The purpose of the project is to define the optimal process to unload a ship carrying containers. During the process, a number of key factors will be taken into account to improve the efficiency and safety. The main aim is to reduce the distance travelled by the cranes, thus reducing the time of the process. This project also answers the need to switch from human driven processes to rational patterns.

The elements involved in the process are:

- One container ship: The containers in the ship have been defined with a matrix in rows and columns, assuming they form a rectangle on the deck.
- The crane: The crane used has mainly the following movements: linear movement in the X and Y axis, rotation around the Z axis.
- The port (storage area): The storage area is selected adjacent to the pier where the ship is docked.

The ship's unloading process was simulated with a C+ program. This program analyzed the unloading process according to three patterns: by rows, by columns and randomly. Within each pattern the program computed the distance traveled by each container one by one checking if the unload of the container alters the heel of the ship and exceeds the maximum angles.

This program and some DERIVE functions allow us to know at every moment the position of the crane through its position in space. This allows us to calculate the sum of the distances travelled by the containers during the entire process.

After running the program a large amount of times, varying the size of the ships and the cargo inside them, a series of results arose.

In small ships (size less than 3x3) there is no significant improvement with a particular pattern to unload the ship. But, as the ships increases in size, the unloading process using the columns pattern requires less distance to fully unload the ships.

Conclusions

After analyzing the results of the autonomous learning experiment the main conclusions are the following:

- Technology provides the students with material for enhancing the apprenticeship and improving the motivation. In the experiment, we have succeeded in improving the academic performance and students' satisfaction.
- The use of the CAS (in this case DERIVE) allows to solve real and interesting problems, saving computing time. We can apply the methodology proposed using other CAS and dealing with other topics. In fact in García et al. (2014) a similar experience with Maxima is shown.
- Working with the CAS has contributed to improve self-learning, team work and the use of technology.
- The best students have seen in the Spring Competition a challenge that has made them work hard in

their project and gain confidence in themselves. Perceived efficacy fosters motivational commitment to their missions, resistance to adversity and performance accomplishments. Obviously, their generic competences have improved. A good planning of projects allows them to stay competitive in every academic and professional environment.

- The methodology for a course of mathematics for engineering students must combine lectures, work sessions, collaborative work for solving problems (sometimes using a CAS) and developing projects.

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