Signals and Systems: Introduction

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Signals and Systems: Introduction

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2- Continuous-Time Signals
3- Discrete-Time Signals
4- Description of Systems
5- Time-Domain System Analysis
6- Spectral Method
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1- Introduction to Signals and Systems

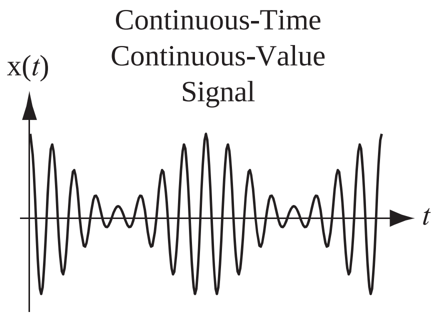
Introduction

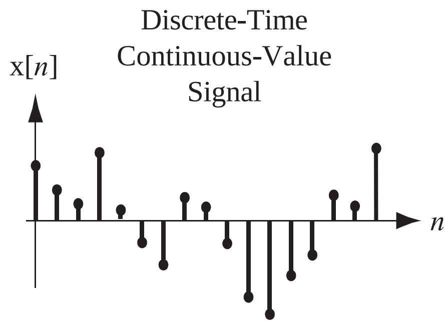
-A **signal** is any physical phenomenon which conveys information.

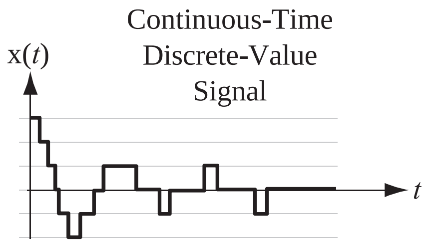
-**Systems** respond to signals and produce new signals.

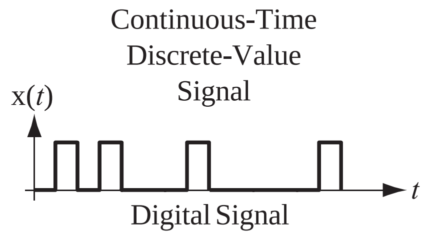
-**Excitation** signals are applied at system **inputs** and **response** signals are produced at system **outputs.**

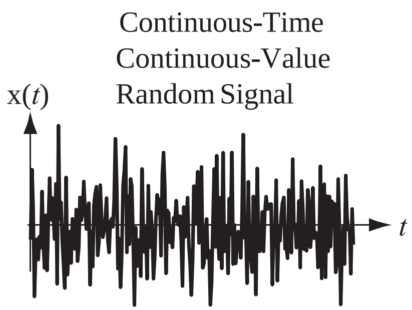
Signal types

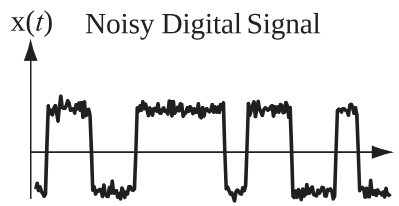


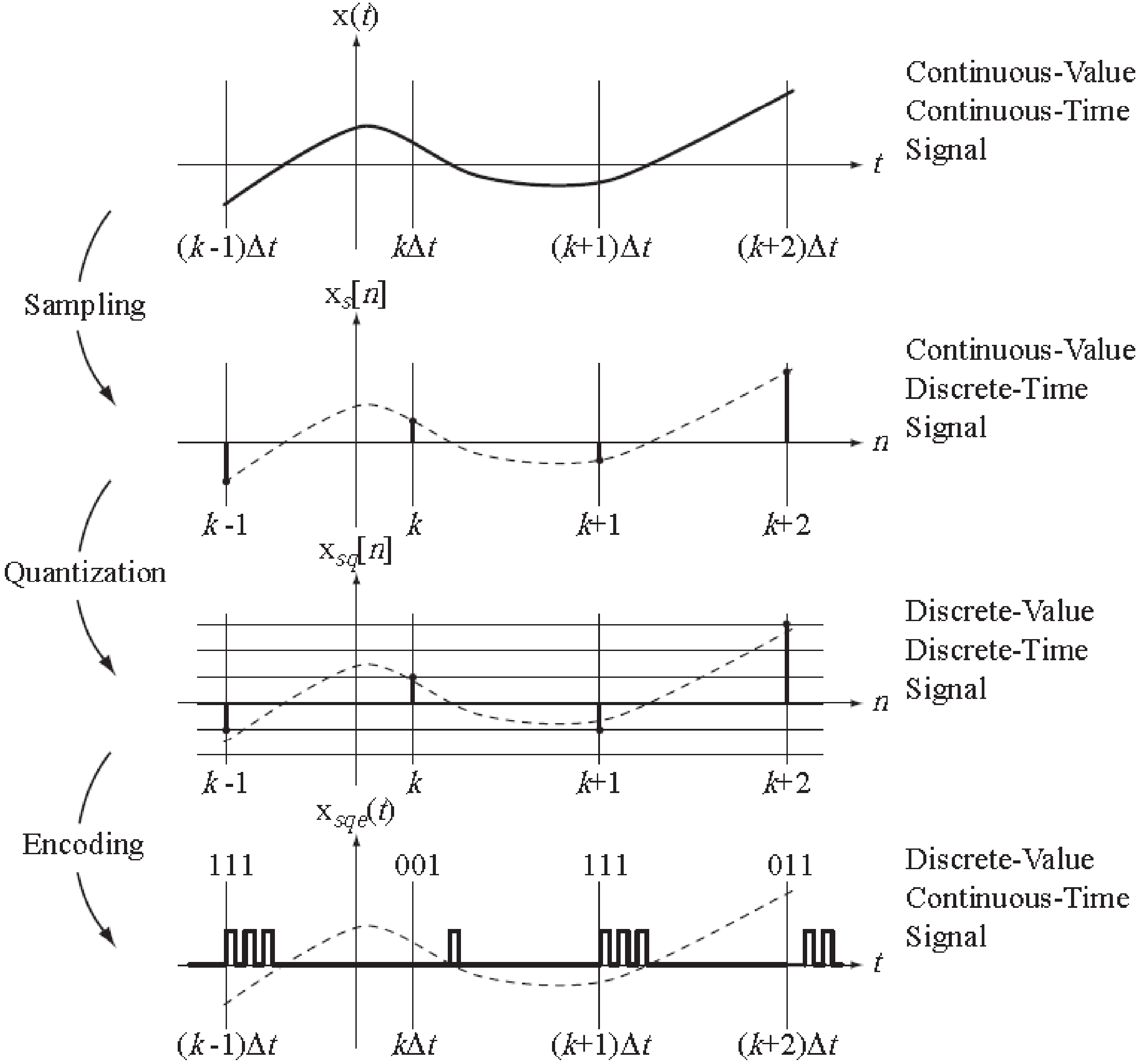








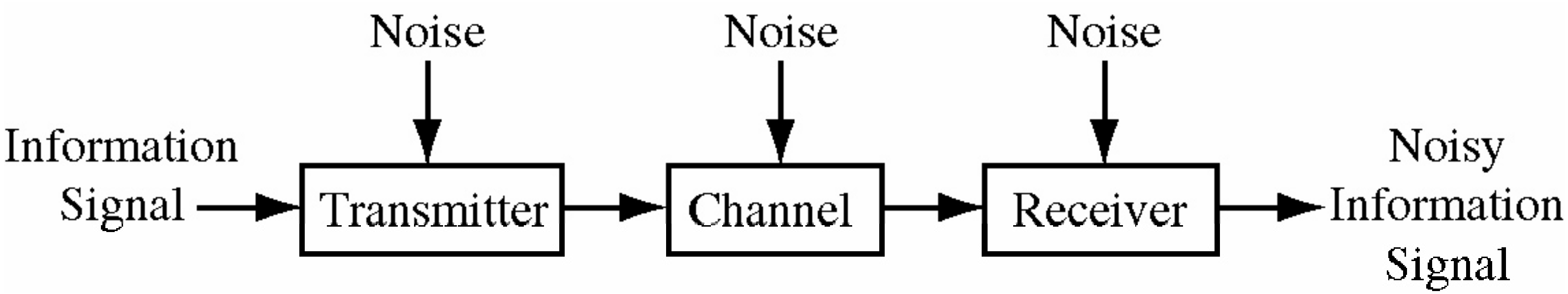




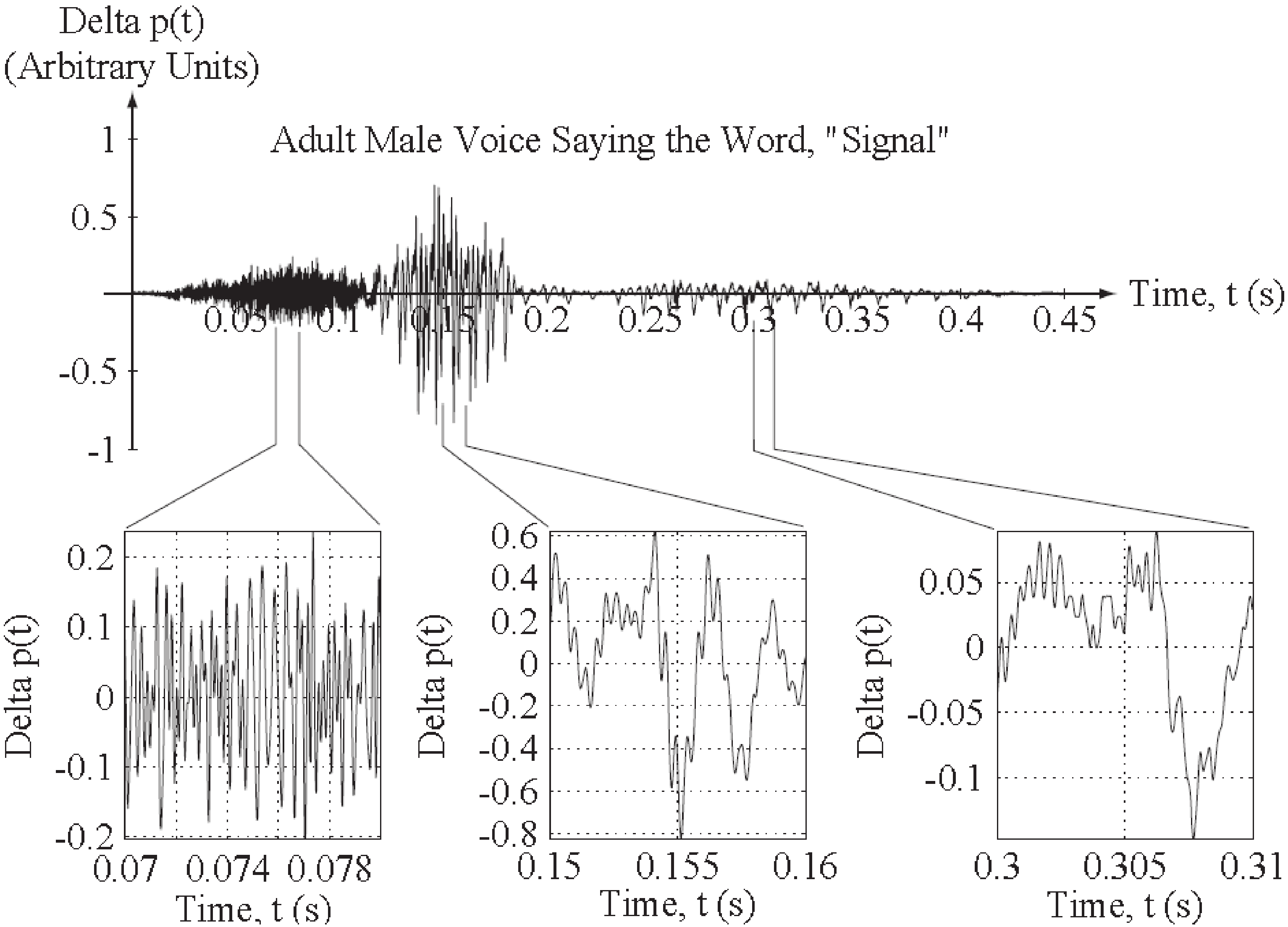
Conversions Between Signal Types

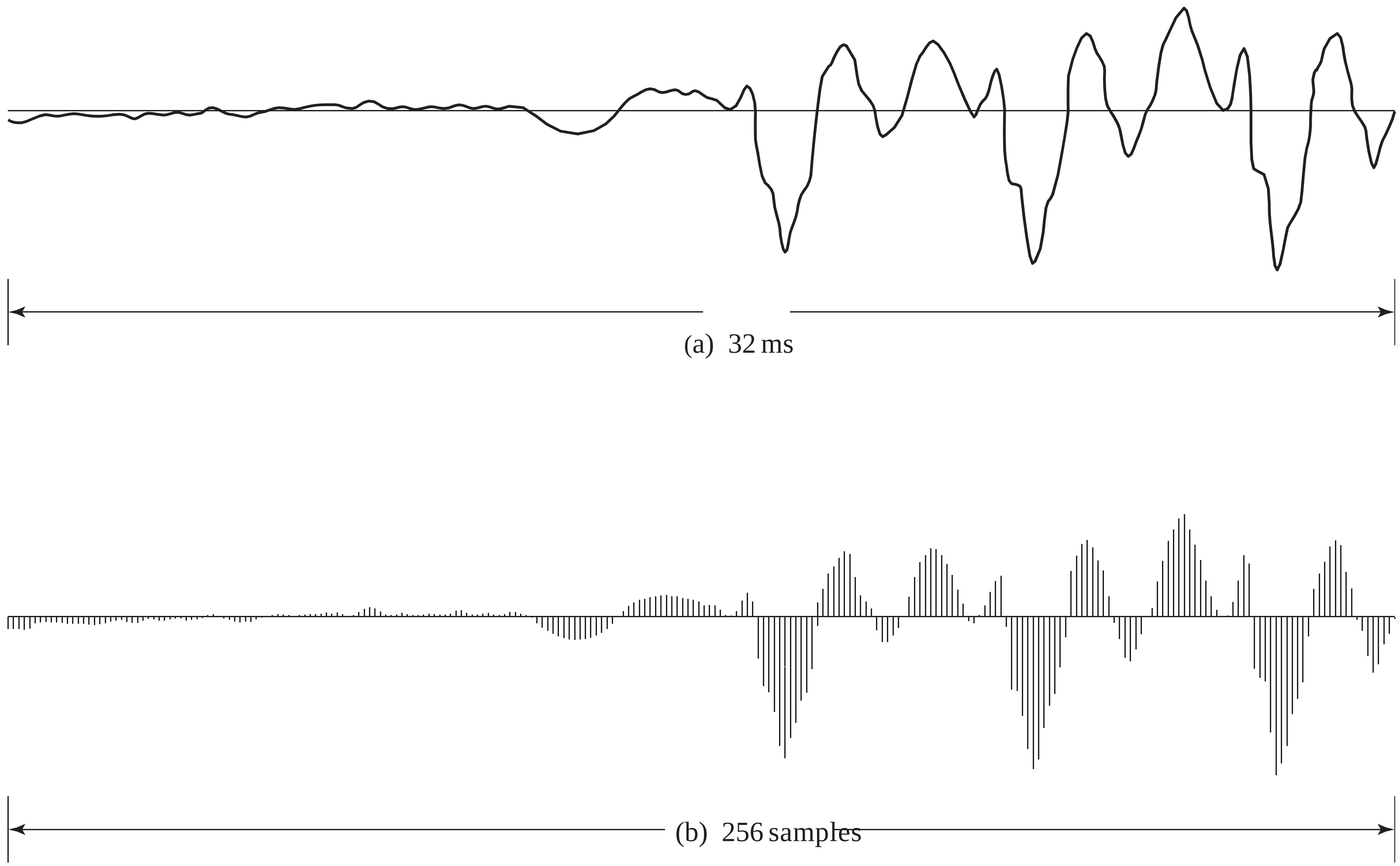
System example

-A communication system has an **information** signal plus **noise** signals.



Voice Signal



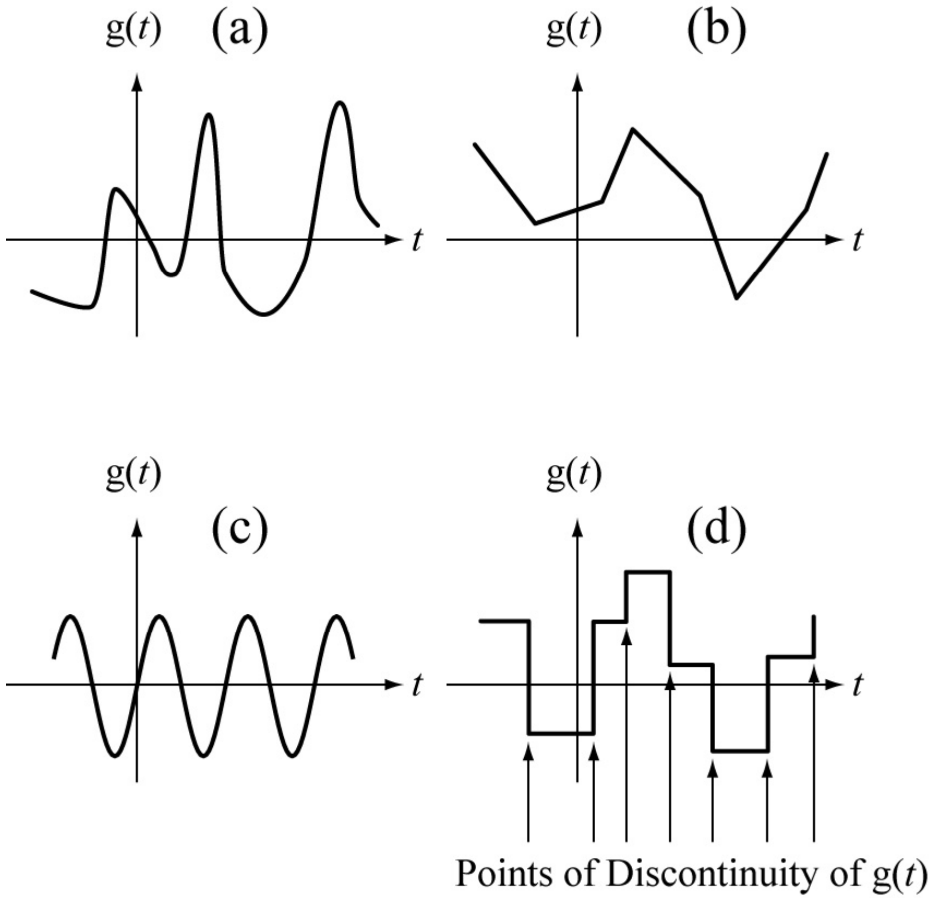


(a) Segment of a continuous-time speech signal $x(t)$.

(b) Sequence of samples $x[n]=x(nT)$ obtained from the signal in part (a) with $T=125\,\mu$s.

2- Continuous-Time Signals

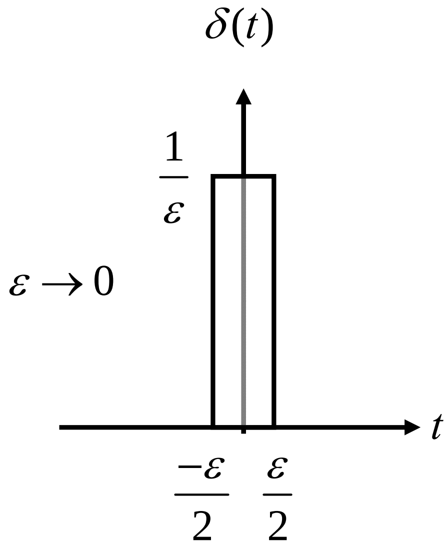
-All continuous signals that are functions of time are continuous-time , but not all continuous-time signals are continuous functions of time.



Impulse

$$\delta(t) = 0, \quad t \neq 0$$

$$\int_{-\infty}^{\infty} \delta(t)\,dt = 1$$

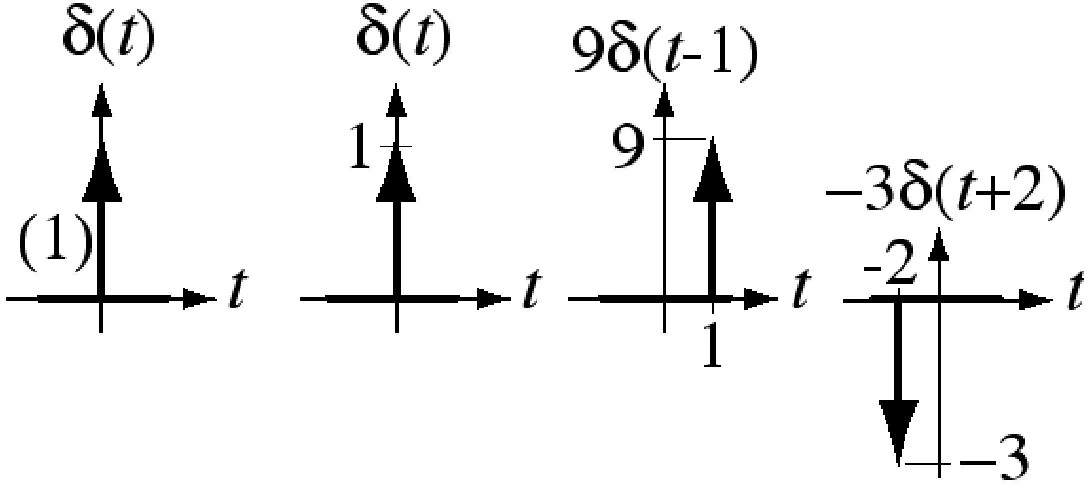


-The impulse is not a function in the ordinary sense because its value at the time of its occurrence is not defined.

-It is a functional.

-It is represented graphically by a vertical arrow.

-Its strength is either written beside it or is represented by its length.



Properties of the Impulse

Sampling Property

$$\int_{-\infty}^{\infty} f(t)\delta(t)\,dt = f(0)$$

$$\int_{-\infty}^{\infty} f(t)\delta(t-t_0)\,dt = f(t_0)$$

-The sampling property “extracts” the value of a function at a point.

Scaling Property

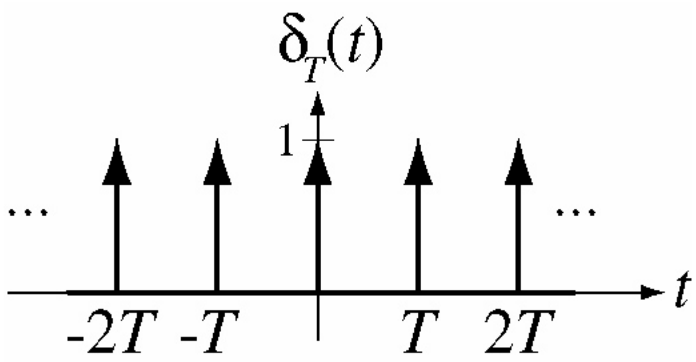
$$\delta\left(a\left(t-t_0\right)\right)=\frac{1}{|a|}\delta\left(t-t_0\right)$$

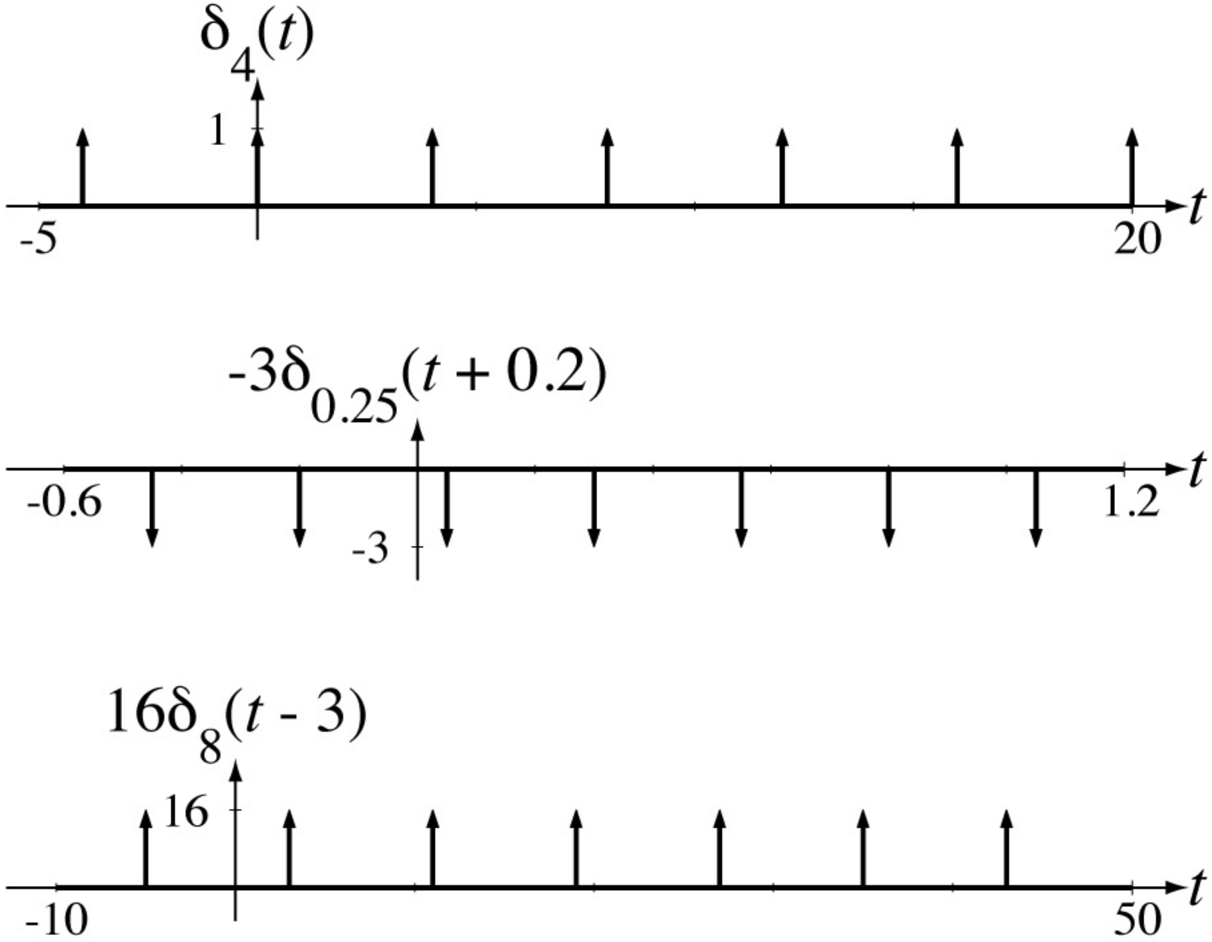
-This property illustrates that the impulse is different from ordinary mathematical functions.

Periodic Impulse

$$\delta_T(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT) \quad [n \text{ an integer}]$$

-The periodic impulse is a sum of infinitely many uniformly-spaced impulses.

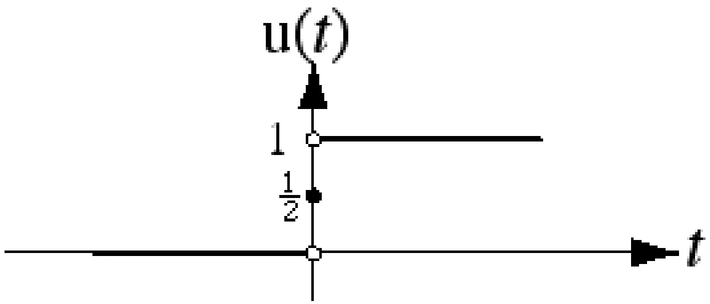



δ4(t)
1
-5
20
t
-3δ0.25(t + 0.2)
-0.6
1.2
-3
t
16δ8(t - 3)
16
-10
50
t

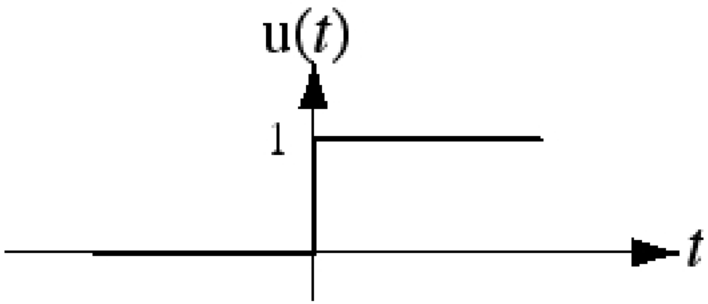
Step

$$u(t) = \begin{cases} 1 & , \quad t > 0 \\ 1/2 & , \quad t = 0 \\ 0 & , \quad t < 0 \end{cases}$$

Precise Graph



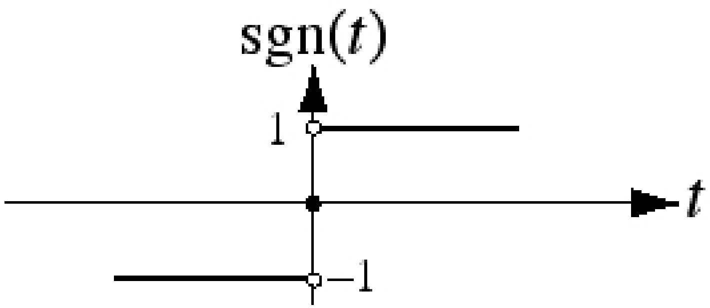
Commonly-Used Graph



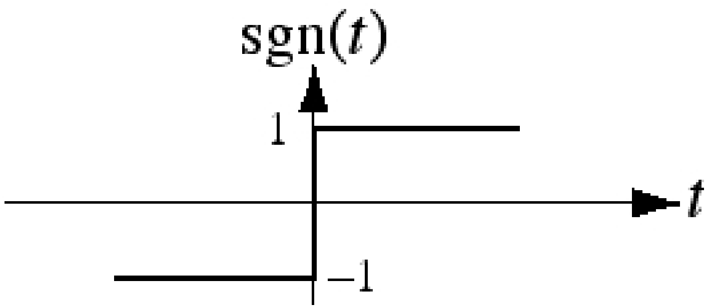
Signum

$$\text{sgn}(t) = \begin{cases} 1 & , \ t > 0 \\ 0 & , \ t = 0 \\ -1 & , \ t < 0 \end{cases}$$

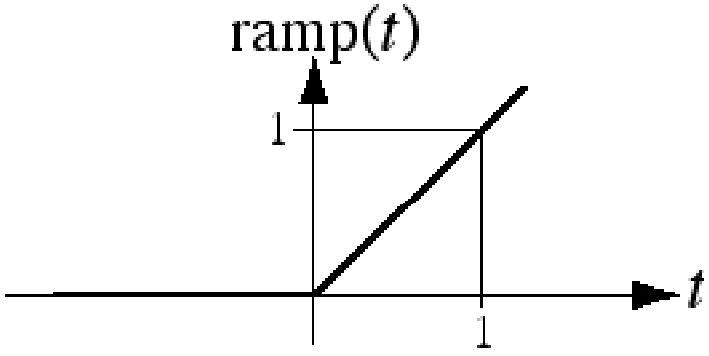
Precise Graph



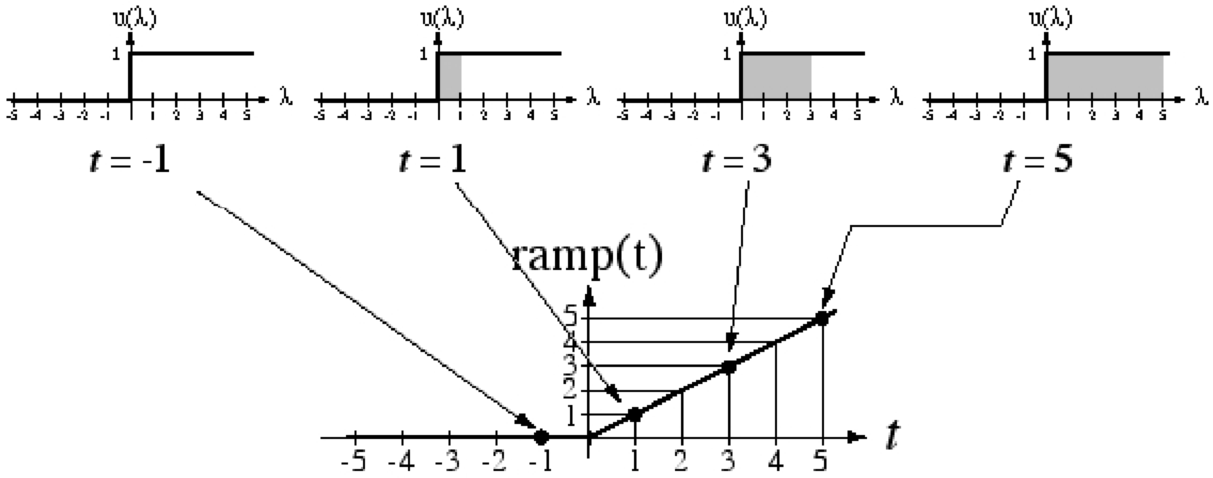
Commonly-Used Graph



Ramp

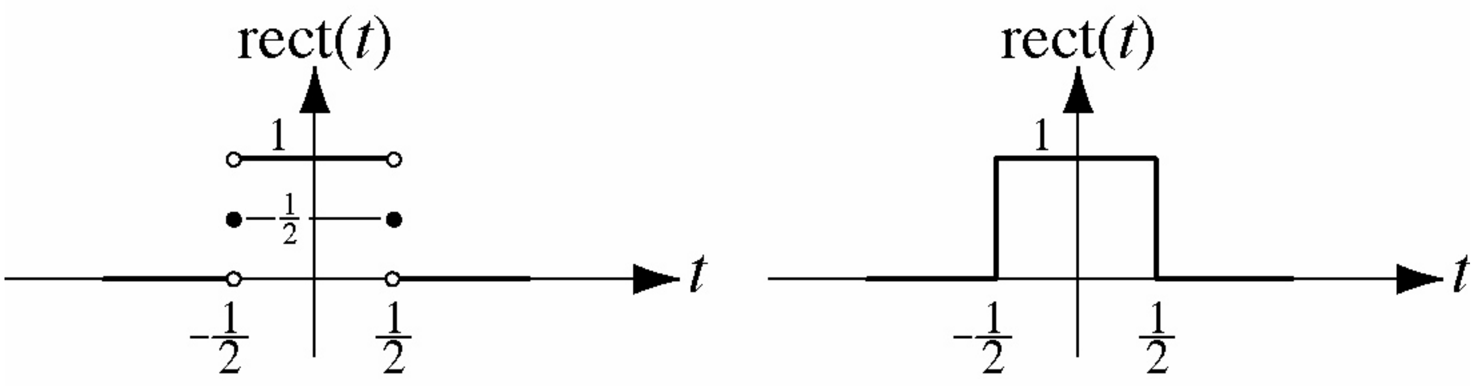


$$\boxed{\text{ramp}(t) = \begin{cases} t \;, & t > 0 \\ 0 \;, & t \le 0 \end{cases}} = \int_{-\infty}^{t} u(\tau)\, d\tau = t u(t)$$

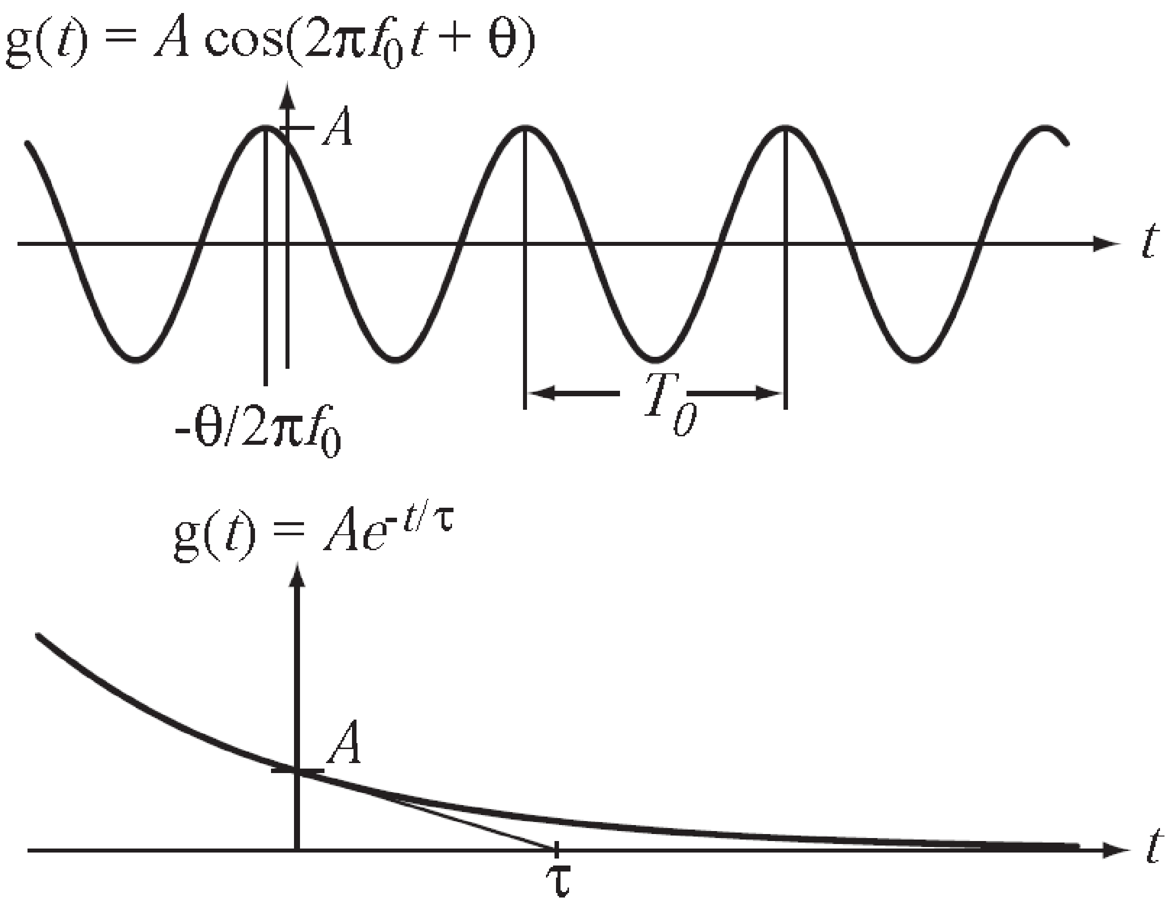


Pulse (Rectangle)

$$\boxed{\operatorname{rect}(t)=\begin{cases}1 & , & |t|<1/2 \\ 1/2 & , & |t|=1/2 \\ 0 & , & |t|>1/2\end{cases}}=u(t+1/2)-u(t-1/2)$$



Real Sinusoid and Real Exponential



Complex Exponential

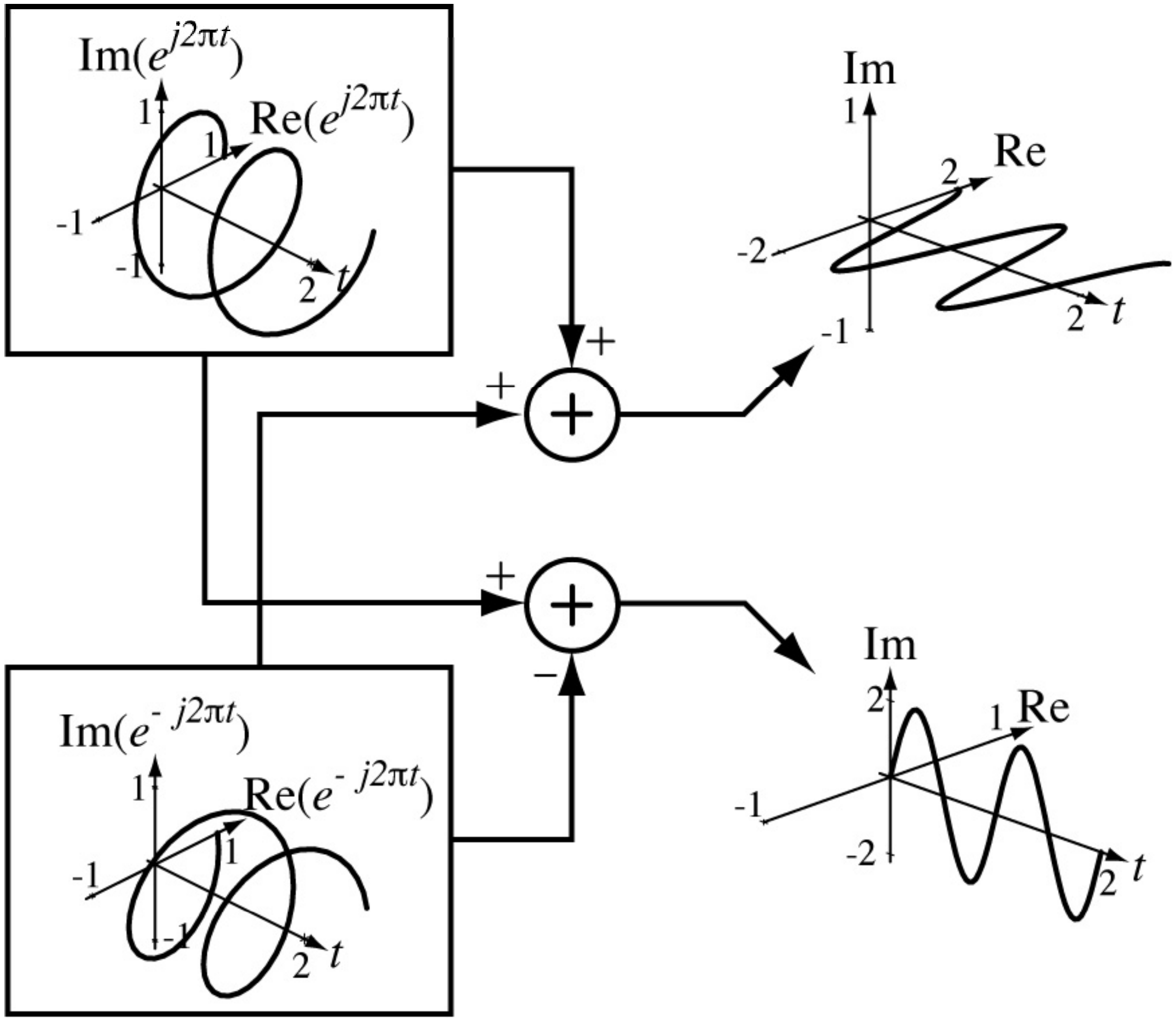
$$g(t) = Ae^{-\frac{t}{\tau}} e^{j2\pi ft}$$

$$e^{j2\pi ft} = \cos\left(2\pi ft\right) + j\sin\left(2\pi ft\right)$$

$$e^{-j2\pi ft} = \cos\left(2\pi ft\right) - j\sin\left(2\pi ft\right)$$

$$\cos\left(2\pi ft\right) = \frac{e^{j2\pi ft} + e^{-j2\pi ft}}{2}$$

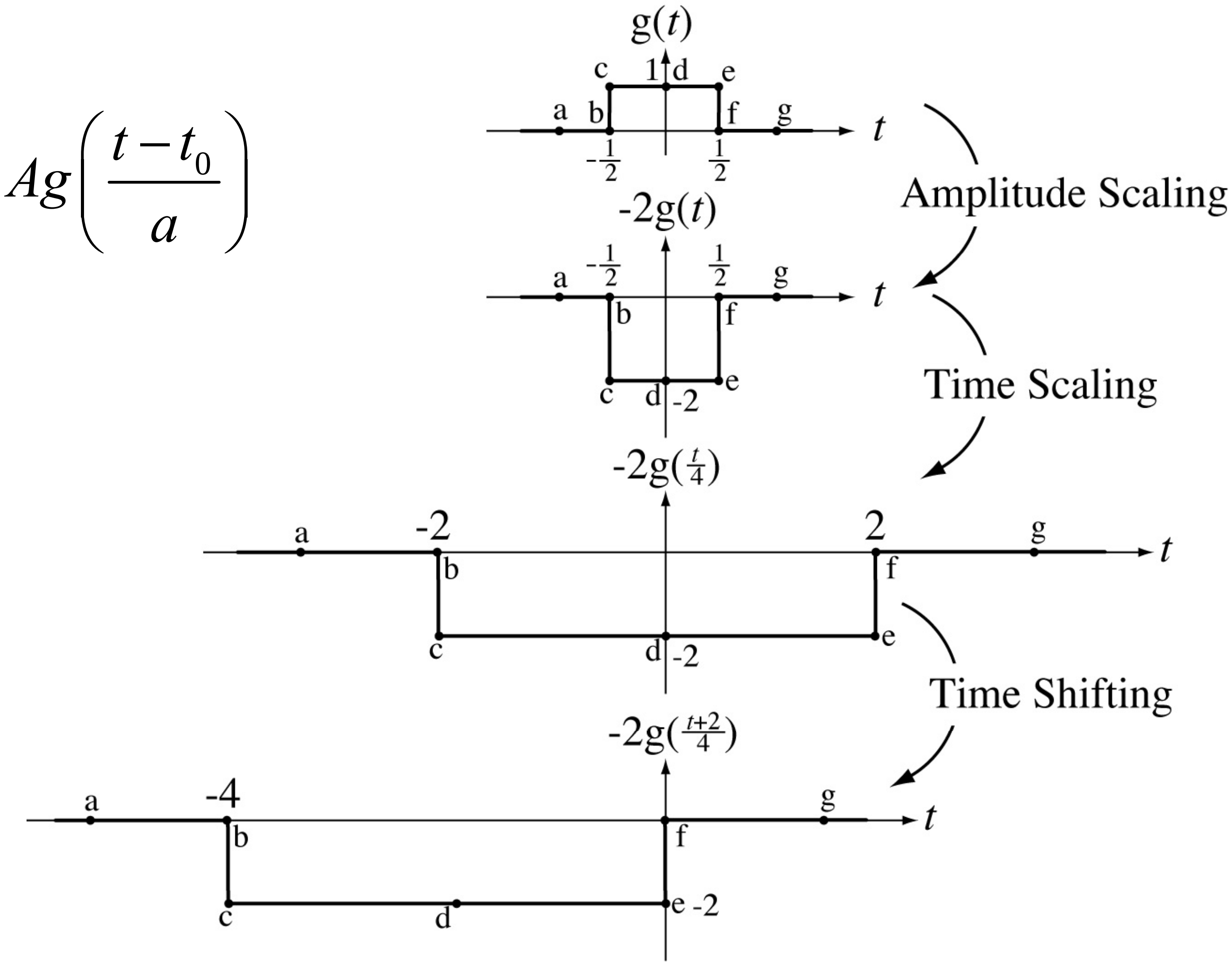
$$\sin\left(2\pi ft\right) = \frac{e^{j2\pi ft} - e^{-j2\pi ft}}{2j}$$



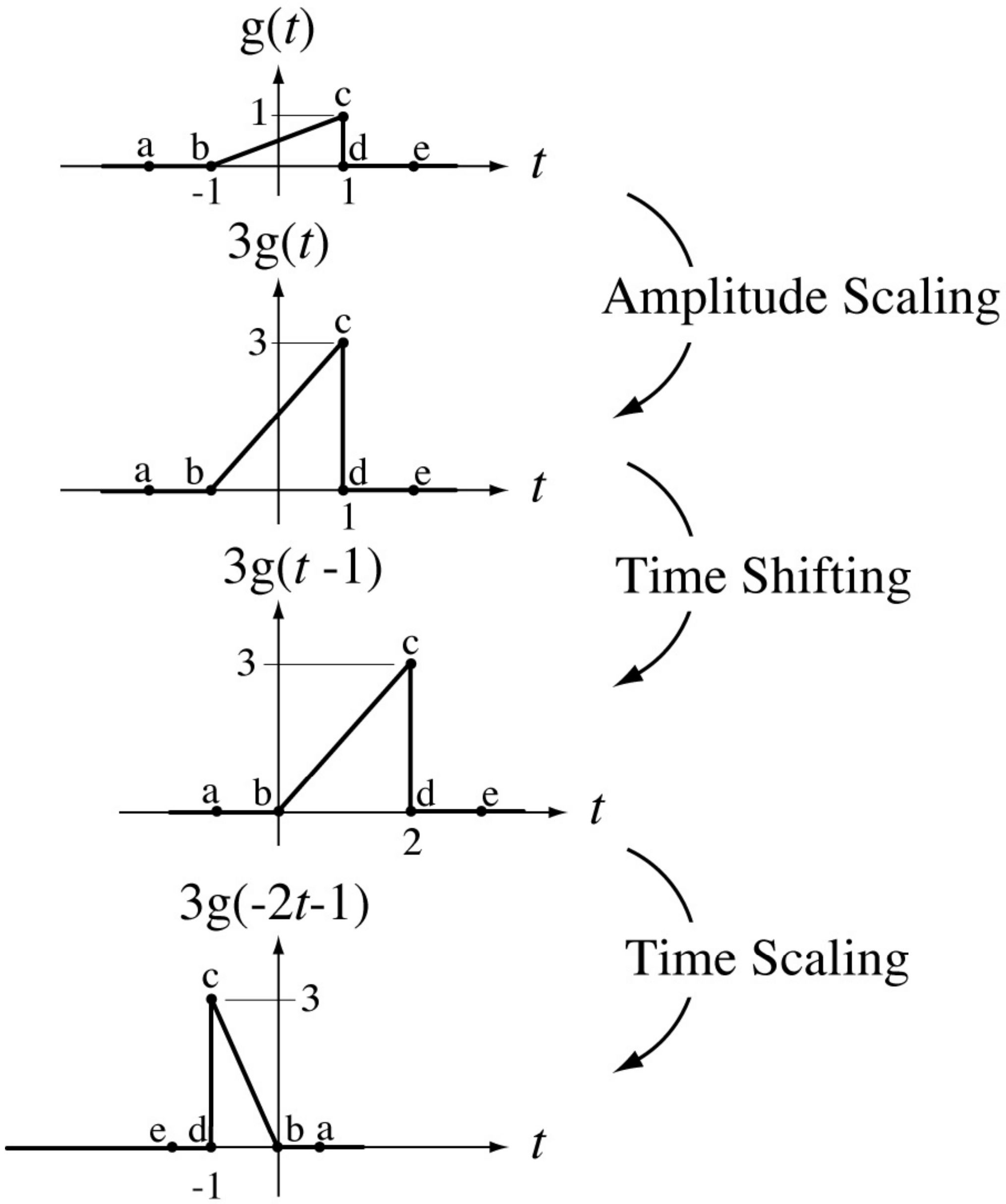
Im($e^{j2\pi t}$)
Re($e^{j2\pi t}$)
1
1
-1
-1
2
t
Im($e^{-j2\pi t}$)
Re($e^{-j2\pi t}$)
1
1
-1
-1
2
t
+
+
+
+
−
Im
Re
1
2
-2
-1
2
t
Im
Re
2
1
-1
-2
2
t

Scaling and Shifting

$$g(t) \to Ag\left(\frac{t-t_0}{a}\right)$$

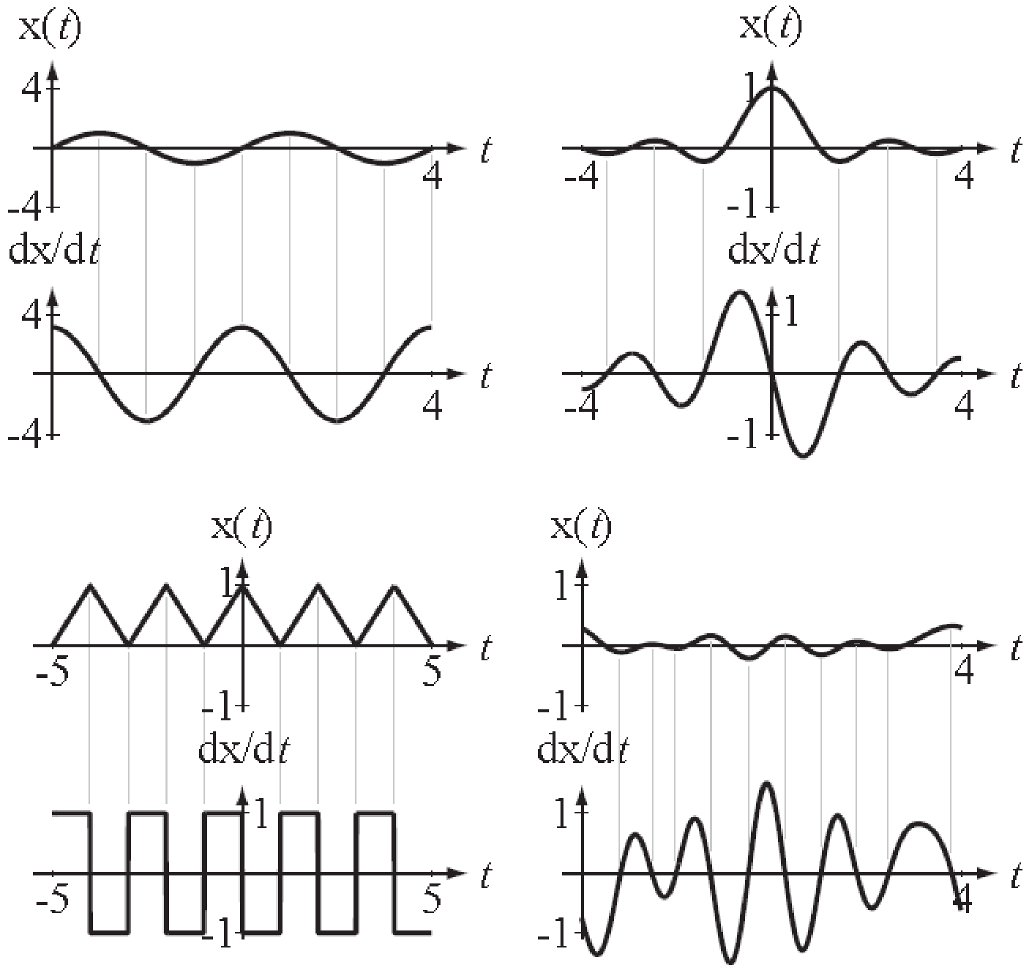


$$g(t) \rightarrow Ag(bt - t_0)$$



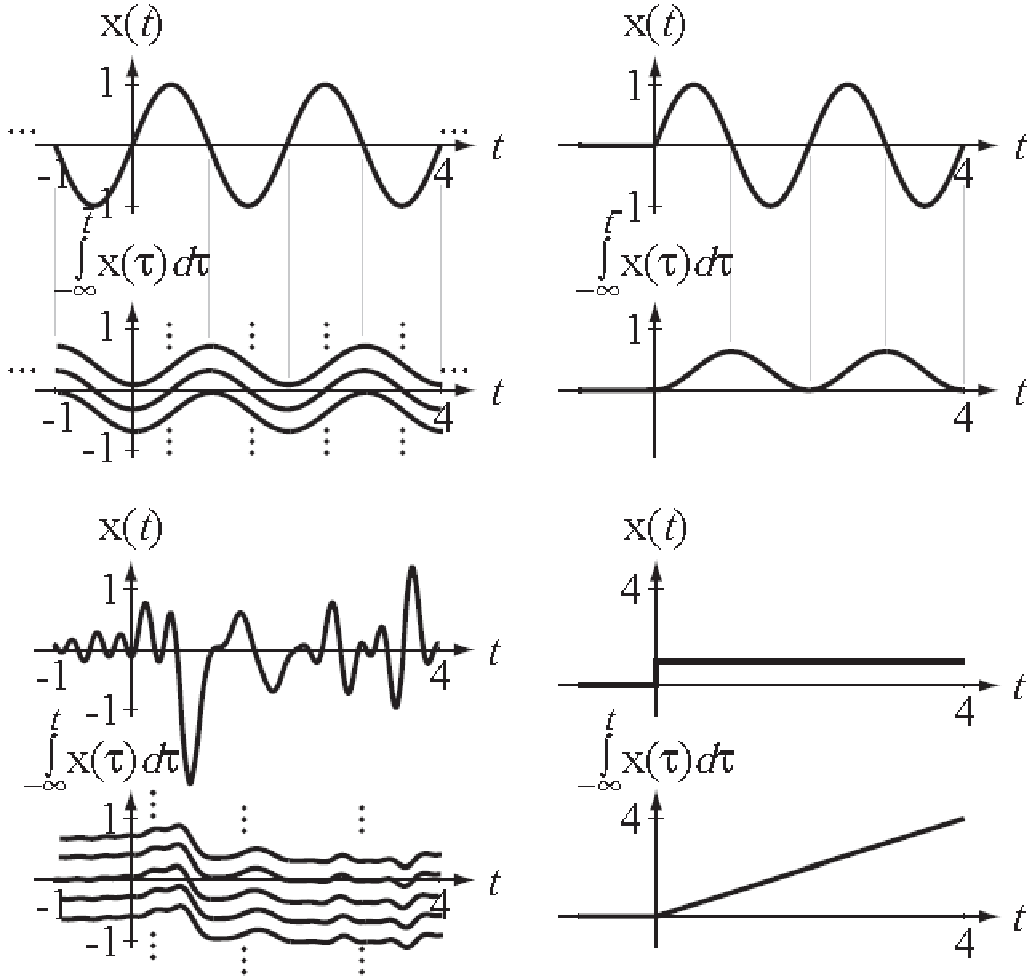
Differentiation

$$g(t) = \frac{dx(t)}{dt}$$



Integration

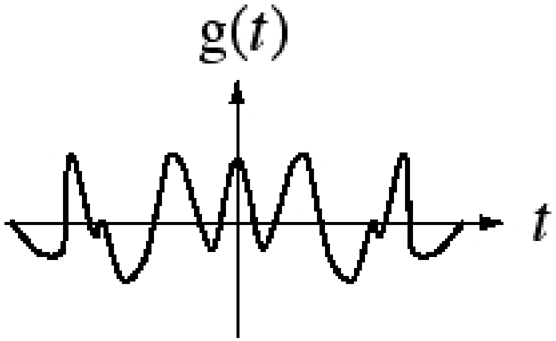
$$g(t) = \int_{-\infty}^{t} x(\tau)\,d\tau$$

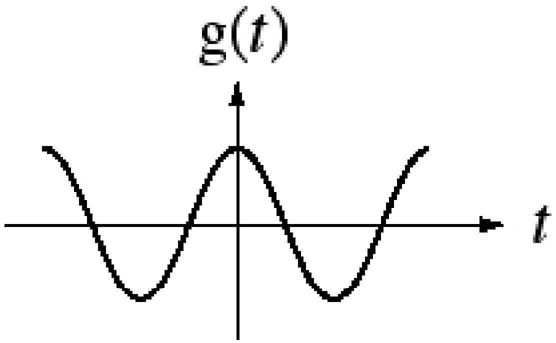


Even and Odd Signals

Even Functions

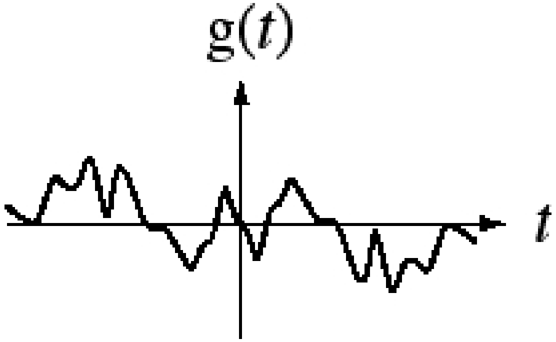
$$g(t) = g(-t)$$

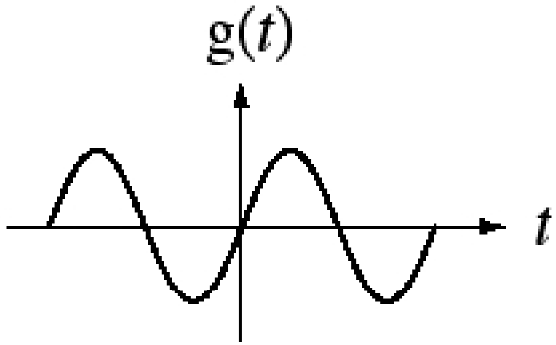




Odd Functions

$$g(t) = -g(-t)$$





Even and Odd Parts of Signals

$$g(t) = g_e(t) + g_o(t)$$

The **even part** of a function is $\boxed{g_e(t) = \dfrac{g(t) + g(-t)}{2}}$

The **odd part** of a function is $\boxed{g_o(t) = \dfrac{g(t) - g(-t)}{2}}$

-The derivative of an even/odd function is odd/even.

-The integral of an even/odd function is an odd/even function, plus a constant.

Periodic Signals

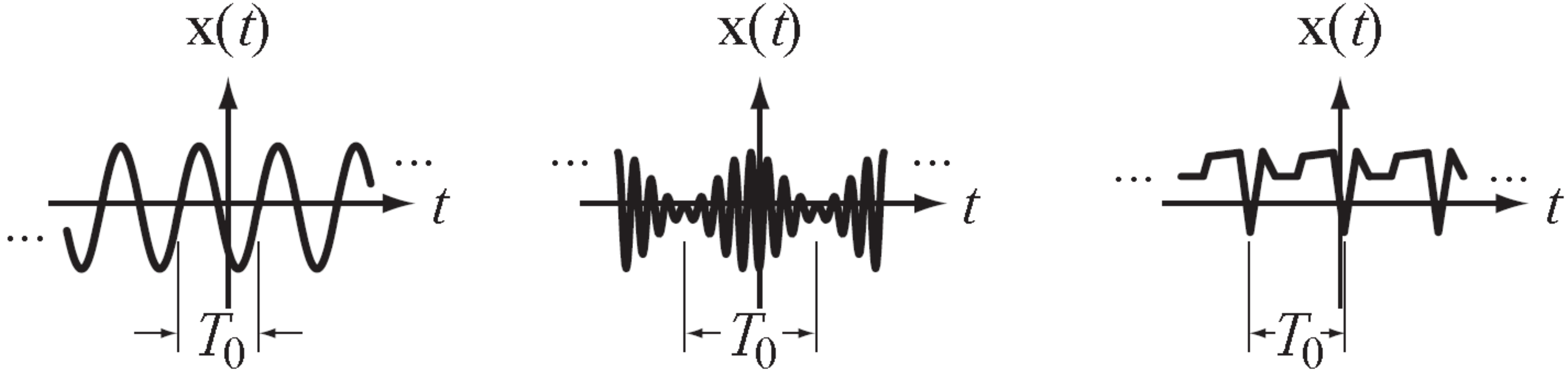
$$g(t) = g(t + nT)$$

n is any integer

T is a **period** of the function

-The minimum positive value of T for which $g(t) = g(t+T)$ is called the **fundamental period** T_0 of the function.

-The reciprocal of the fundamental period is the **fundamental frequency** $f_0 = 1/T_0$

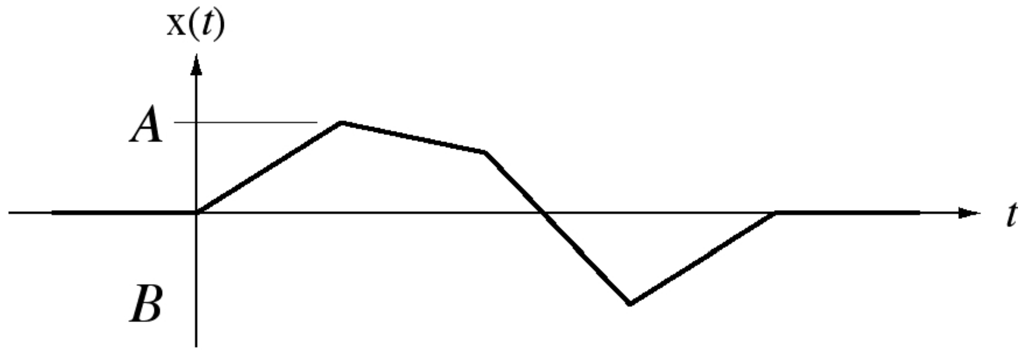


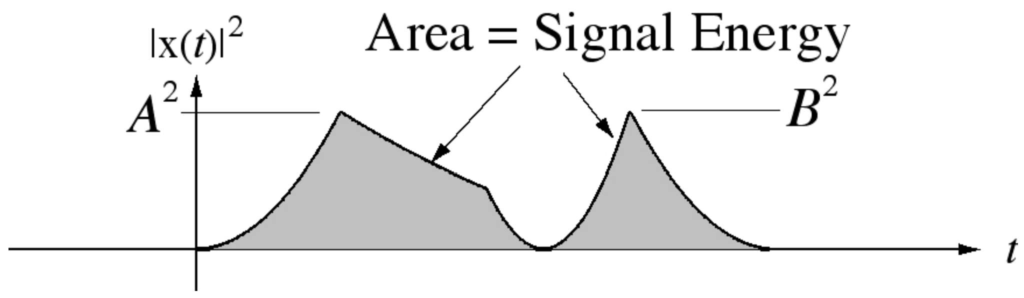
Examples of periodic functions with fundamental period T_0

Signal Energy

The signal energy of a signal $x(t)$ is

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 \, dt$$





Signal Power

-Some signals have infinite signal energy.

-This usually occurs because the signal is not time limited.

-In that case it is more convenient to deal with average signal power.

-The average signal power of a signal $x(t)$ is:

$$\boxed{P_x = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} \left| x(t) \right|^2 dt}$$

-For a periodic signal $x(t)$ the average signal power is:

$$P_x = \frac{1}{T}\int_T |x(t)|^2 \, dt$$

where T is any period of $|x(t)|^2$

-A signal with finite signal energy is called an **energy signal**.

-A signal with infinite signal energy and finite average signal power is called a **power signal**.

-*All periodic signals are power signals.*

3- Discrete-Time Signals

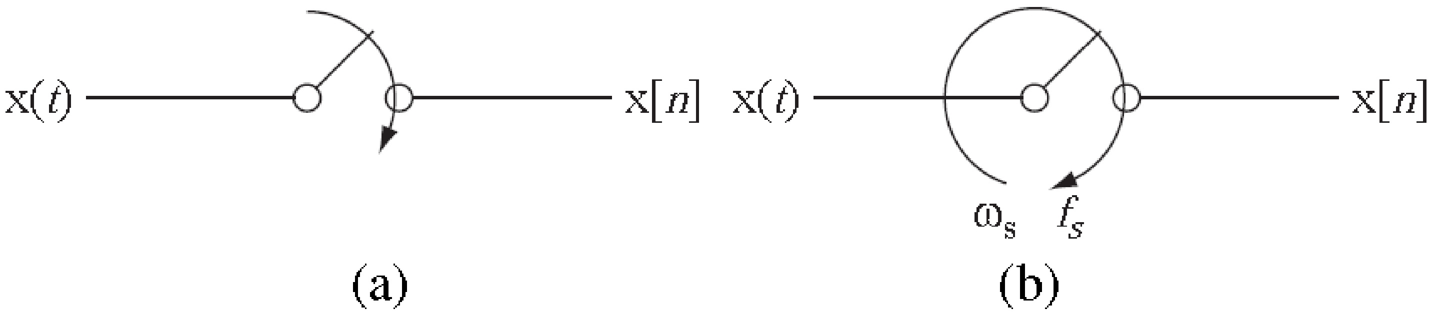
Sampling and Discrete Time

-Sampling is the acquisition of the values of a continuous-time signal at discrete points in time.

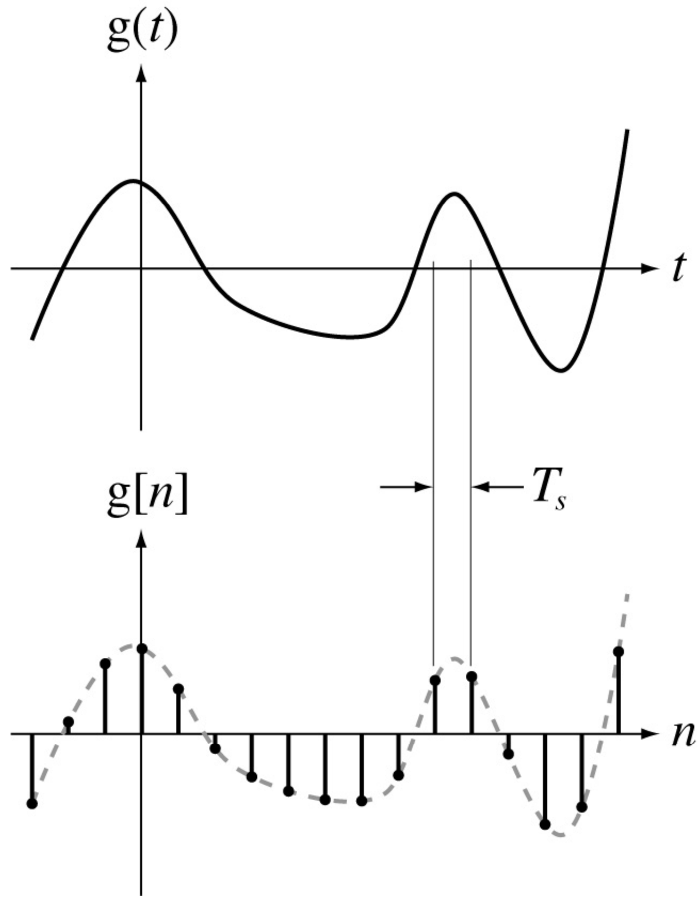
-$x(t)$ is a continuous-time signal, $x[n]$ is a discrete-time signal.

$$\boxed{x[n] = x(nT_s)}$$

$[T_s$ is the time between samples$]$

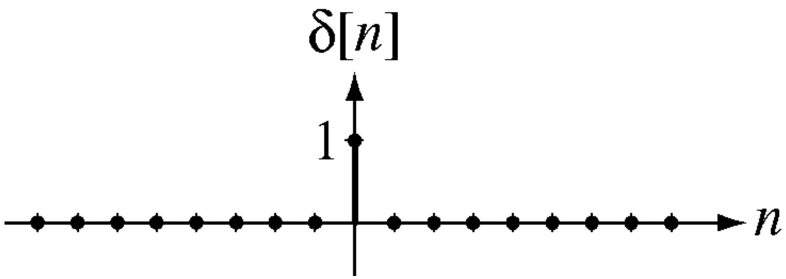


(a) An ideal sampler (b) An ideal sampler sampling uniformly



Creating a discrete-time signal by sampling a continuous-time signal

Unit Impulse



$$\delta[n] = \begin{cases} 1 \ , & n = 0 \\ 0 \ , & n \neq 0 \end{cases}$$

-The discrete-time unit impulse is a function in the ordinary sense (in contrast with the continuous-time impulse).

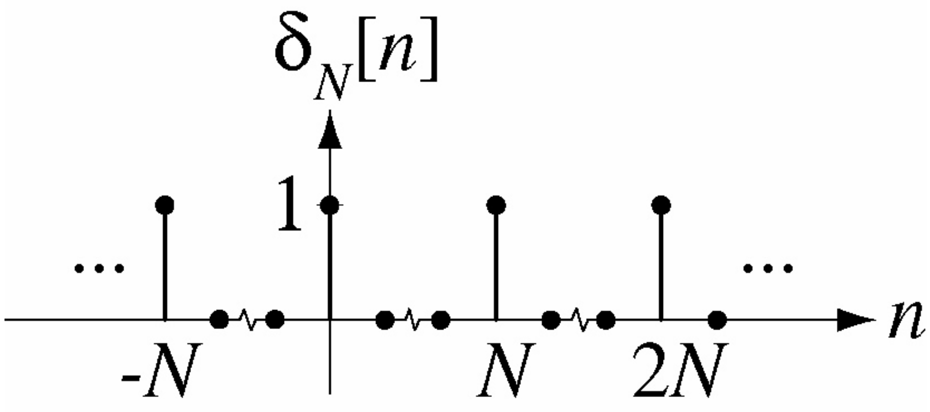
-It has a sampling property

$$\sum_{n=-\infty}^{\infty} A\delta[n-n_0]x[n] = Ax[n_0]$$

-But no scaling property, $\delta[n] = \delta[an]$ for any non-zero, finite integer a.

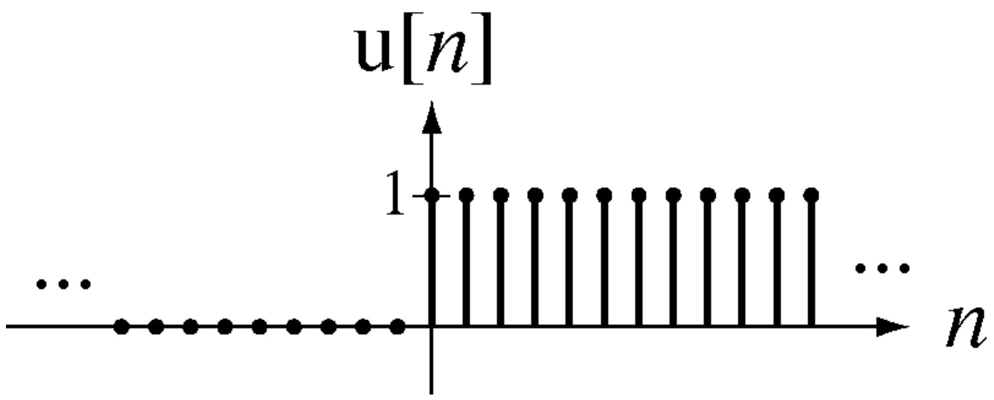
Periodic Impulse

$$\delta_N[n] = \sum_{m=-\infty}^{\infty} \delta[n - mN]$$



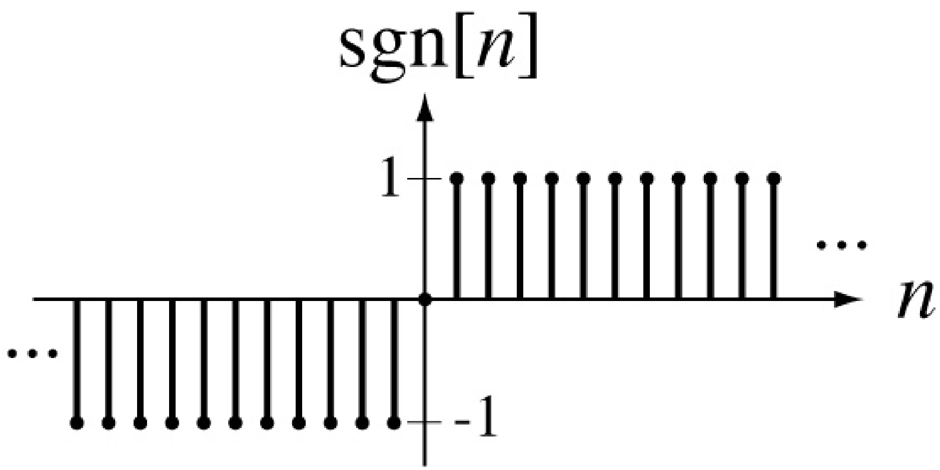
Unit Sequence

$$u[n]=\begin{cases}1 \;, & n\geq 0\\ 0 \;, & n<0\end{cases}$$



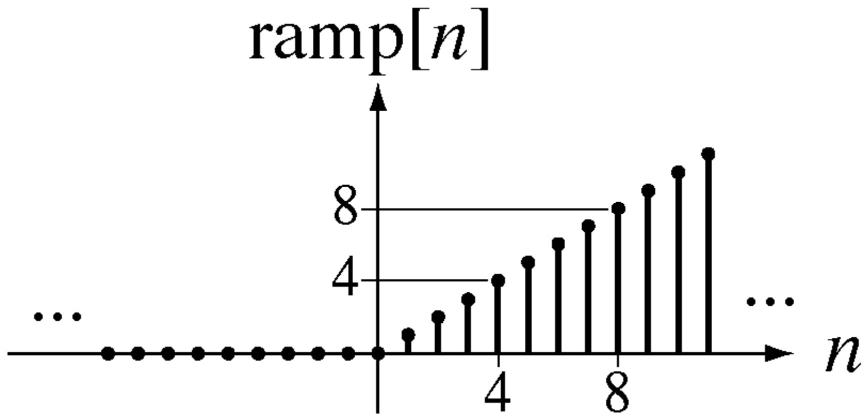
Signum

$$\operatorname{sgn}[n] = \begin{cases} 1 & , \quad n > 0 \\ 0 & , \quad n = 0 \\ -1 & , \quad n < 0 \end{cases}$$



Unit Ramp

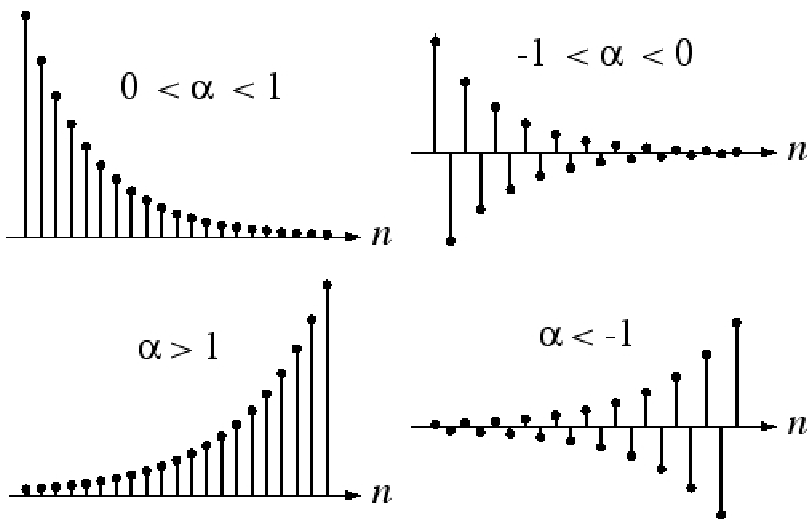
$$\boxed{ramp[n] = \begin{cases} n \;, & n \geq 0 \\ 0 \;, & n < 0 \end{cases}} = nu[n]$$



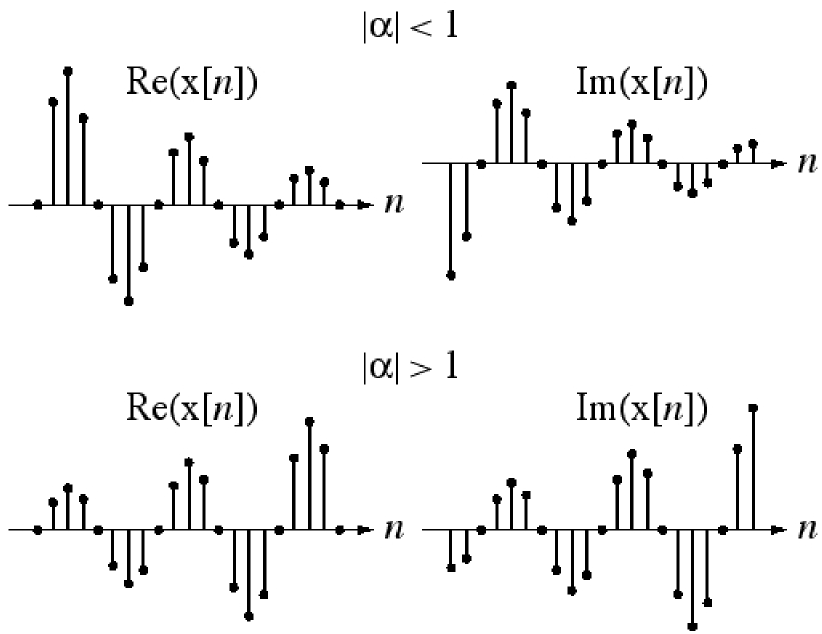
Exponentials

$$\boxed{x[n] = A\alpha^n} \qquad [A \text{ is a real constant}]$$

Real α



Complex α



Sinusoids

$$\boxed{g[n] = A\cos(2\pi F_0 n + \theta)}$$

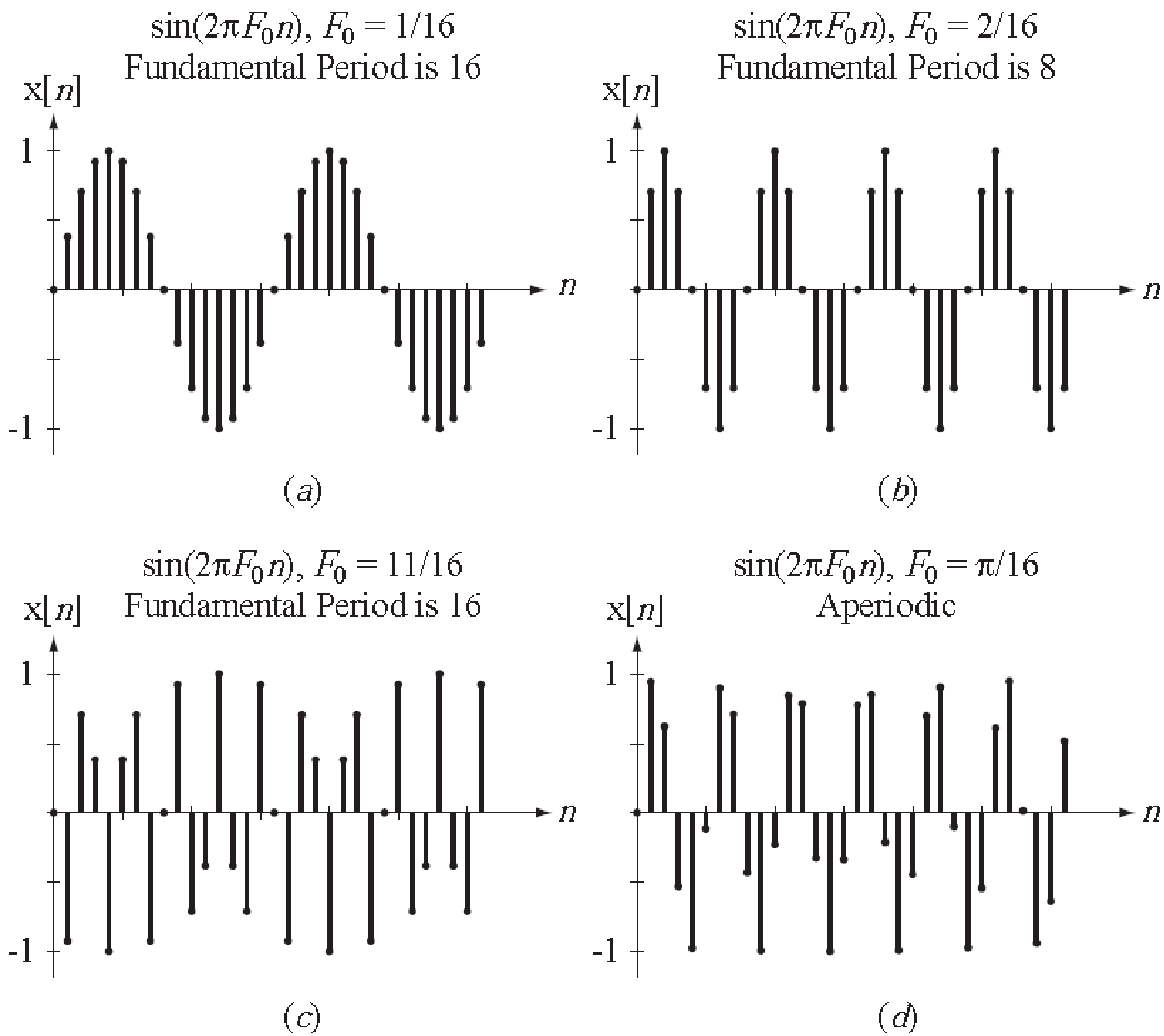
A is a real constant, θ is a real phase shift in radians

F_0 is a real number, and n is discrete time

-Unlike a continuous-time sinusoid,

a discrete-time sinusoid is <u>not necessarily periodic.</u>

- $g[n]$ periodic $\rightarrow F_0$ must be a ratio of integers (a rational number).



Four discrete-time sinusoids

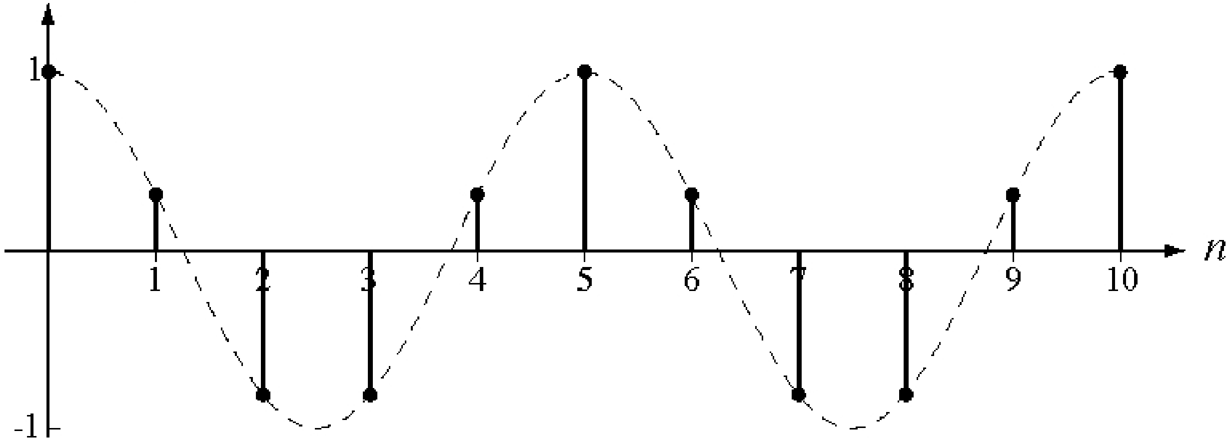
-Two sinusoids whose analytical expressions look different

$$g_1[n] = A\cos(2\pi F_{01}n + \theta) \text{ and } g_2[n] = A\cos(2\pi F_{02}n + \theta)$$

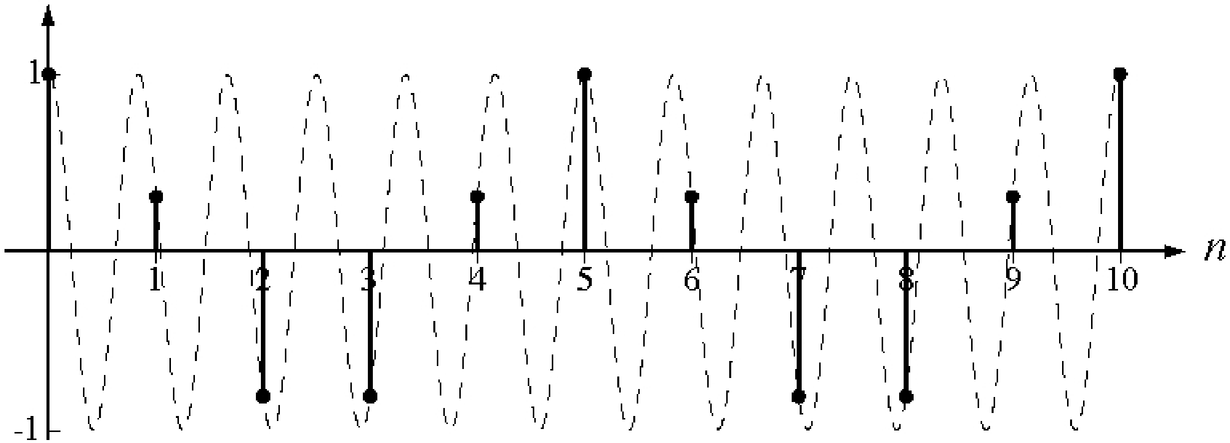
may actually be the same if $\underline{F_{02} = F_{01} + m,}$ where m is an integer.

$$A\cos(2\pi F_{02}n + \theta) = A\cos(2\pi F_{01}n + 2\pi mn + \theta) = A\cos(2\pi F_{01}n + \theta)$$

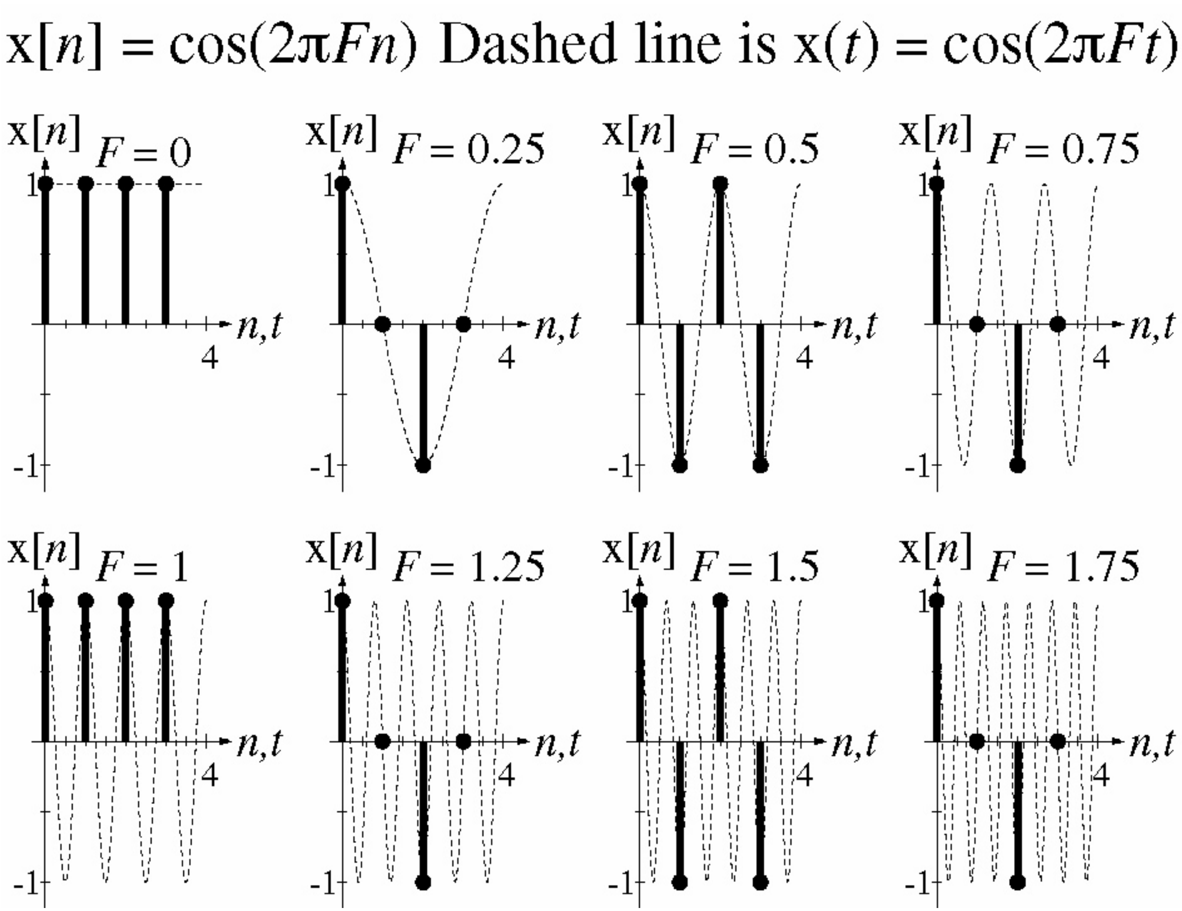
$$g_1[n] = \cos\left(\frac{2\pi n}{5}\right)$$



$$g_2[n] = \cos\left(\frac{12\pi n}{5}\right)$$



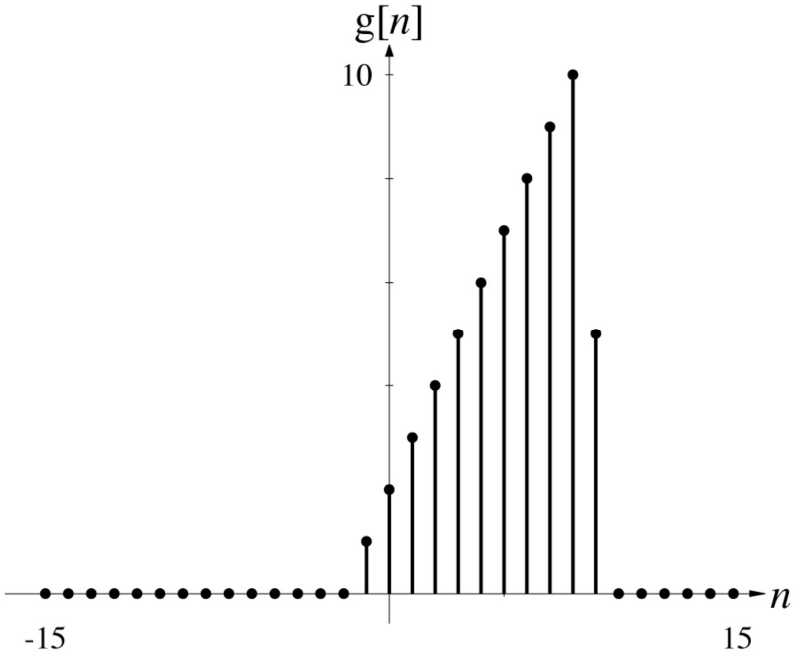
Two cosines with different *F's but the same functional behavior*



A discrete-time sinusoid with frequency *F repeats* every time *F changes by one*

Scaling and Shifting Functions

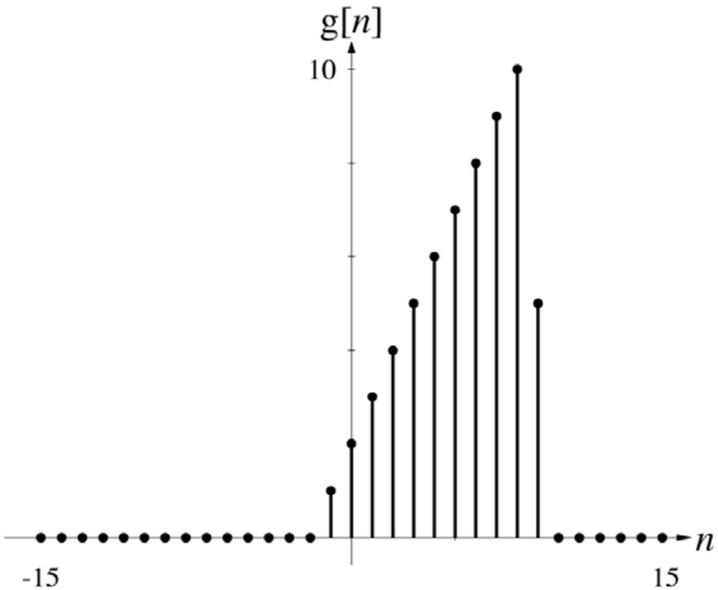
Let $g[n]$ be graphically defined by:



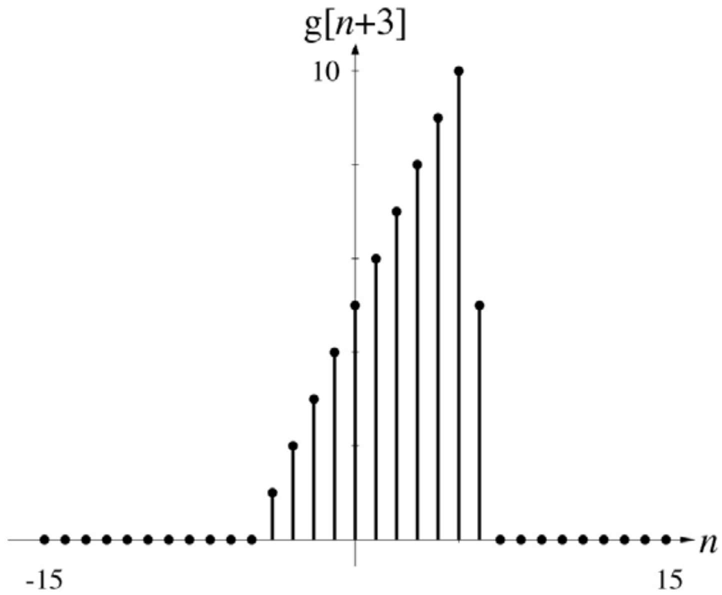
n	-1	0	1	2	3	4	5	6	7	8	9	10
$g[n]$	1	2	3	4	5	6	7	8	9	10	5	0

$$g[n]=0\ ,\ |n|>15$$

Time shifting $\quad n \to n + n_0 \quad [n_0 \text{ an integer}]$



n	-1	0	1	2	3	4	5	6	7	8	9	10
$g[n]$	1	2	3	4	5	6	7	8	9	10	5	0



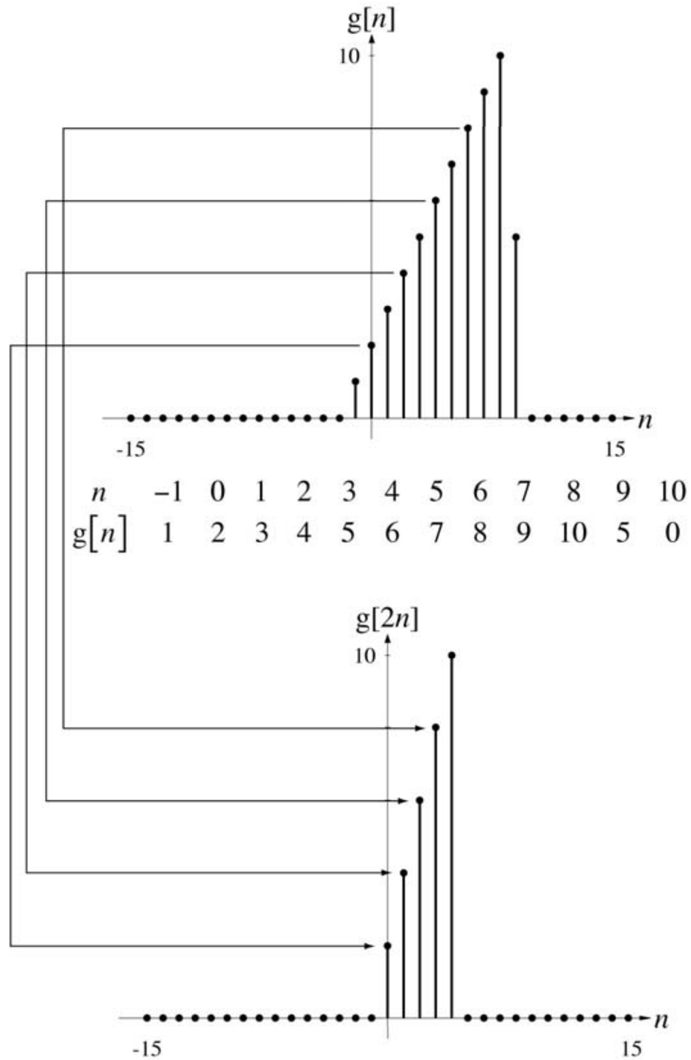
n	-4	-3	-2	-1	0	1	2	3	4	5	6	7
$n+3$	-1	0	1	2	3	4	5	6	7	8	9	10
$g[n+3]$	1	2	3	4	5	6	7	8	9	10	5	0

Time compression $\quad n \rightarrow Kn \quad \left[K \text{ an integer} > 1\right]$

Time expansion $\quad n \rightarrow n/K \quad \left[K \text{ an integer} > 1\right]$

-For all n such that n/K is an integer, $g\left[n/K\right]$ is defined.

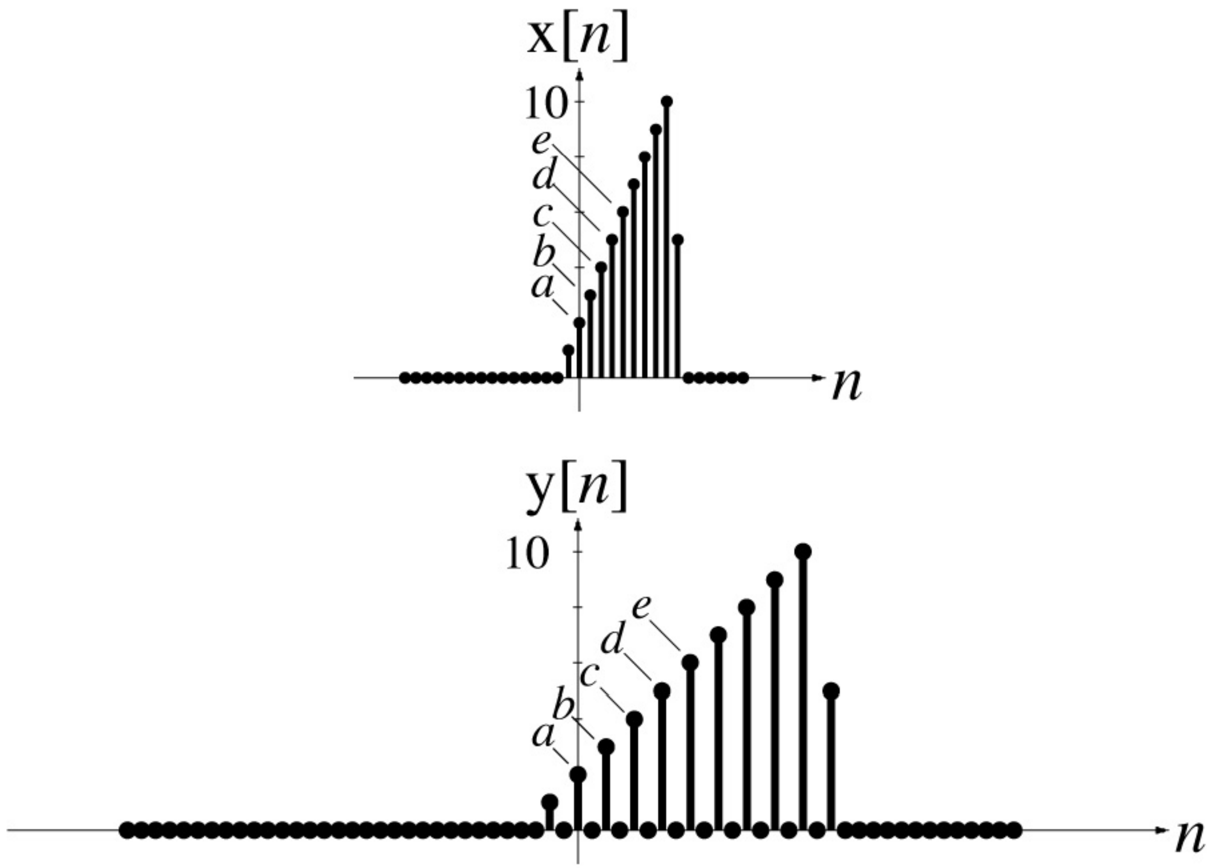
-For all n such that n/K is not an integer, $g\left[n/K\right]$ is not defined.



n	-1	0	1	2	3	4	5	6	7	8	9	10
$g[n]$	1	2	3	4	5	6	7	8	9	10	5	0

n	0	1	2	3	4
$2n$	0	2	4	6	8
$g[2n]$	2	4	6	8	10

Time compression for a discrete-time function



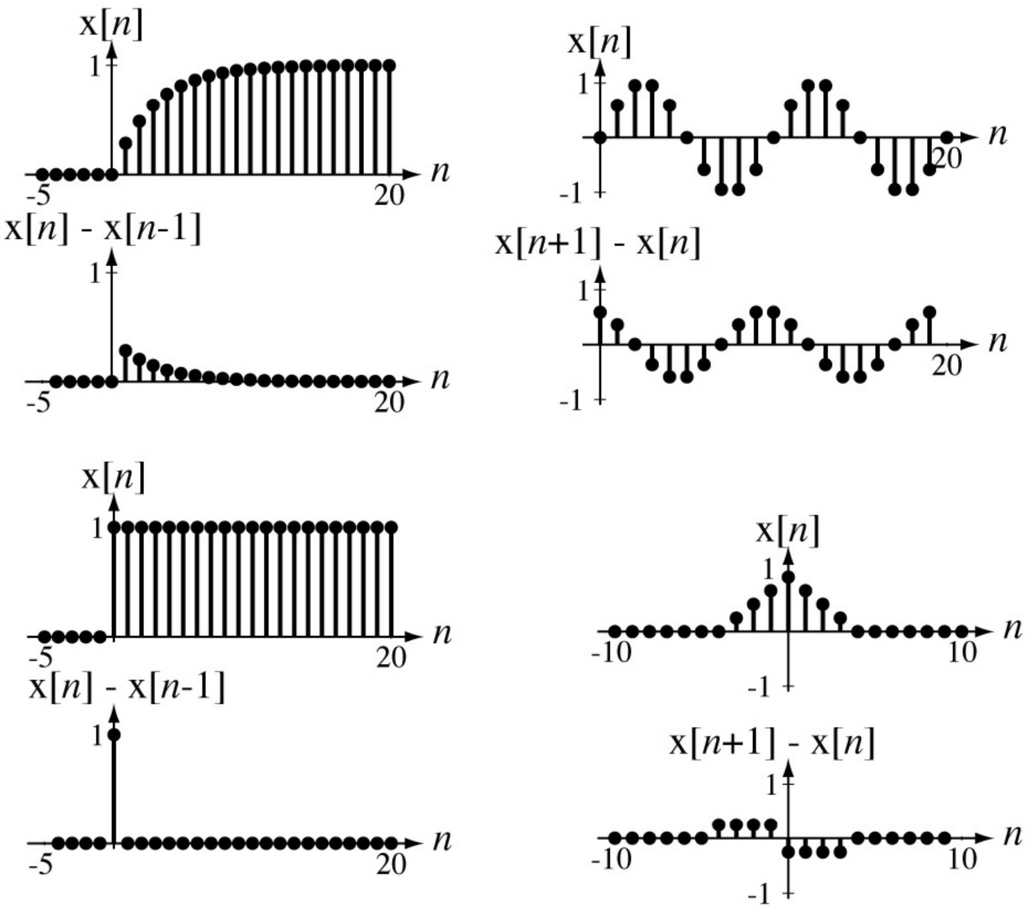
Time expansion for a discrete-time function (K=2)

Differencing

$$g[n] = x[n] - x[n-1]$$

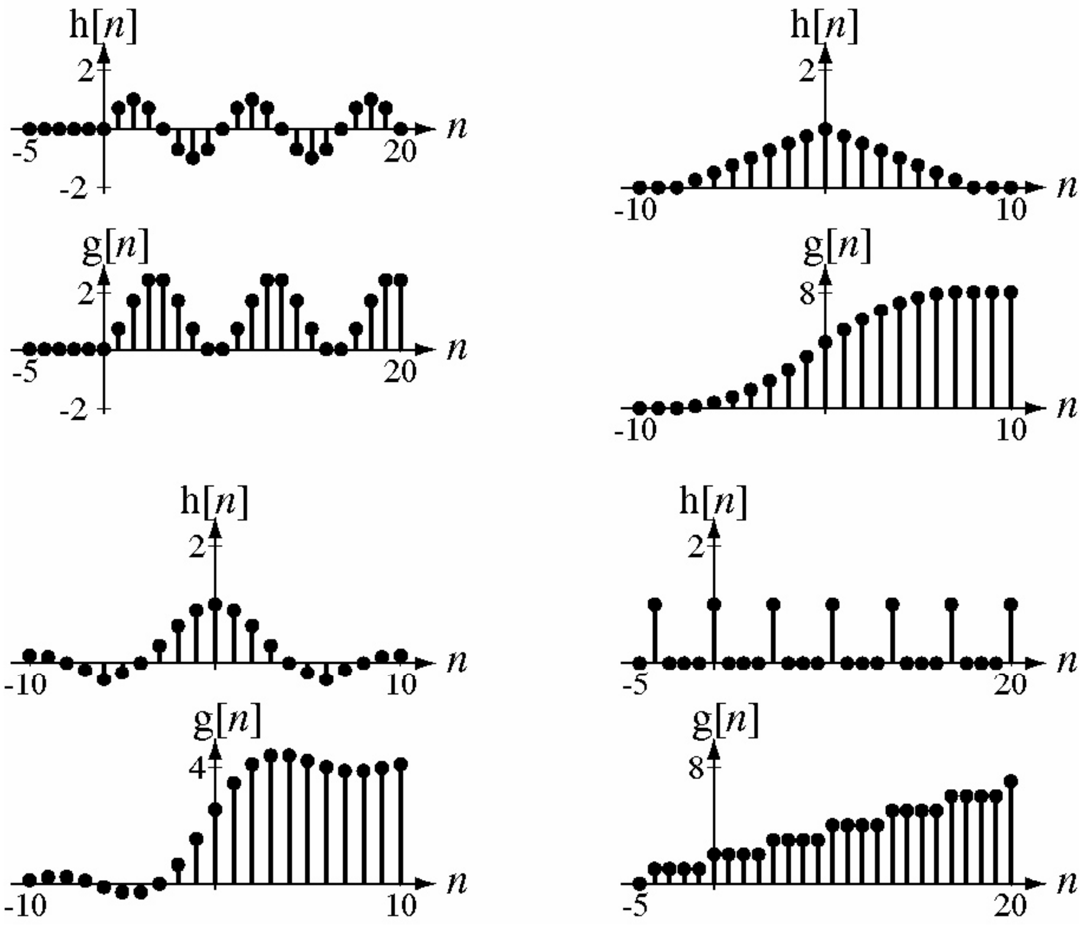
$$g[n] = x[n+1] - x[n]$$

Backward Differences   Forward Differences



Accumulation

$$g[n] = \sum_{m=-\infty}^{n} h[m]$$

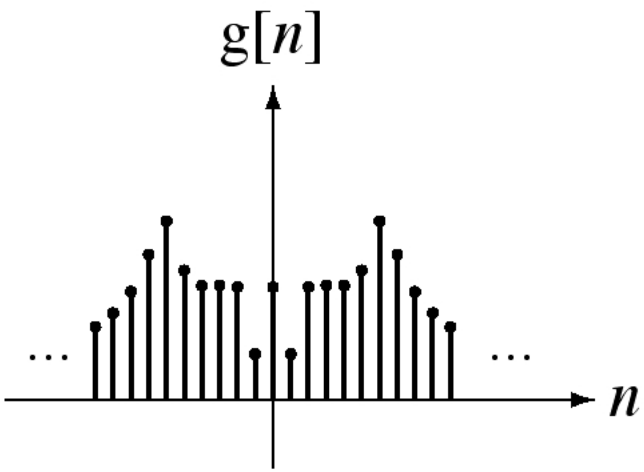


Even and Odd Signals

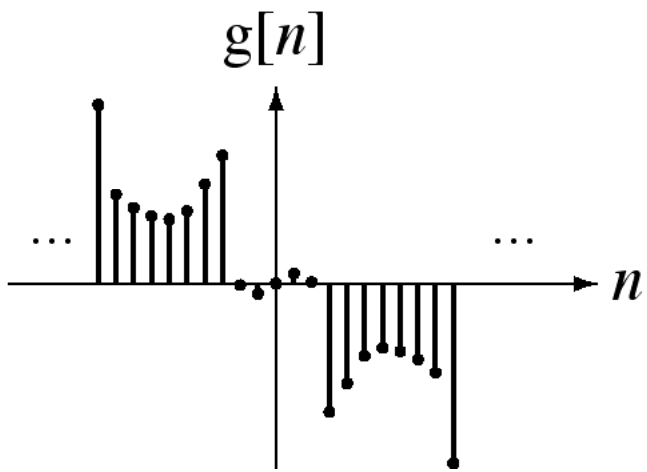
$$g[n] = g[-n]$$

$$g[n] = -g[-n]$$

Even Function



Odd Function



$$g_e[n] = \frac{g[n] + g[-n]}{2}$$

$$g_o[n] = \frac{g[n] - g[-n]}{2}$$

Periodic Signals

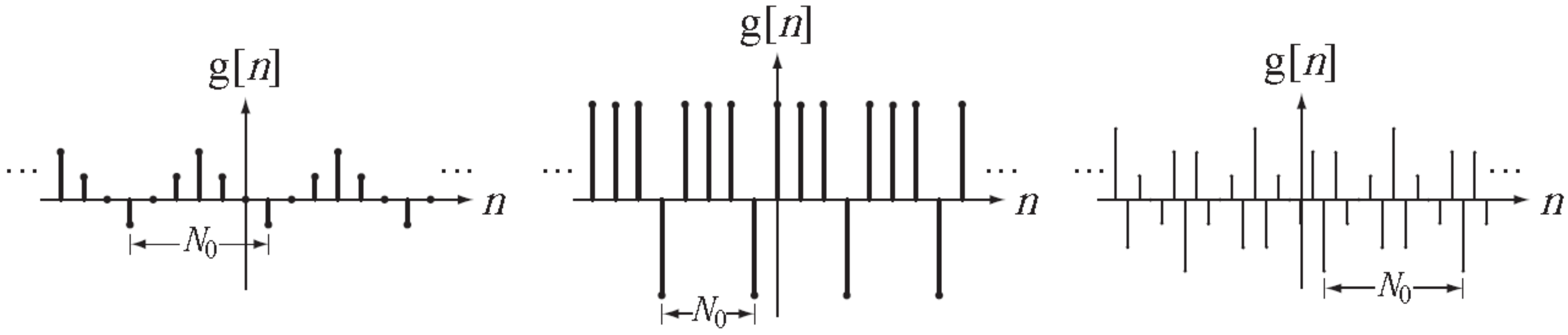
$$\boxed{g[n] = g[n + mN]}$$

m is any integer

N is a **period** of the function

-The minimum positive value of N for which $g[n] = g[n+N]$ is called the **fundamental period** N_0 of the function.

-The reciprocal of the fundamental period is the **fundamental frequency** $F_0 = 1/N_0$



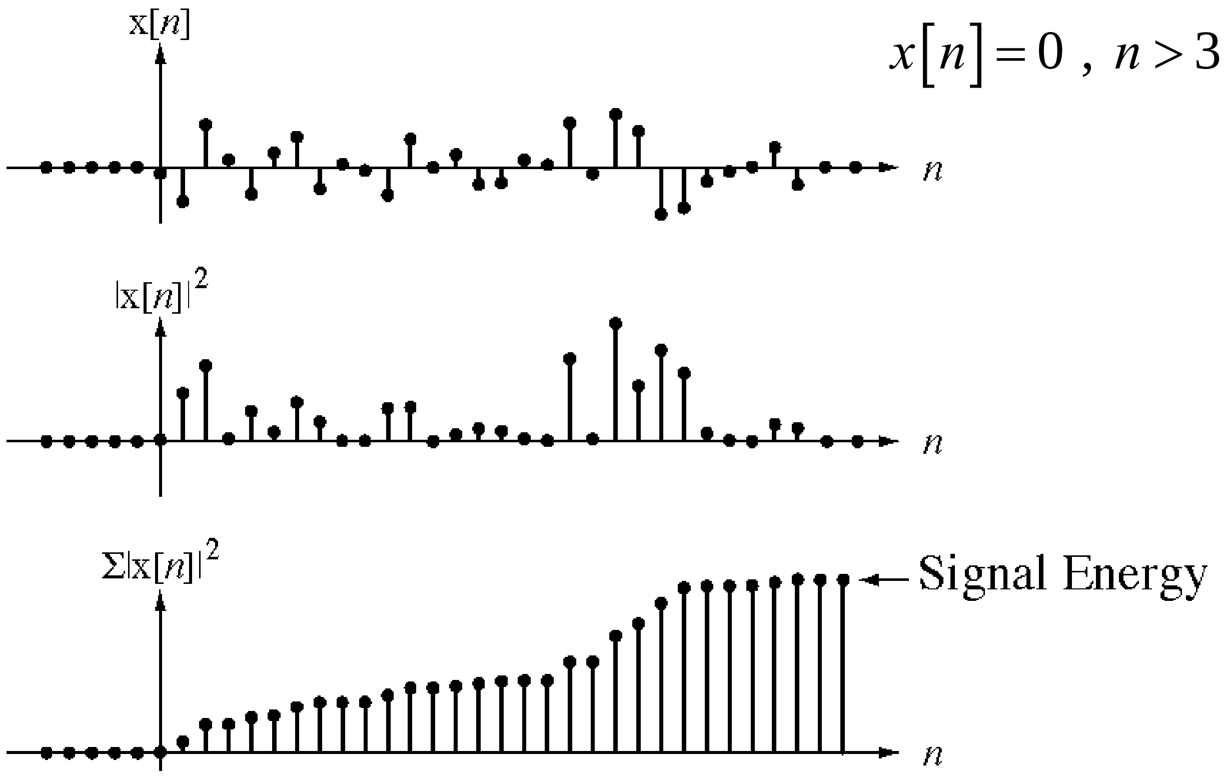
Examples of periodic functions with fundamental period N_0

Signal Energy

The signal energy of a signal $x[n]$ is

$$E_x = \sum_{n=-\infty}^{\infty} \left| x[n] \right|^2$$

$$x[n] = 0 \ , \ n > 31$$



Signal Power

-Some signals have infinite signal energy.

-This usually occurs because the signal is not time limited.

-In that case it is more convenient to deal with average signal power.

-The average signal power of a signal $x[n]$ is

$$P_x = \lim_{N \to \infty} \frac{1}{2N} \sum_{n=-N}^{N-1} \left| x[n] \right|^2$$

-For a periodic signal $x[n]$ the average signal power is:

$$P_x = \frac{1}{N} \sum_{n=\langle N \rangle} \left| x[n] \right|^2$$

$\left(\text{The notation } \sum_{n=\langle N \rangle} \text{ means the sum over any set of consecutive } n\text{'s exactly } N \text{ in length} \right)$

-A signal with finite signal energy is called an **energy signal.**

-A signal with infinite signal energy and finite average signal power is called a **power signal.**

-*All periodic signals are power signals.*

4- Description of Systems

Systems

-Broadly speaking, a system is anything that responds when stimulated or excited.

-Engineering system analysis is the application of mathematical methods to the design and analysis of systems.

-Systems have **inputs** and **outputs.**

-Systems accept **excitations** or **input signals** at their inputs and produce **responses** or **output signals** at their outputs.

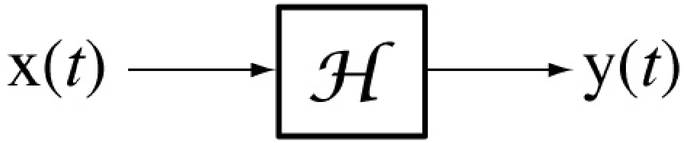
Systems

-Continuous-time systems are usually described by **differential equations.**

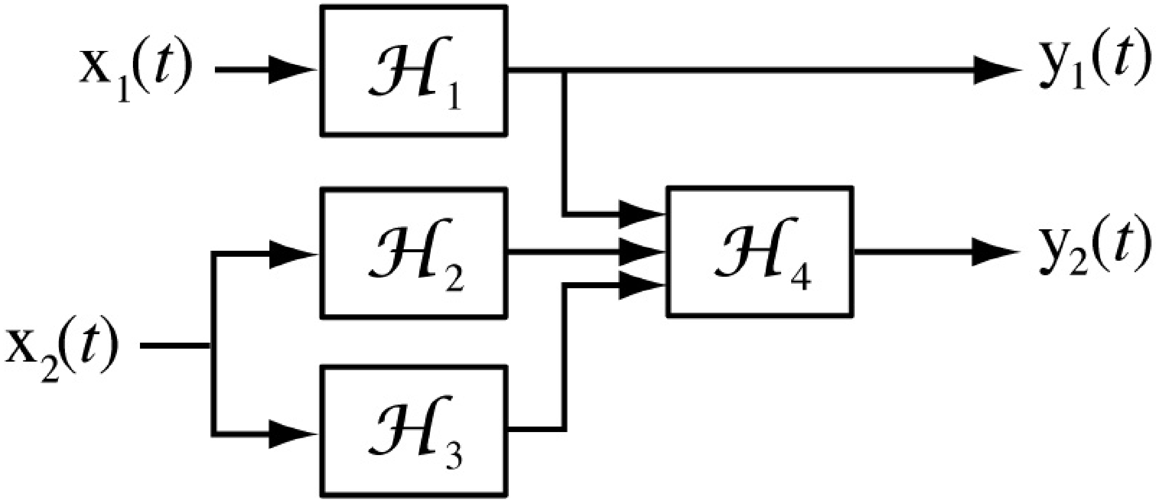
-Discrete-time systems are usually described by **difference equations.**

-The properties of discrete-time systems have the same meaning as they do in continuous-time systems.

-Systems are often represented by **block diagrams**

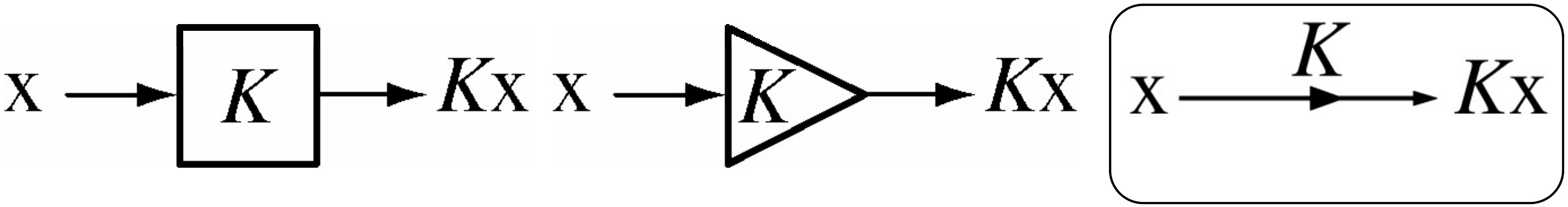


A single-input, single-output system block diagram

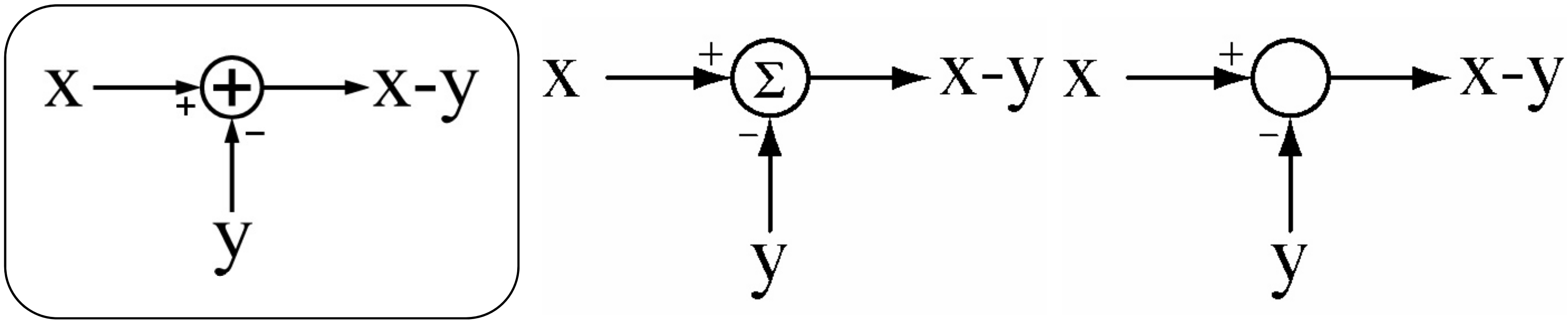


A multiple-input, multiple-output system block diagram

Some Block Diagram Symbols

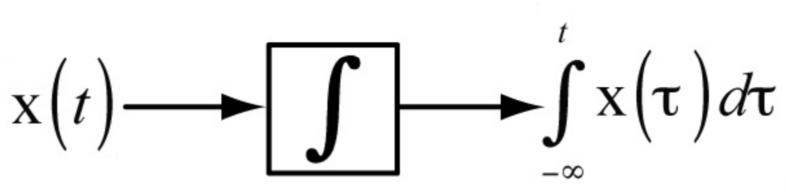


Three common block diagram symbols for an **amplifier**

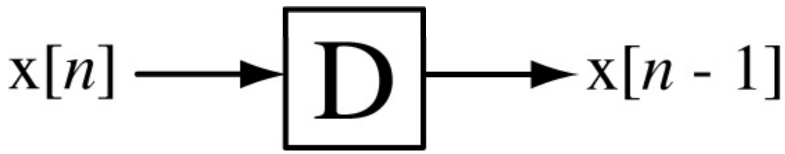


Three common block diagram symbols for a **summing junction**

-The block diagram symbols for a summing junction and an amplifier are the same for discrete-time systems as they are for continuous-time systems.



Block diagram symbol (continuous-time systems) for an **integrator**



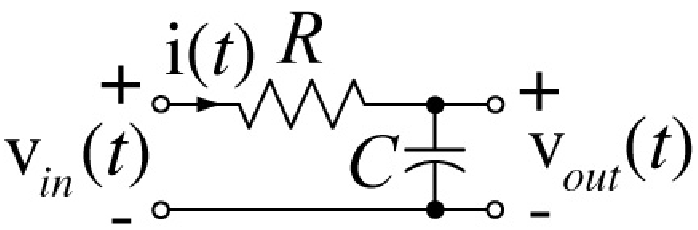
Block diagram symbol (discrete time systems) for a **delay**

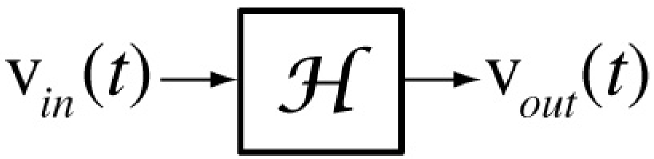
An Electrical Circuit Viewed as a System

-An RC lowpass filter is a simple electrical system.

-It is excited by a voltage $v_{in}(t)$ and responds with a voltage $v_{out}(t)$.

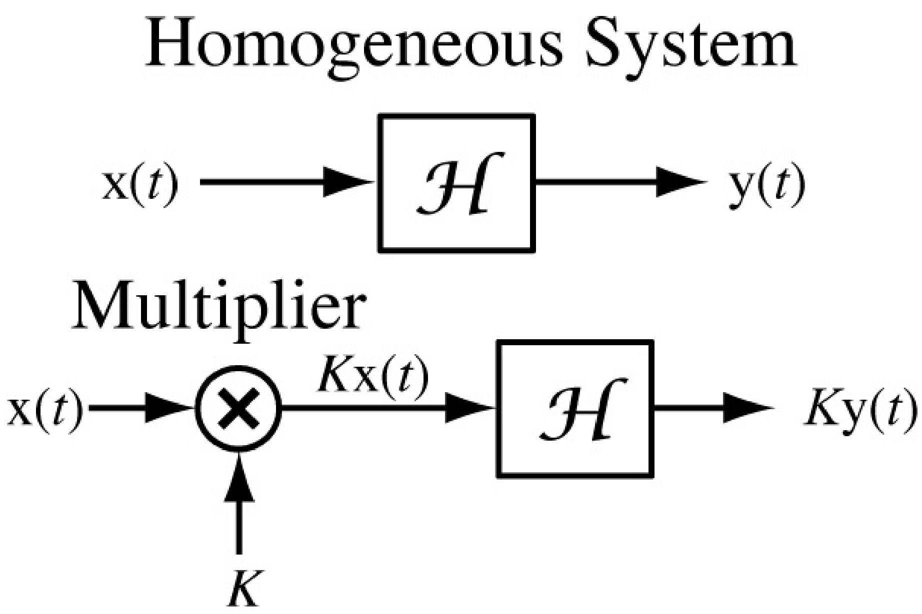
-It can be viewed or modeled as a single-input, single-output system.





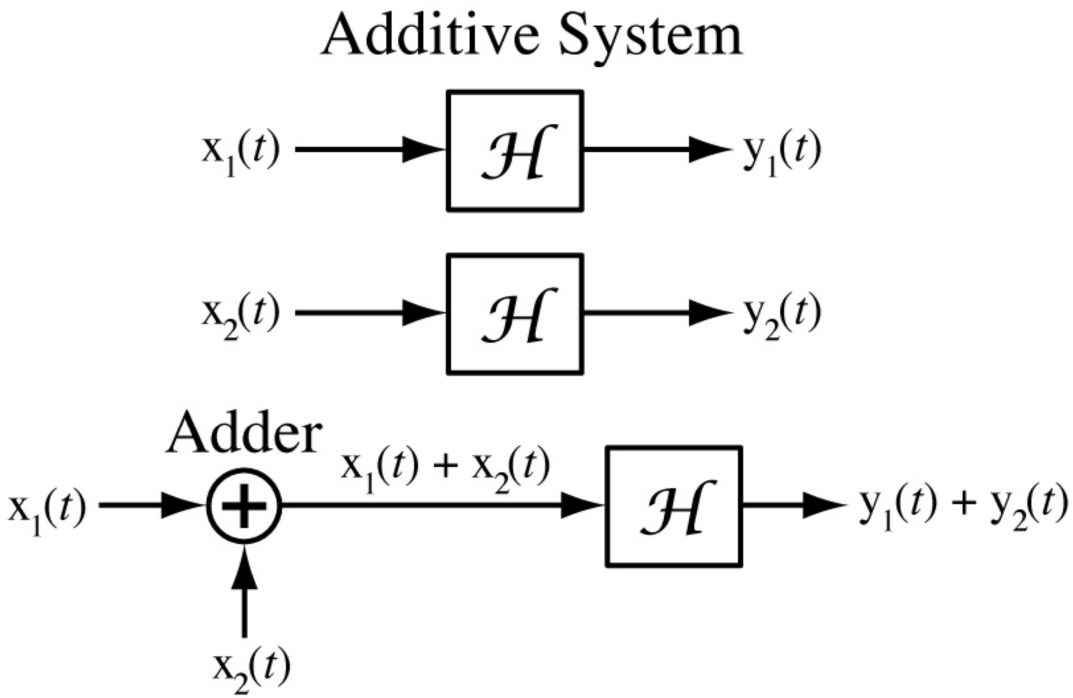
Homogeneity

-In a **homogeneous** system, multiplying the excitation by any constant (including complex constants), multiplies the response by the same constant.



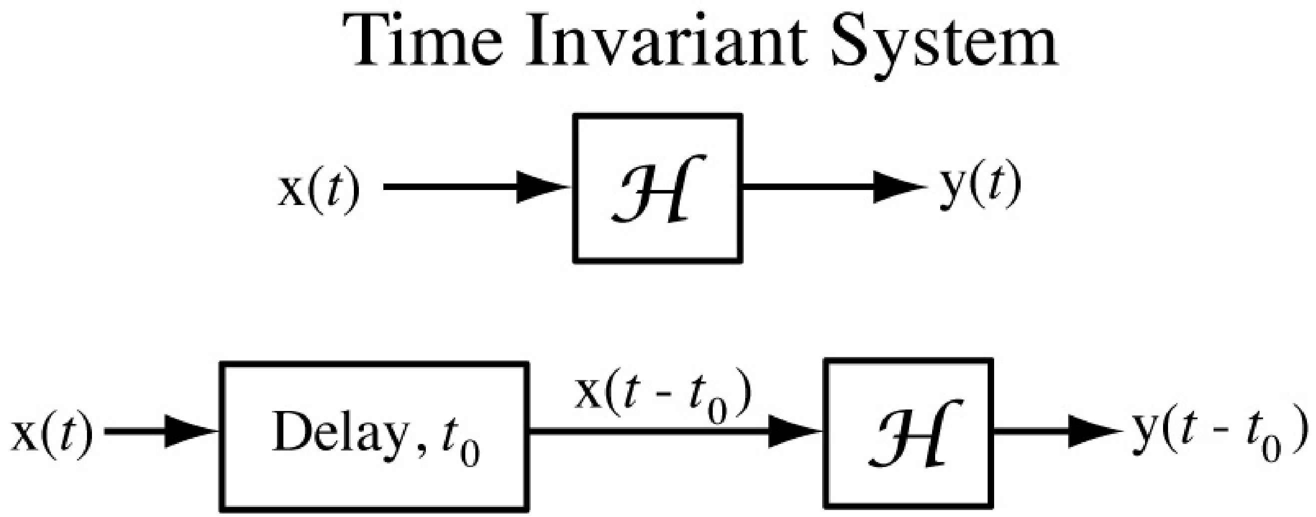
Additivity

-If a system when excited by an arbitrary $x_1(t)$ produces a response $y_1(t)$, and when excited by an arbitrary $x_2(t)$ produces a response $y_2(t)$ and $x_1(t)+x_2(t)$ always produces the zero-state response $y_1(t)+y_2(t)$, the system is **additive**.



Time Invariance

-If an arbitrary input signal $x(t)$ causes a response $y(t)$, and an input signal $x(t - t_0)$ causes a response $y(t - t_0)$, for any arbitrary t_0, the system is said to be **time invariant**.



Linearity and LTI Systems

-If a system is both *homogeneous* and *additive* it is **linear.**

-If a system is both *linear* and *time-invariant* it is called an **LTI** system.

-The **eigenfunctions** of an LTI system are complex exponentials.

-The **eigenvalues** of an LTI system are either real or, if complex, occur in complex conjugate pairs.

-Any LTI system excited by a complex sinusoid respond with another complex sinusoid of the same frequency, but generally with different amplitud and phase (multiplied by a complex constant).

-Using the principle of superposition for LTI systems, if the input signal is an arbitrary function that is a linear combination of complex sinusoids of various frequencies, then the output signal is also a linear combination of complex sinusoids at those same frequencies. This idea is **the basis for the methods of Fourier transform analysis.**

-All these statements are true of both continuous-time and discrete-time systems.

Causality

-Any system for which the response occurs only during or after the time in which the excitation is applied is called a **causal** system.

-Strictly speaking, **all real physical systems are causal.**

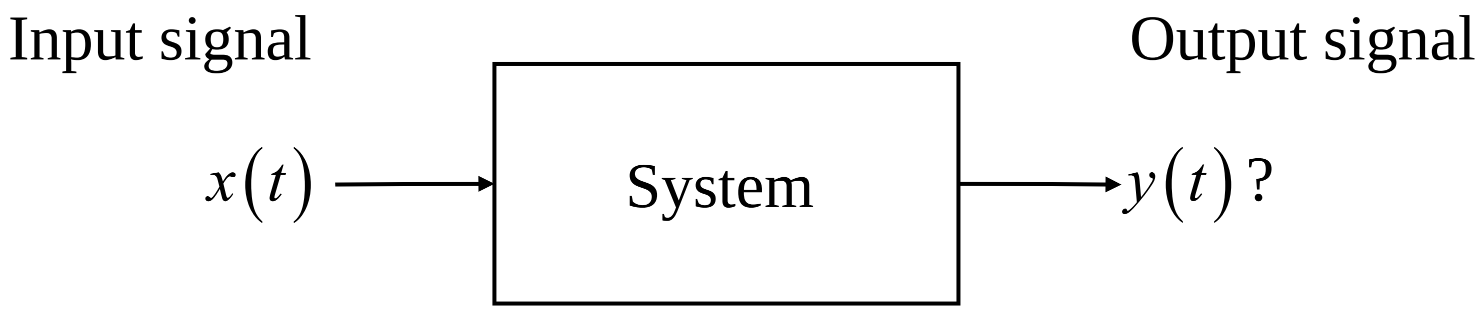
Stability

-Any system for which the response is bounded for any arbitrary bounded excitation, is called a **bounded-input-bounded-output (BIBO)** stable system.

5- Time-Domain System Analysis

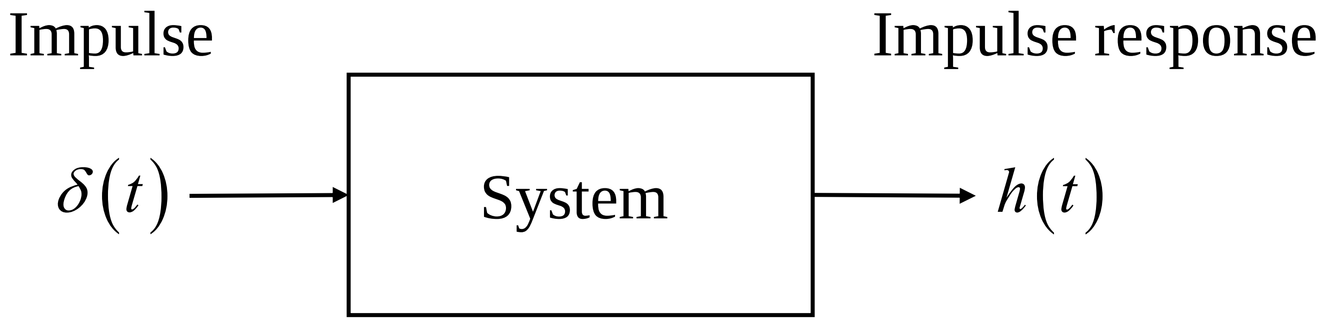
Continuous Time

System Response

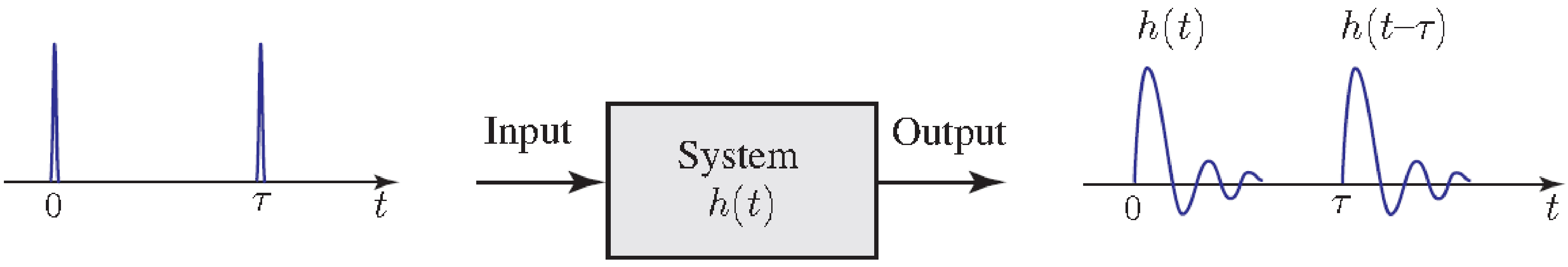


-Once the response to an impulse is known, the response of any LTI system to any arbitrary excitation can be found.

Impulse Response *h(t)*



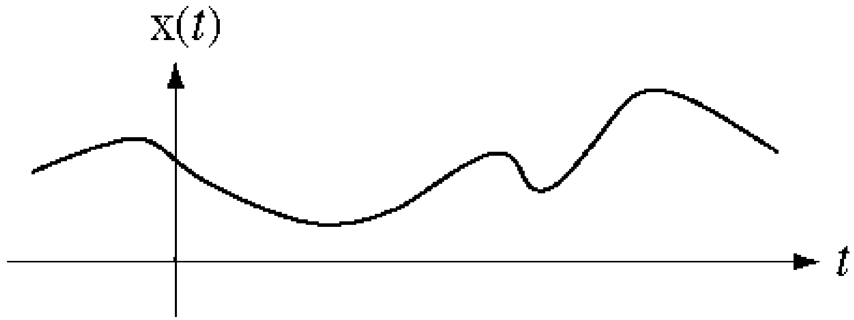
-For an LTI system, the impulse response *h(t)* of the system is a complete description of how it responds to any signal.



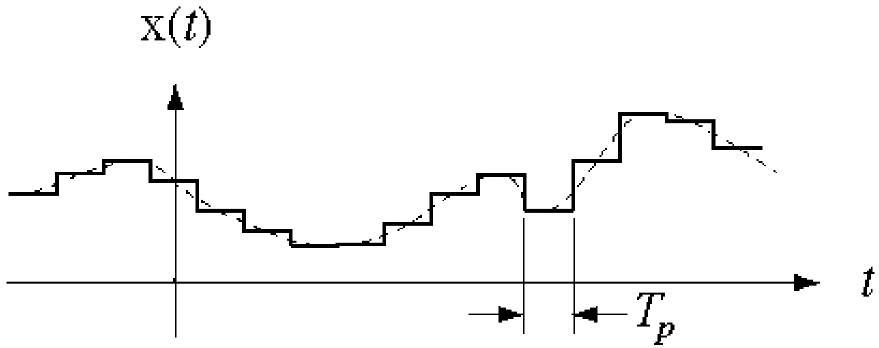
Response of a linear shift-invariant system to impulses

-If a continuous-time LTI system is excited by an arbitrary excitation, the response could be found approximately by approximating the excitation as a sequence of continuous rectangular pulses of width T_p

Exact Excitation

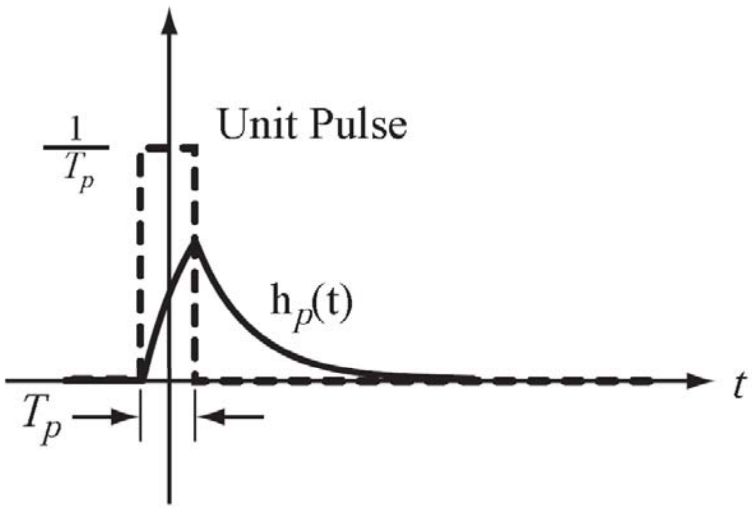


Approximate Excitation



$$x(t) \cong \sum_{n=-\infty}^{\infty} x(nT_p)\, rect\left(\frac{t-nT_p}{T_p}\right)$$

$$x(t) \cong \sum_{n=-\infty}^{\infty} T_p x(nT_p) \underbrace{\frac{1}{T_p} rect\left(\frac{t-nT_p}{T_p}\right)}_{\text{Shifted Unit Pulse}}$$



Unit pulse response

$$y(t) \cong \sum_{n=-\infty}^{\infty} T_p x(nT_p)\, h_p(t-nT_p)$$

$$T_p \to d\tau$$

$$x(t) = \underbrace{\sum_{n=-\infty}^{\infty}}_{\displaystyle\downarrow \atop \displaystyle\int} \underbrace{T_p}_{\displaystyle\downarrow \atop \displaystyle d\tau} x\left(\underbrace{nT_p}_{\tau}\right) \underbrace{\frac{1}{T_p} rect\left(\frac{t-nT_p}{T_p}\right)}_{\displaystyle\downarrow \atop \displaystyle\delta(t\text{-}\tau)}$$

$$\boxed{x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau}$$

Sampling property of $\delta(t)$

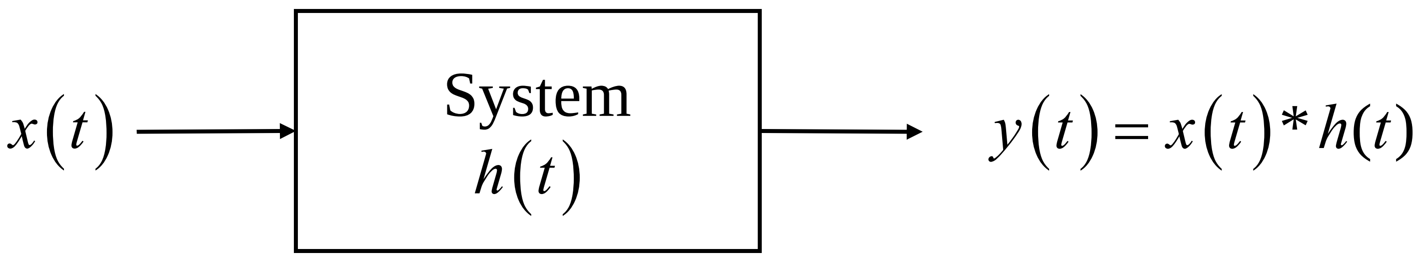
$$y(t) = \underbrace{\sum_{n=-\infty}^{\infty}}_{\displaystyle\downarrow \atop \displaystyle\int} \underbrace{T_p}_{\displaystyle\downarrow \atop \displaystyle d\tau} x\left(\underbrace{nT_p}_{\tau}\right) \underbrace{h_p\left(t-nT_p\right)}_{\displaystyle\downarrow \atop \displaystyle h(t\text{-}\tau)}$$

$$\boxed{y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau}$$

Convolution integral

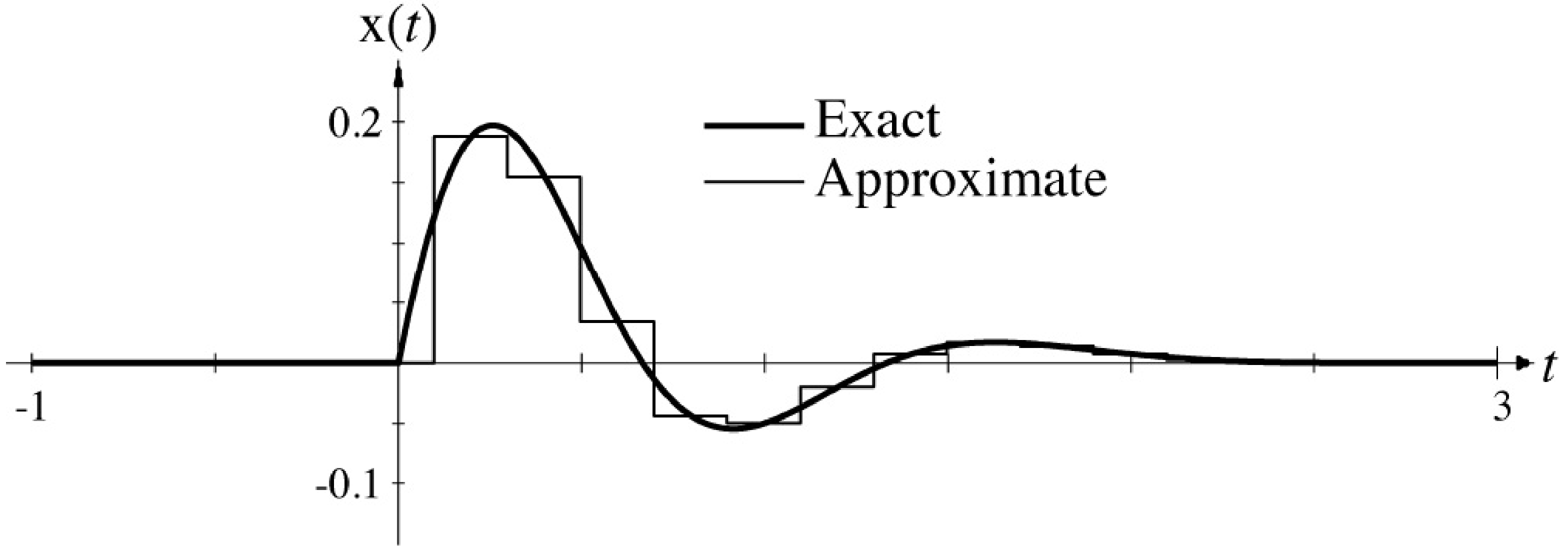
Convolution Integral

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = x(t) * h(t)$$

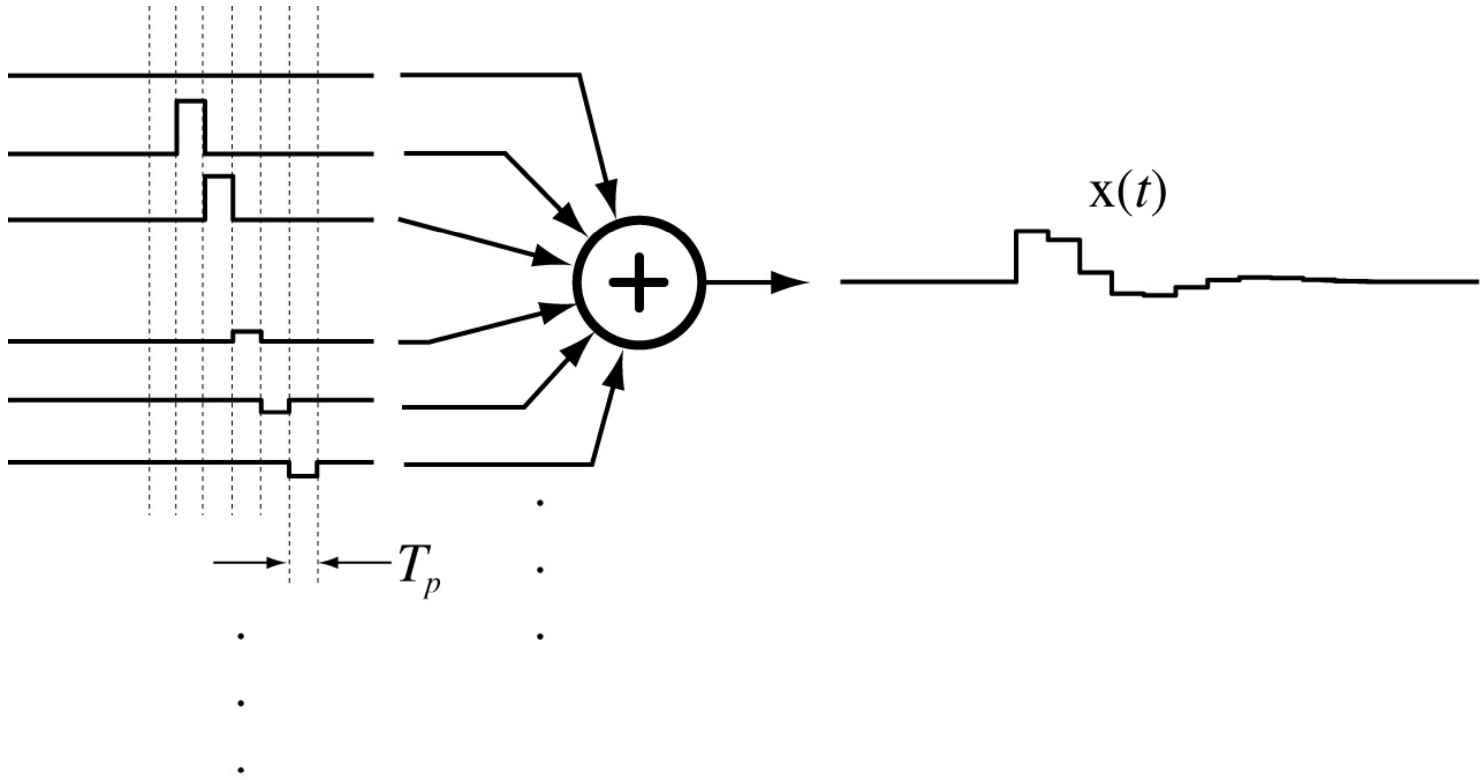


Convolution Graphical Example

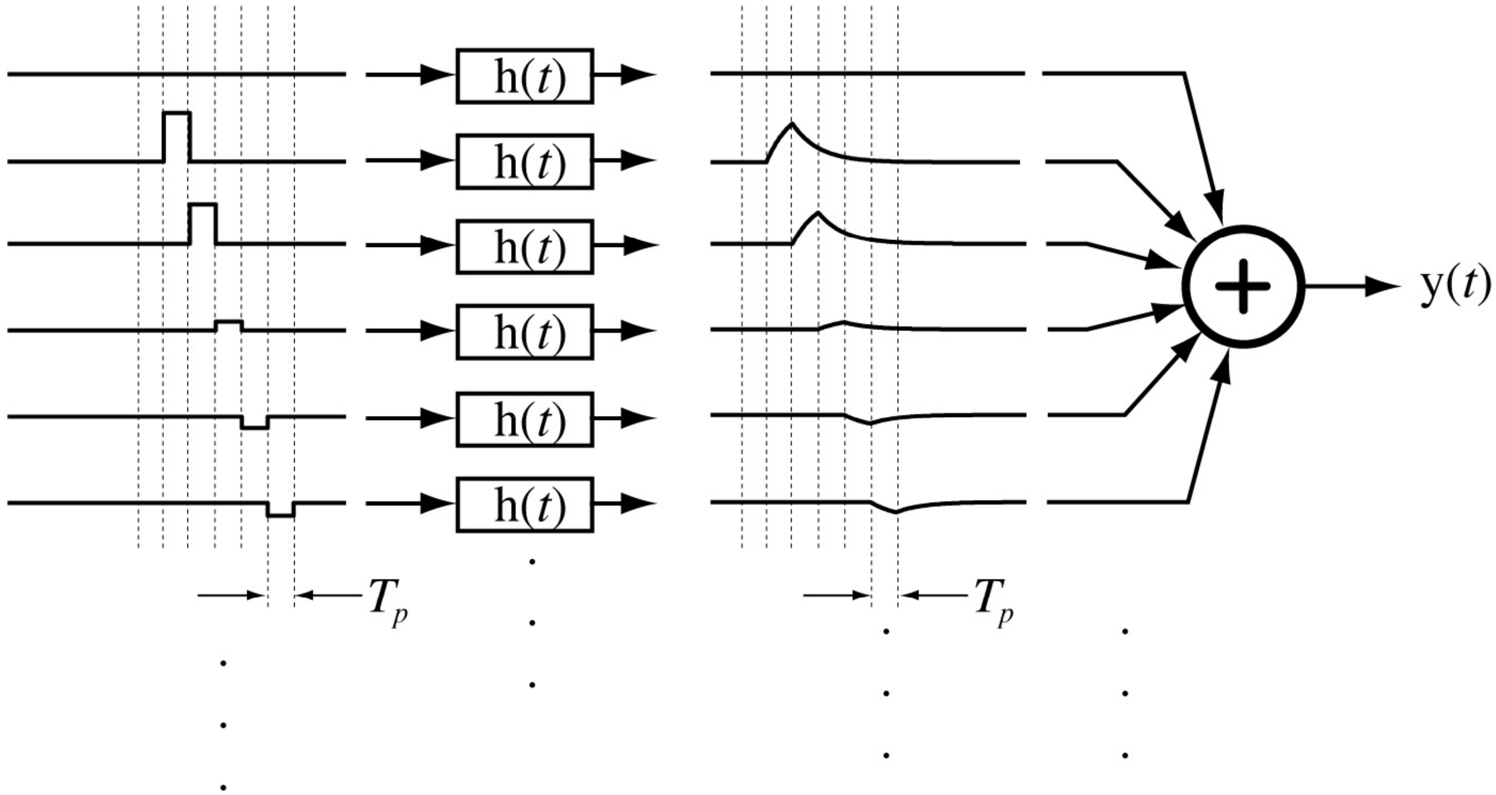
-Let $x(t)$ be this smooth waveform and let it be approximated by a sequence of rectangular pulses.

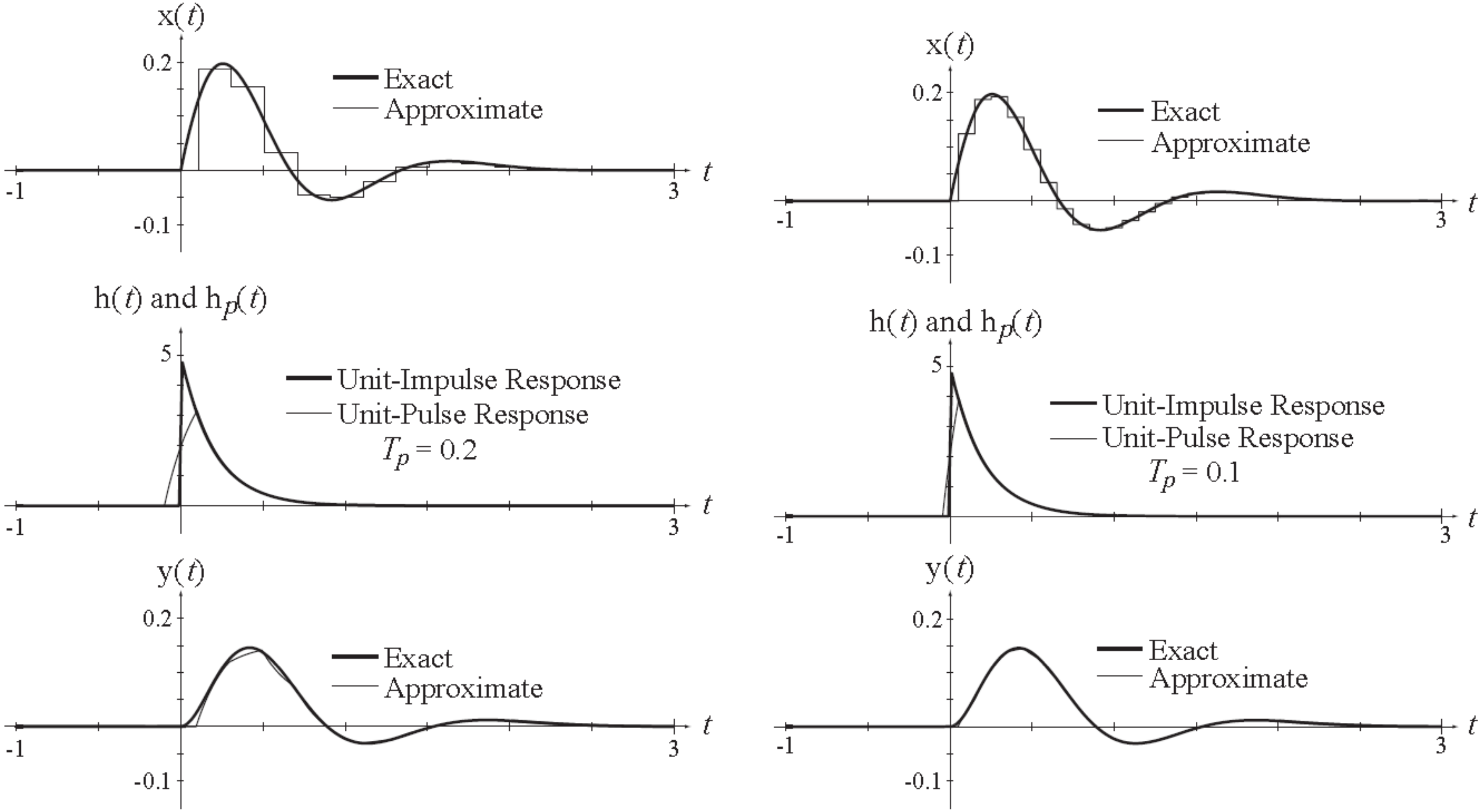


The approximate excitation is a sum of rectangular pulses

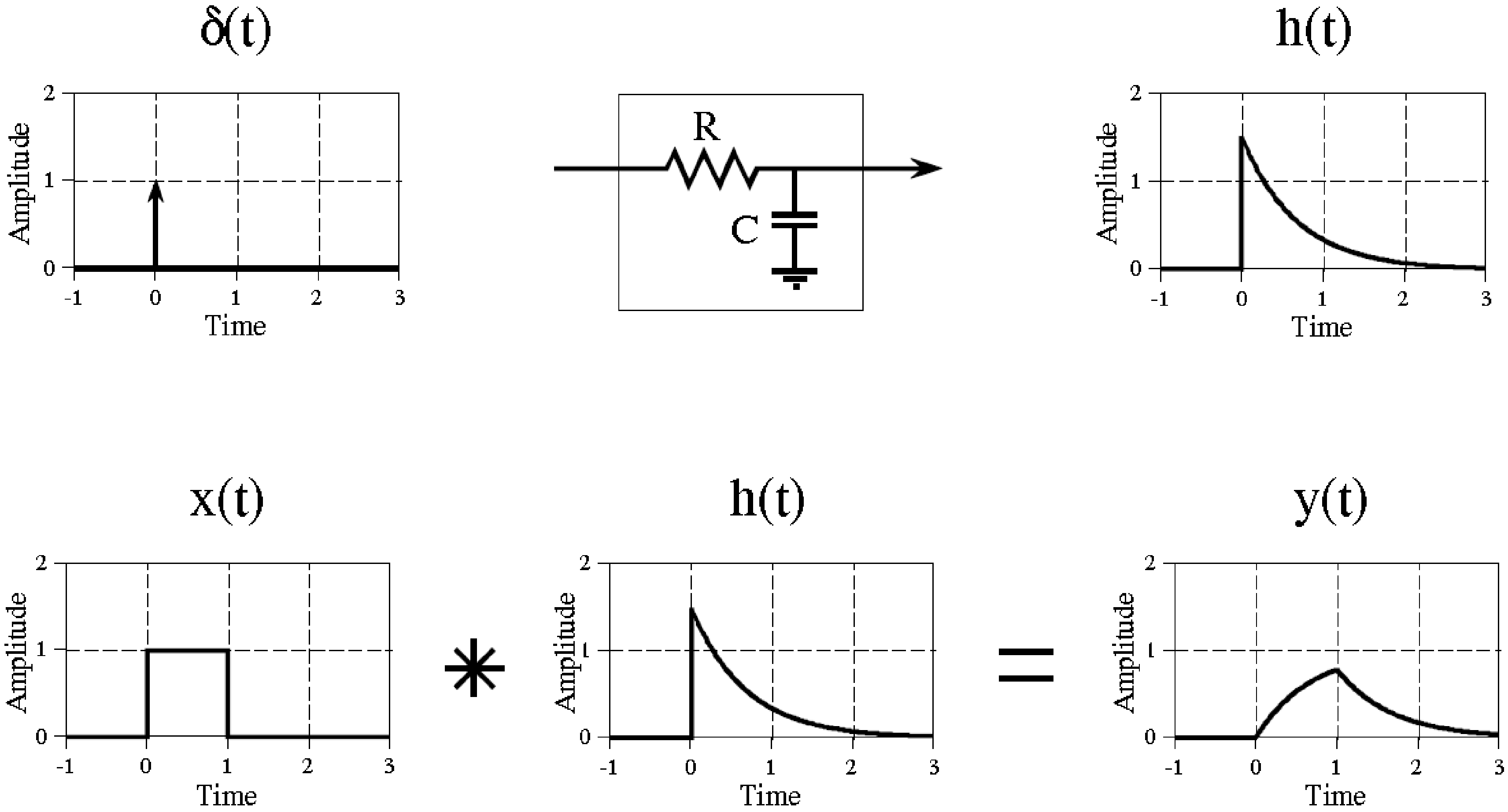


The approximate response is a sum of pulse responses.





Exact and approximate excitation, unit-impulse response, unit-pulse response and exact and approximate system response with *Tp=0,2 and 0.1*

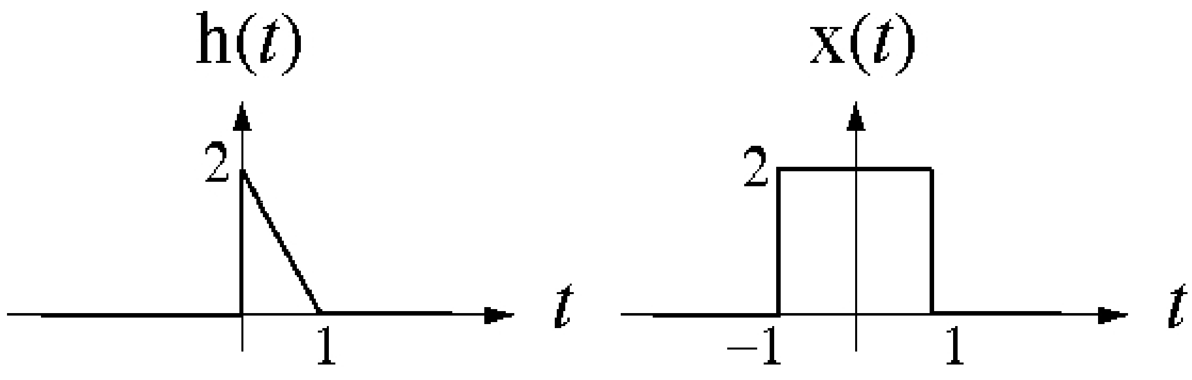

δ(t)
R
C
h(t)
x(t)
h(t)
y(t)
Amplitude
Time
-1 0 1 2 3
0 1 2

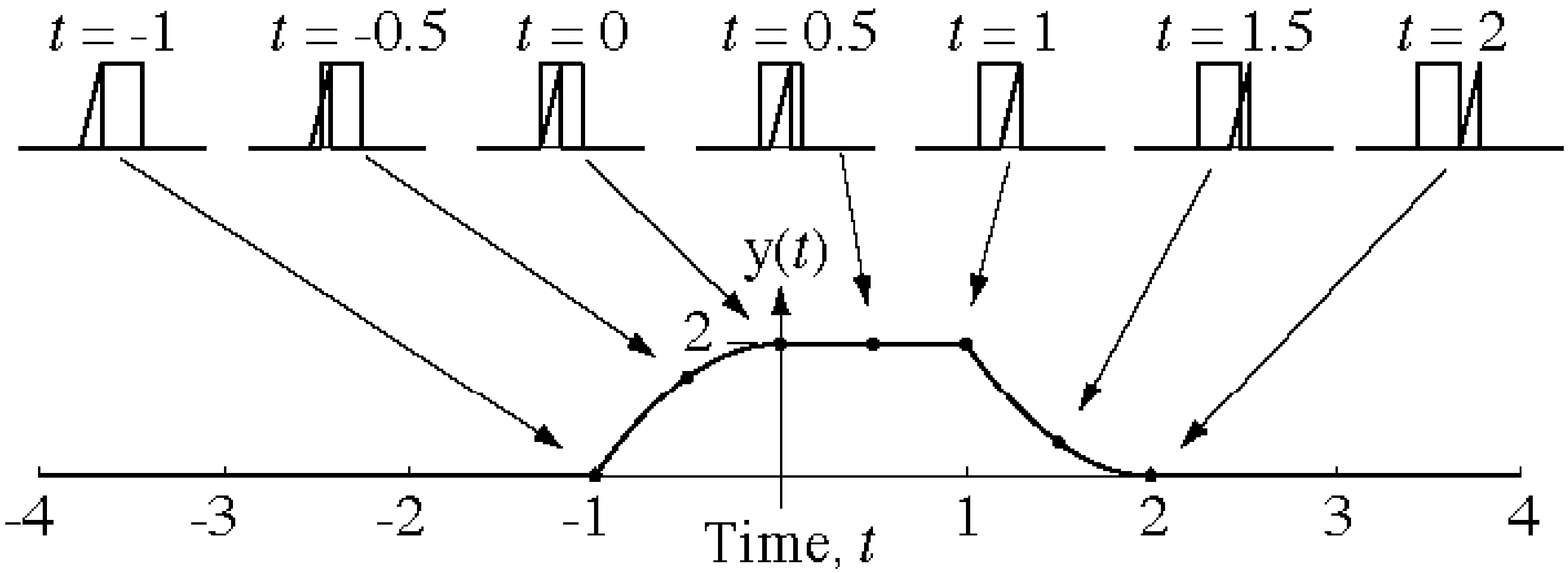
Graphical Illustration of the Convolution Integral

-The convolution integral is defined by

$$x(t)*h(t)=\int_{-\infty}^{\infty}x(\tau)h(t-\tau)d\tau$$

-For illustration purposes let the excitation $x(t)$ and the impulse response $h(t)$ be the two functions below.





Process of convolving

Convolution Integral Properties

Convolution with an shifted impulse

$$x(t) * A\delta(t - t_0) = Ax(t - t_0)$$

Commutativity

$$x(t) * y(t) = y(t) * x(t)$$

Associativity

$$\left[x(t) * y(t)\right] * z(t) = x(t) * \left[y(t) * z(t)\right]$$

Distributivity

$$\left[x(t) + y(t)\right] * z(t) = x(t) * z(t) + y(t) * z(t)$$

If $y(t) = x(t) * h(t)$

Differentiation Property

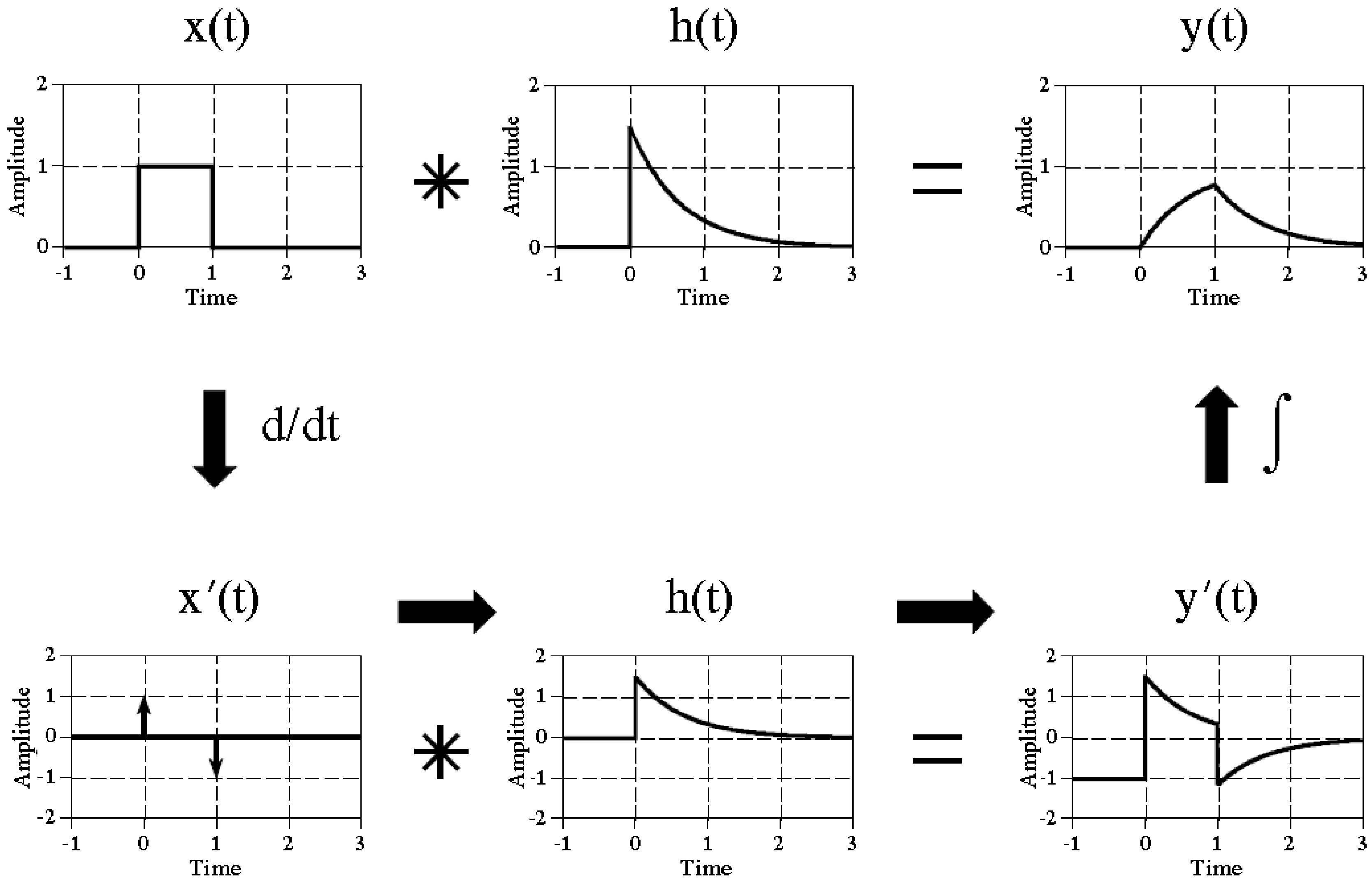
$$y'(t) = x'(t) * h(t) = x(t) * h'(t)$$

Area Property

Area of y = (Area of x) x (Area of h)

Scaling Property

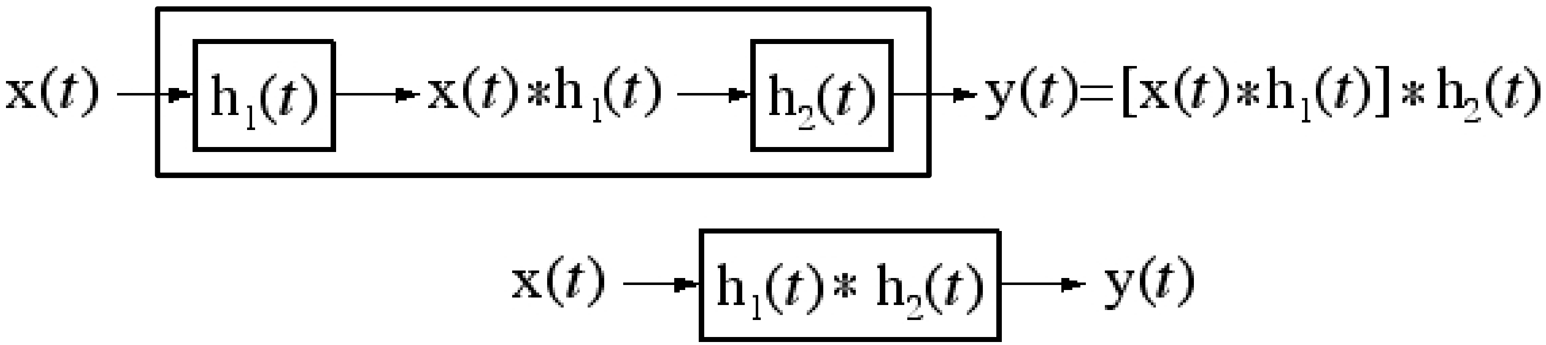
$$y(at) = |a| x(at) * h(at)$$


x(t)
h(t)
y(t)
Amplitude
Time
d/dt
∫
x′(t)
h(t)
y′(t)

System Interconnections

-If the output signal from a LTI system is the input signal to a second LTI system, the systems are said to be **cascade** connected.

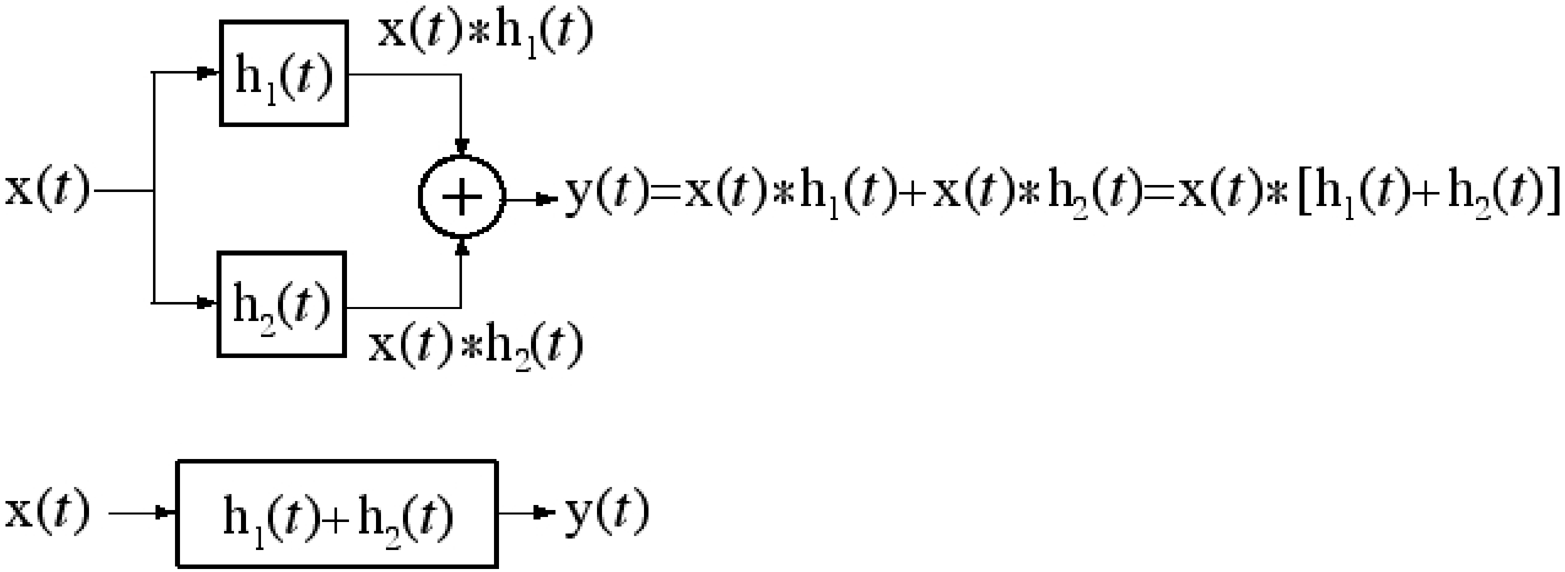
-From the properties of convolution:



Cascade Connection

-If two LTI systems are excited by the same signal and their responses are added they are said to be **parallel** connected.

-From properties of convolution:



Parallel Connection

Step Response

-One of the most common signals used to test systems is the step function. The response of an LTI system to a unit step is:

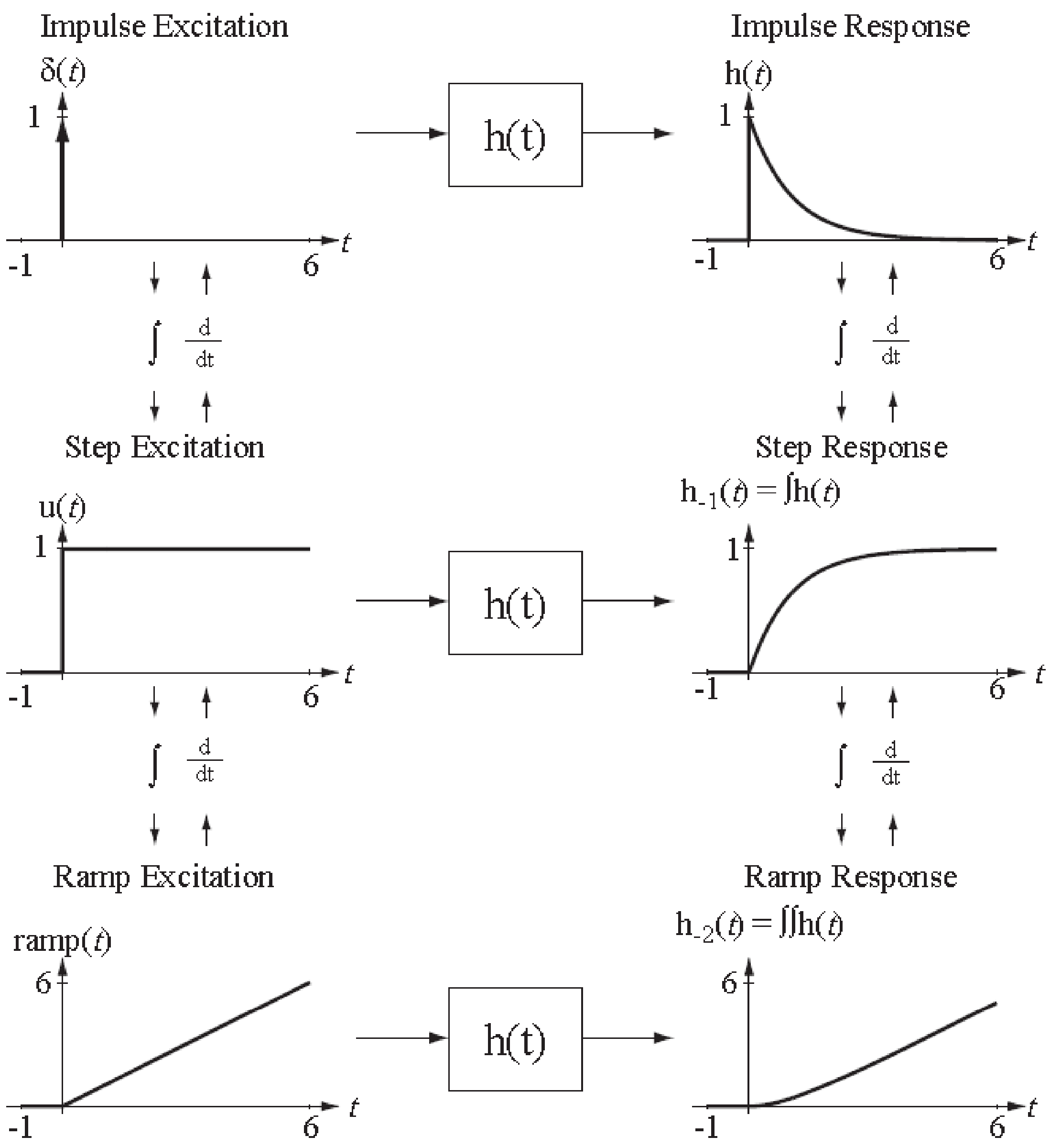
$$h_{-1}(t) = h(t) * u(t) = \int_{-\infty}^{\infty} h(\tau) u(t-\tau) d\tau = \int_{-\infty}^{t} h(\tau) d\tau$$

-The response of an LTI system excited by a unit step is the integral of the impulse response.

-As the unit step is the integral of the impulse, the unit-step response is the integral of the unit-impulse response.

-In fact, this relationship holds for any excitation.

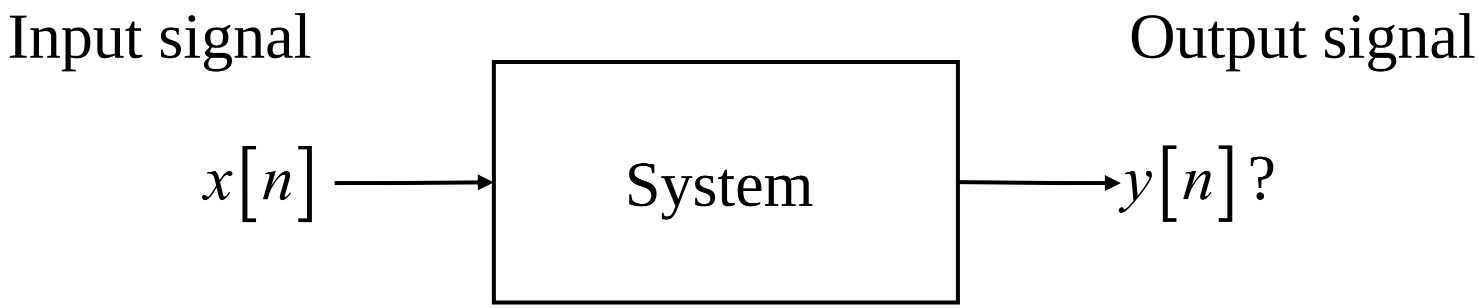
-If any excitation is changed to its integral, the response also changes to its integral.



Relations between integrals and derivatives of excitations and responses for an LTI system

Discrete Time

System Response

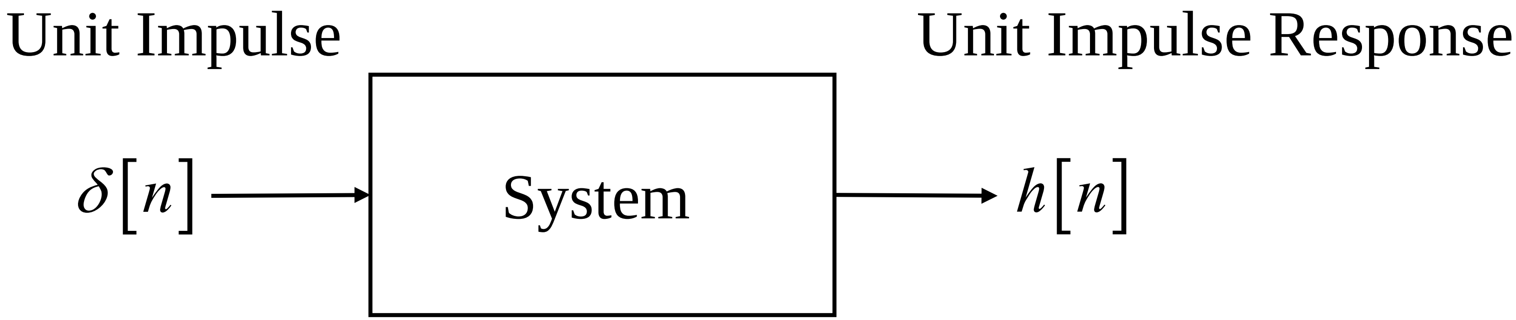


-Once the response to a unit impulse is known, the response of any LTI system to any arbitrary excitation can be found.

-Any arbitrary excitation is simply a sequence of amplitude-scaled and time-shifted impulses.

-Therefore the response is simply a sequence of amplitude-scaled and time-shifted impulse responses.

Unit Impulse Response *h[n]*



-For an LTI system, the unit impulse response *h[n]* of the system is a complete description of how it responds to any signal.

Convolution Sum

-The response $y[n]$ to an arbitrary excitation $\mathrm{x}[n]$ is of the form

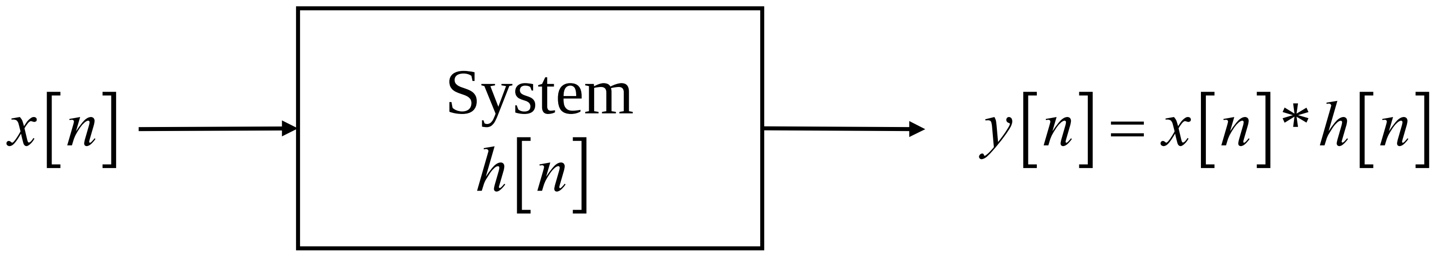
$$y[n] = \cdots x[-1]h[n+1] + x[0]h[n] + x[1]h[n-1] + \cdots$$

-This can be written in a more compact form (Convolution Sum)

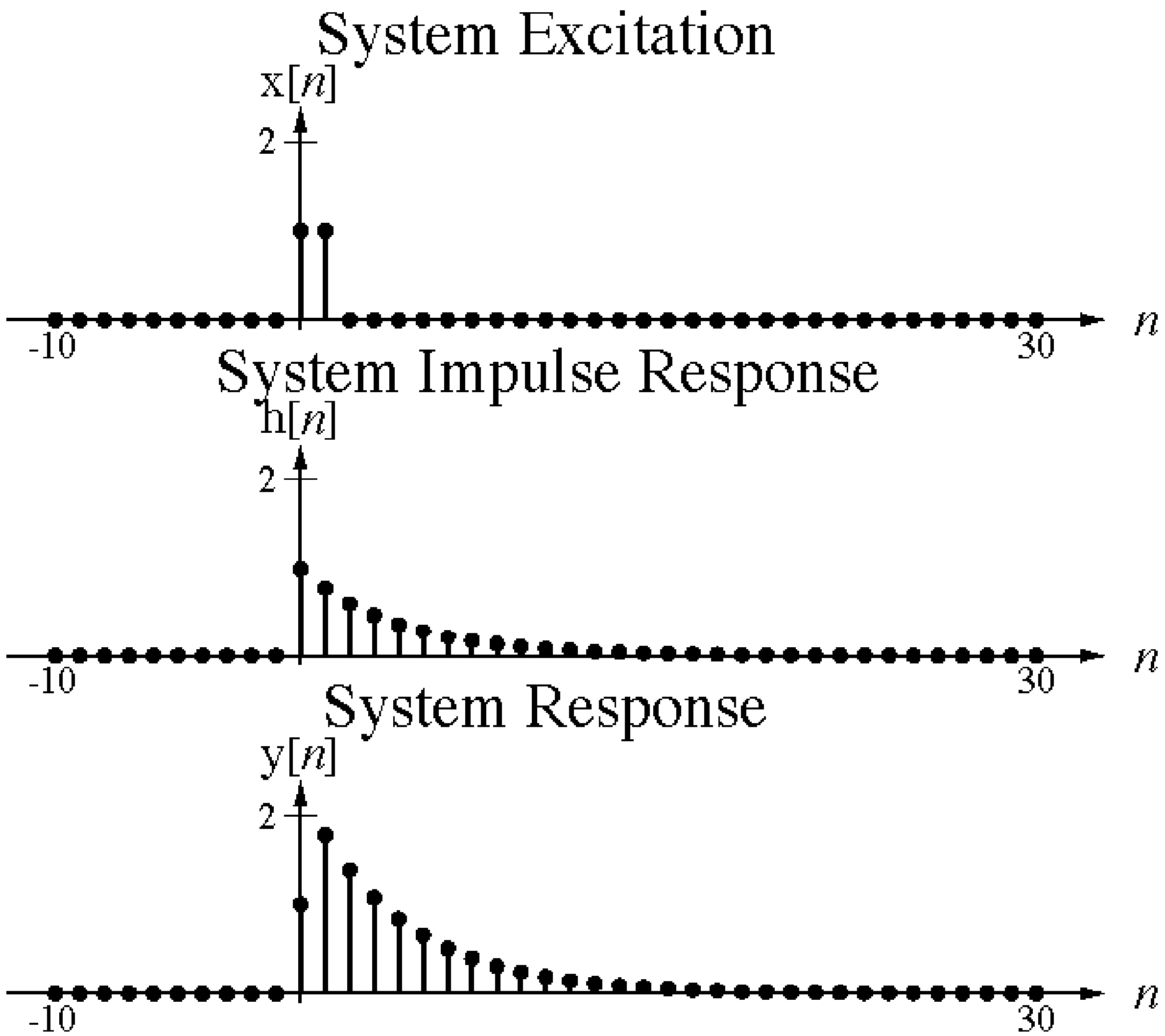
$$\boxed{y[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m]}$$

-Compare with $$\mathrm{y}(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

$$y[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m] = x[n]*h[n]$$



System Response Example



Convolution Sum Properties

Convolution with a shifted unit impulse

$$x[n] * A\delta[n-n_0] = Ax[n-n_0]$$

Commutativity

$$x[n] * y[n] = y[n] * x[n]$$

Associativity

$$\left(x[n] * y[n]\right) * z[n] = x[n] * \left(y[n] * z[n]\right)$$

Distributivity

$$\left(x[n] + y[n]\right) * z[n] = x[n] * z[n] + y[n] * z[n]$$

If $y[n] = x[n] * h[n]$

Differentiation Property

$$y[n] - y[n-1] = x[n] * \left(h[n] - h[n-1]\right) = \left(x[n] - x[n-1]\right) * h[n]$$

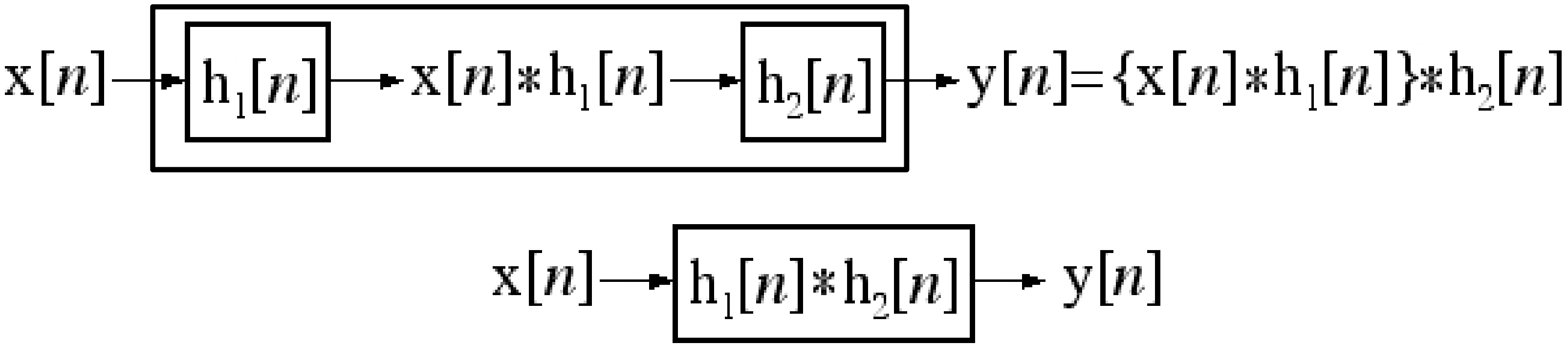
Sum Property (Sum of y = (Sum of x) x (Sum of h))

$$\sum_{n=-\infty}^{\infty} y[n] = \left(\sum_{n=-\infty}^{\infty} x[n]\right) \text{ x } \left(\sum_{n=-\infty}^{\infty} h[n]\right)$$

System Interconnections

-If the output signal from a LTI system is the input signal to a second LTI system, the systems are said to be **serie/cascade** connected.

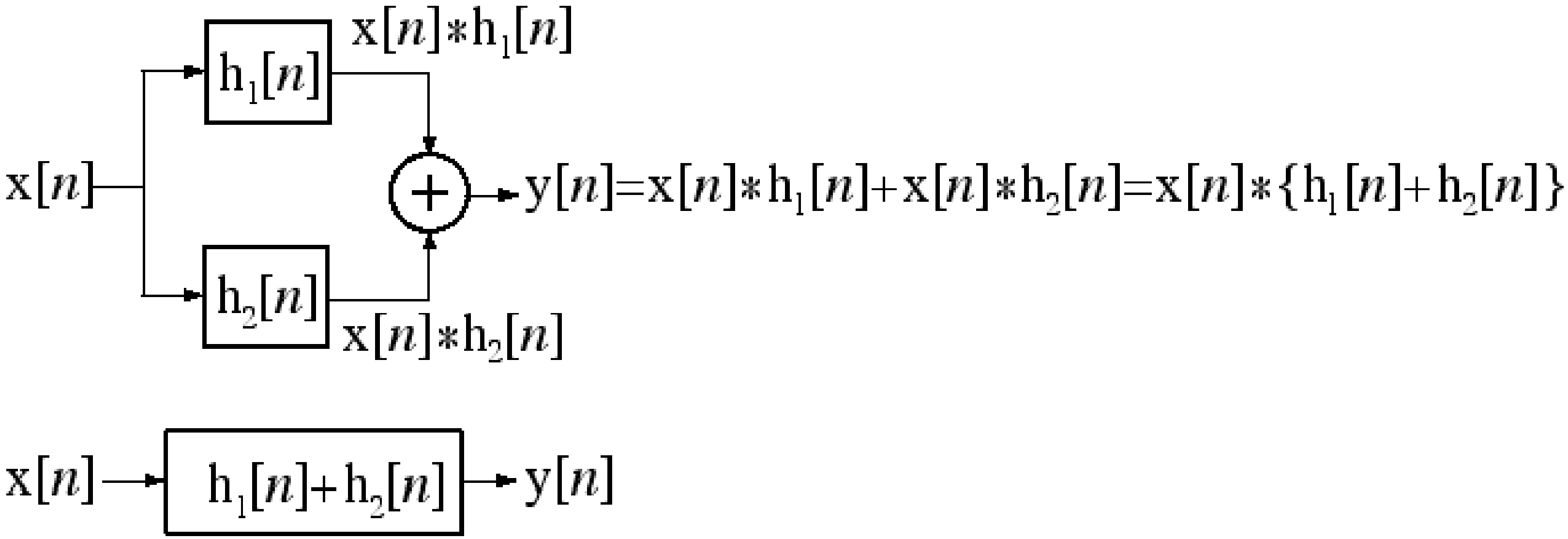
-From the properties of convolution:



Serie/Cascade Connection

-If two LTI systems are excited by the same signal and their responses are added they are said to be **parallel** connected.

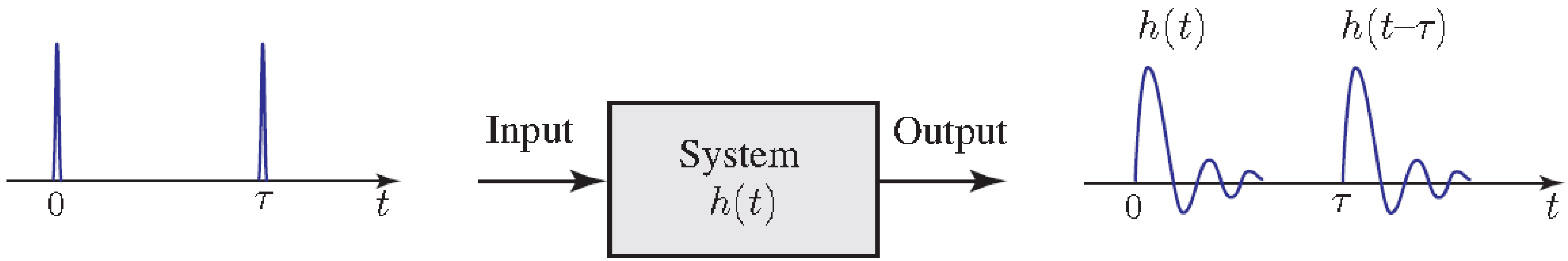
-From properties of convolution:



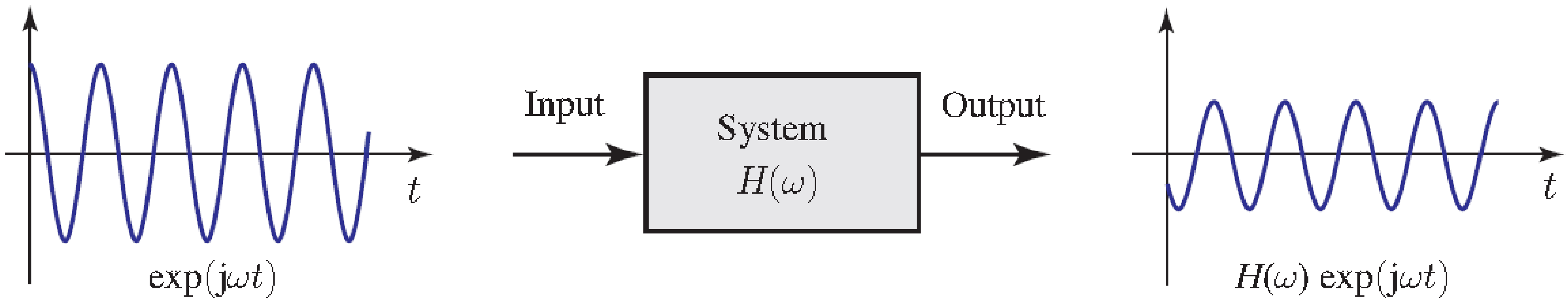
Parallel Connection

6- Spectral Method

Introduction



Response of a linear shift-invariant system to impulses



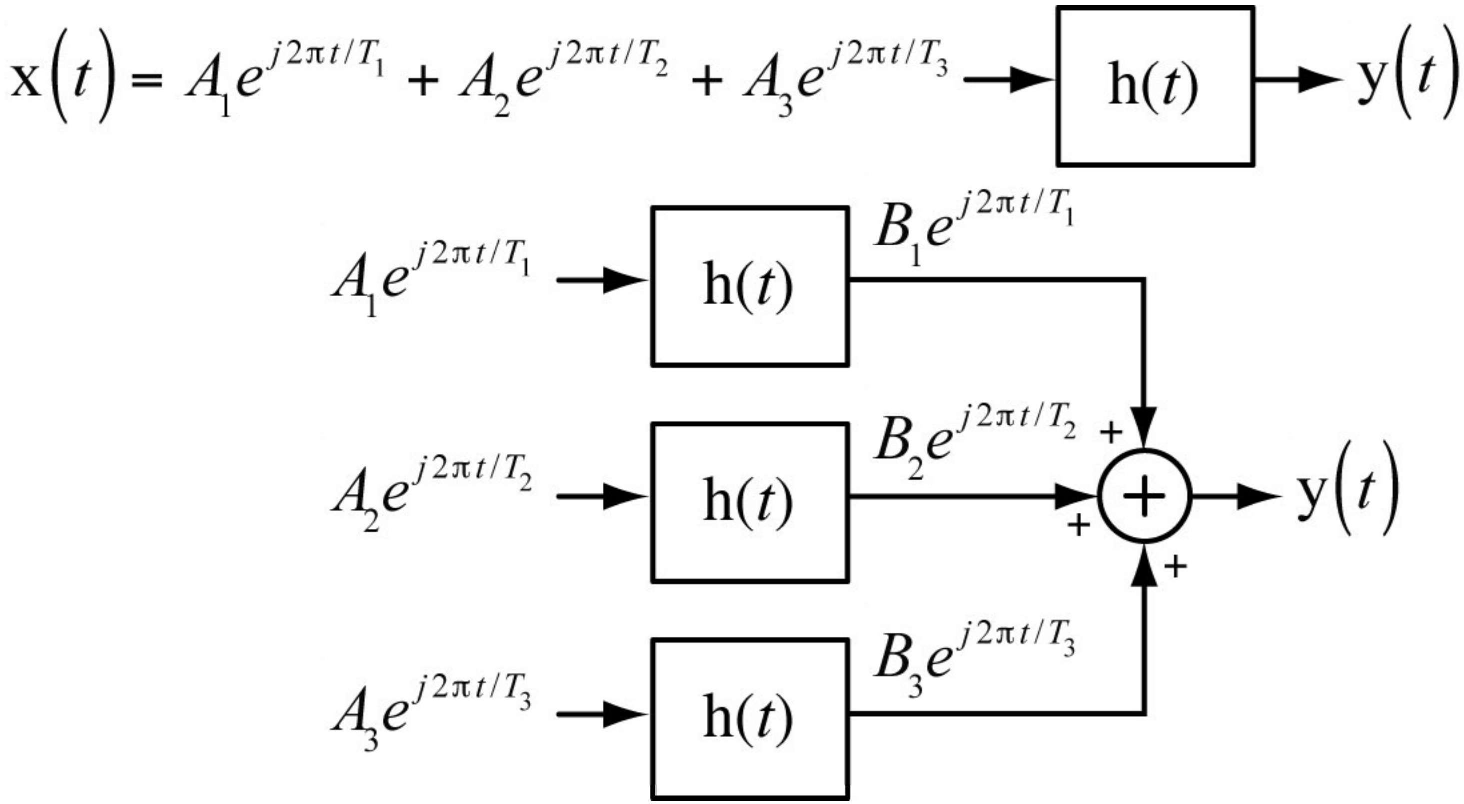
Response of a linear shift-invariant system to a harmonic function

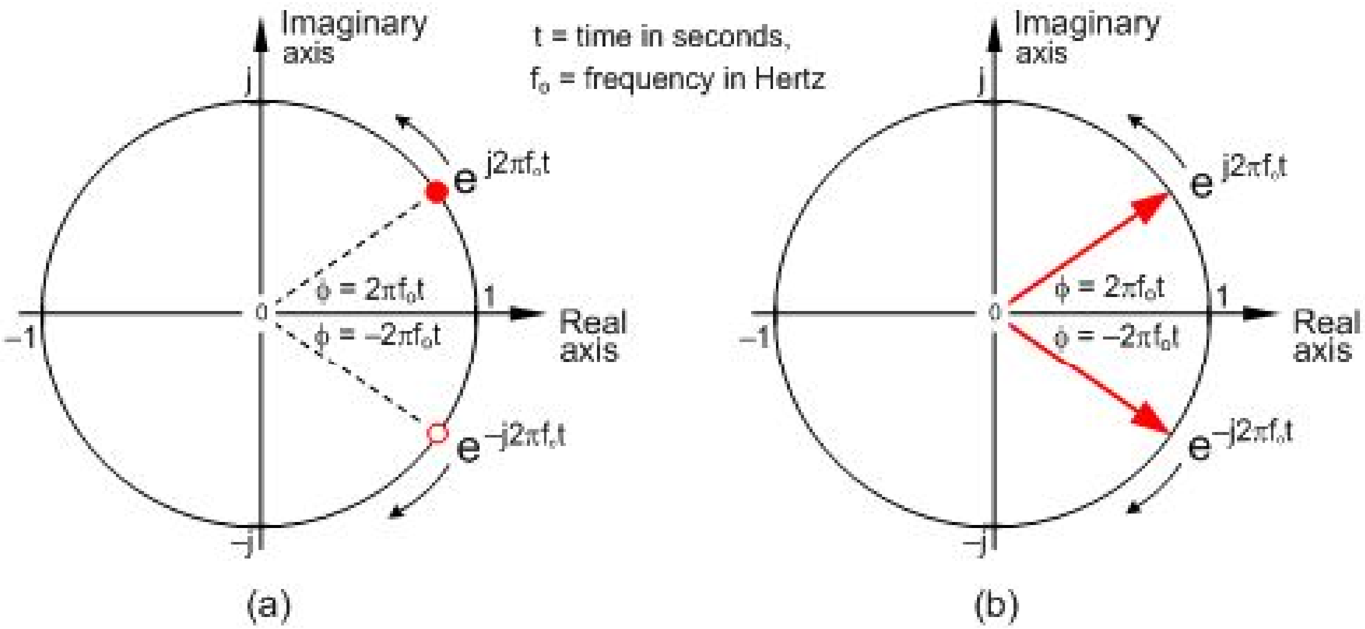
Representing a Signal

-The **Spectral (Fourier) method** represents a signal as a linear combination of **complex exponentials (Harmonic functions).**

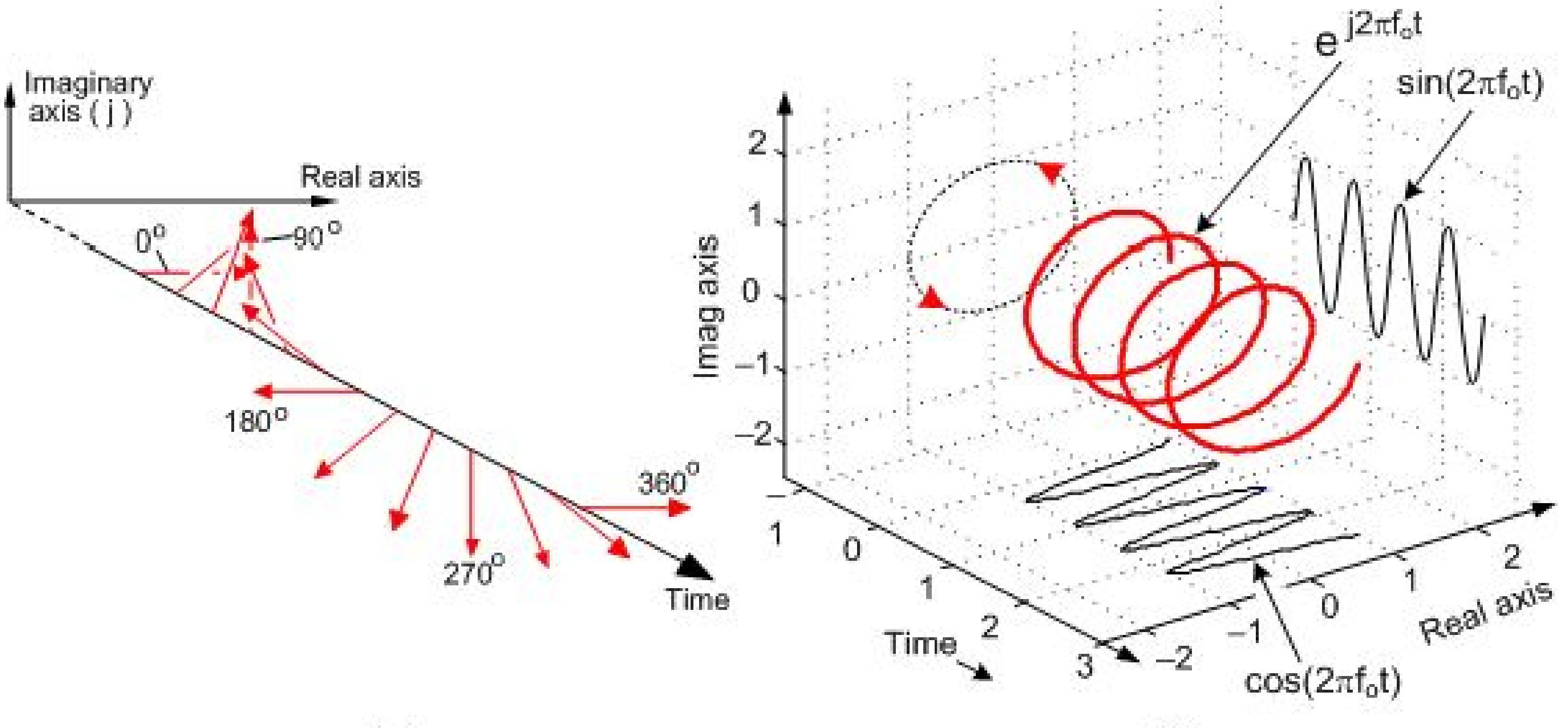
-If an excitation can be expressed as a sum of complex exponentials, the response of an LTI system can be expressed as the sum of responses to complex sinusoids too.

-This is the fundament of the Spectral Method.


x(t) = A1e^(j2πt/T1) + A2e^(j2πt/T2) + A3e^(j2πt/T3)
h(t)
y(t)
A1e^(j2πt/T1)
A2e^(j2πt/T2)
A3e^(j2πt/T3)
h(t)
h(t)
h(t)
B1e^(j2πt/T1)
B2e^(j2πt/T2)
B3e^(j2πt/T3)
+
+
+
+
y(t)

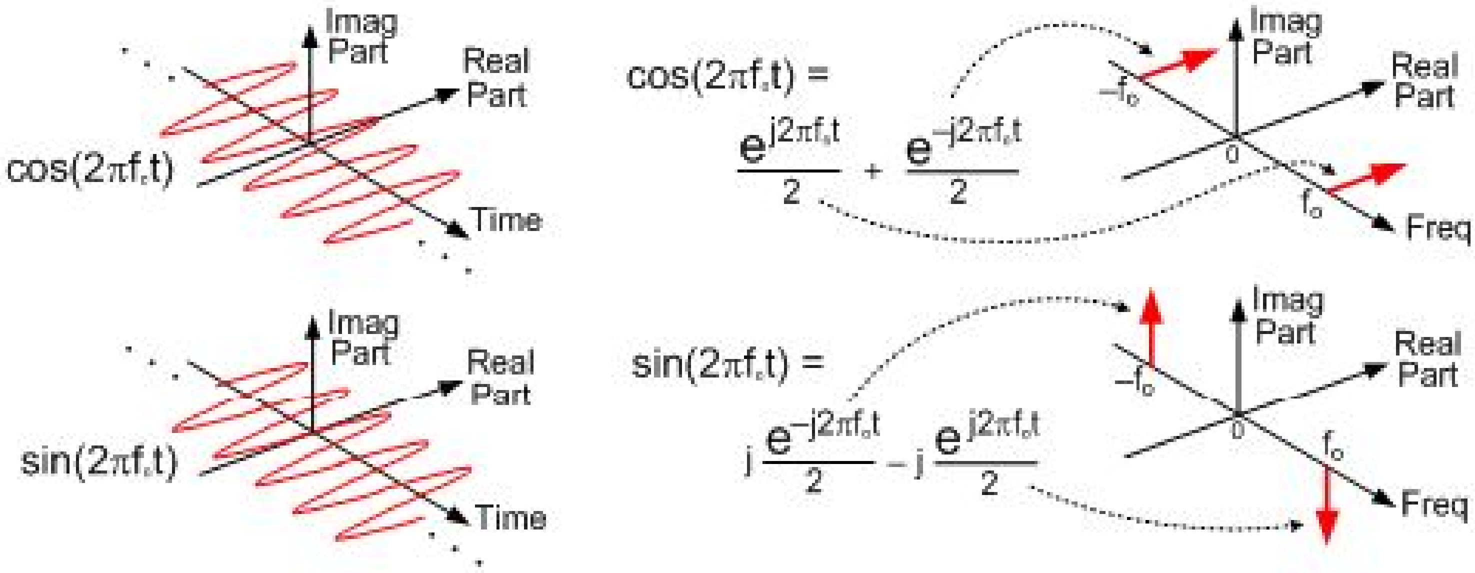


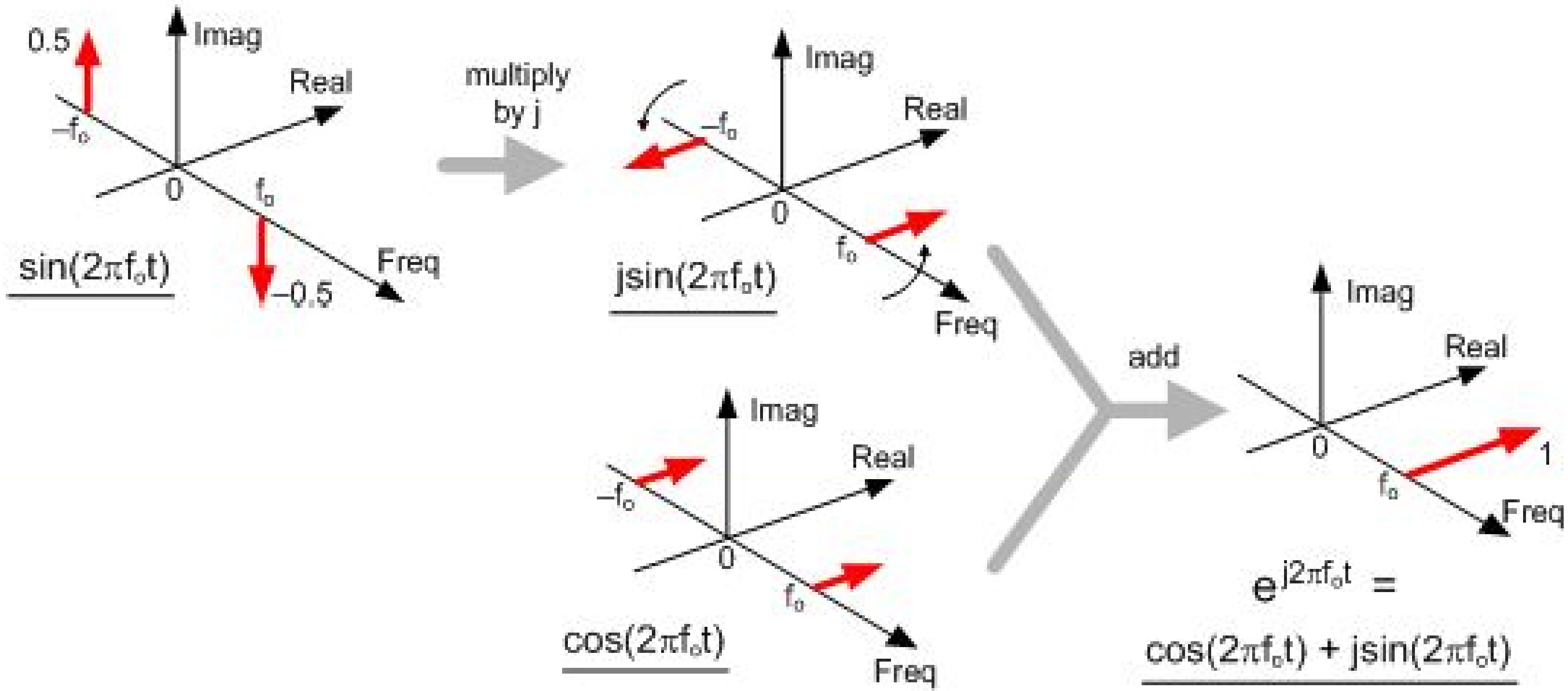
Imaginary axis
t = time in seconds,
f_o = frequency in Hertz
j
$e^{j2\pi f_o t}$
$\phi = 2\pi f_o t$
$\phi = -2\pi f_o t$
0
1
–1
Real axis
$e^{-j2\pi f_o t}$
–j
(a)
(b)



Imaginary axis (j)
Real axis
0°
90°
180°
270°
360°
Time
$e^{j2\pi f_o t}$
$\sin(2\pi f_o t)$
$\cos(2\pi f_o t)$
Imag axis
Real axis
Time
2
1
0
−1
−2
1
0
1
2
3
−2
−1
0
1
2

(a) (b)


Imag
Part
Real
Part
$\cos(2\pi f_o t)$
Time
$\cos(2\pi f_o t) =$
$\frac{e^{j2\pi f_o t}}{2} + \frac{e^{-j2\pi f_o t}}{2}$
Imag
Part
Real
Part
$-f_o$
0
f_o
Freq
Imag
Part
Real
Part
$\sin(2\pi f_o t)$
Time
$\sin(2\pi f_o t) =$
$j\frac{e^{-j2\pi f_o t}}{2} - j\frac{e^{j2\pi f_o t}}{2}$
Imag
Part
Real
Part
$-f_o$
0
f_o
Freq



www.dsprelated.com

Inner Product

-The inner product of two functions, is **the integral of the product of one function and the complex conjugate of the other function** over an interval.

-Inner Product of $x_1(t)$ and $x_2(t)$ on the interval $t_0 < t < t_0 + T$

$$\underbrace{\left(x_1(t), x_2(t)\right)}_{\text{Inner Product}} = \int_{t_0}^{t_0+T} x_1(t) x_2^{*}(t) dt$$

Orthogonality

-**Orthogonal** means that the **inner product** of the two functions of time, on some time interval, **is zero**.

-Two functions $x_1(t)$ and $x_2(t)$ are orthogonal on the interval $t_0 < t < t_0 + T$, if :

$$\underbrace{\left(x_1(t), x_2(t)\right)}_{\text{Inner Product}} = \int_{t_0}^{t_0+T} x_1(t) x_2^{\,*}(t) dt = 0$$

Orthogonality of complex exponentials

$$x_1(t) = e^{j\omega_1 t} = e^{j2\pi f_1 t}$$

$$x_2(t) = e^{j\omega_2 t} = e^{j2\pi f_2 t}$$

$$\int_{-\infty}^{\infty} e^{-j\omega t}\,dt = \delta(\omega) \rightarrow \begin{cases} \left(e^{j\omega_1 t}, e^{j\omega_2 t}\right) \neq 0 & [\omega_1 = \omega_2] \\ \left(e^{j\omega_1 t}, e^{j\omega_2 t}\right) = 0 & [\omega_1 \neq \omega_2] \end{cases}$$

$\boxed{e^{j\omega t}} \rightarrow$ Orthonormal (Orthonormal and Normalized) Basis

7- Fourier Transform

Definition

-Fourier Basis $\rightarrow$ $\boxed{e^{j\omega t}}$

-The Fourier Transform of $x(t)$ is the projection of this function onto the Fourier basis.

-The Fourier Transform of $x(t)$ is defined as:

$$\mathrm{F}\left(x(t)\right) = \boxed{X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt}$$

$$x(t) \overset{\mathrm{F}}{\leftrightarrow} X(\omega)$$

-It follows that the Inverse Fourier Transform of $X(\omega)$ is:

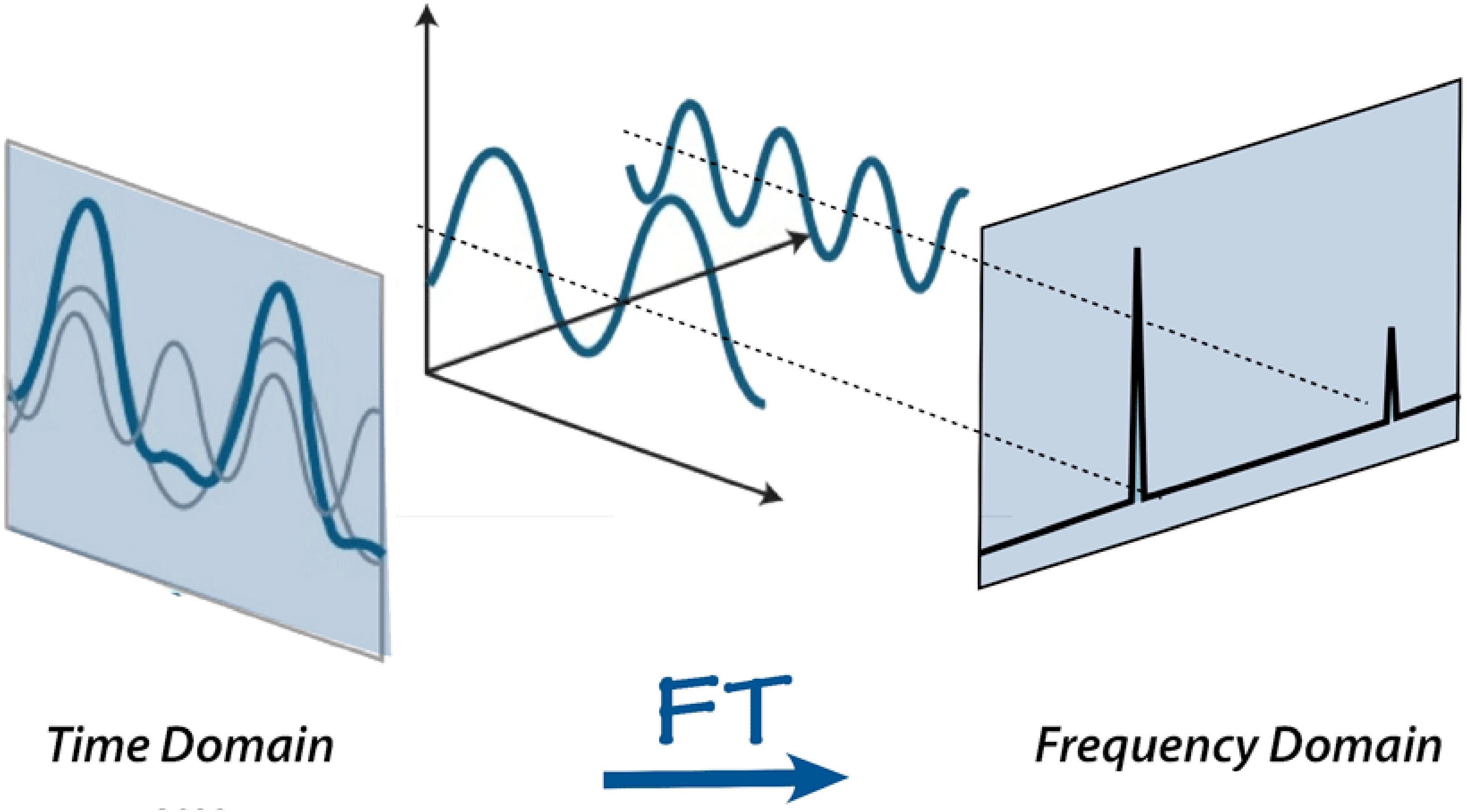
$$\mathcal{F}^{-1}\left(X(\omega)\right) = \boxed{x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\,d\omega}$$

$$X(\omega) \overset{\mathcal{F}^{-1}}{\leftrightarrow} x(t)$$

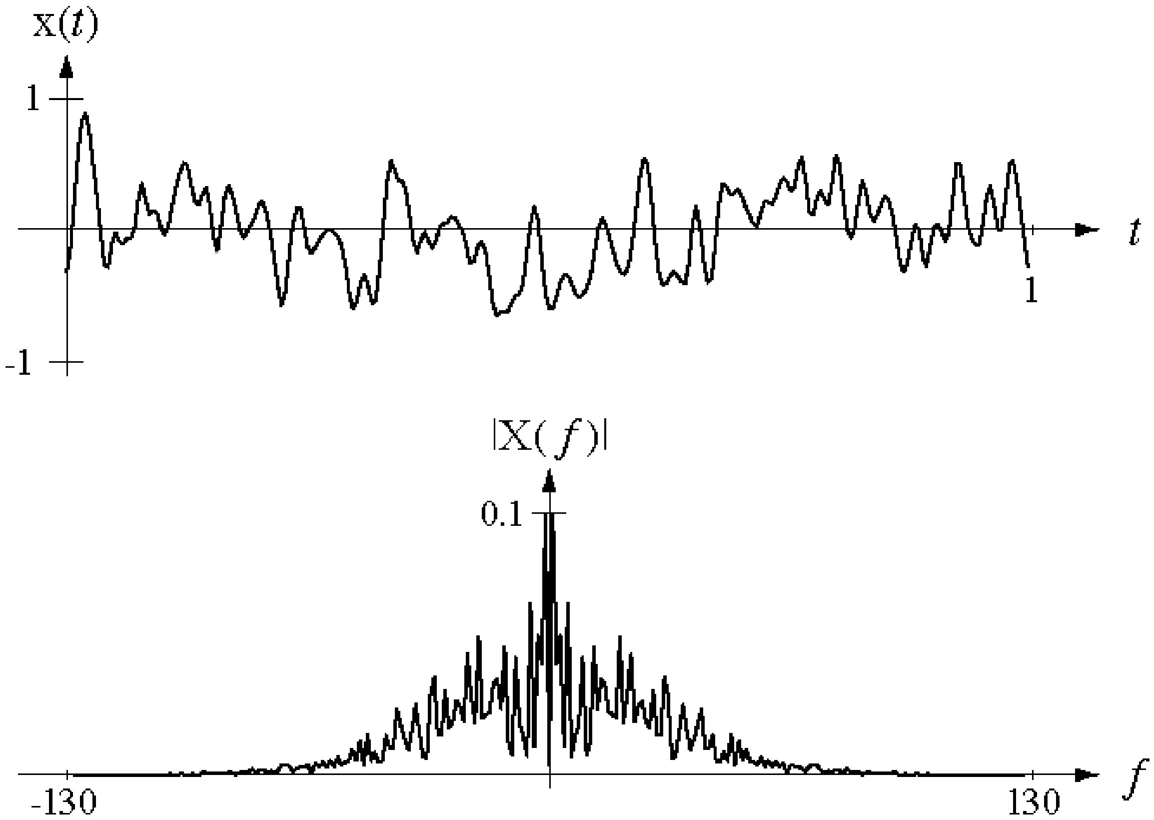
-The definitions based on frequency terms are:

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} dt$$

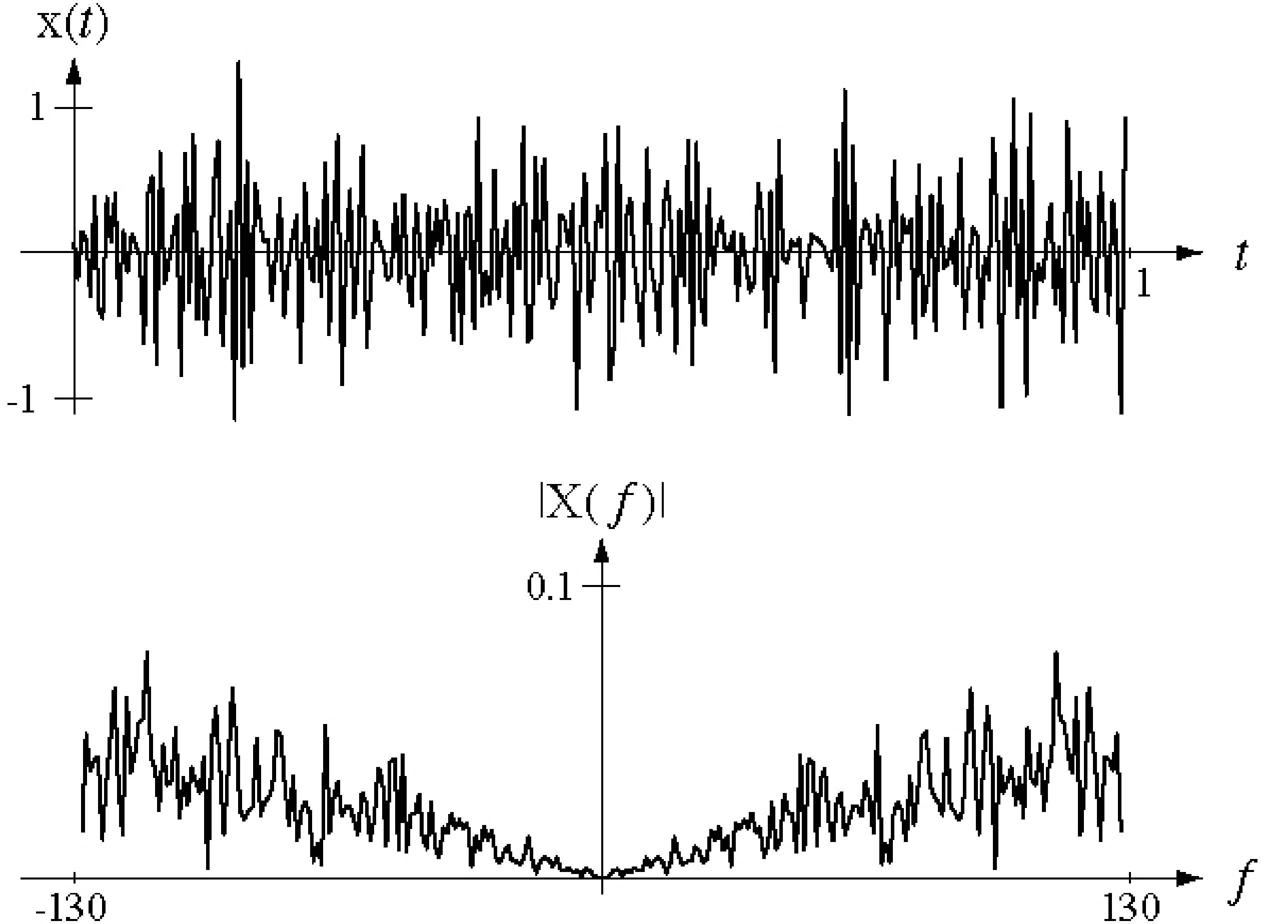
$$x(t) = \int_{-\infty}^{\infty} X(f) e^{j\omega t} df$$


FT
Time Domain
Frequency Domain

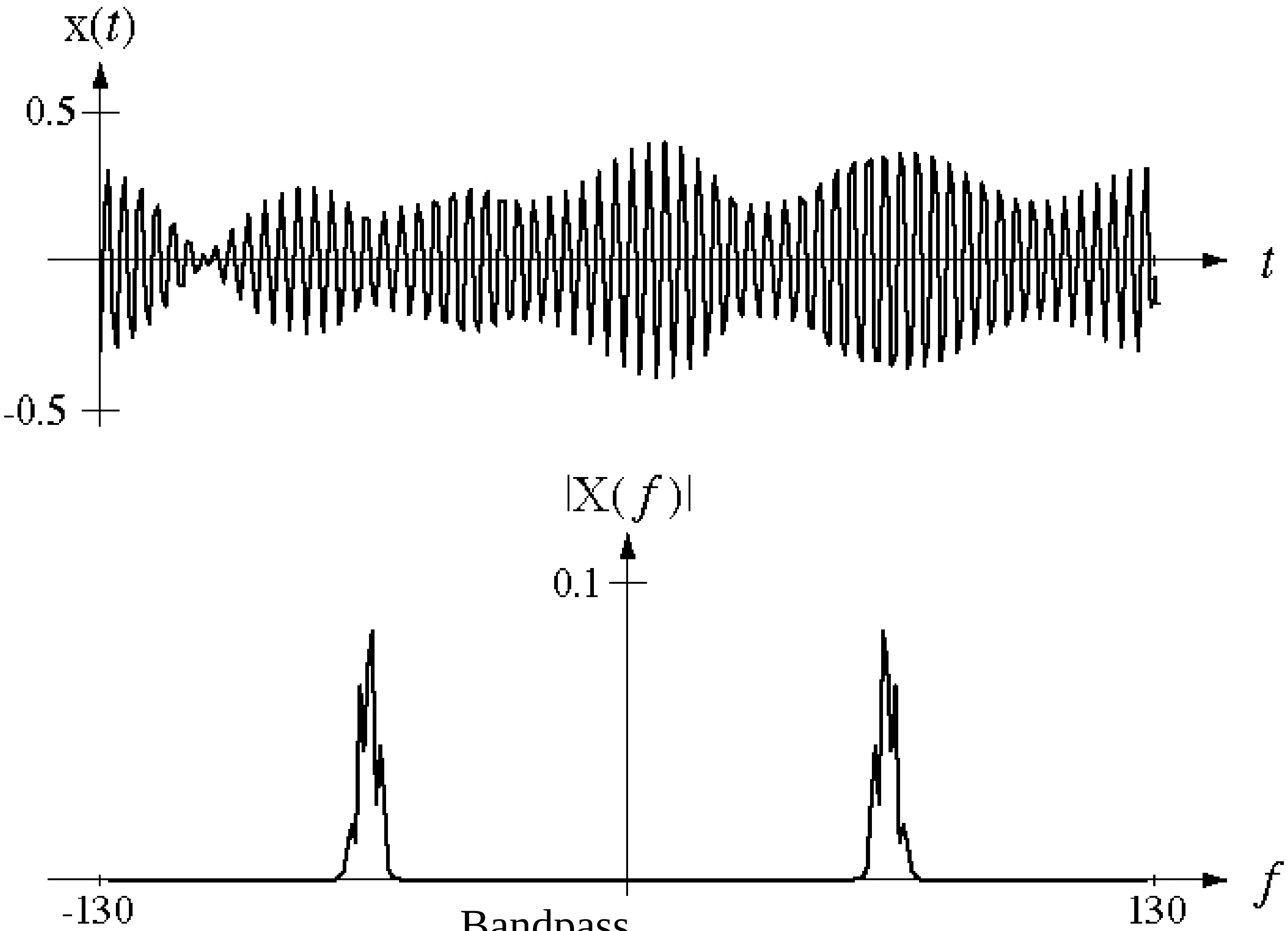
Examples



Lowpass

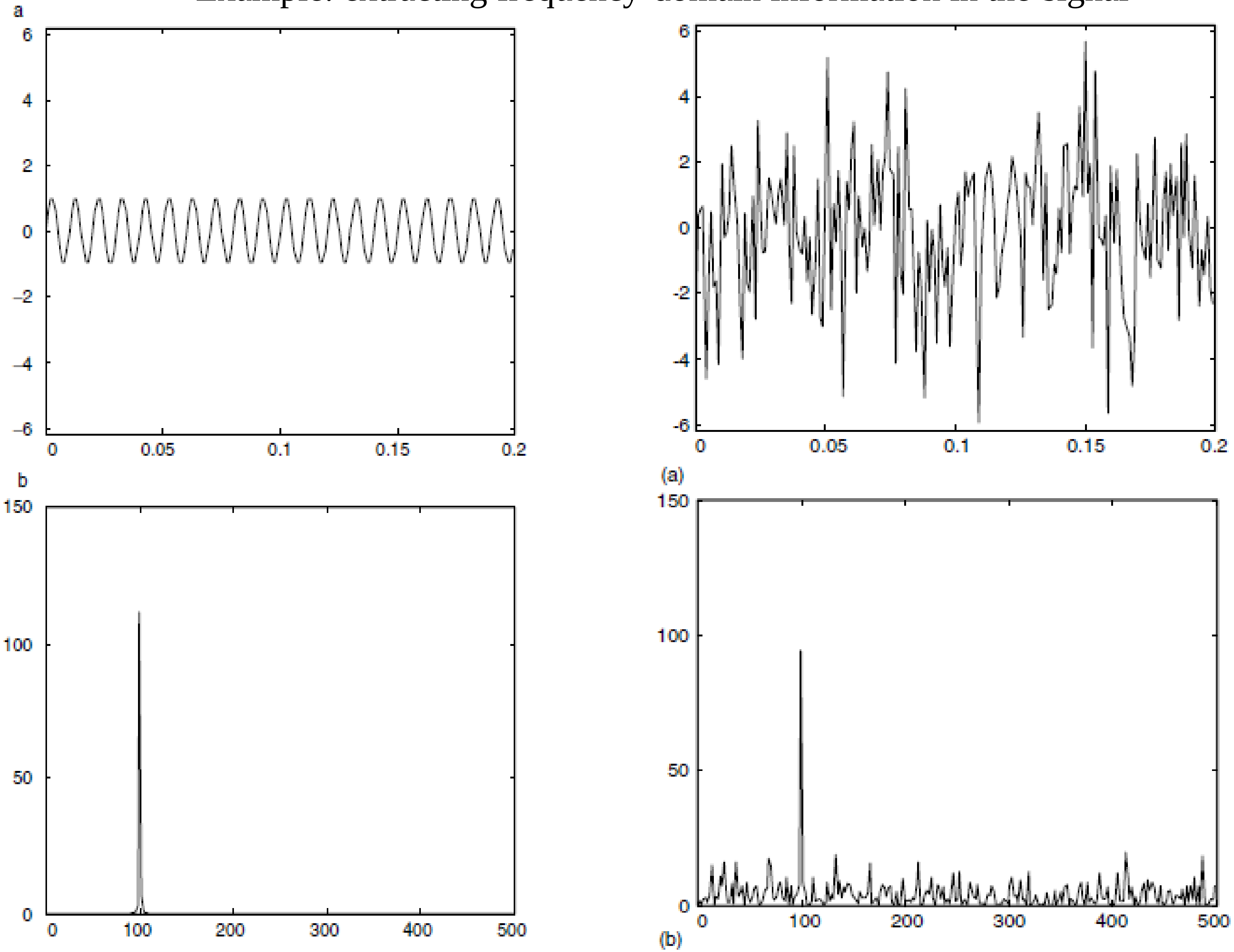


Highpass

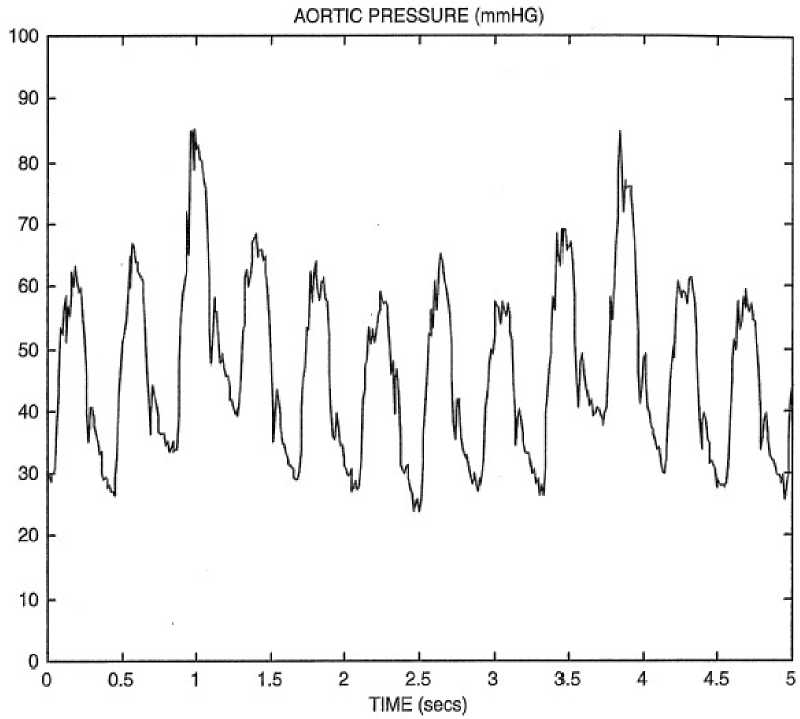


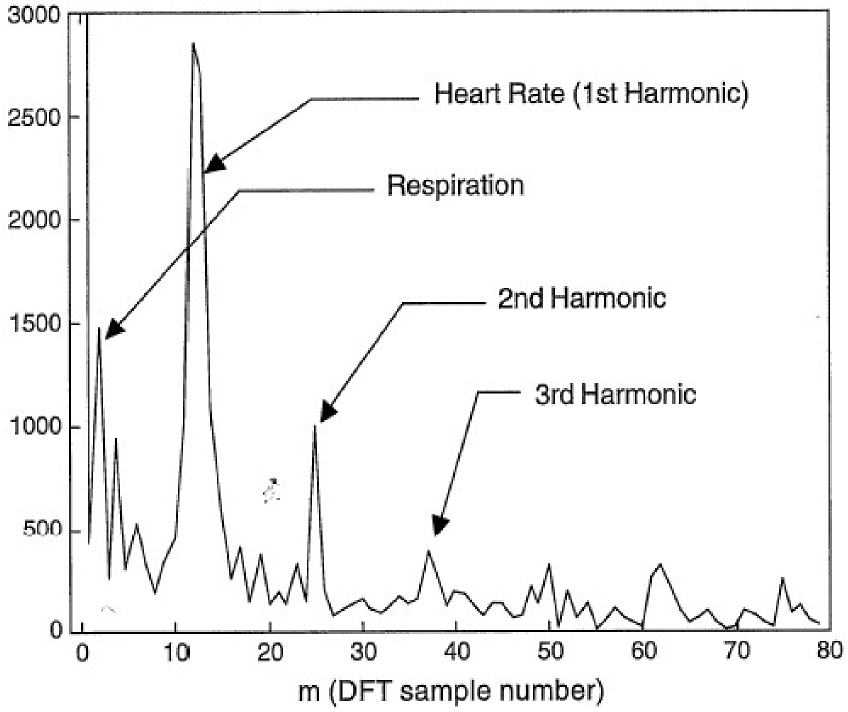
Bandpass

Example: extracting frequency-domain information in the signal

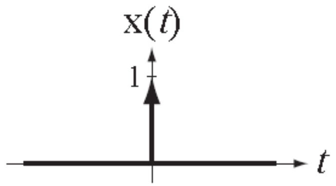


Example: blood pressure waveform (sampled at 200 points/s)

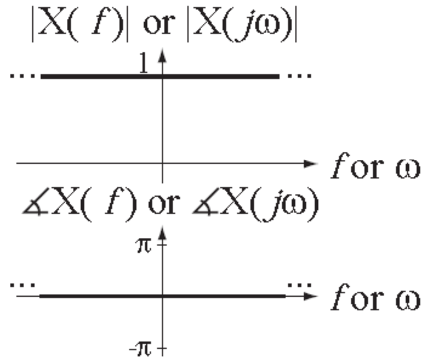


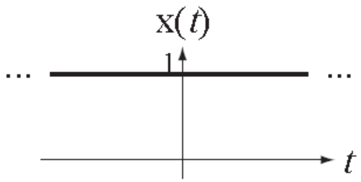


FT Pairs

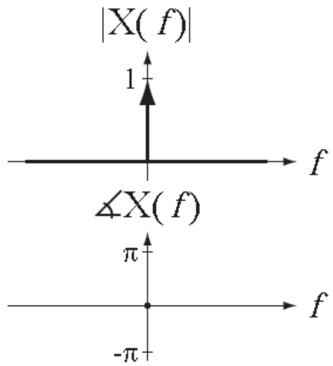


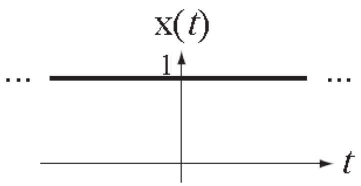
$$\delta(t) \overset{\mathcal{F}}{\longleftrightarrow} 1$$



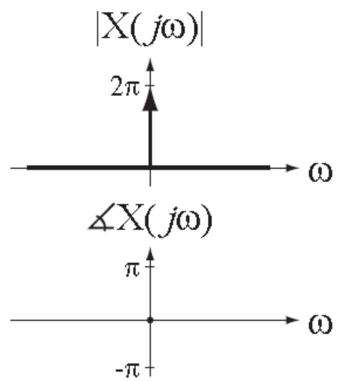


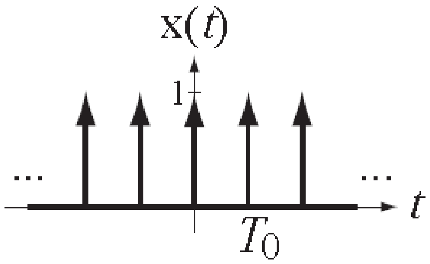
$$1 \overset{\mathcal{F}}{\longleftrightarrow} \delta(f)$$





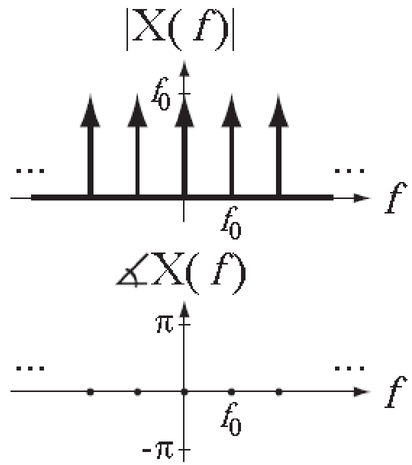
$$1 \overset{\mathcal{F}}{\longleftrightarrow} 2\pi\delta(\omega)$$

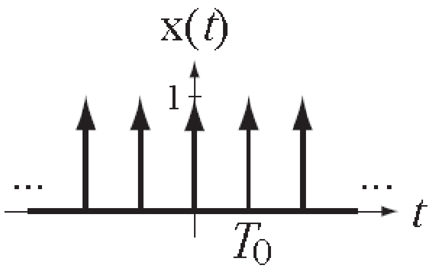




$$\delta_{T_0}(t) \overset{\mathcal{F}}{\longleftrightarrow} f_0 \delta_{f_0}(f)$$

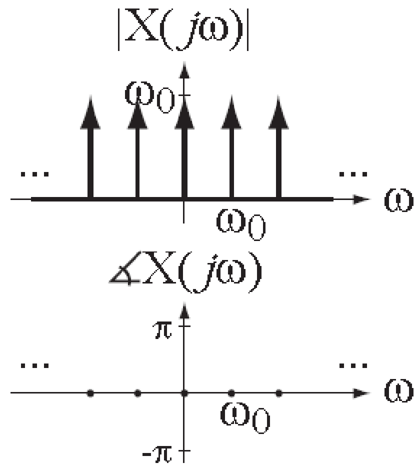
$$f_0 = 1/T_0$$

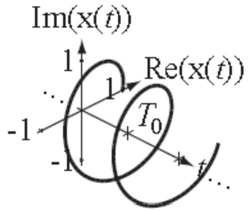




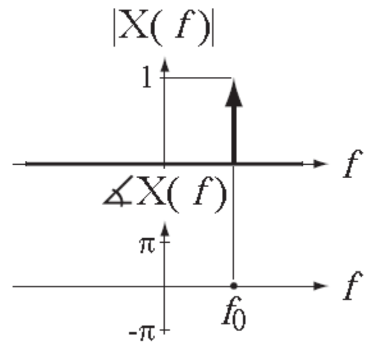
$$\delta_{T_0}(t) \overset{\mathcal{F}}{\longleftrightarrow} \omega_0 \delta_{\omega_0}(\omega)$$

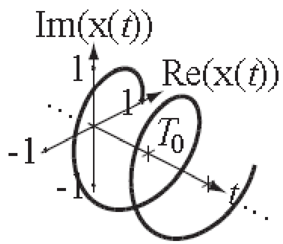
$$\omega_0 = 2\pi/T_0$$



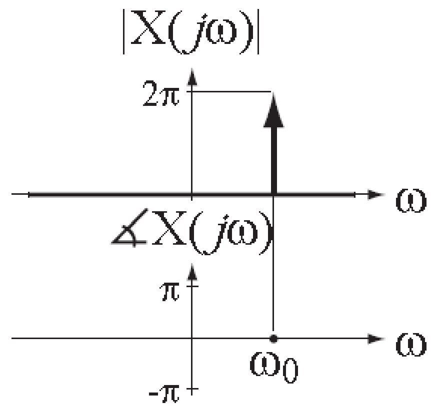


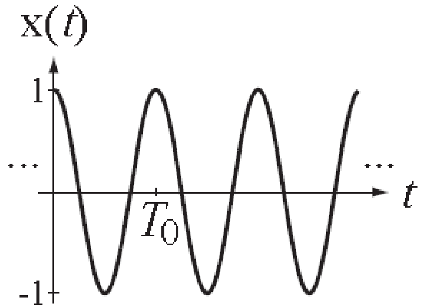
$$e^{j2\pi f_0 t} \overset{\mathcal{F}}{\longleftrightarrow} \delta(f - f_0)$$



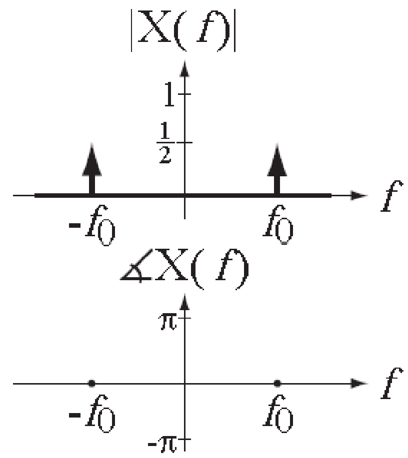


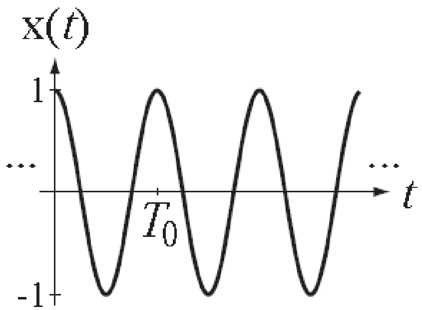
$$e^{j\omega_0 t} \overset{\mathcal{F}}{\longleftrightarrow} 2\pi\delta(\omega - \omega_0)$$



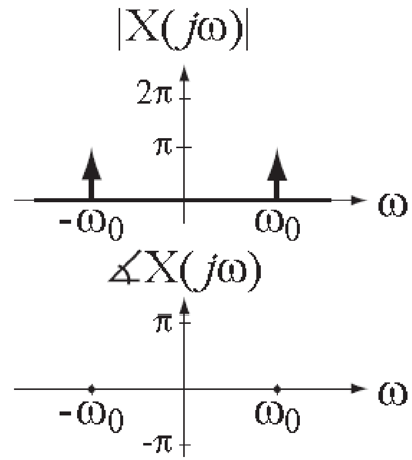


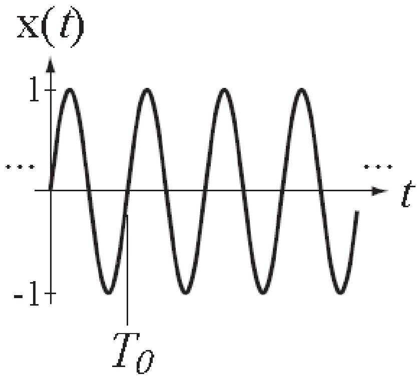
$$\cos(2\pi f_0 t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{2}[\delta(f - f_0) + \delta(f + f_0)]$$



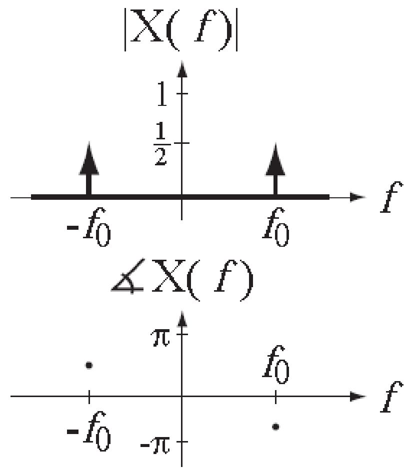


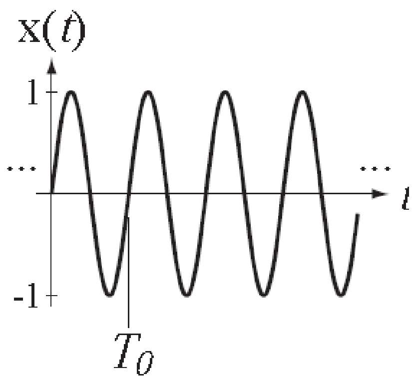
$$\cos(\omega_0 t) \overset{\mathcal{F}}{\longleftrightarrow} \pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$$



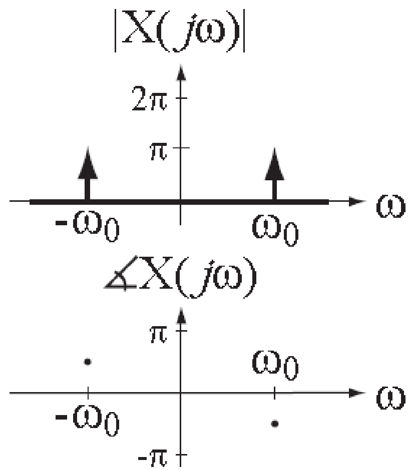


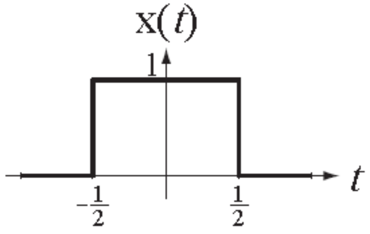
$$\sin(2\pi f_0 t) \xleftrightarrow{\mathcal{F}} \frac{j}{2}[\delta(f + f_0) - \delta(f - f_0)]$$





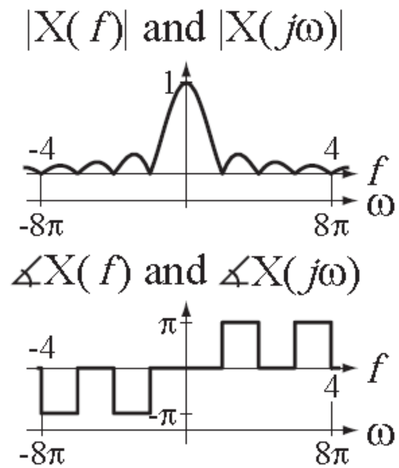
$$\sin(\omega_0 t) \xleftrightarrow{\mathcal{F}} j\pi[\delta(\omega + \omega_0) - \delta(\omega - \omega_0)]$$

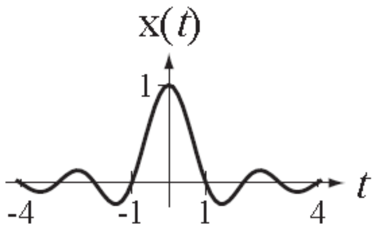




$$\text{rect}(t) \overset{\mathcal{F}}{\longleftrightarrow} \text{sinc}(f)$$

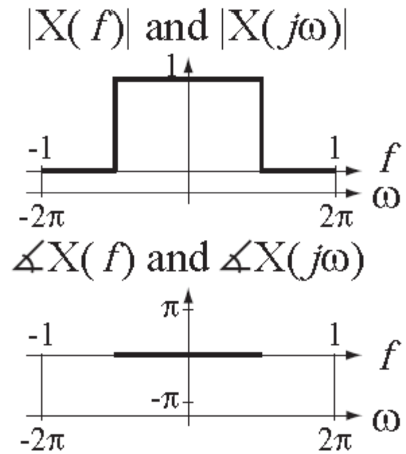
$$\text{rect}(t) \overset{\mathcal{F}}{\longleftrightarrow} \text{sinc}(\omega/2\pi)$$





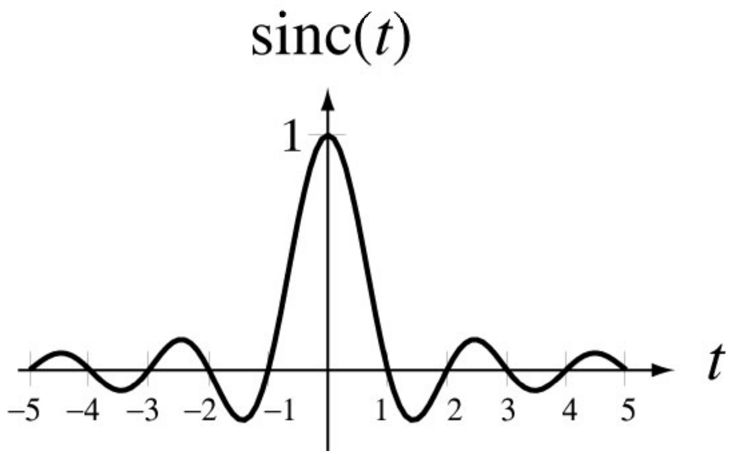
$$\text{sinc}(t) \overset{\mathcal{F}}{\longleftrightarrow} \text{rect}(f)$$

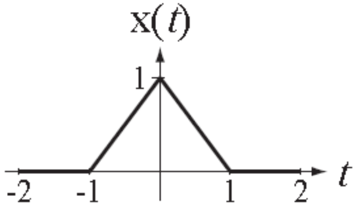
$$\text{sinc}(t) \overset{\mathcal{F}}{\longleftrightarrow} \text{rect}(\omega/2\pi)$$



Sinc Function

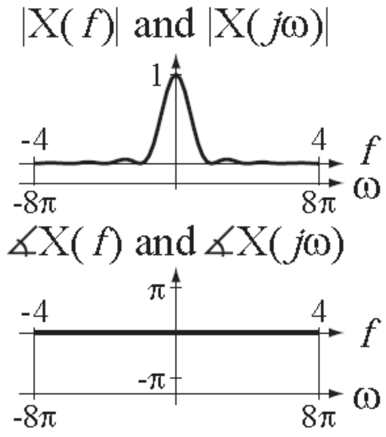
$$\text{sinc}(t) = \frac{\sin(\pi t)}{\pi t}$$

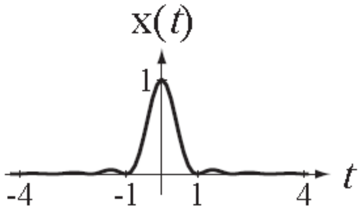




$$\mathrm{tri}(t) \overset{\mathcal{F}}{\longleftrightarrow} \mathrm{sinc}^2(f)$$

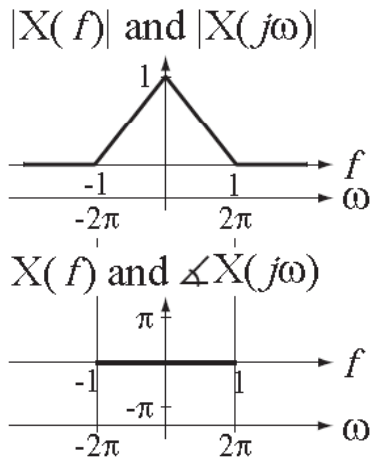
$$\mathrm{tri}(t) \overset{\mathcal{F}}{\longleftrightarrow} \mathrm{sinc}^2(\omega/2\pi)$$

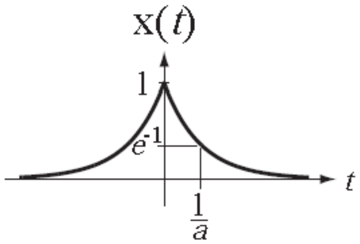




$$\mathrm{sinc}^2(t) \overset{\mathcal{F}}{\longleftrightarrow} \mathrm{tri}(f)$$

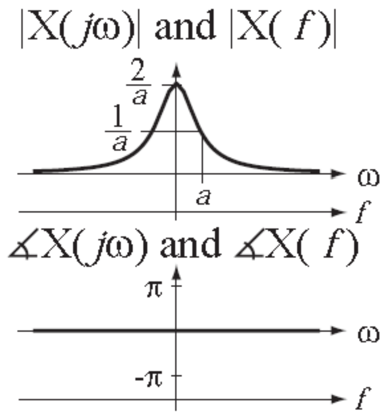
$$\mathrm{sinc}^2(t) \overset{\mathcal{F}}{\longleftrightarrow} \mathrm{tri}(\omega/2\pi)$$

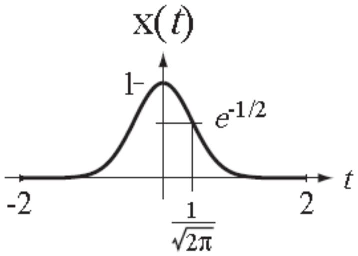




$$e^{-a|t|} \overset{\mathcal{F}}{\longleftrightarrow} \frac{2a}{\omega^2 + a^2}, \quad \mathrm{Re}(a) > 0$$

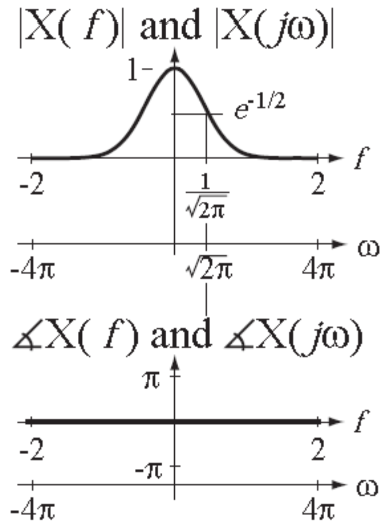
$$e^{-a|t|} \overset{\mathcal{F}}{\longleftrightarrow} \frac{2a}{(2\pi f)^2 + a^2}, \quad \mathrm{Re}(a) > 0$$

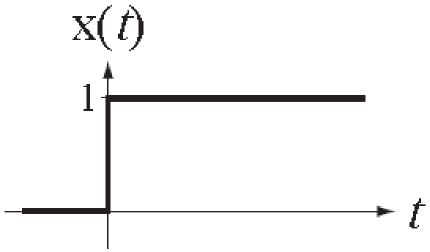




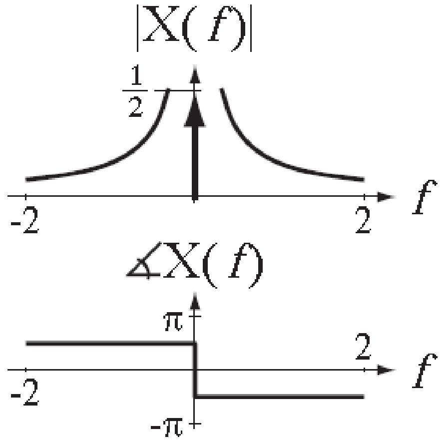
$$e^{-\pi t^2} \overset{\mathcal{F}}{\longleftrightarrow} e^{-\pi f^2}$$

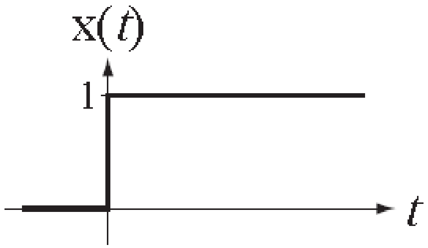
$$e^{-\pi t^2} \overset{\mathcal{F}}{\longleftrightarrow} e^{-\omega^2/4\pi}$$



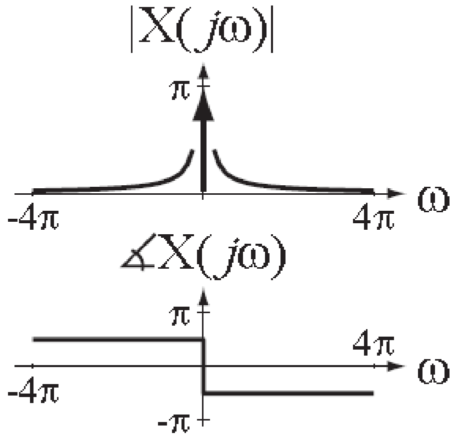


$$u(t) \stackrel{\mathcal{F}}{\longleftrightarrow} (1/2)\delta(f) + 1/j2\pi f$$





$$u(t) \stackrel{\mathcal{F}}{\longleftrightarrow} \pi\delta(\omega) + 1/j\omega$$

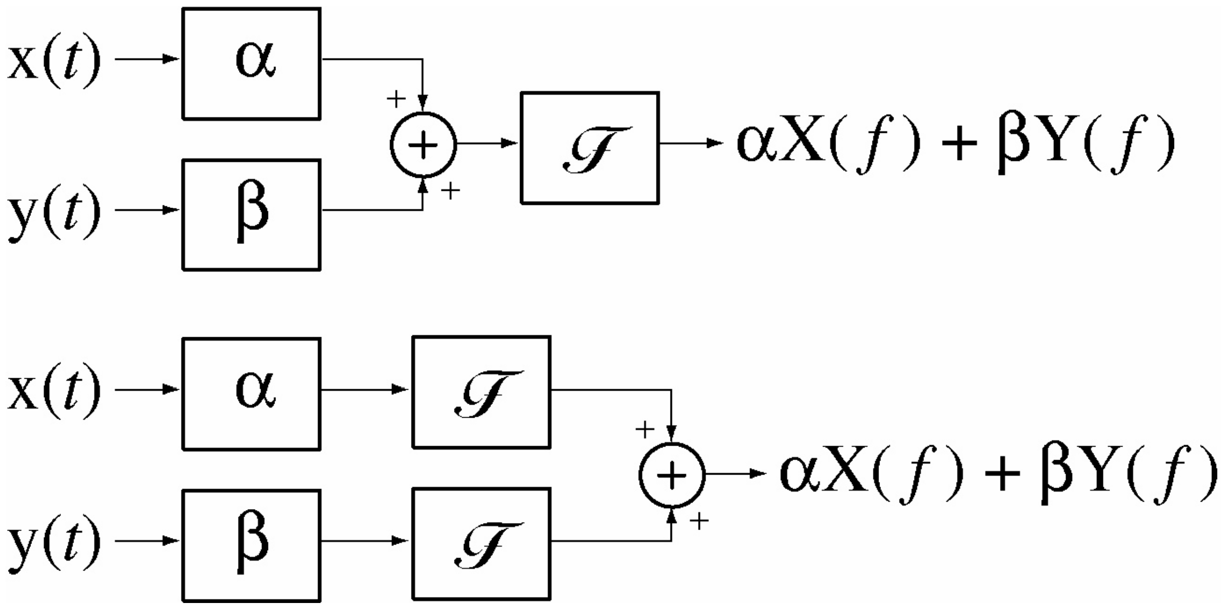


FT Properties

Lineality

$$\alpha x(t)+\beta y(t) \stackrel{F}{\longleftrightarrow} \alpha X(f)+\beta Y(f)$$

$$\alpha x(t)+\beta y(t) \stackrel{F}{\longleftrightarrow} \alpha X(j\omega)+\beta Y(j\omega)$$



Domains Duality

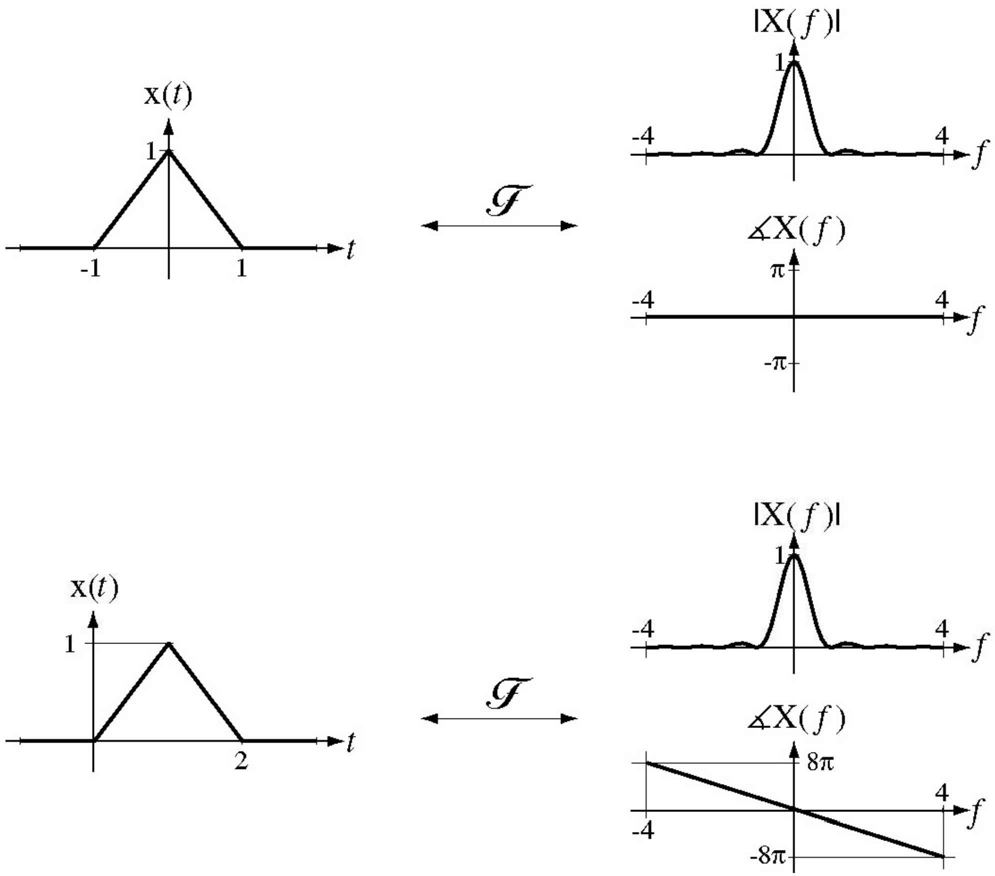
$$X(t) \xleftrightarrow{\mathrm{F}} x(-f) \quad \text{and} \quad X(-t) \xleftrightarrow{\mathrm{F}} x(f)$$

$$X(jt) \xleftrightarrow{\mathrm{F}} 2\pi x(-\omega) \quad \text{and} \quad X(-jt) \xleftrightarrow{\mathrm{F}} 2\pi x(\omega)$$

Time Shifting

$$x(t-t_0) \overset{F}{\longleftrightarrow} X(f)e^{-j2\pi f t_0}$$

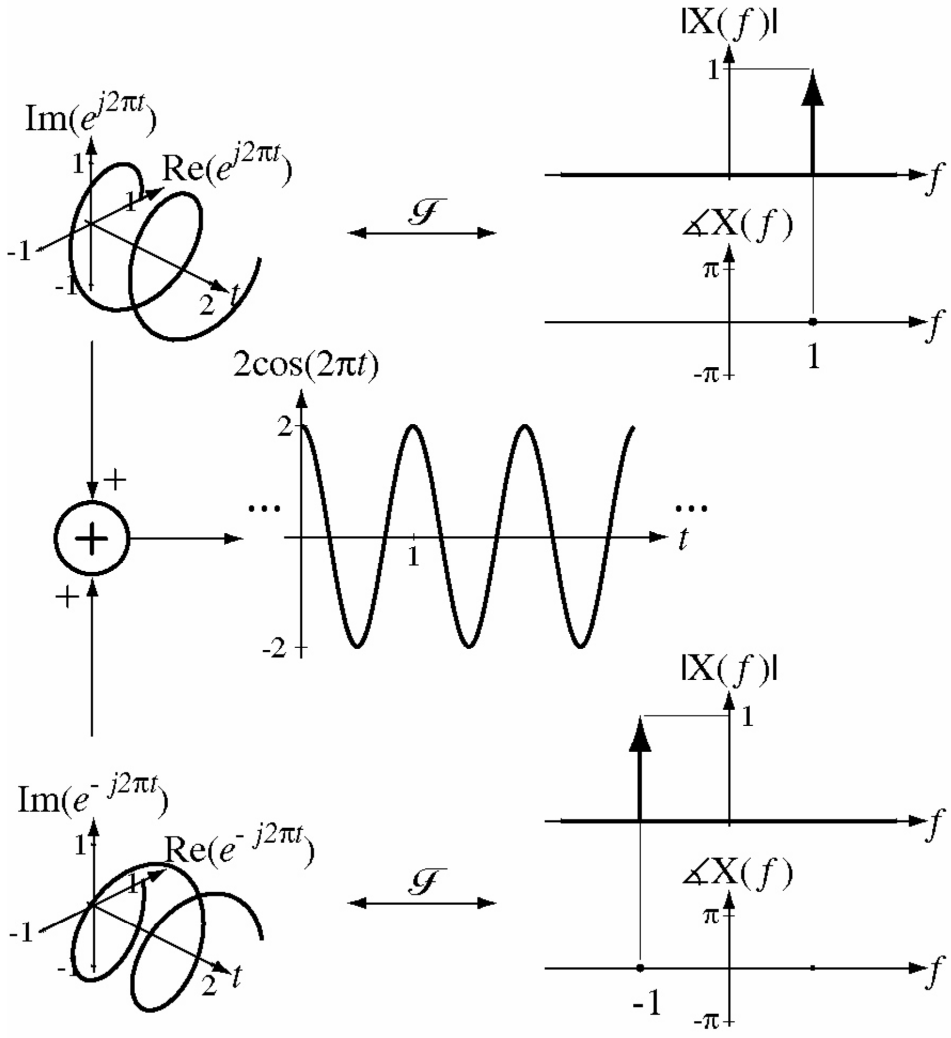
$$x(t-t_0) \overset{F}{\longleftrightarrow} X(j\omega)e^{-j\omega t_0}$$



Frequency Shifting

$$x(t)e^{+j2\pi f_0 t} \overset{F}{\longleftrightarrow} X(f - f_0)$$

$$x(t)e^{+j\omega_0 t} \overset{F}{\longleftrightarrow} X(j(\omega - \omega_0))$$

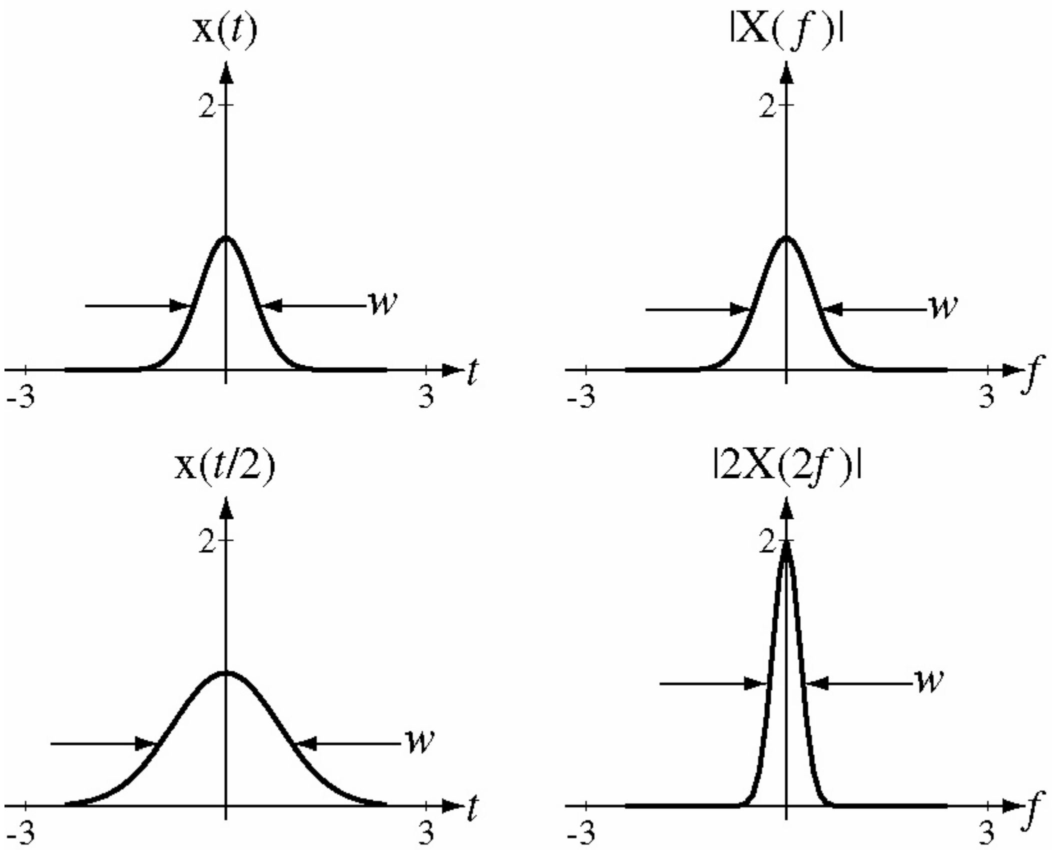


The "Uncertainty" Principle

-The time and frequency scaling properties indicate that if a signal is expanded in one domain it is compressed in the other domain.
-This is called the "uncertainty principle" of Fourier analysis.

$$e^{-\pi t^2} \xleftrightarrow{\mathcal{F}} e^{-\pi f^2}$$

$$e^{-\pi (t/2)^2} \xleftrightarrow{\mathcal{F}} 2e^{-\pi (2f)^2}$$



Time Scaling (Uncertainty Principle)

$$x(at) \overset{F}{\longleftrightarrow} \frac{1}{|a|} X\left(\frac{f}{a}\right)$$

$$x(at) \overset{F}{\longleftrightarrow} \frac{1}{|a|} X\left(j\frac{\omega}{a}\right)$$

Frequency Scaling (Uncertainty Principle)

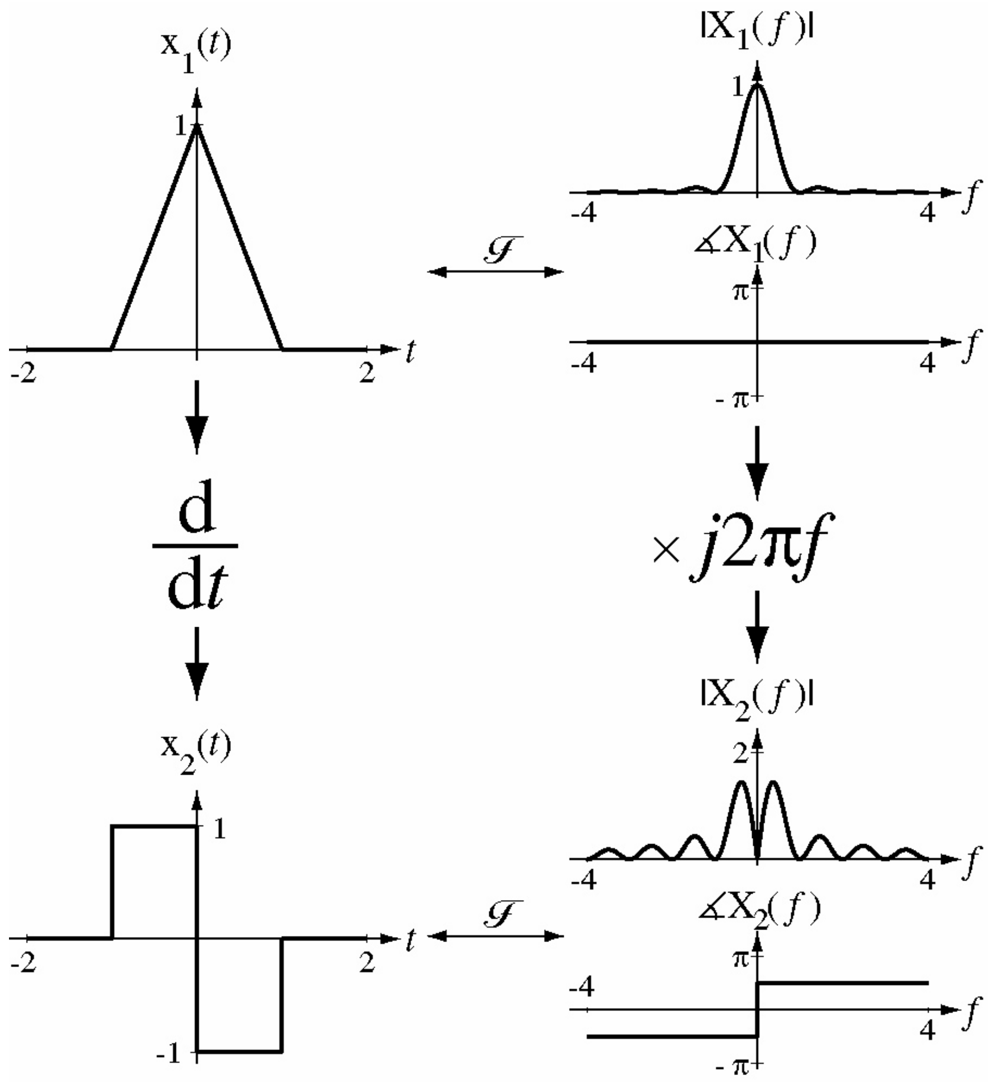
$$\frac{1}{|a|}x\left(\frac{t}{a}\right)\xleftrightarrow{\mathrm{F}}X(af)$$

$$\frac{1}{|a|}x\left(\frac{t}{a}\right)\xleftrightarrow{\mathrm{F}}X(ja\omega)$$

Time Differentiation

$$\frac{d}{dt}\left(x\left(t\right)\right)\xleftrightarrow{\text{F}} j2\pi f X\left(f\right)$$

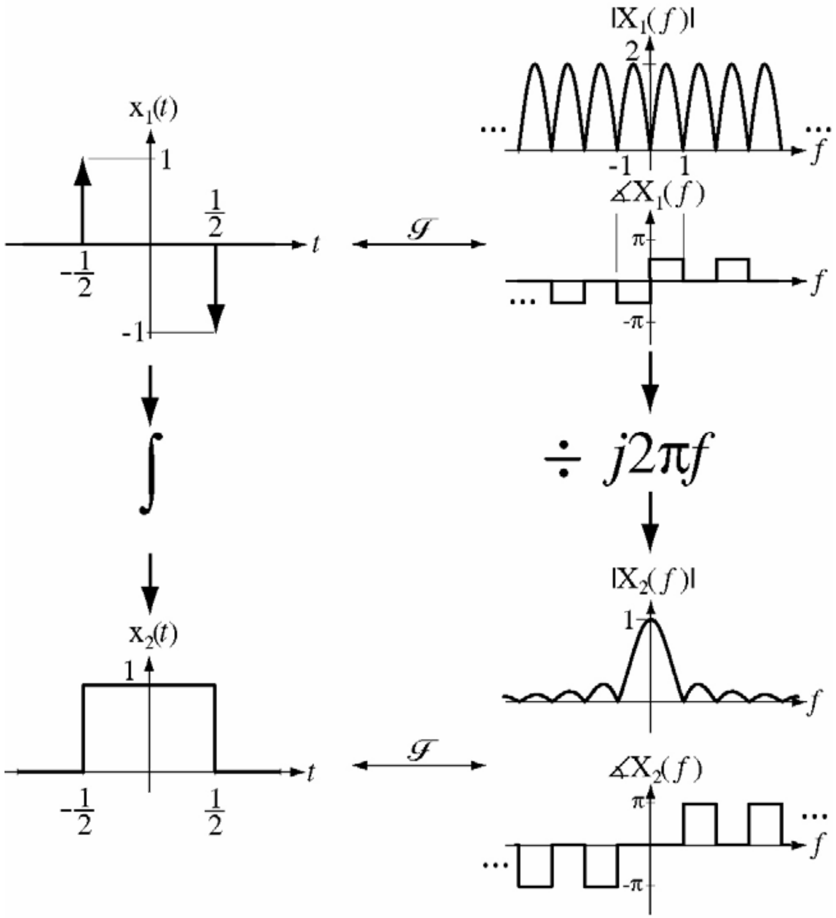
$$\frac{d}{dt}\left(x\left(t\right)\right)\xleftrightarrow{\text{F}} j\omega X\left(j\omega\right)$$



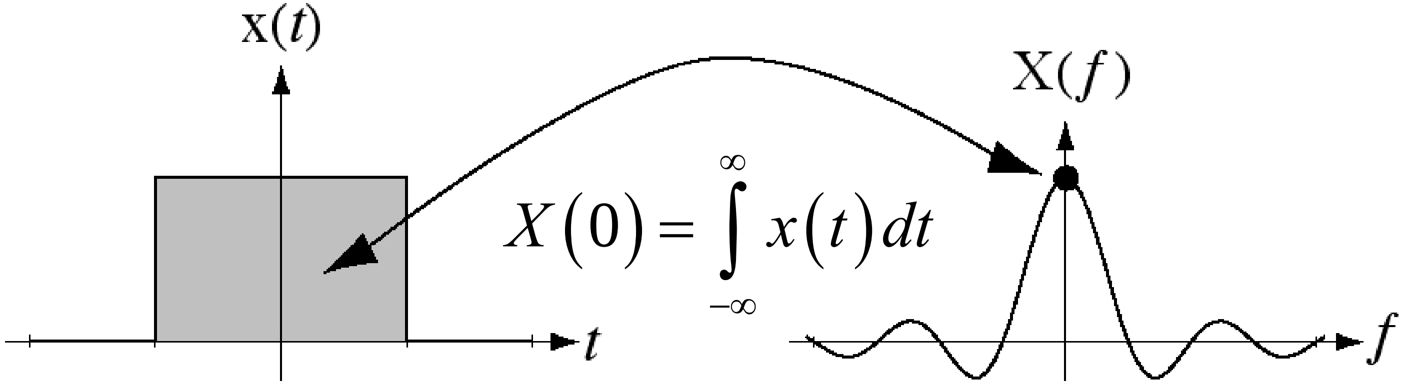
Time Integration

$$\int_{-\infty}^{t} x(\tau)\,d\tau \overset{F}{\longleftrightarrow} \frac{X(f)}{j2\pi f} + \frac{1}{2}X(0)\delta(f)$$

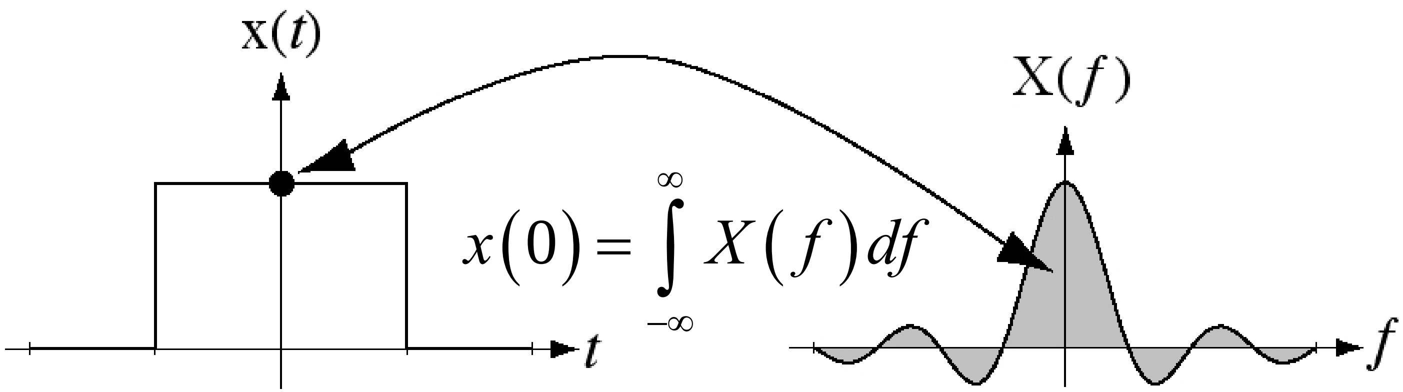
$$\int_{-\infty}^{t} x(\tau)\,d\tau \overset{F}{\longleftrightarrow} \frac{X(j\omega)}{j\omega} + \pi X(0)\delta(\omega)$$



Total - Area Integrals



$$X(0)=\int_{-\infty}^{\infty} x(t)\,dt$$



$$x(0)=\int_{-\infty}^{\infty} X(f)\,df$$

Parseval's Theorem

$$\int_{-\infty}^{\infty} \left| x(t) \right|^2 dt = \int_{-\infty}^{\infty} \left| X(f) \right|^2 df$$

$$\int_{-\infty}^{\infty} \left| x(t) \right|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| X(j\omega) \right|^2 df$$

Multiplication Convolution Duality

$$x(t) * y(t) \overset{\mathrm{F}}{\longleftrightarrow} X(f)Y(f)$$

$$x(t) * y(t) \overset{\mathrm{F}}{\longleftrightarrow} X(j\omega)Y(j\omega)$$

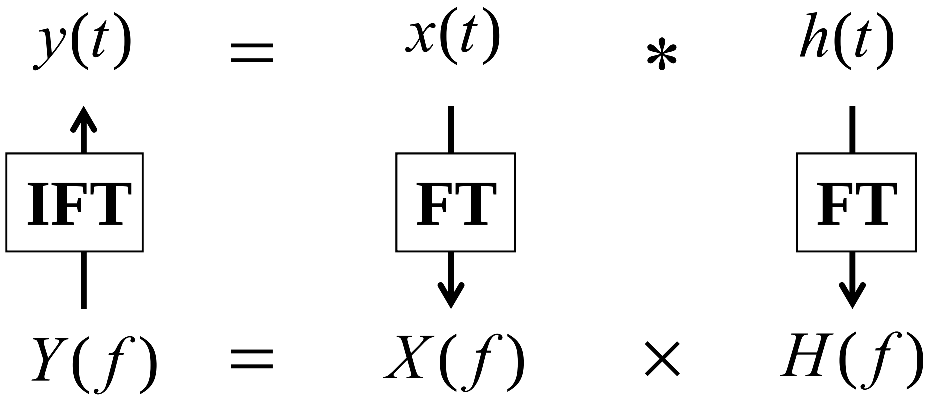
$$x(t)y(t) \overset{\mathrm{F}}{\longleftrightarrow} X(f) * Y(f)$$

$$x(t)y(t) \overset{\mathrm{F}}{\longleftrightarrow} \frac{1}{2\pi} X(j\omega) * Y(j\omega)$$

Time Domain (t) | Frequency Domain (f)

$$y = x * h \quad \longleftrightarrow \quad Y = XH$$

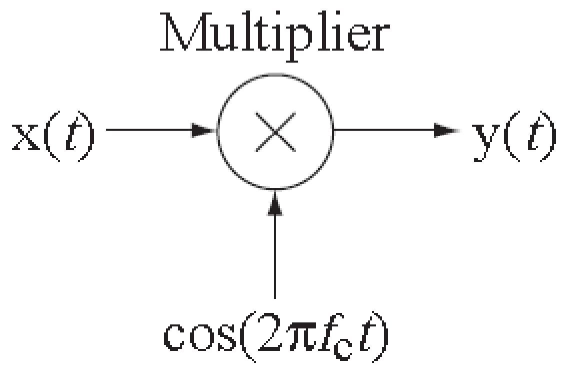
$$y = xh \quad \longleftrightarrow \quad Y = X * H$$

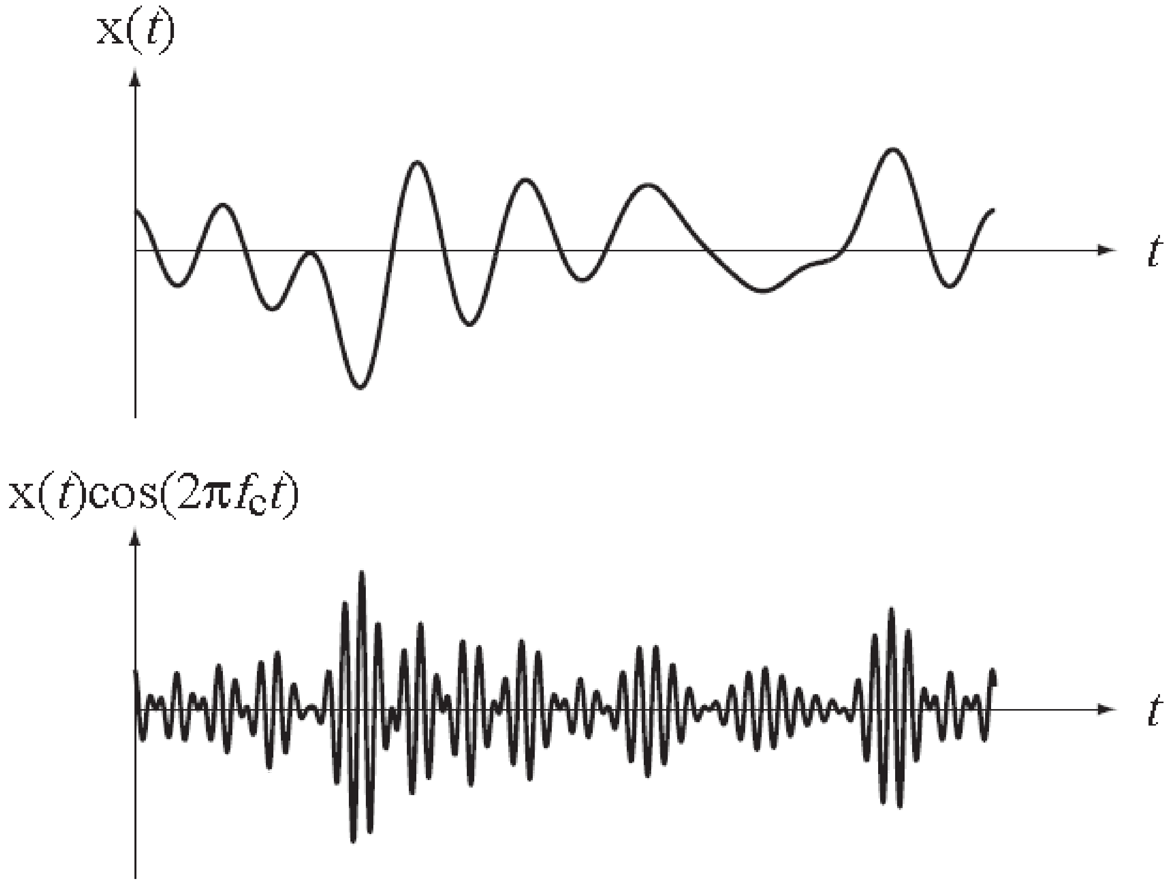

y(t) = x(t) * h(t)
IFT
FT
FT
Y(f) = X(f) × H(f)

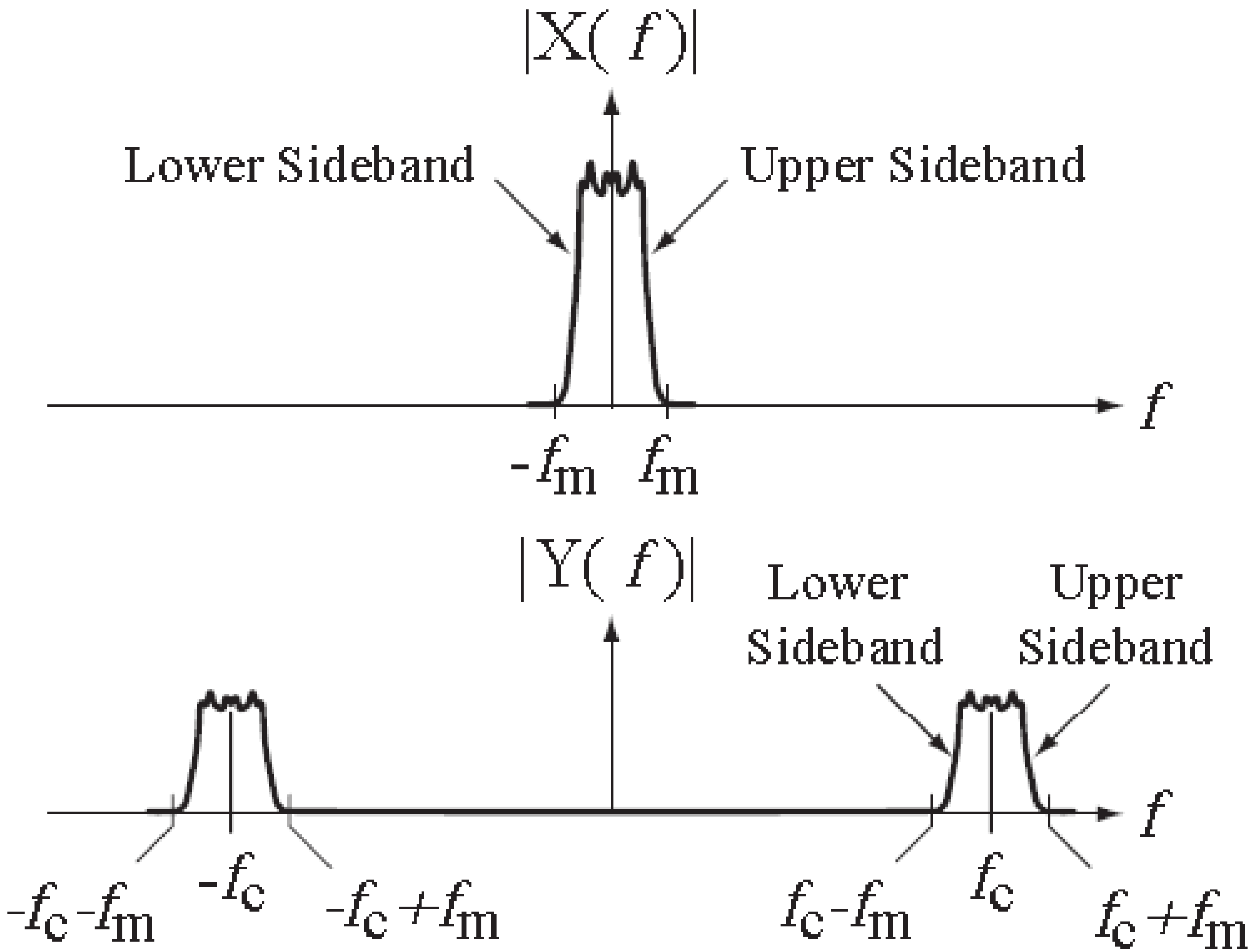
Modulation

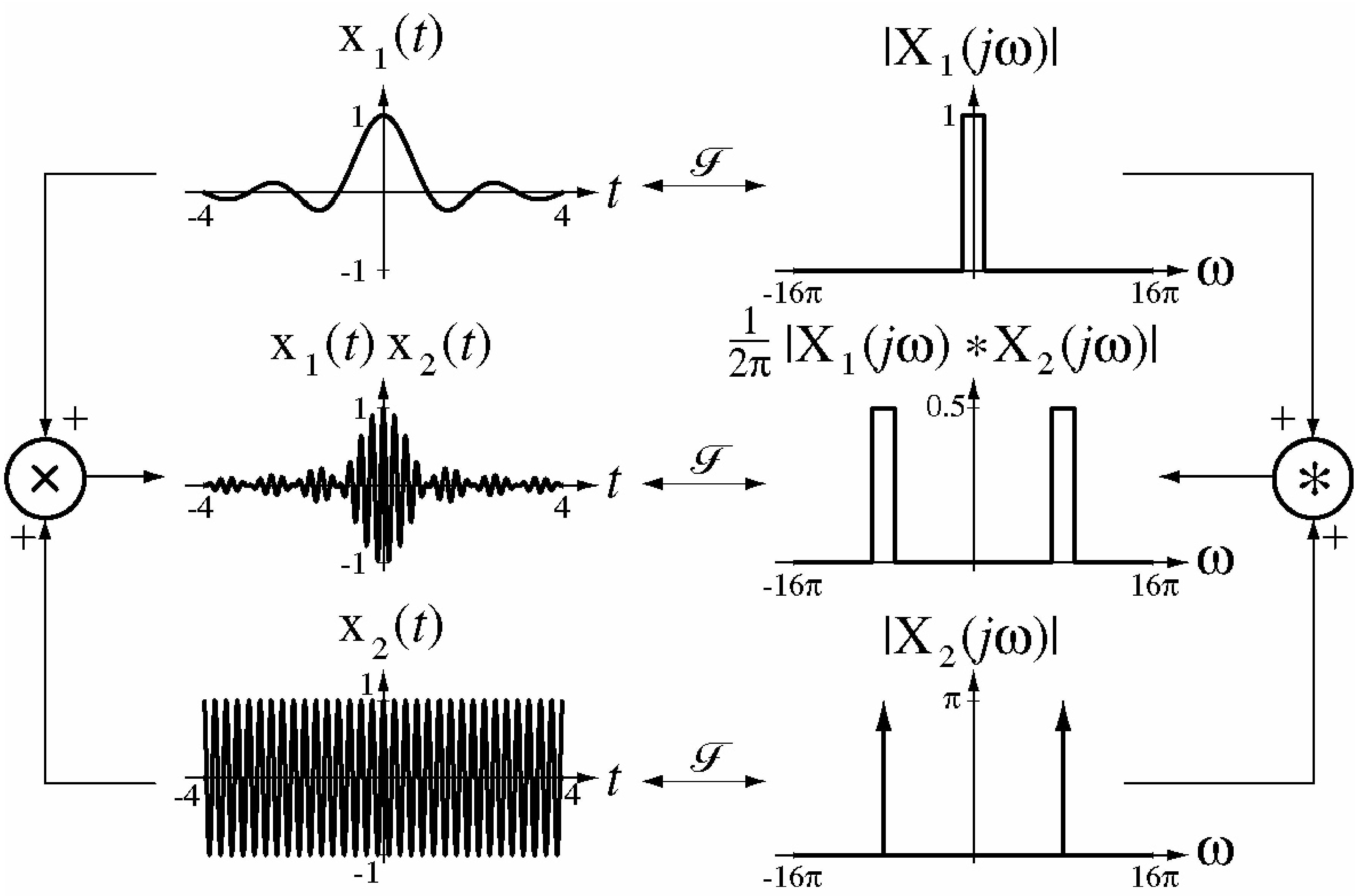
$$x(t)\cos(2\pi f_0 t) \stackrel{F}{\longleftrightarrow} \frac{1}{2}\left[X(f-f_0)+X(f+f_0)\right]$$

$$x(t)\cos(\omega_0 t) \stackrel{F}{\longleftrightarrow} \frac{1}{2}\left[X\left(j(\omega-\omega_0)\right)+X\left(j(\omega+\omega_0)\right)\right]$$

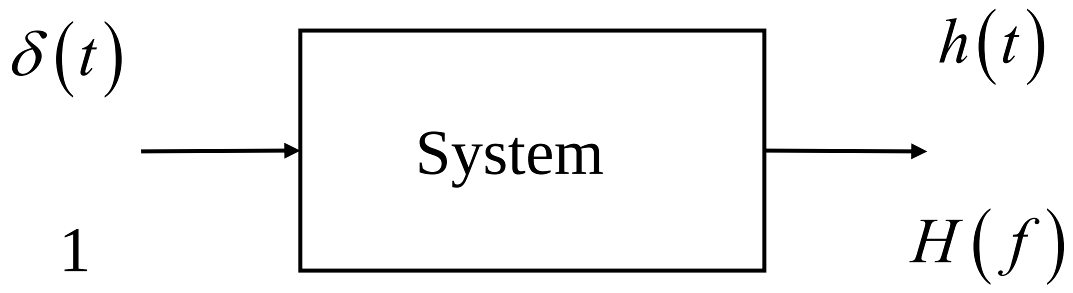

Multiplier
x(t)
y(t)
cos(2πfct)


x(t)
t
x(t)cos(2πfct)
t

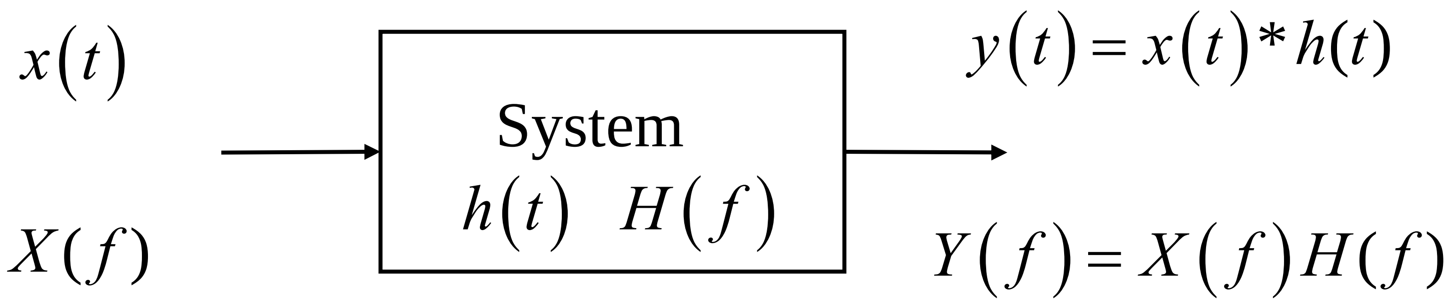

|X(f)|
Lower Sideband
Upper Sideband
f
-fm
fm
|Y(f)|
Lower
Sideband
Upper
Sideband
f
-fc-fm
-fc
-fc+fm
fc-fm
fc
fc+fm



x 1(t)
1
-1
-4
4
t
|X 1(jω)|
1
-16π
16π
ω
x 1(t) x 2(t)
1/2π |X 1(jω) * X 2(jω)|
0.5
x 2(t)
|X 2(jω)|
π
+



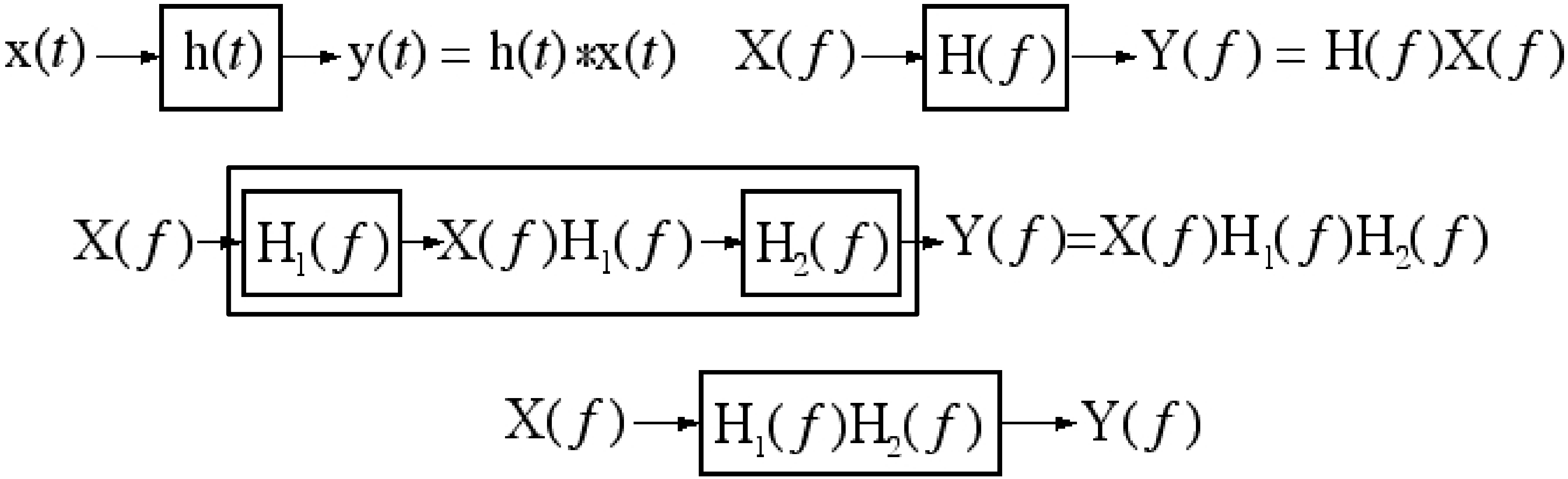
δ(t)
1
System
h(t)
H(f)


$x(t)$
$X(f)$
System
$h(t)$ $H(f)$
$y(t) = x(t) * h(t)$
$Y(f) = X(f)H(f)$

System Interconnections

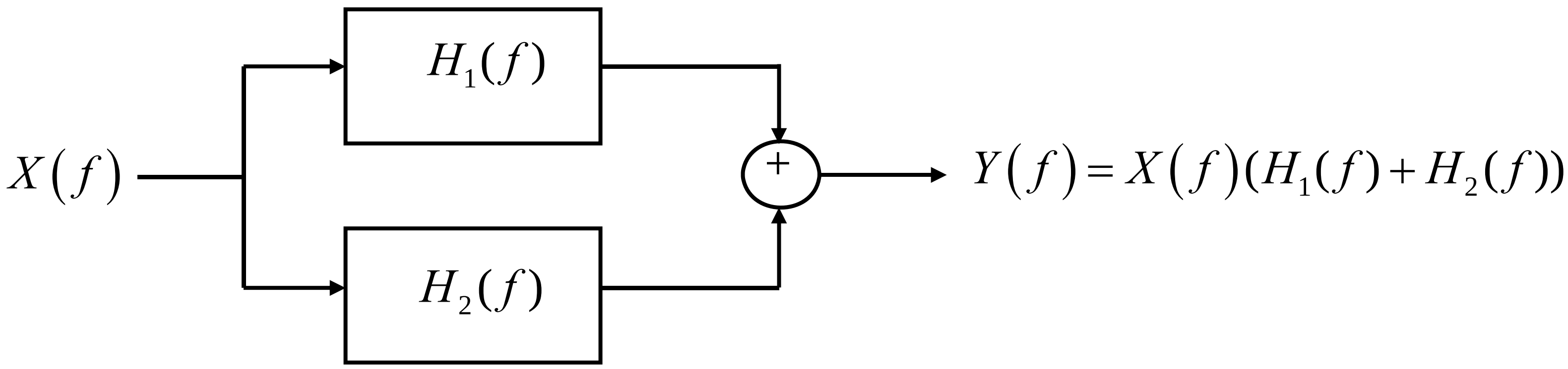
-If the output signal from a LTI system is the input signal to a second LTI system, the systems are said to be **serie/cascade** connected.

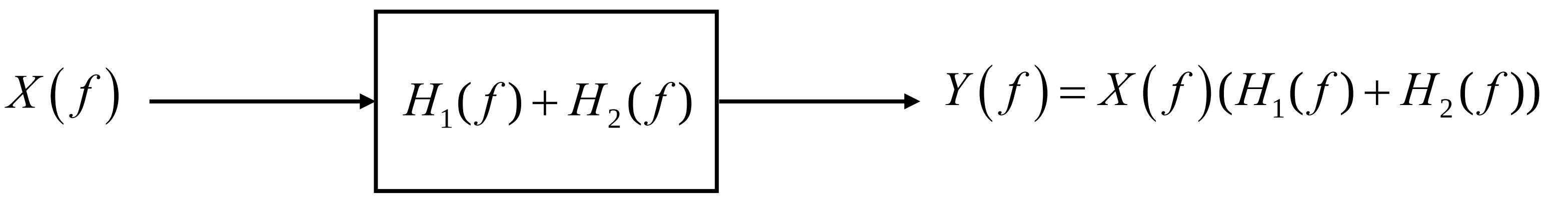
-In the frequency domain, the **serie/cascade connection** multiplies the frequency responses instead of convolving the impulse responses.



Serie/Cascade Connection

-If two LTI systems are excited by the same signal and their responses are added they are said to be **parallel** connected.



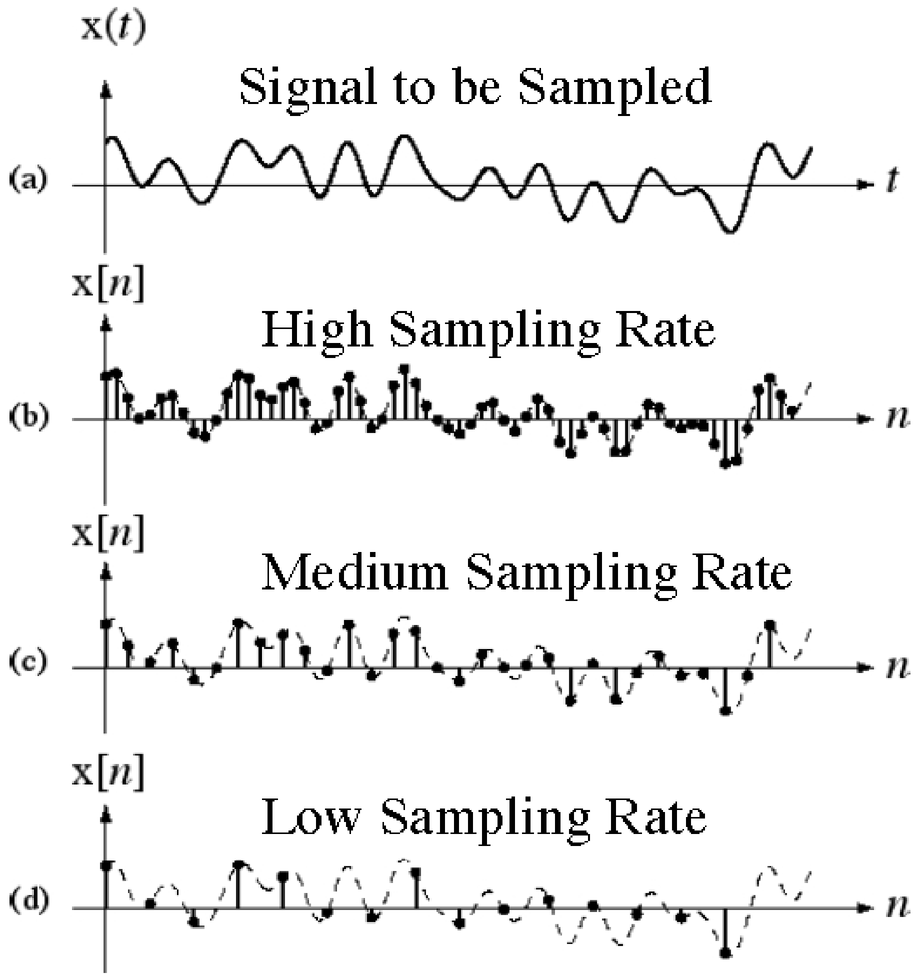


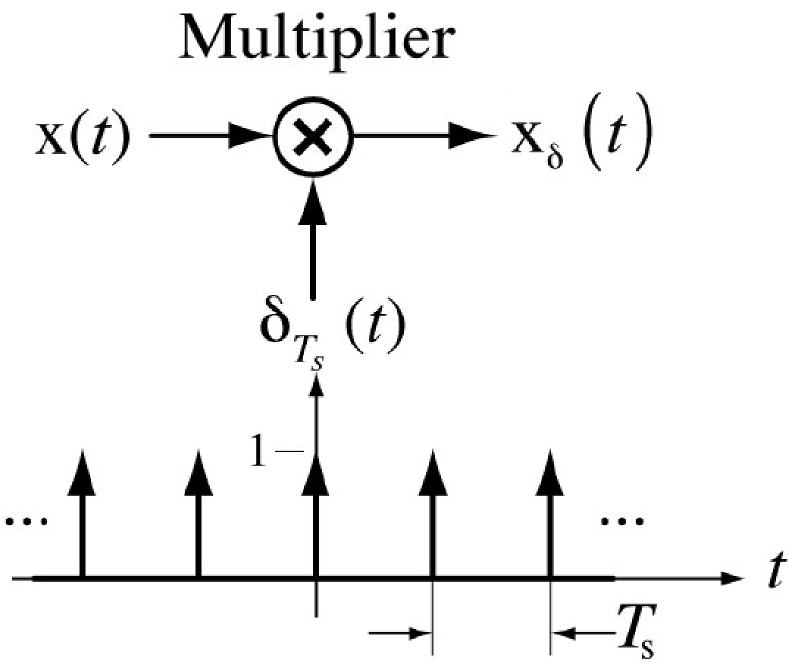
Parallel Connection

8- Sampling

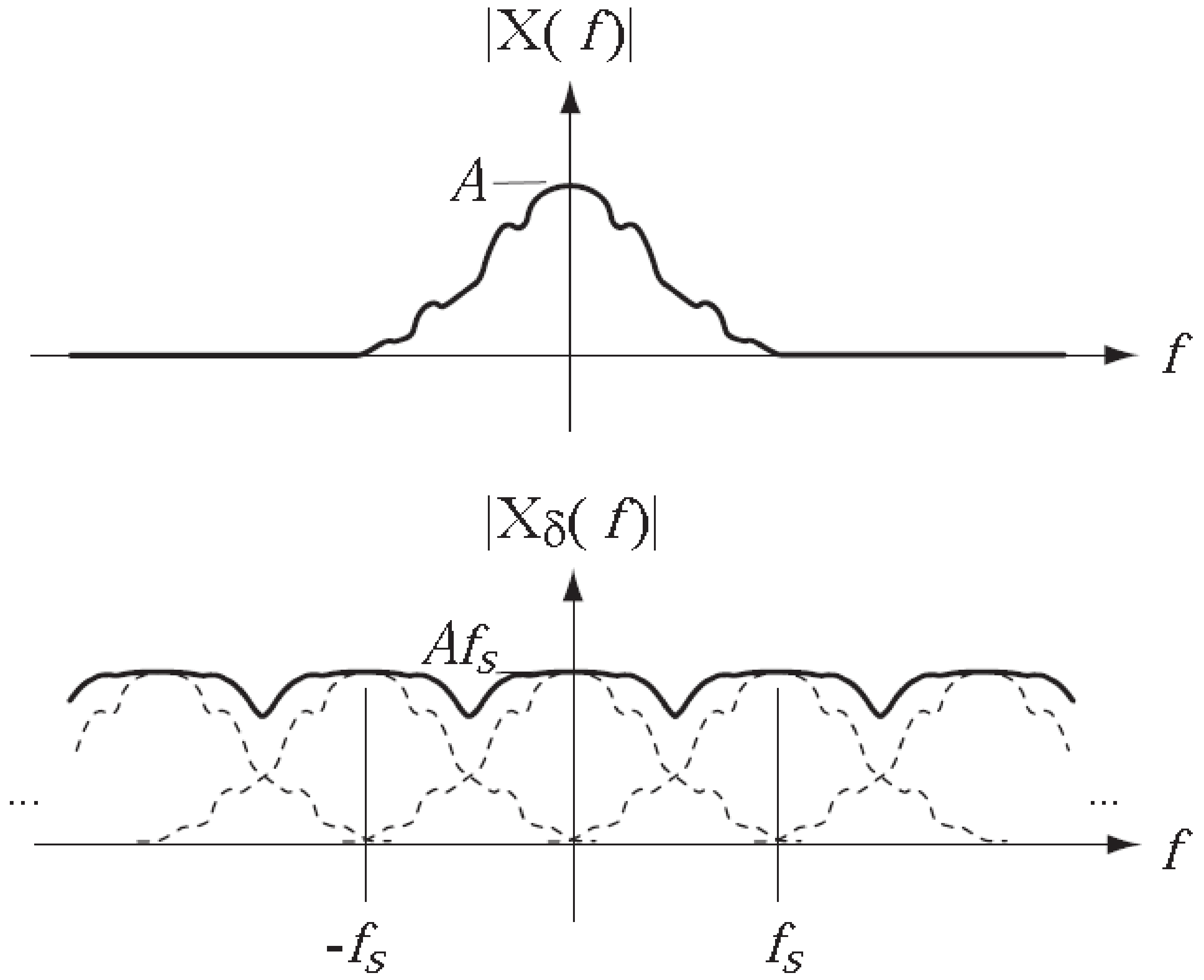
Introduction

-The fundamental consideration in sampling theory is how fast to sample a signal to be able to reconstruct the signal from the samples.

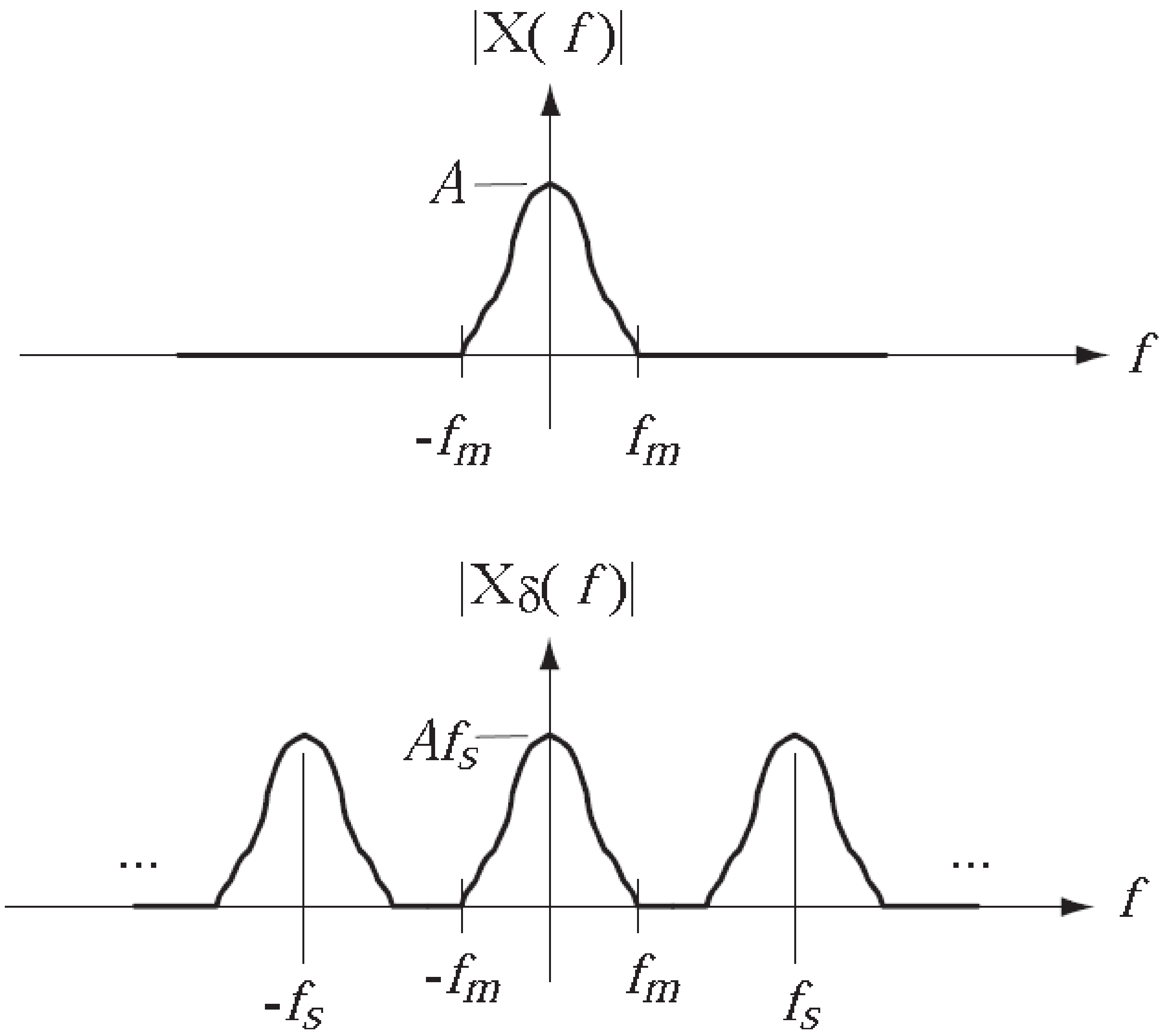



Multiplier
x(t)
x_δ(t)
δ_Ts(t)
1
...
...
t
Ts

Aliasing



$$f_S = \frac{1}{T_S}$$


|X(f)|
A
f
-fm
fm
|Xδ(f)|
Afs
...
...
f
-fs
-fm
fm
fs

Nyquist rate

-If a continuous-time signal is sampled for all time at a rate f_s that is more than twice the bandlimit f_m of the signal, *the original continuous-time signal can be recovered exactly from the samples*.

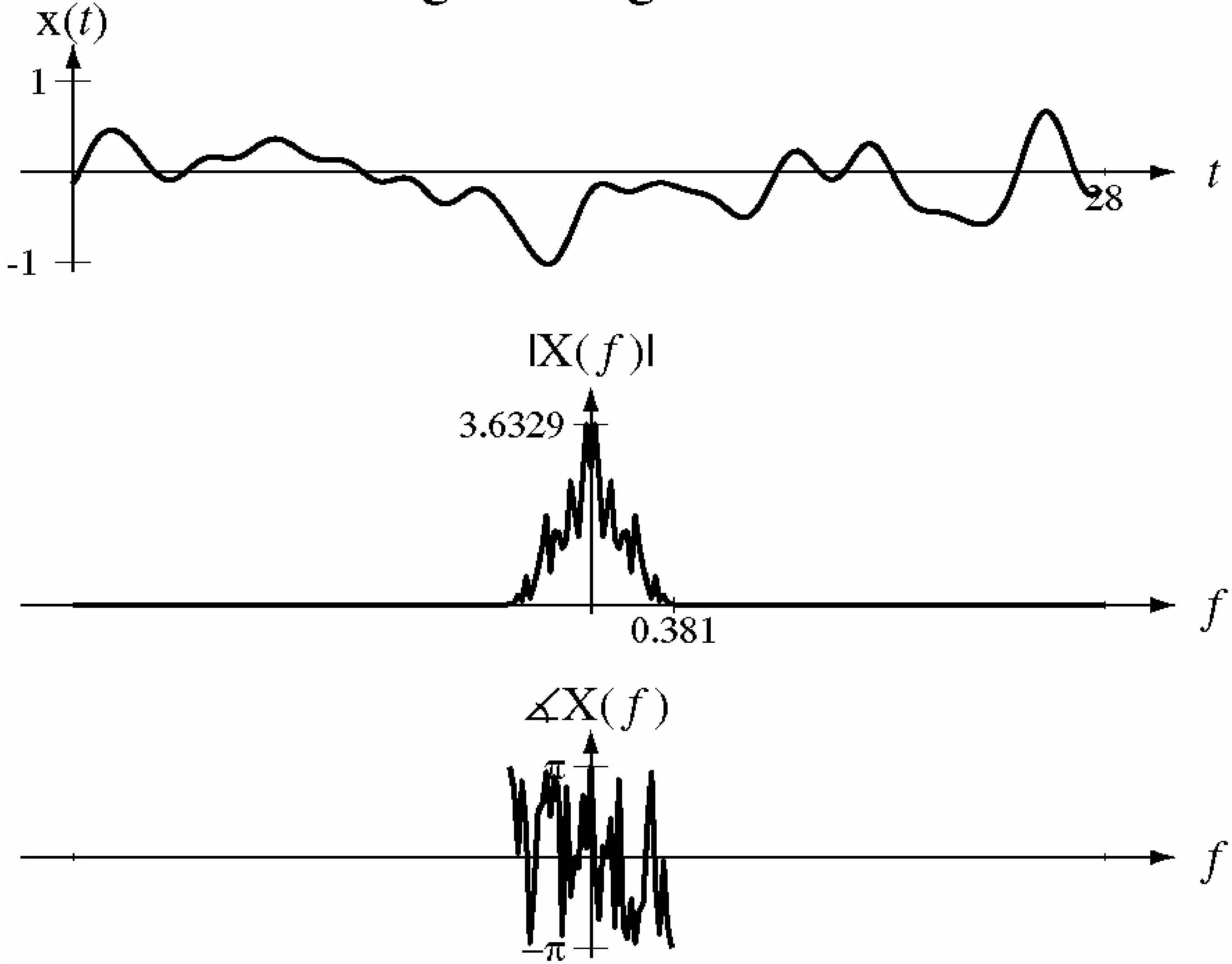
$$\boxed{f_S \geq 2 f_m}$$

-The frequency $2 f_m$ is called the **Nyquist rate**.

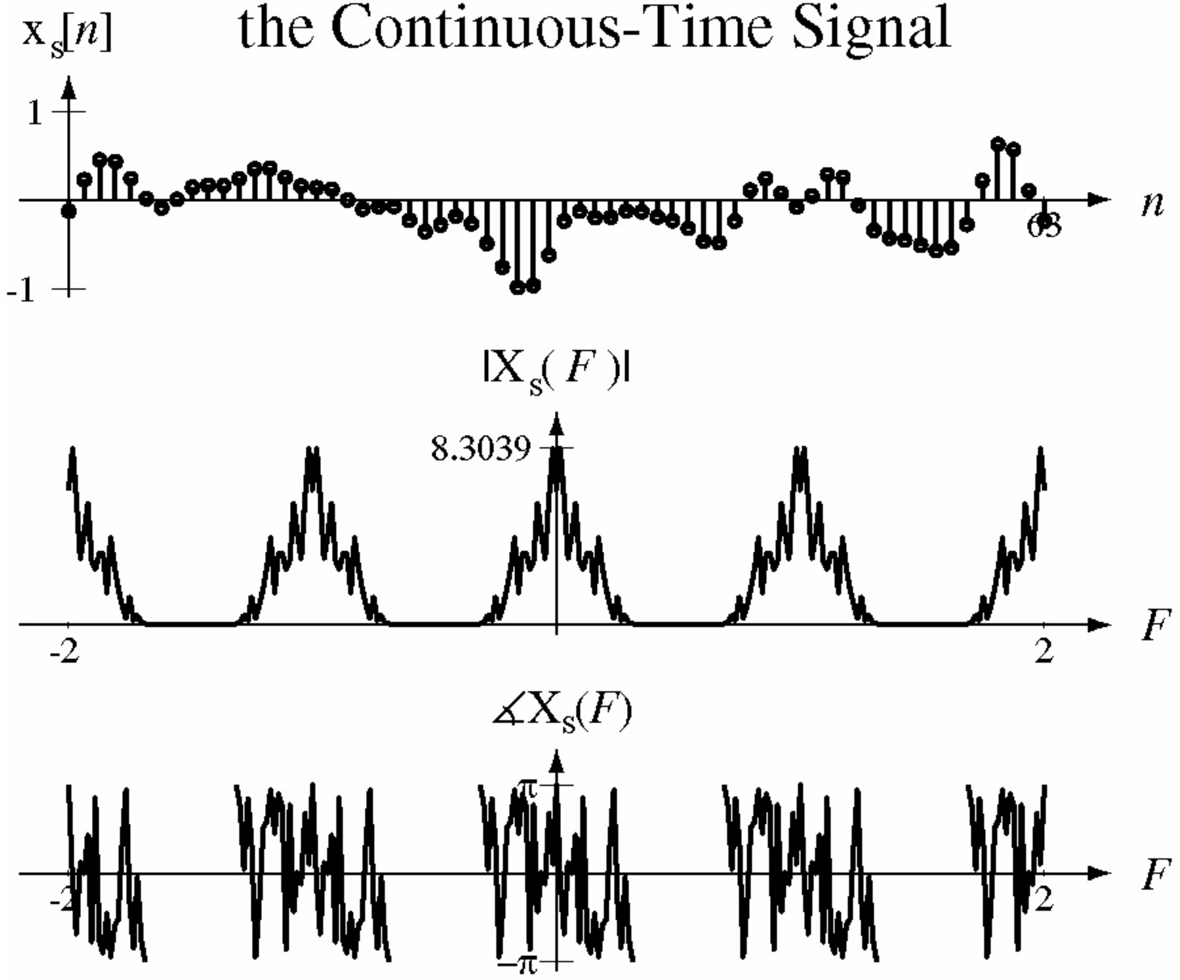
-A signal sampled at a rate less than the Nyquist rate is **undersampled**

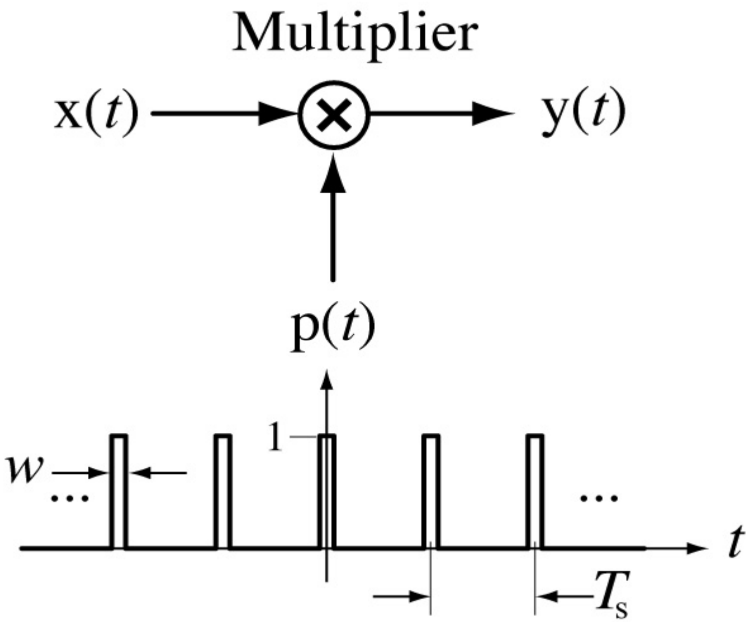
-A signal sampled at a rate greater than the Nyquist rate is **oversampled**.

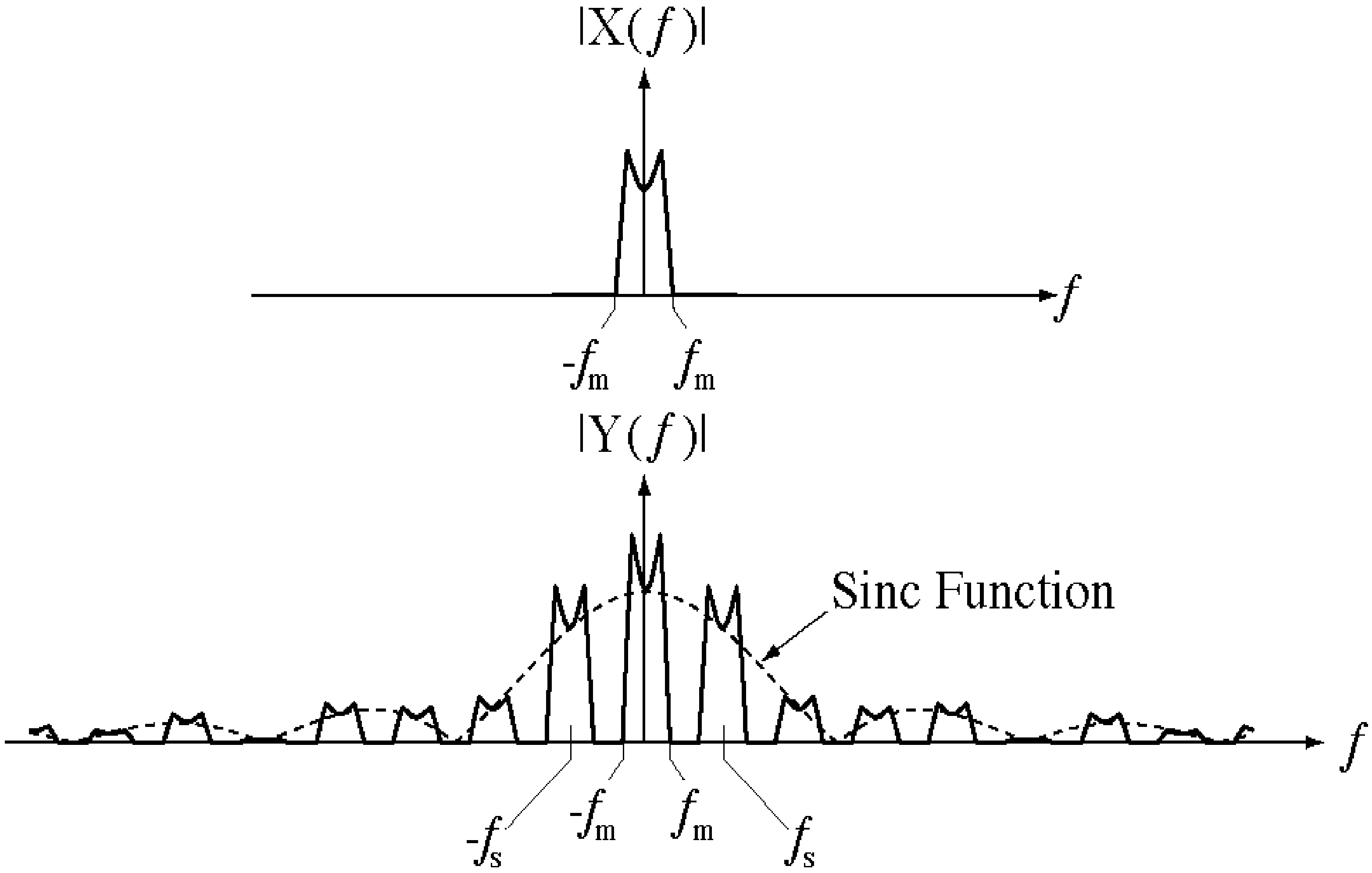
Original Signal



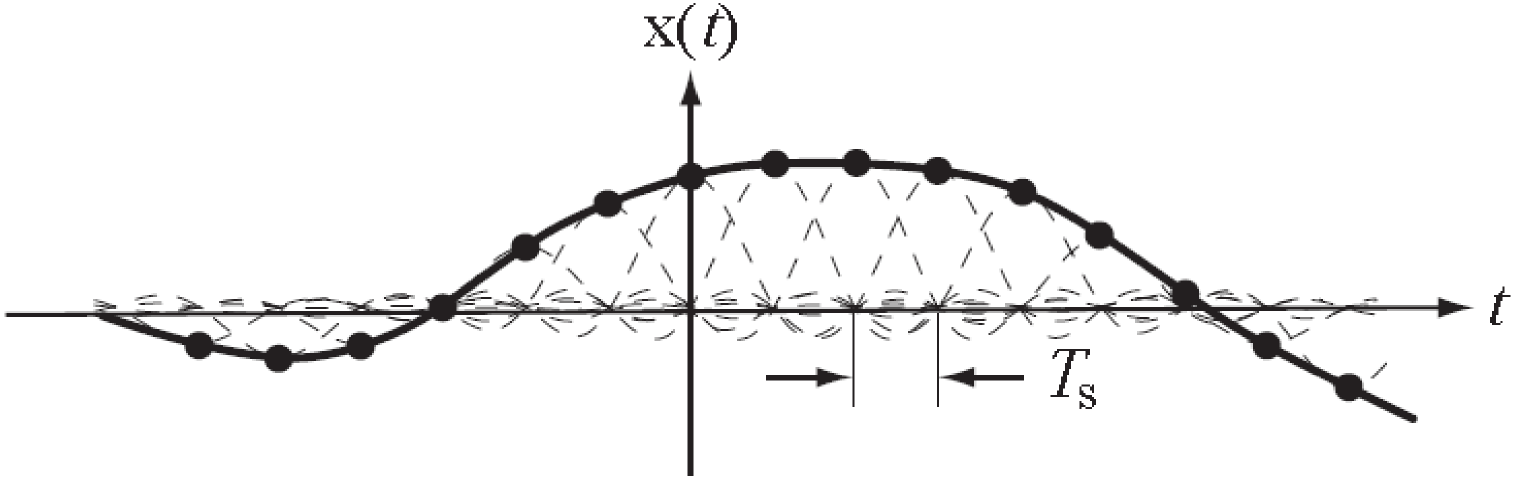
Discrete-time Signal Formed by Sampling the Continuous-Time Signal




Multiplier
x(t)
y(t)
p(t)
1
w
...
...
t
T_s


|X(f)|
f
-fm
fm
|Y(f)|
Sinc Function
f
-fs
-fm
fm
fs

Interpolation

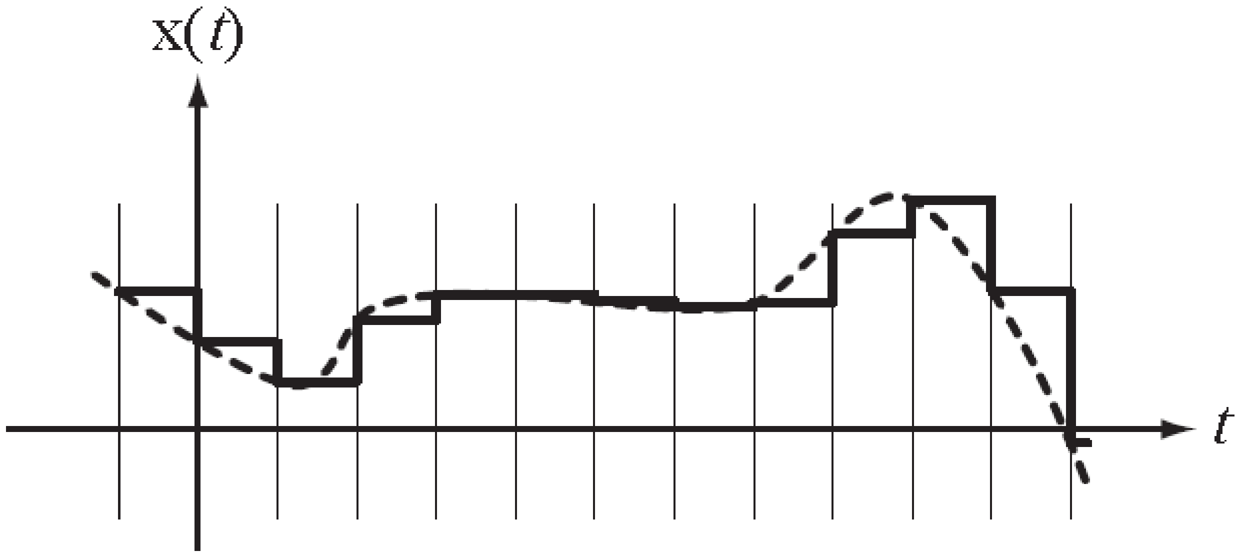


-Interpolation process for an ideal lowpass filter
-The corner frequency set to half the sampling rate

-Sinc-function interpolation is theoretically perfect but it can never be done in practice because it requires samples from the signal for all time.

-Real interpolation must make causal compromises.

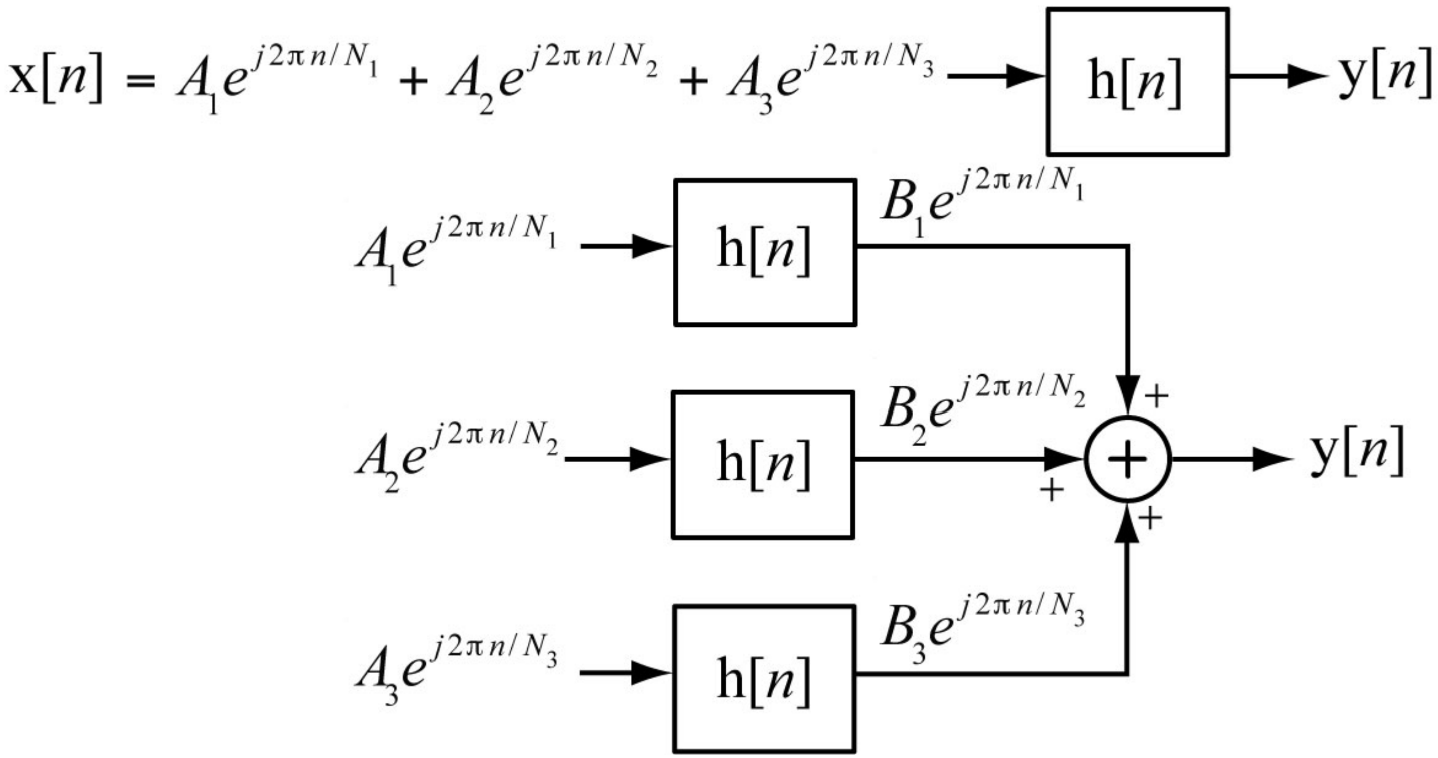
-The simplest realizable interpolation technique is what a DAC does.



9- Discrete Fourier Transform (DFT)

Intoduction

-A signal can be represented as a linear combination of harmonic complex exponentials.



Orthonormal (Orthogonal and Normalized Basis)

$$\boxed{e^{j(\frac{2\pi}{N})nk}}$$

$$0 \le n \le N-1 \qquad (t) \rightarrow [n]$$

$$0 \le k \le N-1 \qquad (f) \rightarrow [k]$$

Definition

-The most common definition of the Discrete Fourier Transform (DFT) of $x[n]$ is:

$$X[k]=\sum_{n=0}^{N-1} x[n]e^{-j(\frac{2\pi}{N})nk}$$

$$x[n]\overset{\text{DFT}}{\underset{N}{\longleftrightarrow}} X[k]$$

$x[n] \rightarrow N$ real values

$X[k] \rightarrow N$ complex values $\rightarrow 2N$ real values

-It follows that the Inverse Discrete Fourier Transform (IDFT) of $X[k]$ is:

$$x[n]=\frac{1}{N}\sum_{n=0}^{N-1}X[k]e^{j(\frac{2\pi}{N})nk} \qquad X[k]\xleftrightarrow[N]{\text{DFT}^{-1}}x[n]$$

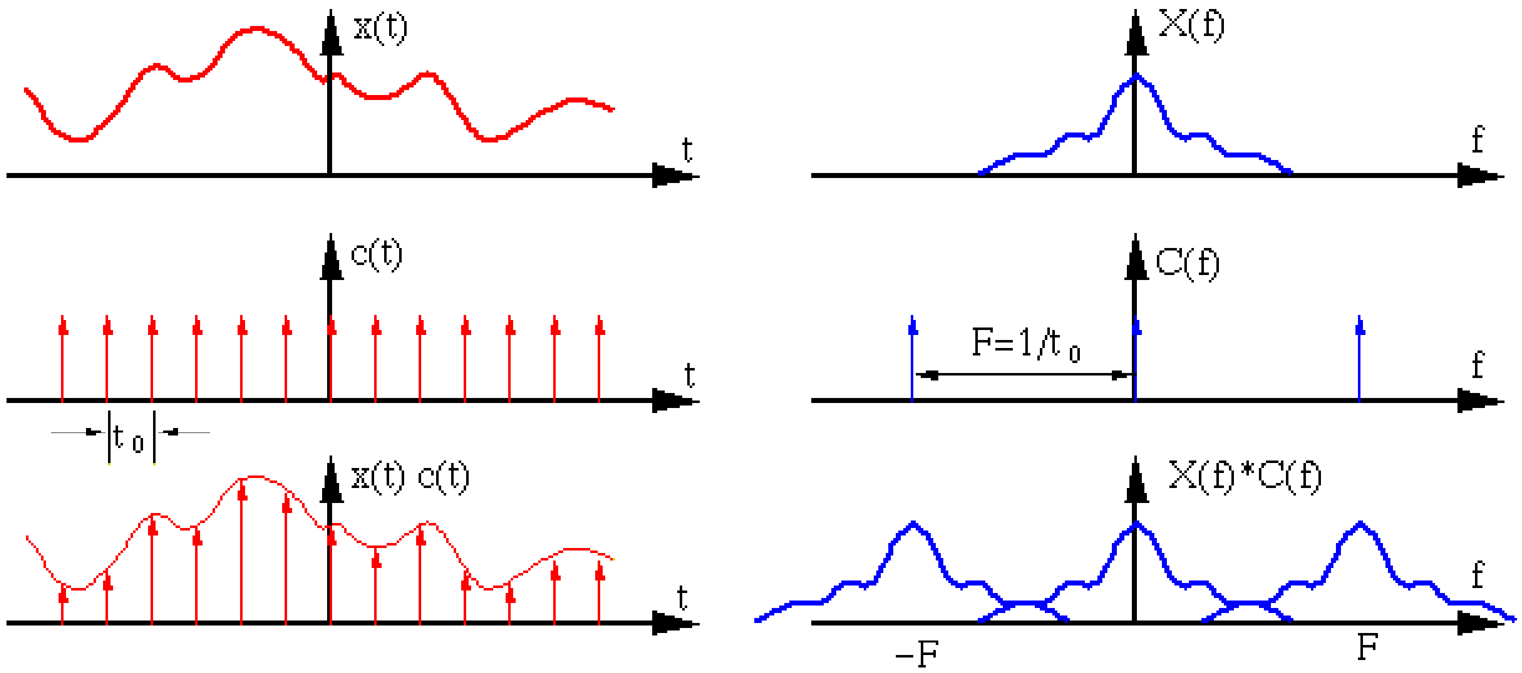
DFT Matrix

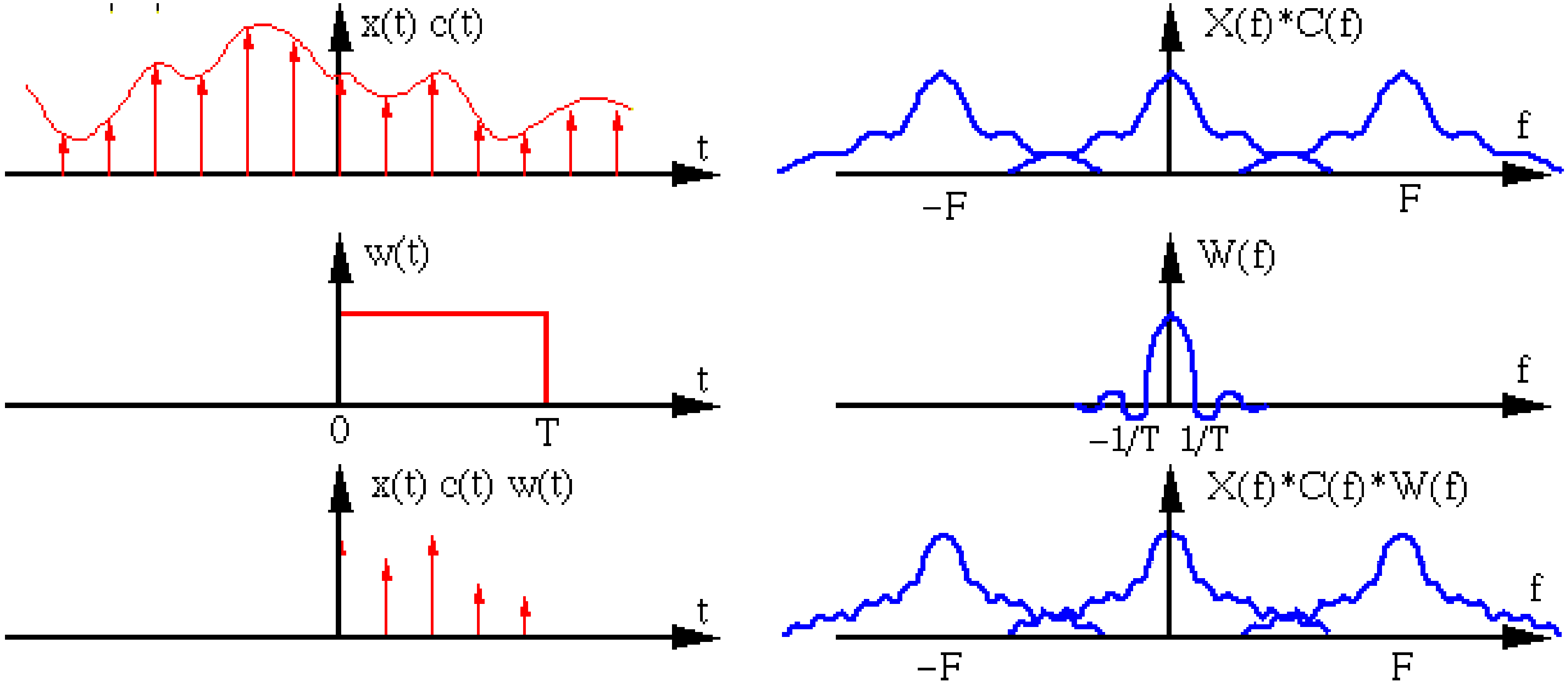
$$W_N = e^{-j(\frac{2\pi}{N})} \qquad X[k] = \sum_{n=0}^{N-1} x[n] W_N^{nk}$$

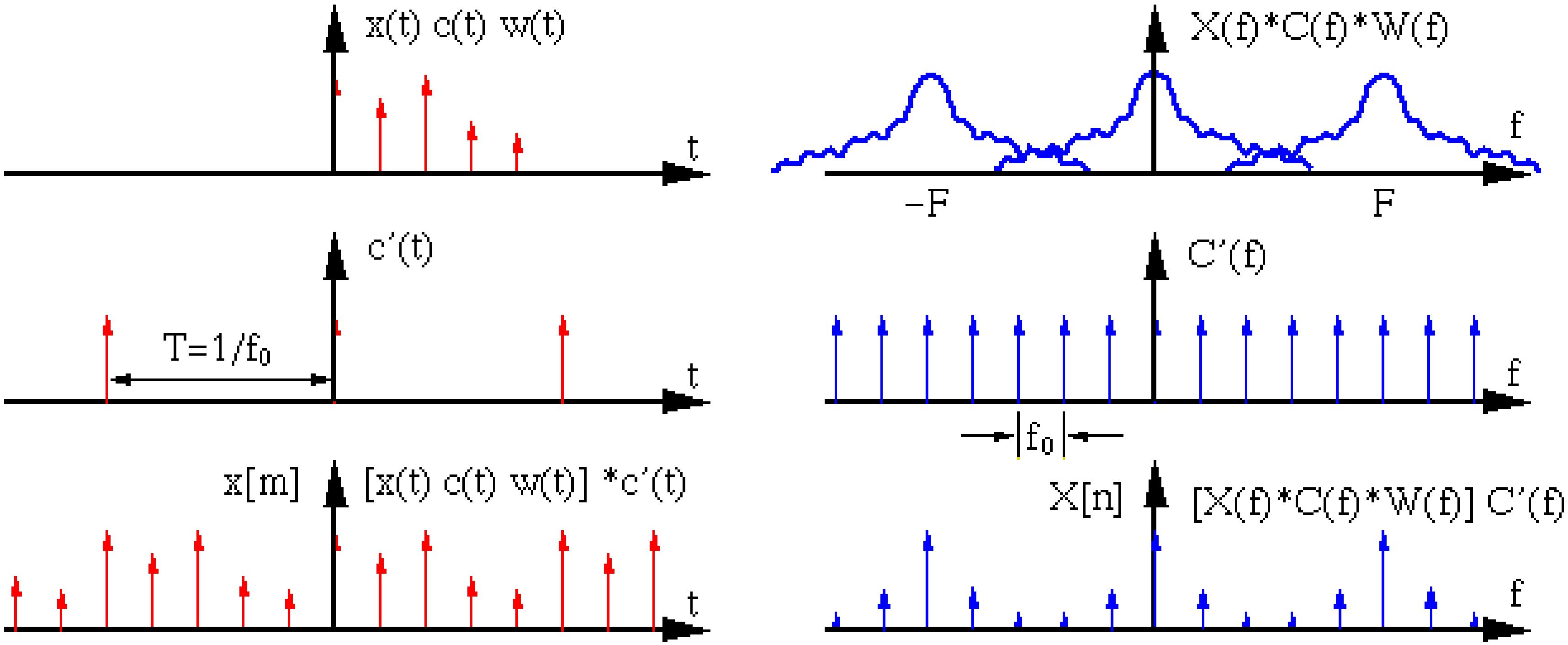
$$0 \rightarrow n \rightarrow N-1 \text{ (across columns)}$$

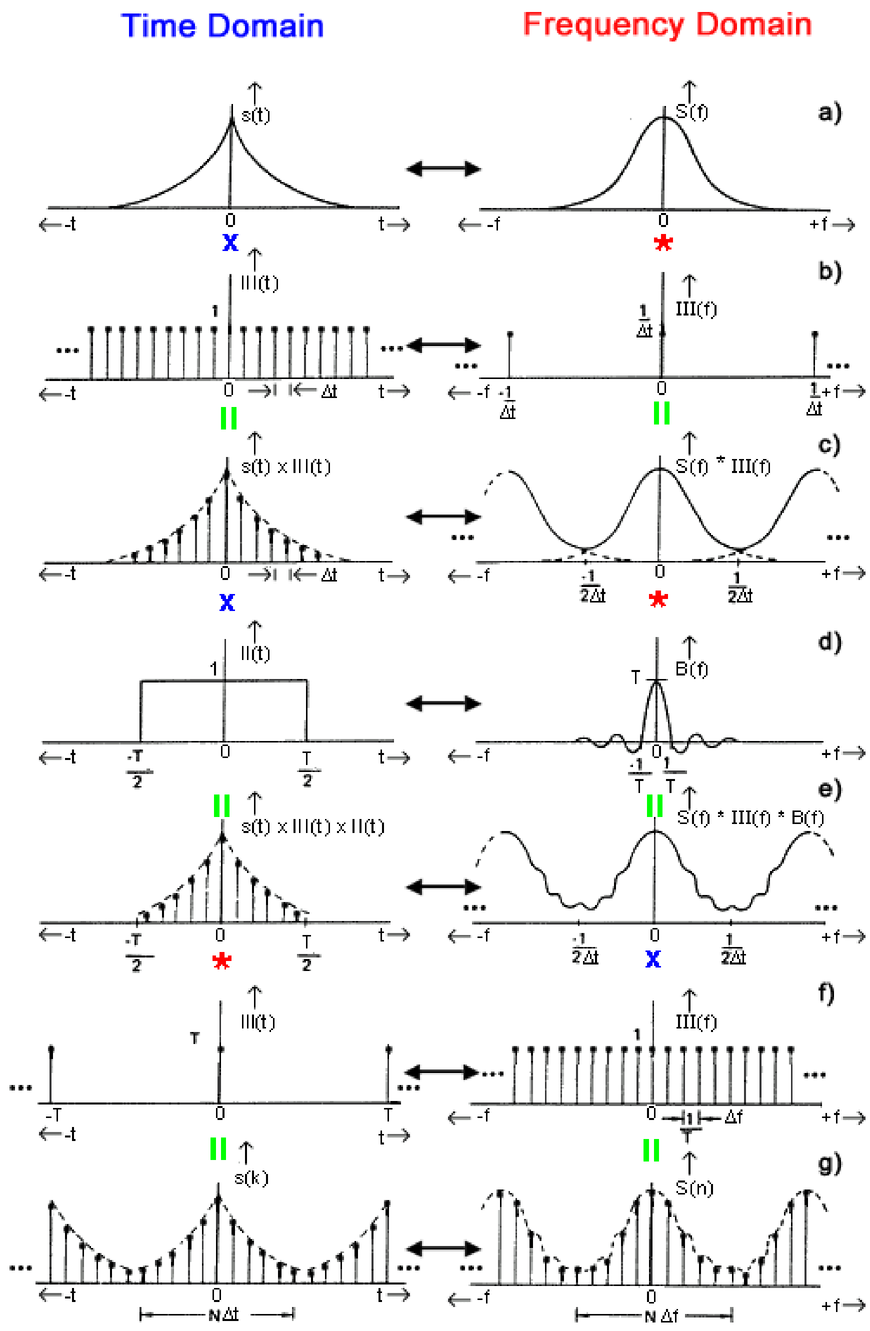
$$\begin{pmatrix} X[0] \\ X[1] \\ X[2] \\ \vdots \\ X[N-1] \end{pmatrix} = \begin{pmatrix} W_N^0 & W_N^0 & W_N^0 & \cdots & W_N^0 \\ W_N^0 & W_N^1 \cdots & W_N^2 & \cdots & W_N^{N\text{-}1} \\ W_N^0 & W_N^2 & W_N^4 & \cdots & W_N^{2(N\text{-}1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ W_N^0 & W_N^{N\text{-}1} \cdots & W_N^{2(N\text{-}1)} & \cdots & W_N^{(N\text{-}1)(N\text{-}1)} \end{pmatrix} \begin{pmatrix} x[0] \\ x[1] \\ x[2] \\ \vdots \\ x[N-1] \end{pmatrix}$$

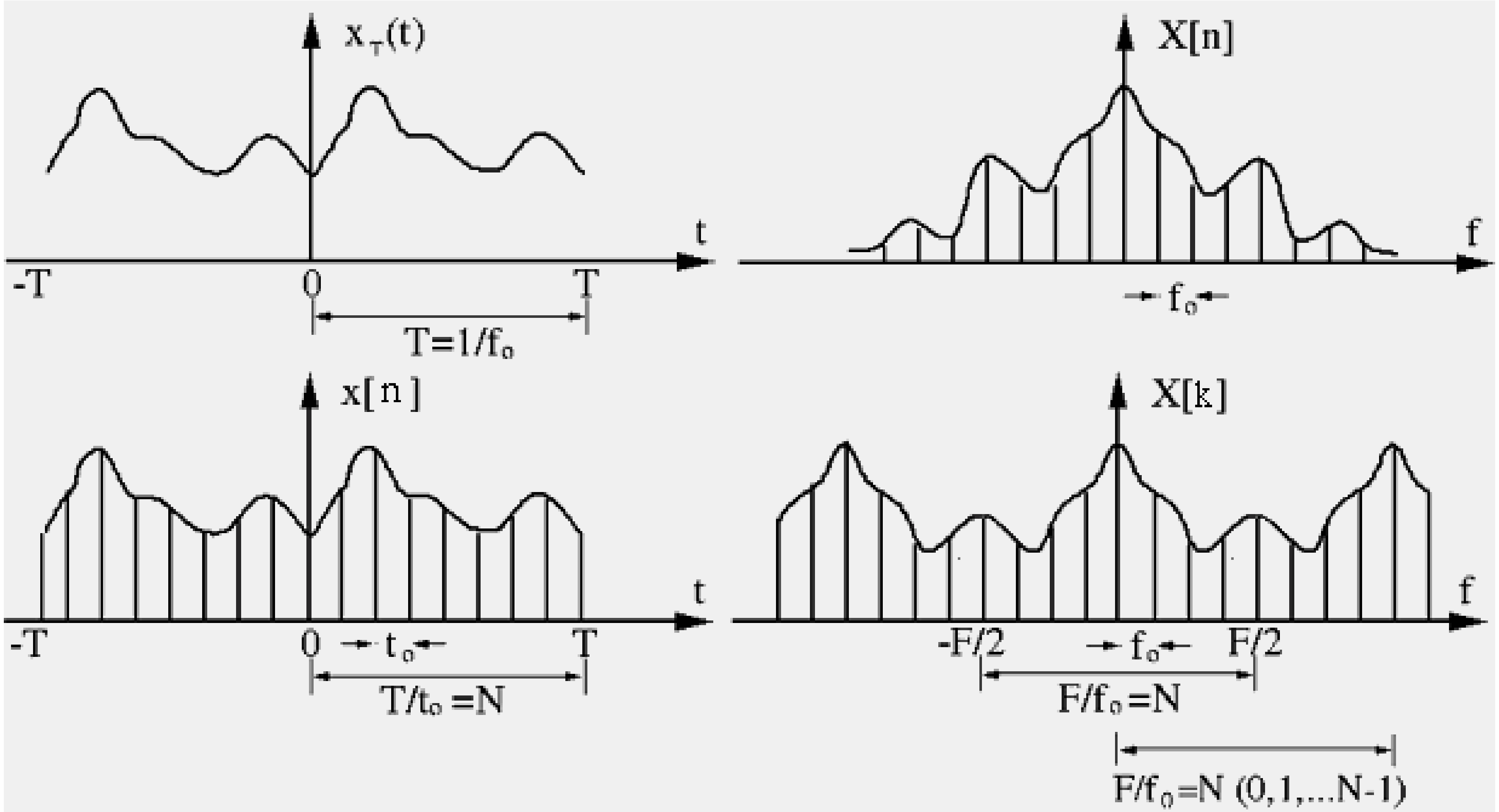
$$0 \rightarrow k \rightarrow N-1 \text{ (down rows)}$$


x(t)
t
X(f)
f
c(t)
t
C(f)
F=1/t 0
f
t 0
x(t) c(t)
t
X(f)*C(f)
f
−F
F

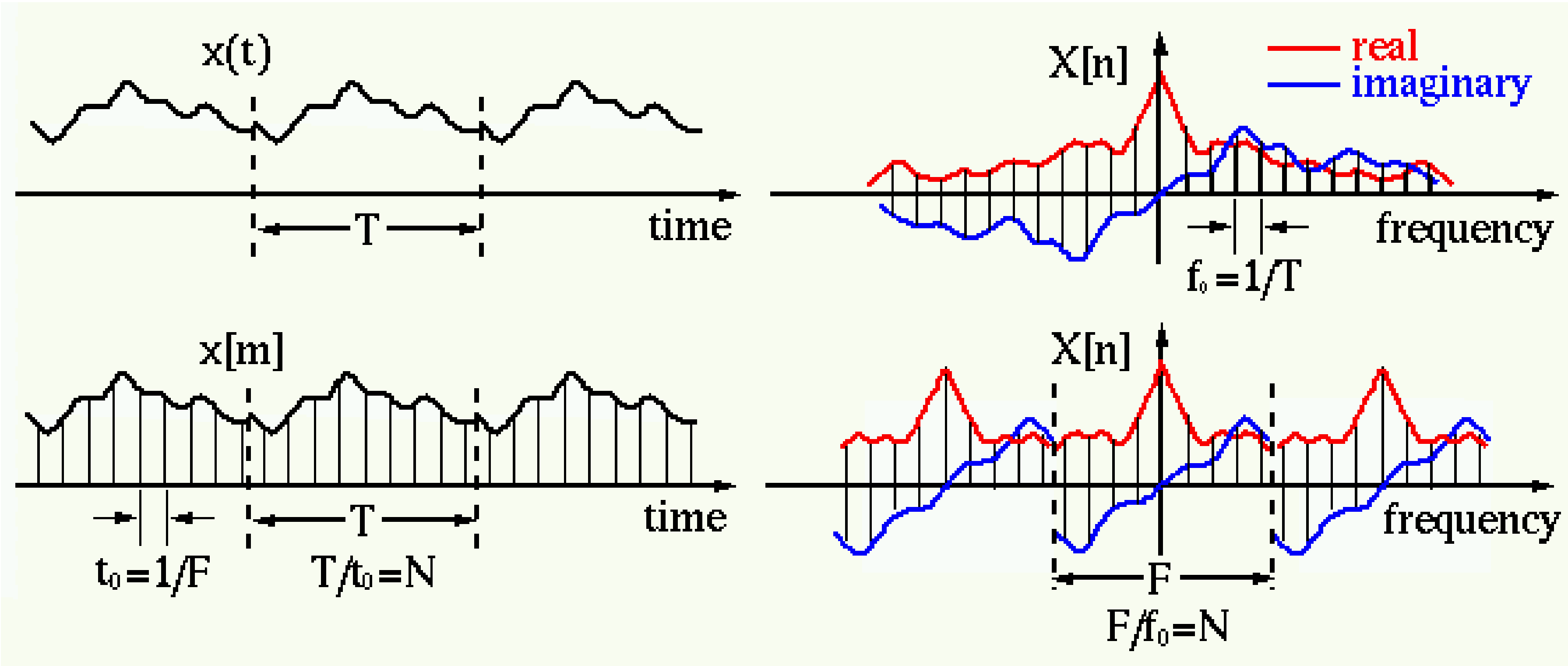

x(t) c(t)
t
X(f)*C(f)
f
−F
F
w(t)
t
0
T
W(f)
f
−1/T
1/T
x(t) c(t) w(t)
t
X(f)*C(f)*W(f)
f
−F
F


x(t) c(t) w(t)
t
X(f)*C(f)*W(f)
f
−F
F
c′(t)
T=1/f0
t
C′(f)
f
f0
x[m]
[x(t) c(t) w(t)] *c′(t)
t
X[n]
[X(f)*C(f)*W(f)] C′(f)
f

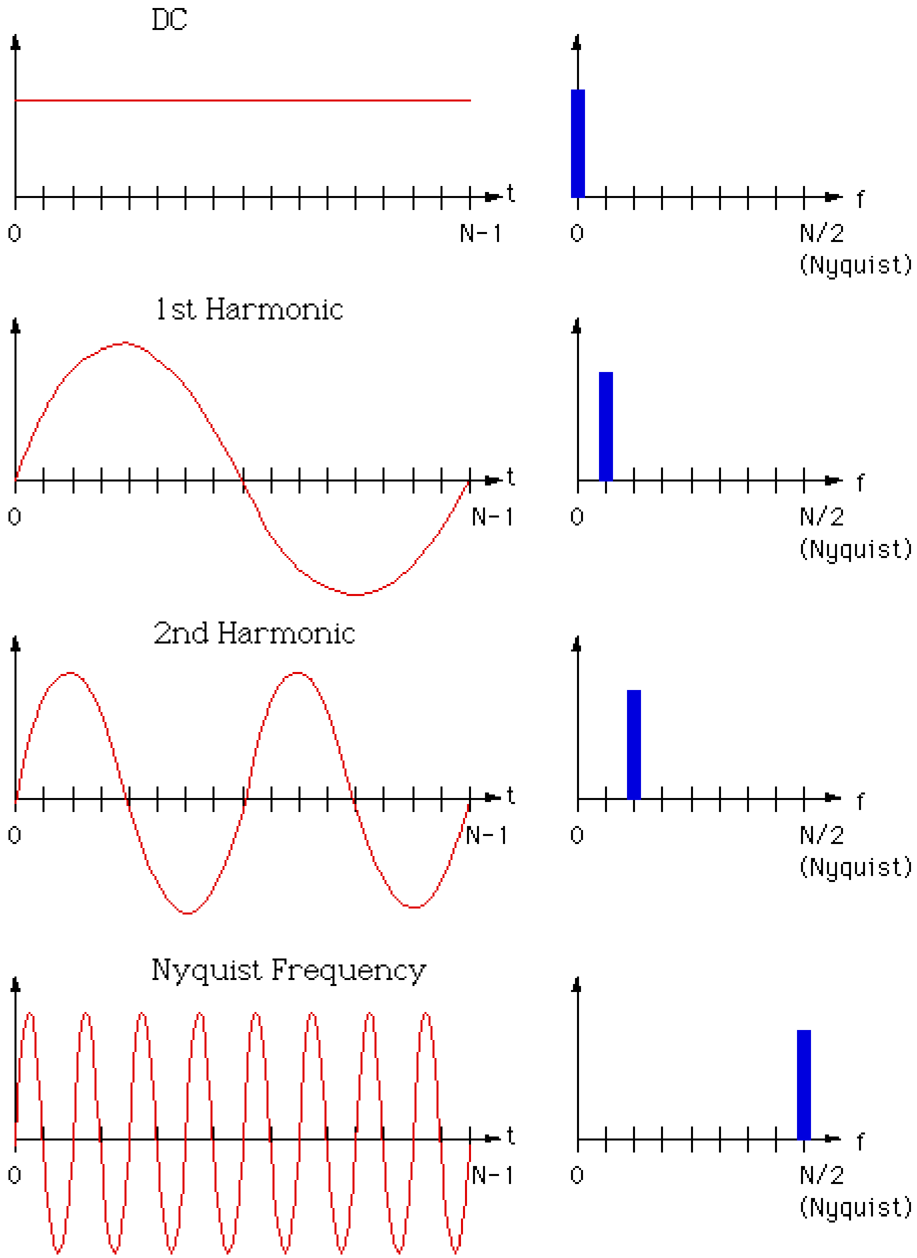

Time Domain
Frequency Domain
a)
s(t)
S(f)
b)
III(t)
III(f)
c)
s(t) x III(t)
S(f) * III(f)
d)
II(t)
B(f)
e)
s(t) x III(t) x II(t)
S(f) * III(f) * B(f)
f)
III(t)
III(f)
g)
s(k)
S(n)
N∆t
N ∆f

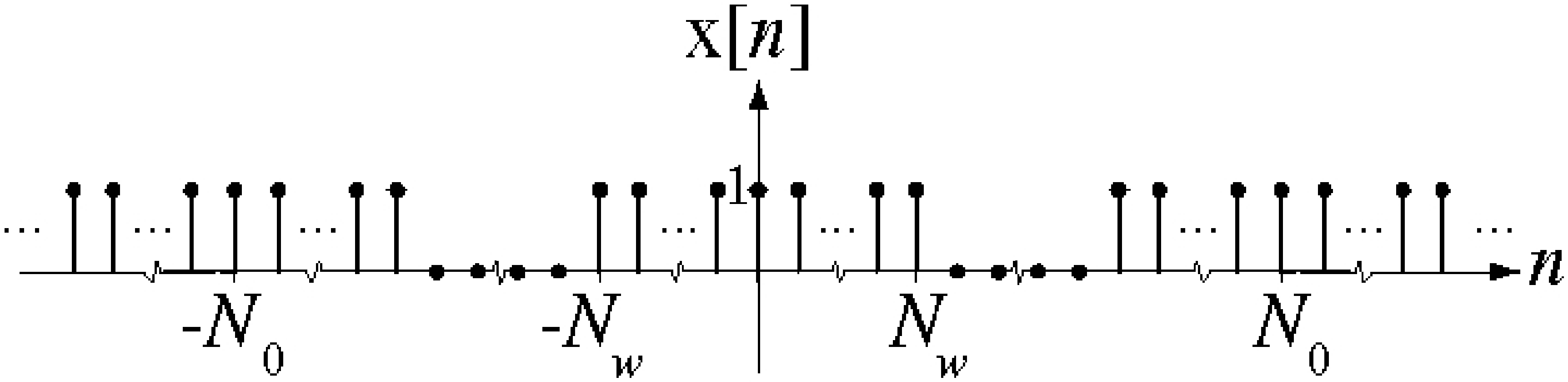


$x_T(t)$
t
-T
0
T
T=1/f₀
X[n]
f
f₀
x[n]
t
-T
0
t₀
T
T/t₀ =N
X[k]
f
-F/2
f₀
F/2
F/f₀ =N
F/f₀ =N (0,1,...N-1)

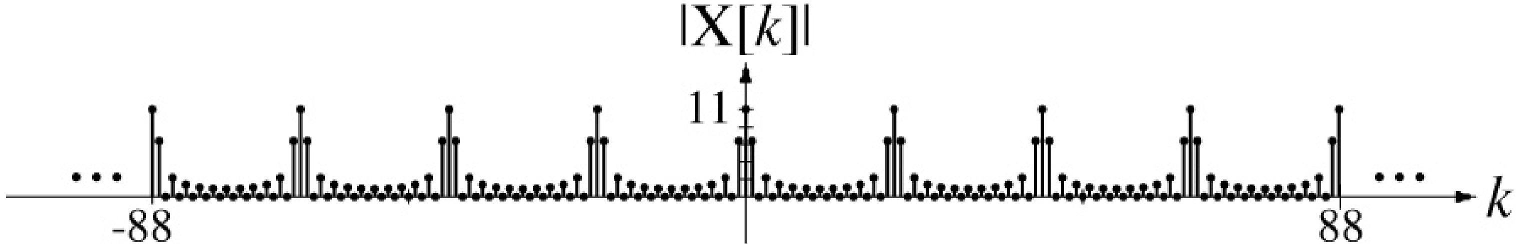


x(t)
T
time
X[n]
real
imaginary
frequency
f₀=1/T
x[m]
T
time
t₀=1/F
T/t₀=N
X[n]
frequency
F
F/f₀=N

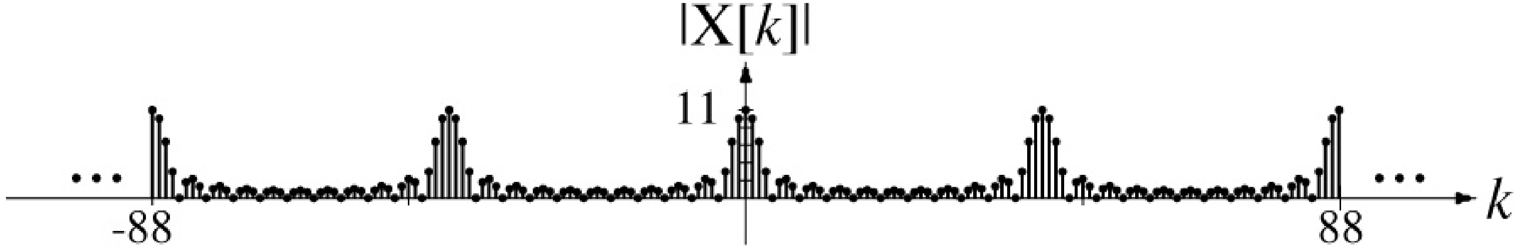

DC
t
0
N-1
f
0
N/2
(Nyquist)
1st Harmonic
t
0
N-1
f
0
N/2
(Nyquist)
2nd Harmonic
t
0
N-1
f
0
N/2
(Nyquist)
Nyquist Frequency
t
0
N-1
f
0
N/2
(Nyquist)


x[n]
1
n
-N0
-Nw
Nw
N0

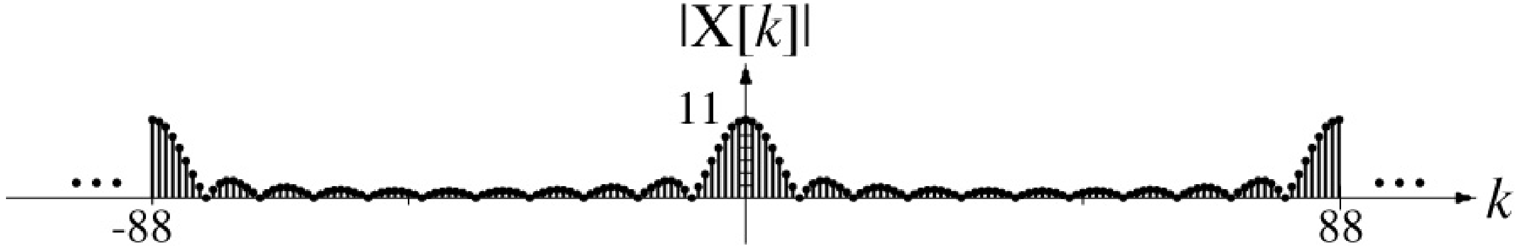
$$N_w = 5, N_0 = 22$$



$$N_w = 5, N_0 = 44$$

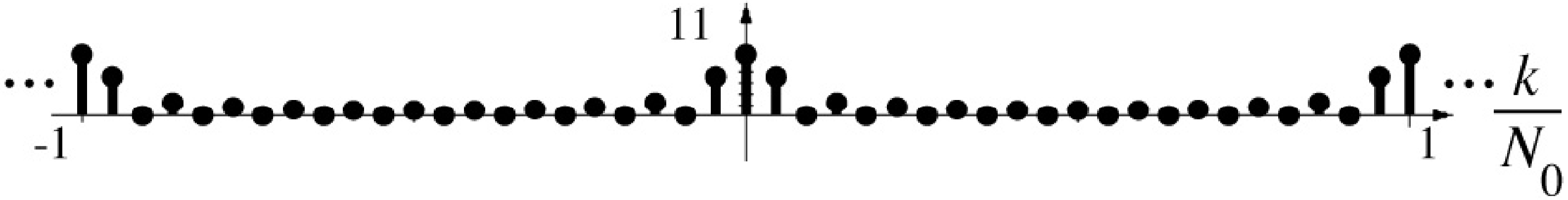


$$N_w = 5, N_0 = 88$$



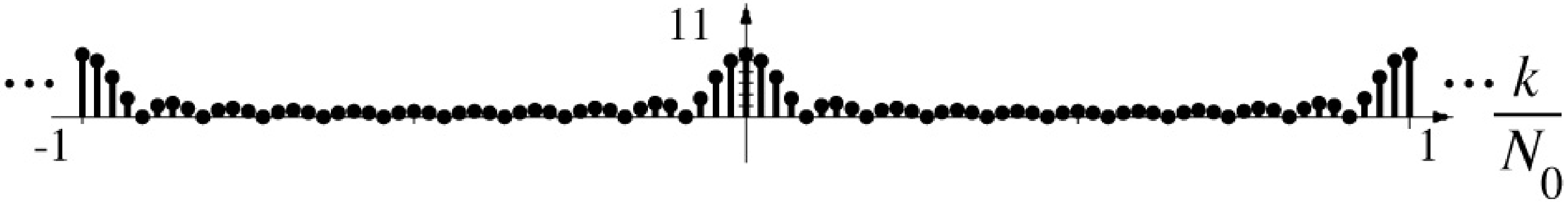
$N_w = 5, N_0 = 22$

$|X[k]|$



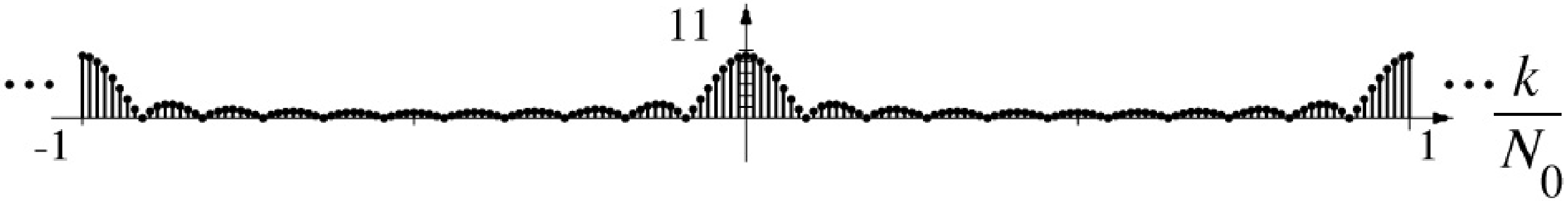
$N_w = 5, N_0 = 44$

$|X[k]|$



$N_w = 5, N_0 = 88$

$|X[k]|$



2-point DFT

DFT of vector (a,b)

$$N = 2 \;\rightarrow W_2 = e^{-j\pi} = -1$$

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a+b \\ a-b \end{pmatrix}$$

IDFT

$$\frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} a+b \\ a-b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

4-points DFT

DFT of vector (a,b,c,d)

$$N = 4 \;\rightarrow W_4 = e^{-j\frac{\pi}{2}} = -j$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} a+b+c+d \\ (a-c)-j(b-d) \\ (a+c)-(b+d) \\ (a-c)+j(b-d) \end{pmatrix}$$

IDFT

$$\frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{pmatrix}\begin{pmatrix} a+b+c+d \\ (a-c)-j(b-d) \\ (a+c)-(b+d) \\ (a-c)+j(b-d) \end{pmatrix}=\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

DFT Properties

Linearity $\alpha\,\mathrm{x}[n]+\beta\,\mathrm{y}[n] \xleftrightarrow[N]{\text{DFT}} \alpha\,\mathrm{X}[k]+\beta\,\mathrm{Y}[k]$

Time Shifting $\mathrm{x}[n-n_0] \xleftrightarrow[N]{\text{DFT}} \mathrm{X}[k]e^{-j2\pi k n_0/N}$

Frequency Shifting $\mathrm{x}[n]e^{j2\pi k_0 n/N} \xleftrightarrow[N]{\text{DFT}} \mathrm{X}[k-k_0]$

Time Reversal $\mathrm{x}[-n]=\mathrm{x}[N-n] \xleftrightarrow[N]{\text{DFT}} \mathrm{X}[-k]=\mathrm{X}[N-k]$

Conjugation $\mathrm{x}^*[n] \xleftrightarrow[N]{\text{DFT}} \mathrm{X}^*[-k]=\mathrm{X}^*[N-k]$

⋮ $\mathrm{x}^*[-n]=\mathrm{x}^*[N-n] \xleftrightarrow[N]{\text{DFT}} \mathrm{X}^*[k]$

Time Scaling

$$\mathrm{z}[n]=\begin{cases}\mathrm{x}[n/m] & , \ n/m \text{ an integer} \\ 0 & , \ \text{otherwise}\end{cases}$$

⋮ $N \rightarrow mN \ , \ \mathrm{Z}[k]=(1/m)\mathrm{X}[k]$

Change of Period $N \to qN,\ q$ a positive integer

$$X_q[k] = \begin{cases} X[k/q] & , \ k/q \text{ an integer} \\ 0 & , \ \text{otherwise} \end{cases}$$

Multiplication - Convolution Duality

$$x[n]y[n] \xleftrightarrow[N]{\text{DFT}} (1/N)\, Y[k] \circledast X[k]$$

$$x[n] \circledast y[n] \xleftrightarrow[N]{\text{DFT}} Y[k]X[k]$$

$$\text{where } x[n] \circledast y[n] = \sum_{m=\langle N \rangle} x[m]y[n-m]$$

Parseval's Theorem

$$\sum_{n=\langle N \rangle} \left|x[n]\right|^2 = \frac{1}{N} \sum_{k=\langle N \rangle} \left|X[k]\right|^2$$

DFT Pairs

$$e^{j2\pi n/N} \xleftrightarrow[mN]{\text{DFT}} mN\delta_{mN}[k-m]$$

$$\cos(2\pi qn/N) \xleftrightarrow[mN]{\text{DFT}} (mN/2)\left(\delta_{mN}[k-mq]+\delta_{mN}[k+mq]\right)$$

$$\sin(2\pi qn/N) \xleftrightarrow[mN]{\text{DFT}} (jmN/2)\left(\delta_{mN}[k+mq]-\delta_{mN}[k-mq]\right)$$

$$\delta_N[n] \xleftrightarrow[mN]{\text{DFT}} m\delta_{mN}[k]$$

$$1 \xleftrightarrow[N]{\text{DFT}} N\delta_N[k]$$

$$\left(u[n-n_0]-u[n-n_1]\right)*\delta_N[n] \xleftrightarrow[N]{\text{DFT}} \frac{e^{-j\pi k(n_1+n_0)/N}}{e^{-j\pi k/N}}(n_1-n_0)\,drcl(k/N,n_1-n_0)$$

$$tri(n/N_w)*\delta_N[n] \xleftrightarrow[N]{\text{DFT}} N_w drcl^2(k/N,N_w) \quad , \quad N_w \text{ an integer}$$

$$\operatorname{sinc}(n/w)*\delta_N[n] \xleftrightarrow[N]{\text{DFT}} w\,rect(wk/N)*\delta_N[k]$$

Fast Fourier Transform (FFT)

-The DFT requires N^2 complex multiplies and $N(N-1)$ complex additions.

-Algorithms that exploit computational savings are collectively called *Fast Fourier Transforms*.

-They take advantage of the symmetry and periodicity of the complex exponential.

$$W_N = e^{-j\left(\frac{2\pi}{N}\right)} \qquad X[k] = \sum_{n=0}^{N-1} x[n] W_N^{nk}$$

Symmetry $\rightarrow \quad W_N^{[N-n]k} = W_N^{-nk} = (W_N^{nk})^*$

Periodicity $\rightarrow \quad W_N^{nk} = W_N^{[n+N]k} = W_N^{n[k+N]}$

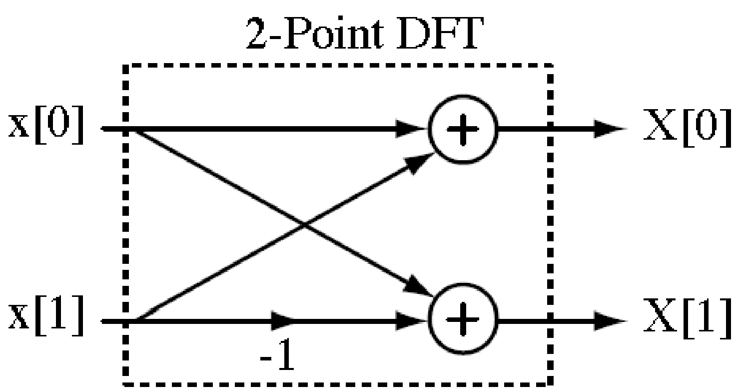
N length DFT $\rightarrow N^2$ multiplications

$\mathbf{2}\left(\frac{\mathbf{N}}{\mathbf{2}}\right)$ lengths DFT $\rightarrow 2\left(\frac{N}{2}\right)^2 = \frac{N^2}{2}$ multiplications

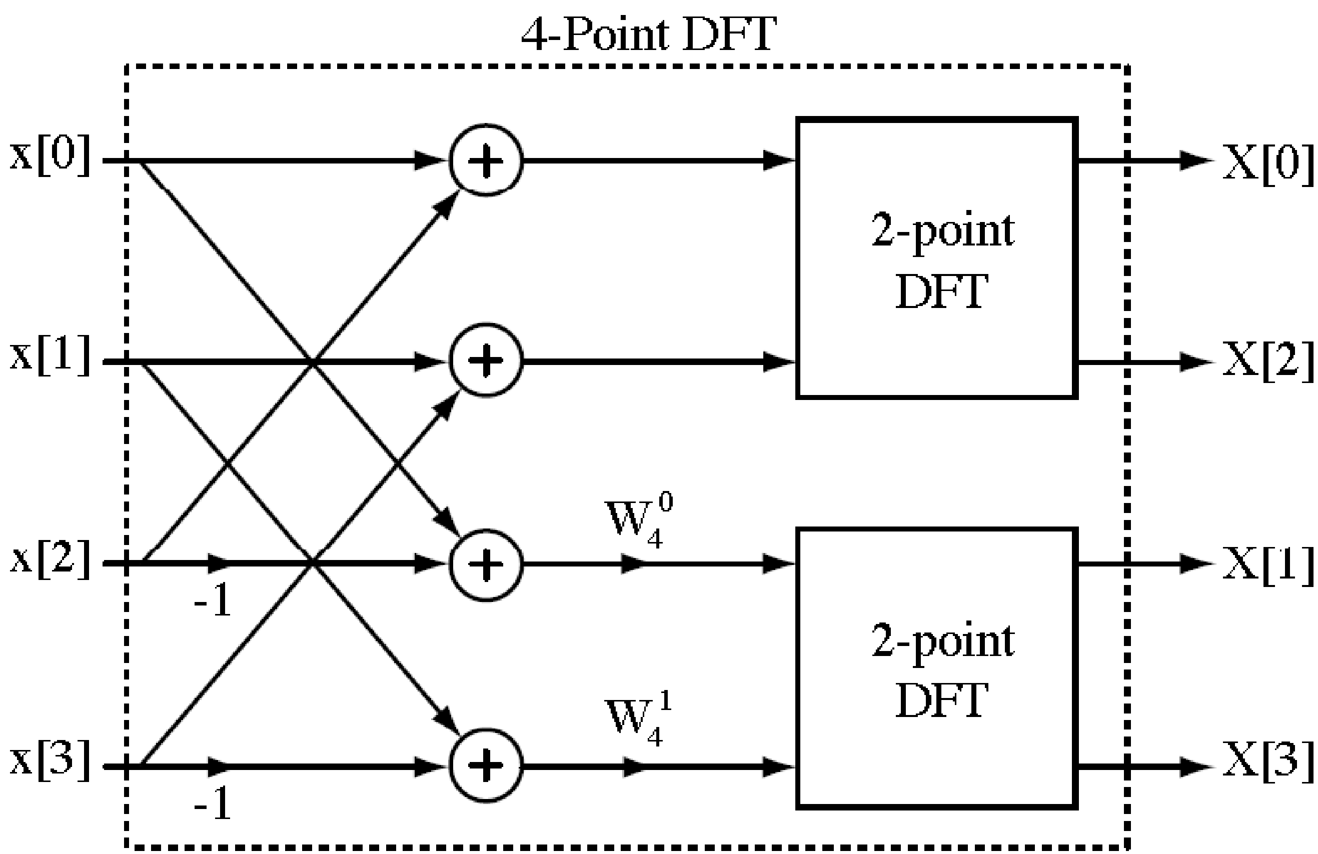
$$\boxed{N=2} \rightarrow W_2 = e^{-j\pi} = -1$$

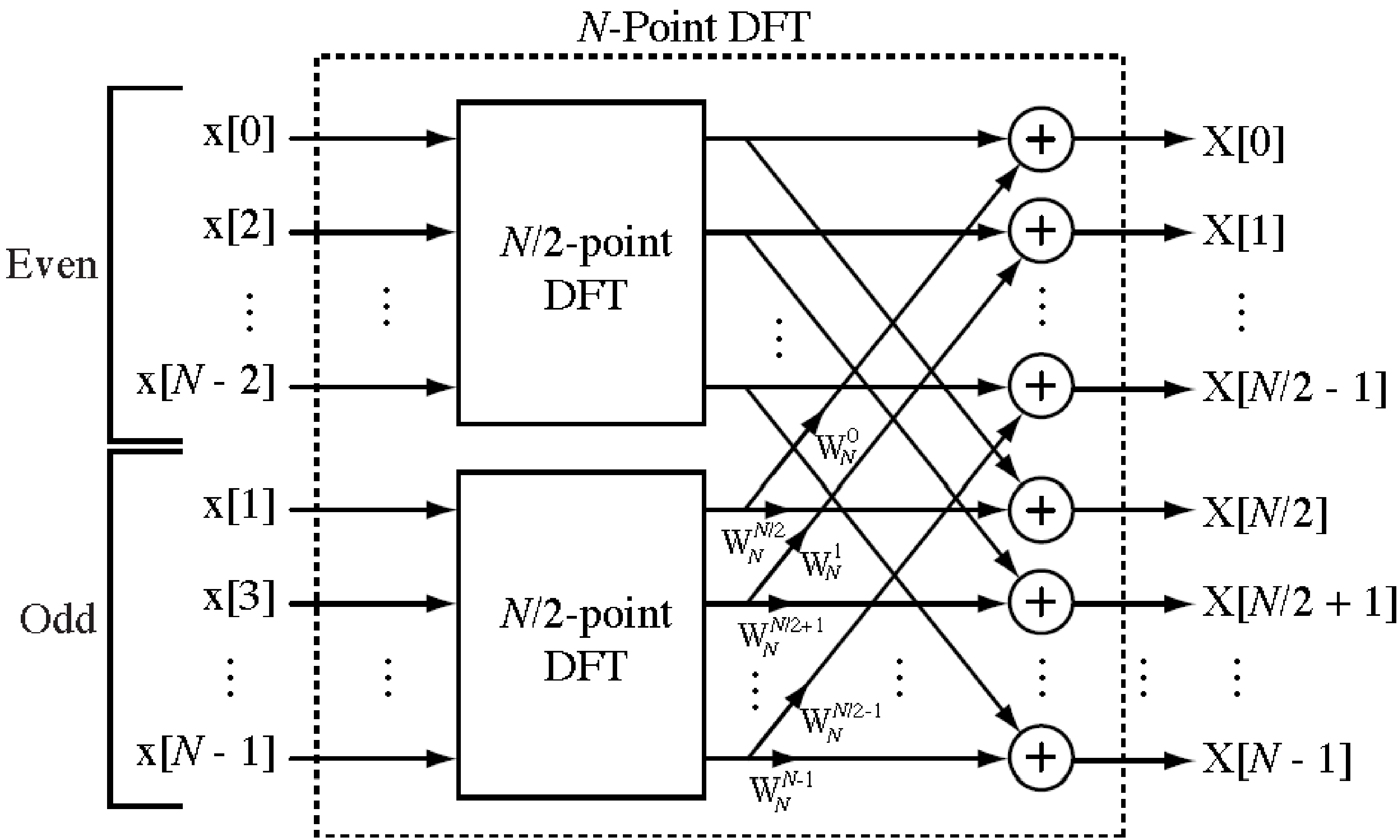
$$\mathrm{X}\left[k\right] = \sum_{n_0=0}^{1} \mathrm{x}\left[n\right] W_2^{nk}$$

$$\begin{bmatrix} \mathrm{X}\left[0\right] \\ \mathrm{X}\left[1\right] \end{bmatrix} = \begin{bmatrix} W_2^0 & W_2^0 \\ W_2^0 & W_2^1 \end{bmatrix} \begin{bmatrix} \mathrm{x}\left[0\right] \\ \mathrm{x}\left[1\right] \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \mathrm{x}\left[0\right] \\ \mathrm{x}\left[1\right] \end{bmatrix}$$



A 2-point DFT


4-Point DFT
x[0]
x[1]
x[2]
x[3]
-1
-1
W_4^0
W_4^1
2-point DFT
2-point DFT
X[0]
X[2]
X[1]
X[3]



An *N-point DFT computed using two N/2-point DFT's*

Textbook

-M.J. Roberts, "Signals and Systems Analysis Using Transform Methods and MATLAB® ", 2nd Ed, McGraw-Hill (2012).

(Content and Figures are mainly from Textbook)