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SHEAR STRESS DISTRIBUTION ON BEAM CROSS SECTIONS UNDER SHEAR LOADING

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Abstract. *In this work, some results obtained by Trabucho and Viaño for the shear stress distribution in beam cross sections using asymptotic expansions of the three-dimensional elasticity equations are compared with those calculated by the classical formulae of the Strength of Materials. We use beams with rectangular and circular cross section to compare the degree of accuracy reached by each method.*

1 A classical example of the Strength of Materials.

It is considered a straight beam with constant, solid and simply connected section Ω and length L. It is assumed a system of coordinates $Ox_1x_2x_3$ such that Ox_3 pass through the gravity center of cross sections and the origin O is the gravity center of one end so the axes Ox_1 and Ox_2 are the principal inertia axes of the section Ω (figure 1). To simplify, it is supposed that the section is symmetrical with respect to the plane $x_1 = 0$. On the cross section $x_3 = L$ are applied loads of density $p_i(x_1, x_2, L)$ $(i = 1, 2, 3)$, whose resultant is denoted by $(0, P_2, 0)$, i.e., only the component in the direction Ox_2 does not vanish.

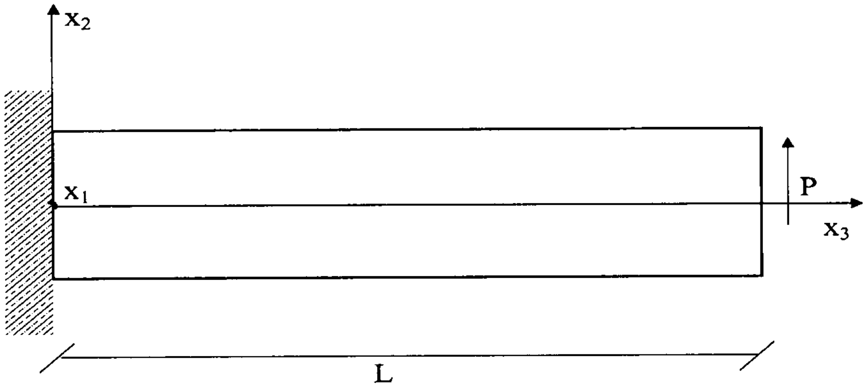

Figure 1:

From geometrical properties of a beam, it is known that the order of magnitude of L is greater than the diameter of Ω, which will be denoted by d, and following the Saint Venant's principle, it can be admitted that the stress distribution in the sections $x_3 \in (0, L - kd)$, where k is a little integer number, is independent to the distribution of the forces $p_i(x_1, x_2, L)$, and only depending on its resultant force P_2 at section $x_3 = L$.
In such a beam, the shear loading are constant through the axes $x_3 \in (0, L)$, and they are $Q_1 = 0$ and $Q_2 = P_2$ respectively. Assuming the material of the beam is elastic homogeneous and isotropic and the validity of the Navier-Bernoulli hypothesis, which states that the normal, plane and straight sections of the initial beam, continue to be normal and plane sections in the deformated beam, then it can be obtained the well-known formula of the shear stress distribution on a section due to the shear force Q_2 (Samartín [4]):

$$\sigma_{32}^R(x_1, x_2) = \sigma_{32}^R(x_2) = \frac{M_{est}(x_2)}{b(x_2)I_2} Q_2 \tag{1}$$

where $M_{est}(x_2)$ is the static moment of the section

$$\Omega(x_2) = \{(\xi_1, \xi_2) \in \Omega : \xi_2 \geq x_2\}, \tag{2}$$

defined by:

$$M_{est}(x_2) = \int_{\Omega(x_2)} \xi_2 d\xi_1 d\xi_2, \tag{3}$$

$b(x_2)$ is the length (width) of the lower boundary of Ω or, equivalently, the length of the segment $\{(\xi_1, \xi_2) \in \Omega : \xi_2 = x_2\}$ (figure 2), and I_2 is the inertia moment of Ω with respect to the axesOx_1:

$$I_2 = \int_{\Omega} x_2^2 dx_1 dx_2. \tag{4}$$

It should be pointed out that the expression (1) is obtained by applying exclusively the equilibrium equations in the cross section ("slice"). By other hand, it can be observed that the shear stress $\sigma_{32}^R(x_1, x_2)$ is independent of the x_1 coordinate, since (1) only express the mean value of the shear stresses through lines of the section parallel to the Ox_1 axes (figure 3). Therefore, these formulae give suitable results for slender beams, i.e., with small values $b(x_2)$. Moreover, these results deteriorate at line level in large width sections cases. Finally, it is observed that the formulae (1) only gives the Ox_2 component of the shear stress σ_{32}, but it is not possible to deduce from it the Ox_1 components. Usually, these components are approximated by some equilibrium considerations at the ends of the segment $\{(\xi_1, \xi_2) \in \Omega : \xi_2 = x_2\}$. For example, by supposing the lateral surface free of stress in these end points A and B (figure 4), the normal component is zero. This condition allows to obtain the components σ_{31} in these points and then these values can be used to estimate the values of σ_{31} in the intermediate points by a simple interpolation.

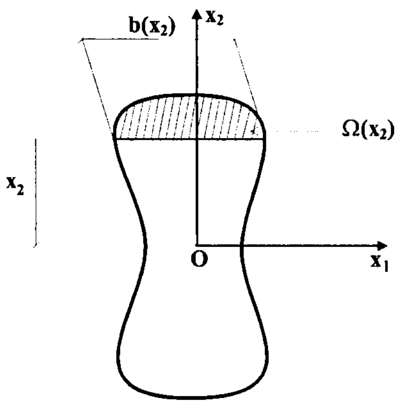

Figure 2:

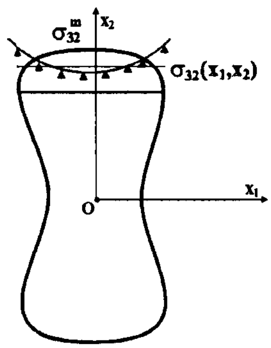

Figure 3:

Finally, it seems to be relevant a remark about the generalization of formula (1) to the case where the coordinates axes Ox_1 and Ox_2 are not coincident with the principal axes of inertia of the cross section, but the origin O remains to be its gravity center. Let $\Omega(C)$ be the subset of points (ξ_1, ξ_2) of Ω supported by the curve C (figure 5). By supposing that the curve C is determined

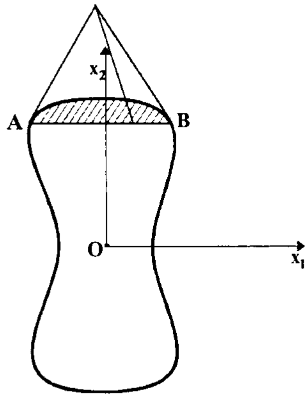

Figure 4:

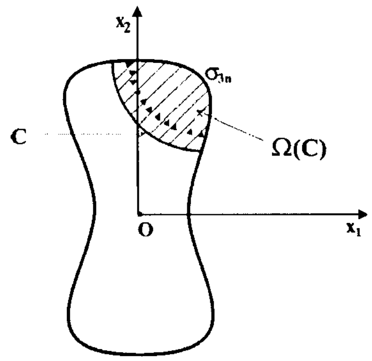

Figure 5:

by its parametric equations depending on its arc length $x_\alpha = x_\alpha(s),\ s_1 \leq s \leq s_2,\ \ \alpha = 1, 2,$ then:

$$\Omega(C) \equiv \Omega \cap \{(\xi_1, \xi_2) \in \mathbb{R}^2 / \xi_1 \geq x_1(s), \xi_2 \geq x_2(s)\}. \tag{5}$$

So it is obtained the following generalization of the formula (1), which represents the mean value σ_{3n}^m of the shear stress σ_{3n} through C, contained in the cross section Ω and directed in the normal direction to the curve C:

$$\sigma_{3n}^m = [-I_{12} M_1^{est}(C) + I_2 M_2^{est}(C)] \frac{Q_2}{(I_1 I_2 - I_{12}^2) b(C)}, \tag{6}$$

where $b(C)$ is the length of the curve C and $M_\alpha^{est}(C)$ $(\alpha = 1, 2)$ is the static moment of the section $\Omega(C)$, i.e.:

$$b(C) = \oint_C ds, \qquad M_\alpha^{est}(C) = \int_{\Omega(C)} \xi_\alpha d\xi_1 d\xi_2. \tag{7}$$

and the following notation is introduced:

$$I_\alpha = \int_\Omega x_\alpha^2 dx_1 dx_2, \qquad I_{12} = \int_\Omega x_1 x_2 dx_1 dx_2. \tag{8}$$

By definition it can be written:

$$\sigma_{3n}^m = \frac{1}{b(C)} \oint_C \sigma_{3n}(x_1, x_2) ds, \tag{9}$$

and thus, it represents the mean shear stress through the lower boundary of $\Omega(C)$. In the extreme points of the curve C, as before, it is possible to consider the equilibrium equations which express that the lateral surface of the beam is free of stresses, and consequently, to deduce the values in these points of the stress components σ_{3t}, parallel to the tangent to the curve C.

2 The beam as a three-dimensional elastic solid.

The prismatic elastic beam defined in the introduction, can be calculated using the general theory of the elasticity, without resort "a priori" additional hypothesis to simplify those of the three-dimensional elasticity. The following notation is introduced: V the beam domain, i.e., $V = \Omega \times (0, L)$, Γ_0 and Γ_L the extreme cross sections corresponding to $x_3 = 0$ and $x_3 = L$, and Γ the lateral section of beam defined by the expression $\gamma \times (0, L)$ with γ the boundary of Ω. Let (n_1, n_2) be the normal unit vector to γ and outward to Ω. Then $(n_1, n_2, 0)$ is the normal unit vector to the lateral surface Γ and outward to V.
The repeated index convention for the sum will be used. Moreover, the Greek index $\alpha, \beta, \gamma, \ldots$ will belong to the set $\{1, 2\}$. The partial derivatives with respect to the variable x_i of a function ϕ is denoted by $\phi_{,i}$. For functions z only depending on the variable x_3, the derivatives will be denoted by $z', z'', \ldots$
A general shape for the cross section of the beam is considered (may be with cross section not symmetrical), which is clamped in Γ_0. In its extreme Γ_L, surface forces of density $p_i = p_i(x_1, x_2, L)$ are given. Furthermore, for a most general case, it will suppose a body force of density $f_i = f_i(x_1, x_2, x_3)$ acting on V and a surface force of density $g_i = g_i(x_1, x_2, x_3)$, $(x_1, x_2) \in \gamma$ acting on the lateral surface Γ .
Then the equations which solve the 3D linear elasticity problem on V, corresponding to the cantilever beam, can be written as follows:

$$\begin{aligned} -\sigma_{ij,j} &= f_i \quad in\ V, \\ \sigma_{i3} &= p_i \quad on\ \Gamma_L, \\ \sigma_{i\beta} n_\beta &= g_i \quad on\ \Gamma, \\ u_i &= 0 \quad on\ \Gamma_{0,} \end{aligned} \tag{10}$$

where u_i are the displacements in the Ox_i direction and σ_{ij} the components of the stress tensor. The beam is supposed to be composed by an homogeneous and isotropic material with elasticity modulus E and Poisson's ratio ν. The Hooke's law states that:

$$\sigma_{ij} = \frac{E}{2(1+\nu)}(u_{i,j} + u_{j,i}) + \frac{E\nu}{(1+\nu)(1-2\nu)}(u_{kk})\delta_i^j \tag{11}$$

where δ_i^j is the Kronecker's symbol.
The resultant actions, corresponding to the concentrated loads in the free end of beam, P_i and M_i^L, and the loads through the line L of intensities F_i and M_i depending on x_3:

$$\begin{aligned} P_i &= \int_\Omega p_i dx_1 dx_2, \\ M_\alpha^L &= \int_\Omega x_\alpha p_3 dx_1 dx_2, \\ M_3^L &= \int_\Omega (x_2 p_1 - x_1 p_2) dx_1 dx_2. \end{aligned} \tag{12}$$

$$
\begin{aligned}
F_i &= \int_\Omega f_i dx_1 dx_2 + \int_\gamma g_i d\gamma, \\
M_\alpha &= \int_\Omega x_\alpha f_3 dx_1 dx_2 + \int_\gamma x_\alpha g_3 d\gamma, \\
M_3 &= \int_\Omega (x_2 f_1 - x_1 f_2) dx_1 dx_2 + \int_\gamma (x_2 g_1 - x_1 g_2) d\gamma.
\end{aligned} \tag{13}
$$

It is interesting to remark that the classical case considered in §1 corresponds to the particular case where $F_i = M_i = M_i^L = P_1 = P_3 = 0$ and besides Ox_2 is a symmetrical axe of the beam cross section.

As it has already printed out the diameter of the cross section $d = d(\Omega)$ is very small respect to the length L and it implies that three-dimensional problem (10) is difficult to solve since it is ill conditioned. Then it is convenient to approximate the displacement field u_i and the stress field σ_{ij} by other methods. Trabucho–Viaño [7] have used the cross section area as the little parameter in an asymptotic expansion, and as a byproduct they have justified an explicit formula which approximates the stress σ_{31} and σ_{32} produced by the arbitrary actions f_i, g_i and p_i, and axes Ox_α which do not need to be of symmetry of cross section. This explicit formula represents, in principle, a better approximation than formulae given in the Strength of Materials for particular cases. It is written as follows:

$$
\begin{aligned}
\sigma_{31}^0(x_1, x_2, x_3) &= \frac{E}{2(1+\nu)}\{-\psi_{,2}(x_1, x_2)\theta_3'(x_3) \\
&+ [(1+\nu)r_{\beta,1}(x_1, x_2) + \nu s_{\beta,1}(x_1, x_2) + \nu\phi_{1\beta}(x_1, x_2)]U_\beta'''(x_3)\} + w_{,1}^0 \\
\sigma_{32}^0(x_1, x_2, x_3) &= \frac{E}{2(1+\nu)}\{\psi_{,1}(x_1, x_2)\theta_3'(x_3) \\
&+ [(1+\nu)r_{\beta,2}(x_1, x_2) + \nu s_{\beta,2}(x_1, x_2) + \nu\phi_{2\beta}(x_1, x_2)]U_\beta'''(x_3)\} + w_{,2}^0
\end{aligned} \tag{14}
$$

where $\phi_{\alpha\beta} = \phi_{\alpha\beta}(x_1, x_2)$ are the following functions defined in the cross section domain:

$$
\phi_{11} = -\phi_{22} = \frac{1}{2}(x_1^2 - x_2^2),\ \phi_{12} = \phi_{21} = x_1 x_2,
$$

and the functions $\psi(x_1, x_2)$ (Saint–Venant's torsion function), $r_\beta(x_1, x_2)$ and $s_\beta(x_1, x_2)$ (Trabucho–Viaño's shear functions) are still functions which only depend on the cross section geometry. They solve the following five classical Laplace's problems (by simplicity it is assumed that Ω is simply connected):

$$\begin{cases} -\Delta\psi &= 2, \quad \text{in } \Omega, \\ \psi &= 0, \quad \text{on } \gamma \end{cases}$$
$$\begin{cases} \Delta r_\beta &= 2x_\beta, \quad \text{in } \Omega, \\ r_{\beta,n} &= 0, \quad \text{on } \gamma, \\ \int_\Omega r_\beta &= 0, \end{cases} \tag{15}$$
$$\begin{cases} -\Delta s_\beta &= 2x_\beta, \quad \text{in } \Omega, \\ s_{\beta,n} &= -\phi_{\beta\alpha} n_\alpha, \quad \text{on } \gamma, \\ \int_\Omega s_\beta &= 0. \end{cases}$$

The warping function w is defined as the only solution of the following Neumann's problem in Ω:

$$\begin{cases} -\Delta w &= 0 \quad \text{in } \Omega \\ w_{,n} &= x_2 n_1 - x_1 n_2 \quad \text{on } \gamma \\ \int_\Omega w &= 0 \end{cases} \tag{16}$$

and the action of the warping due to the axial load is:

$$R = \int_\Omega w f_3 dx_1 dx_2 + \int_\gamma w g_3 d\gamma. \tag{17}$$

The following constants of the section geometry are associated to these functions:

$$\begin{aligned} H_\alpha &= \frac{1}{2}\int_\Omega x_\alpha (x_1^2 + x_2^2) dx_1 x_2, \qquad I_\beta^w = \int_\Omega x_\beta w dx_1 x_2, \qquad J = -\int_\Omega x_\alpha \psi_{,\alpha} dx_1 x_2 \\ I_1^\psi &= -\int_\Omega x_2^2 \psi_{,2} dx_1 dx_2, \qquad I_2^\psi = \int_\omega x_1^2 \psi_{,1} dx_1 x_2, \qquad I_\alpha = \int_\Omega x_\alpha^2 dx_1 x_2. \end{aligned} \tag{18}$$

In formula (14) $U_\beta = U_\beta(x_3)$ are the bending displacements which satisfy the following boundary problem in (0,L), with fourth order derivatives respect to x_3 (no sum on β):

$$\begin{aligned} EI_\beta U_\beta'''' &= F_\beta + M_\beta', \quad in\ (0, L), \\ U_\beta &= 0, \quad at\ x_3 = 0, \\ U_\beta' &= 0, \quad at\ x_3 = 0, \\ -EI_\beta U_\beta'' &= M_\beta^L, \quad at\ x_3 = L, \\ -EI_\beta U_\beta''' &= P_\beta, \quad at\ x_3 = L. \end{aligned} \tag{19}$$

In (14) $\theta_3 = \theta_3(x_3)$ represents the rotation of the section x_3 and it is the solution of the following problem in $(0, L)$ with derivatives of second order respect to x_3:

$$\begin{aligned} -\frac{EJ}{2(1+\nu)}\theta_3'' &= M_3 + R' - \frac{1}{2(1+\nu)I_\alpha}[(1+\nu)I_\alpha^w + \nu I_\alpha^\psi](F_\alpha + M_\alpha') \; in \;\; (0,L), \\ \frac{EJ}{2(1+\nu)}\theta_3' &= -\frac{1}{2(1+\nu)I_\alpha}\left[(1+\nu)I_\alpha^w + \nu I_\alpha^\psi\right] P_\alpha + M_3^L - R(L) \; at \;\; x_3 = L, \\ \theta_3 &= \frac{\nu}{I_1 + I_2}[H_2 u_1''(0) - H_1 u_2''(0)] \; at \;\; x_3 = 0. \end{aligned} \tag{20}$$

Finally, the function $w^0 = w^0(x_1, x_2, x_3)$ is the solution of the following problem in each section x_3:

$$\left\{ \begin{aligned} -\Delta w^0 &= f_3 - \frac{1}{\text{Area}(\Omega)}F_3 \quad in \;\; \Omega, \\ w^0_{,n} &= g_3 \qquad\qquad\qquad on \;\; \gamma, \\ \int_\Omega w^0 dx_1 dx_2 &= 0. \end{aligned} \right. \tag{21}$$

In formulae (14) of Trabucho–Viaño it can be seem observed that the first term corresponds to the general torsion effects which include the effects of the pure torsion due to the moments M_3 and M_3^L, as well as the effects of the torsion produced only if the resultant of the loads F_α and P_α do not pass through the shear center and also the effects of variation of R or the value $R(L)$. In fact, in order to $\theta_3' = 0$ ($\theta_3 =$ cte) be solution of problem (20) it is not sufficient that M_3 and M_3^L be zero, because the loads R and $R(L)$ participate in the solution, if the moment produced by the loads $(F_1 + M_1', F_2 + M_2')$ and (P_1, P_2) with respect to the points of coordinates $(\widehat{x}_1, \widehat{x}_2)$ is not zero. These coordinates are given by the following expressions:

$$\begin{aligned} \widehat{x}_1 &= -\frac{1}{2(1+\nu)I_2}[(1+\nu)I_2^w + \nu I_2^\psi], \\ \widehat{x}_2 &= \frac{1}{2(1+\nu)I_1}[(1+\nu)I_1^w + \nu I_1^\psi], \end{aligned} \tag{22}$$

which is the shear center according to the Trabucho–Viaño's expressions [8]. The classical formula of the shear center corresponds, approximately, to suppose $\nu = 0$ in (22).
The second, third and fourth terms of the formula (14) correspond to the shear effects of the bending which are incorporated by the functions r_β, s_β y $\phi_{\alpha\beta}$. Finally, the term $w^0_{,\alpha}$ in (14) brings to a shear effect due to the presence of axial loads in each section.
It should be pointed out that the formula (14) preserves the section equilibrium of the tangential stresses with the shear stresses, since it is verified:

$$Q_\beta = \int_\Omega \sigma^0_{3\beta} d\Omega = -EI_\beta U_\beta''' + M_\beta \qquad \text{(no sum on } \beta). \tag{23}$$

A final remark: The formula (14) from Trabucho–Viaño is still available for multiply connected cross sections only by modifying the definition of the torsion function ψ (see Trabucho–Viaño [8]) and it can be applied it to beams with either solid or thin walled cross section.

For the usual cases found in practice, the following values hold, $f_i = 0$ and $g_i = 0$ and the equations (19), (20) and (21) are solved explicitly producing the results:

$$U_\beta(x_3) = \frac{1}{EI_\beta}[\frac{1}{2}(P_\beta L - M_\beta^L)x_3^2 - \frac{1}{6}P_\beta x_3^3], \quad (24)$$

$$\theta_3(x_3) = \left\{-\frac{1}{EI_\alpha J}\left[(1+\nu)I_\alpha^w + \nu I_\alpha^\psi\right]P_\alpha + \frac{2(1+\nu)}{EJ}M_3^L\right\}x_3, + \frac{\nu}{I_1+I_2}\left[\frac{H_2}{EI_1}(P_1L - M_1^L) - \frac{H_1}{EI_2}(P_2L - M_2^L)\right], \quad (25)$$

$$w^0 = 0. \quad (26)$$

The classical case described in §1 corresponds to $f_i = 0$, $g_i = 0$ and moreover $P_1 = P_3 = 0$, $M_i^L = 0$, where Ox_2 is an axe of symmetry of cross section. Therefore, $H_1 = I_2^w = I_2^\psi = 0$ and consequently:

$$U_1 = 0,\ \theta_3 = 0,\ U_2(x_3) = \frac{1}{EI_2}[\frac{1}{2}P_2Lx_3^2 - \frac{1}{6}P_2x_3^3] = \frac{P_2x_3^2}{6EI_2}(3L - x_3). \quad (27)$$

In this way, by substituting in (14) it is found:

$$\sigma_{31}^0(x_1,x_2,x_3) = -\frac{1}{2I_2(1+\nu)}[(1+\nu)r_{2,1}(x_1,x_2) + \nu s_{2,1}(x_1,x_2) + \nu\phi_{12}(x_1,x_2)]P_2,$$
$$\sigma_{32}^0(x_1,x_2,x_3) = -\frac{1}{2I_2(1+\nu)}[(1+\nu)r_{2,2}(x_1,x_2) + \nu s_{2,2}(x_1,x_2) + \nu\phi_{22}(x_1,x_2)]P_2. \quad (28)$$

3 Comparative study of the classical and asymptotic solutions.

In the previous sections it has been shown that the calculus of the tangential stresses in a three-dimensional beam is reduced to two separated analysis. A two-dimensional analysis on the section Ω, represented by the study of the slice of a the beam, which can be carried out according to the classical theories of the Strength of Materials or alternatively by the solution of the six Laplace's problems (15)-(16) in the case of the Trabucho–Viaño formulation. The other analysis corresponds to an unidimensional calculus to determinate the behaviour of the middle line of the beam, $x_1 = 0$, $x_2 = 0$, through Ox_3. In general, the solution of the Laplace's problems is found by means of numerical methods, like finite or boundary elements, which work efficiently since they are well conditioned problems. By other hand, the calculus of stress resultants and displacements along the middle line of the beam may be obtained under its analytical form, at least in many cases.
In order to verify the differences, between the Trabucho–Viaño's and the classical of Strength of Materials formulations, for the stress distribution in a section under a shear load Q_2, the

results obtained for some beams with simple and usual cross sections in practice has been compared. Here we report only the results obtained for the rectangular and circular cross section beams.

3.1 Rectangular section of height h and width b

The formulae of strength of materials applied to the section of figure 6 produce the following results:

$$\sigma_{32}^R(x_1, x_2) = \sigma_{32}^R(x_2) = \frac{6}{bh}\left(\frac{1}{4} - \bar{x}_2^2\right) Q_2, \text{ with } \bar{x}_2 = \frac{x_2}{h}, \tag{29}$$

$$\sigma_{31}^R(x_1, x_2) = 0. \tag{30}$$

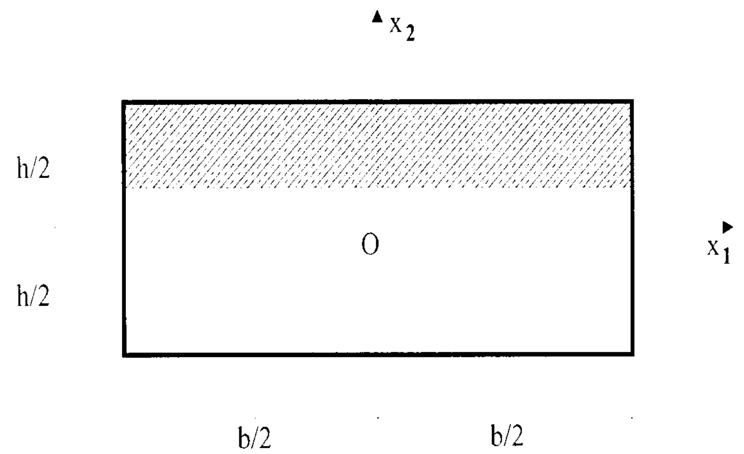

Figure 6: Rectangular section.

Thus, the maximal stress σ_{32}^R is obtained for $\bar{x}_2 = 0$:

$$\sigma_{32,max}^R = \sigma_{32}^R(0) = \frac{Q_2}{A(\Omega)} = \frac{3}{2}\sigma_{32,med}^R \tag{31}$$

where $\sigma_{32,med}^R$ is the stress produced by the shear load Q_2, when it is assumed uniformly distributed on the section, i.e,:

$$\sigma_{32,med}^R = \frac{Q_2}{bh}. \tag{32}$$

and $A(\Omega) = \int_\Omega dx_1 dx_2$ the area of the cross section.
The Trabucho–Viaño's results are deduced by applying the formula (28). In the table 1 and 2 the differences between the results of the two methods for the stress σ_{32} in the lines $(b/2, 0)$ and $(0, 0)$, depending on the ratio h/b and the Poisson's ratio ν, are shown. As it could be expected significant differences between the two methods appear for very width sections and high Poisson's ratio, but they are very small in other cases.

$h/b \rightarrow$ ν $\downarrow$	6	5	4	3	2	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{6}$
0.0	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
0.1	1.00	1.00	1.01	1.01	1.01	1.06	1.13	1.31	1.45	1.58	1.71
0.2	1.00	1.00	1.01	1.01	1.03	1.10	1.33	1.57	1.82	2.07	2.31
0.3	1.00	1.01	1.01	1.02	1.04	1.14	1.46	1.80	2.13	2.45	2.82
0.4	1.00	1.01	1.01	1.02	1.05	1.18	1.56	1.98	2.41	2.83	3.25
0.5	1.00	1.01	1.01	1.02	1.05	1.21	1.66	2.14	2.64	3.13	3.63

Table 1: $\sigma^0_{32}/\sigma^R_{32}$ at point $(b/2, 0)$

$h/b \rightarrow$ ν $\downarrow$	6	5	4	3	2	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{6}$
0.0	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
0.1	1.00	1.00	1.00	1.00	0.99	0.97	0.93	0.92	0.91	0.91	0.90
0.2	1.00	1.00	1.00	0.99	0.99	0.95	0.88	0.85	0.84	0.81	0.80
0.3	1.00	1.00	1.00	0.99	0.98	0.93	0.83	0.79	0.77	0.77	0.76
0.4	1.00	1.00	0.99	0.99	0.98	0.91	0.79	0.74	0.72	0.72	0.70
0.5	1.00	1.00	0.99	0.99	0.97	0.90	0.76	0.70	0.67	0.67	0.66

Table 2: $\sigma^0_{32}/\sigma^R_{32}$ at point $(0, 0)$

In the figures 7 and 8 the functions $\sigma^0_{32}(x_1, x_2)$ and $\sigma^R_{32}(x_1, x_2)$ are represented for the rectangular section of height $h = 1$ and width $b = 6$. From its comparison it is deduced that the two approximations coincide in mean values, along any line defined by $x_2 =$ constant. However, the differences between the stresses values can be very important so at the extreme points of these lines as at their middle points.

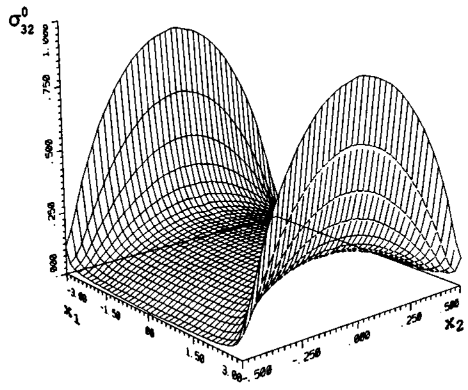

Figure 7: Rectangular section: σ^0_{32}

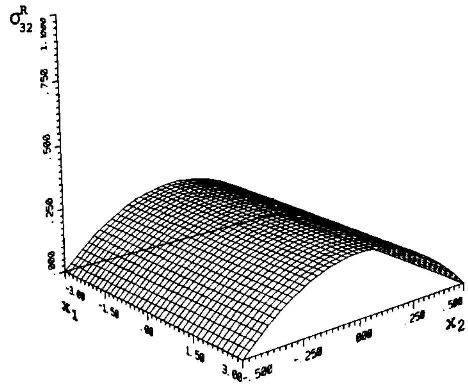

Figure 8: Rectangular section:σ^R_{32}

3.2 Circular section of radius r

Now the distribution of the stresses $\sigma_{3\alpha}$ along the axes $x_2 = 0$ and $x_2 = r/2$ under the action of a shear load Q_2 will be studied. The equations of the Strength of Materials can be used to obtain the following results:

$$\sigma_{32}^R(x_1,x_2) = \sigma_{32}^R(x_2) = \frac{4}{3\pi r^2}[1-\bar{x}_2^2]Q_2, \tag{33}$$

$$\sigma_{31}^R(x_1,x_2) = -\sigma_{32}^R(x_2)tg\varphi(x_1) = -\frac{4\bar{x}_1\bar{x}_2Q_2}{3\pi r^2},\ \bar{x}_\alpha = \frac{x_\alpha}{r}. \tag{34}$$

Therefore

$$\sigma_{32,max}^R = \sigma_{32}^R(0,0) = \frac{4}{3}\frac{Q_2}{\pi r^2}. \tag{35}$$

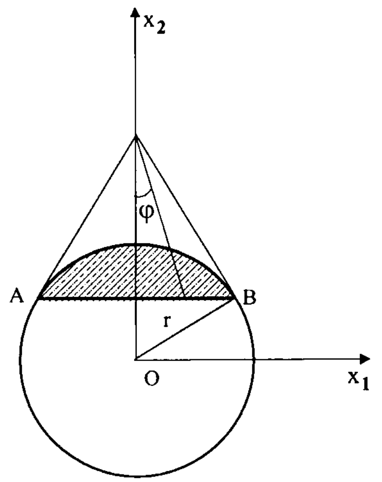

Figure 9: Circular section.

The application of the formula (28) to the particular case of the circle provides the following analytical expression which coincides with the solution deduced by the Linear Elasticity Theory (see [5]):

$$\sigma_{32}^0(x_1,x_2) = \frac{3+2\nu}{8(1+\nu)I_2}[r^2 - x_2^2 - \frac{1-2\nu}{3+2\nu}x_1^2]Q_2, \tag{36}$$

$$\sigma_{31}^0(x_1,x_2) = -\frac{1+2\nu}{4(1+\nu)I_2}x_1x_2Q_2. \tag{37}$$

In table 3, the results obtained in formula (28) are compared with those of the formulae (33)–(34) for different points of the section.

A similar behaviour as in other sections has been observed and the more important discrepancies are produced near the boundary of the circle in the Ox_2 direction, and they increase for small values of Poisson's ratio. This difference is well observed in the figures 10 and 11 corresponding to $\sigma_{32}^0(x_1,x_2)$ for $\nu = 0$, $\nu = 0.5$ and $\sigma_{32}^R(x_1,x_2)$ respectively. The coincidence between σ_{32}^R and σ_{32}^0 for $\nu = 0.5$ is also clear from (33) and (36).

$\nu \rightarrow$	0.0	0.1	0.2	0.3	0.4	0.5
(0.4000, 0.0)	1.06	1.05	1.03	1.02	1.00	0.99
(1.0000, 0.0)	0.76	0.83	0.88	0.93	0.97	1.00
(0.8660, 0.5)	0.78	0.84	0.89	0.94	0.98	1.02
(0.3025, 0.5)	1.08	1.06	1.04	1.02	1.01	0.99

Table 3: Circular section: $\sigma^0_{32}(x_1, x_2)/\sigma^R_{32}(x_1, x_2)$

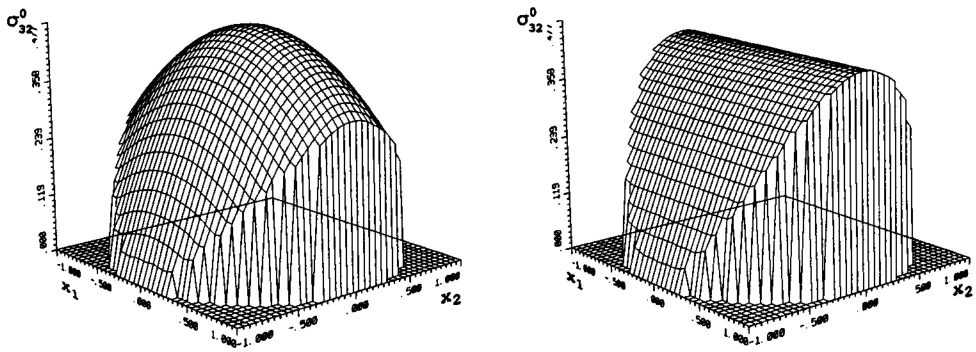

Figure 10: Circular section: σ^0_{32} for $\nu = 0$ and $\nu = 0.5$

4 Conclusions

From numerical and analytical experiments, it is deduced that the formulae of the Strength of Materials which express the distribution of the shear stresses due to a pure shear load, produce results which satisfy the global equilibrium of the acting loads. However, this Strength of Materials formulation does not give any information about the stress variation along lines parallel to the normal to the direction of the shear load, because it is assumed that this stress variation is constant. For thin cross sections, this hypothesis is available, but for solid sections of relatively important width or critical values of the Poisson's ratio, the differences with a distribution obtained by the Elasticity Theory may be important.

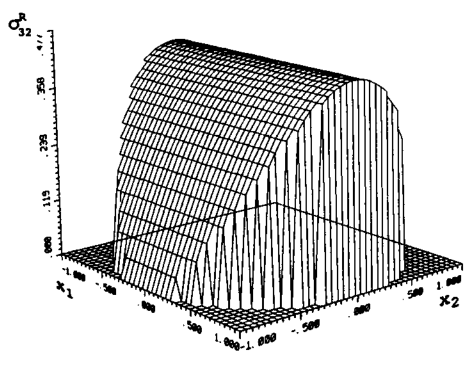

Figure 11: Circular section: σ^R_{32}

The Trabucho–Viaño's formulation allows us to obtain the distribution of the shear stresses dues to a shear load, a torsion moment or more general loads, which approximates the results of the Linear Elasticity Theory in a consistent way.

5 Acknowledgements

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References

[1] Fenner, R.T.: *Mechanics of Solids*. Blackwell Scientific Publications, Oxford, 1989.

[2] Fraejis de Veubecke, B.M.: *A Course in Elasticity*. Springer-Verlag, Berlin, 1979.

[3] Le Dret, H.: *Convergence of displacements and stresses in linearly elastic slender rods as the thickness goes to zero*. Asymptotic Anal. 10, $n^{\underline{o}}$ 4, 367-402, 1993.

[4] Samartín, A.: *Resistencia de Materiales*(Strength of Materials). Servicio de Publicaciones Colegio de Ingenieros de Caminos, Canales y Puertos, Madrid, 1996.

[5] Timoshenko, S.–Goodier, J.N.: *Theory of Elasticity*. McGraw-Hill, New York, 1951.

[6] Trabucho, L–Viaño, J.M.: *Derivation of generalized models for linear elastic beams by asymptotic expansion methods*. Applications of Multiple Scalings in Mechanics, RMA 4, 302-315. Masson, París, 1987.

[7] Trabucho, L–Viaño, J.M.: *Existence and characterization of higher order terms in an asymptotic expansion method for linearized elastic beams* . Asymptotic Analysis 2, 223-255, 1989.

[8] Trabucho, L–Viaño, J.M.: *Mathematical modelling of rods*: Handbook of Numerical Analysis, Vol IV, North Holland, 1996.