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# Application of the Boundary Method to the determination of the properties of the beam cross sections

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**Abstract.** *Using the 3-D equations of linear elasticity and the asymptotic expansion methods in terms of powers of the beam cross-section area as small parameter different beam theories can be obtained [9], according to the last term kept in the expansion. If it is used only the first two terms of the asymptotic expansion the classical beam theories can be recovered without resort to any "a priori" additional hypotheses. Moreover, some small corrections and extensions of the classical beam theories can be found and also there exists the possibility to use the asymptotic general beam theory as a basis procedure for a straightforward derivation of the stiffness matrix and the equivalent nodal forces of the beam. In order to obtain the above results a set of functions and constants only dependent on the cross-section of the beam it has to be computed them as solutions of different 2-D laplacian boundary value problems over the beam cross section domain. In this paper two main numerical procedures to solve these boundary value problems have been discussed, namely the Boundary Element Method (BEM) and the Finite Element Method (FEM). Results for some regular and geometrically simple cross-sections are presented and compared with ones computed analytically. Extensions to other arbitrary cross-sections are illustrated.*

# 1 Introduction

Elastic beams correspond to solids for which two of their characteristic dimensions of its cross section are much smaller than the third one, its length. In this paper only cylindrical straight beams will be considered.

Beam behaviour can be studied by direct application of the general 3-D theory of elasticity. However two main difficulties appear in this approach. First, 3-D elastic analytical solutions are hard to find. Second, the 3-D beam problems are ill conditioned from a numerical point of view due to the characteristics of the beam geometry. In order to avoid these difficulties 1-D models are normally used for beam analysis.

Several well known beam models exist: Navier-Bernoulli, Timoshenko, Saint Venant, Vlasov etc. All of them are based on a set of simplifying "a priori" hypotheses and in this way the beam response under different actions can be obtained.

A more recent approach for obtaining and justifying different elastic beam models corresponds to the application of asymptotic expansion methods. In these methods the solution of the 3-D elasticity equations are approximate trough the successive terms of a power series of a small parameter, namely the diameter of the cross-section. An excellent review of this approach is given in [9].

These asymptotic expansion methods are very general because they have already been applied to different types of structures: shells [5], plates [3] and beams [1]. Besides the mathematical rigor inherent to these methods, they do no use te classical a priori hypotheses. On the contrary they justify the validity of these hypotheses and some introduce small corrections on the results found in classical theories. Finally, the successive terms of the asymptotic expansion can be interpreted and have a physical meaning, because they can be associated separately with various structural effects: stretching, torsion, bending, Poisson's effects and cross-section deformation within its own plane.

Although a summary of the asymptotic expansion method considered in this paper will be given in the following sections, it should be pointed out that in broad terms, one of its main features is the possibility to obtain the elastic beam solution as an union of solutions of 2-D elasticity problems and 1-D solutions along the beam length.

The first group of solutions corresponds to the constants and functions of the beam cross-section. Some of them,like the constants area, second order inertia moments, torsional moment and Timoshenko's constant and the functions of Saint Venant and Prandtl among others, are very well known in the beam literature. However, other constants and functions obtained in the asymptotic expansion method are new or generalization of the classical ones and they will be defined later.

Then this paper is organized as follows. First, a short description of the asymptotic expansion method and its main results are presented. Second, the different constants and functions of the beam cross-section, shown-up in a natural way in the application of the method, are defined. It will be observed that the constants are integrals of the functions over the cross-section domain and the functions are solutions of different 2-D laplacian boundary value problems on the beam cross section.. Third, due to the speciality of the differential operator involved in these problems the Boundary Element Method (BEM) will be proved to be very efficient and the obtained solutions for a given benchmark of cross-sections will be compared to the ones found by the standard Finite Element Method (FEM). Finally, some remarks and conclusions on the comparative advantages and disadvantages of the application of the BEM in comparison to the FEM will be drawn.

## 2 General beam elastic equations

A straight elastic beam is defined as a 3-D cylindrical solid occupying the following reference configuration:

$$V = \Omega \times [0, L] \tag{1}$$

where $\Omega$ is the beam cross-section and $L$ is its span. The beam geometry is characterized by the condition:

$$diameter(\Omega) << L \tag{2}$$

The following definitions and notation are introduced:

The end cross-sections of the beam are

$$\Omega_0 = \Omega \times \{0\} \ ; \ \ \Omega_L = \Omega \times \{L\} \tag{3}$$

and an intermediary cross-section is simply denoted by $\Omega$.

The boundary of a k-th multi-connected cross section is:

$$\gamma = \gamma_0 \cup \gamma_1 \dots \gamma_k \tag{4}$$

where $\gamma_0$ is the exterior part of the boundary and $\gamma_1, \gamma_2 \dots, \gamma_k$ are the interior boundaries of the domain (holes) of sections $\omega_1, \omega_2 \dots, \omega_k$.

It is assumed without lost of generality a system of coordinates $Ox_1x_2x_3$ such that that $Ox_3$ pass through the gravity center of all cross sections and the origin $O$ is the gravity center of one end section so the axes $Ox_1$ and $Ox_2$ are the principal inertia axes of the section $\Omega$. (figure 1).

The lateral boundary of the beam is

$$\Gamma = \gamma \times (0, L) \tag{5}$$

As usual in elasticity theory, Latin indexes takes their values in the set $(1, 2, 3)$ and Greek indexes in the set $(1, 2)$. The Einstein sum convention will also be used unless it is stated the contrary.

The unit outer normal vector to the beam boundary will be denoted by $\mathbf{n} = (n_1, n_2, n_3)$. In particular, in $\Gamma$, $\mathbf{n} = (n_1, n_2, 0)$; in $\Omega_0$, $\mathbf{n} = (0, 0, -1)$ and in $\Omega_L$, $\mathbf{n} = (0, 0, 1)$ .

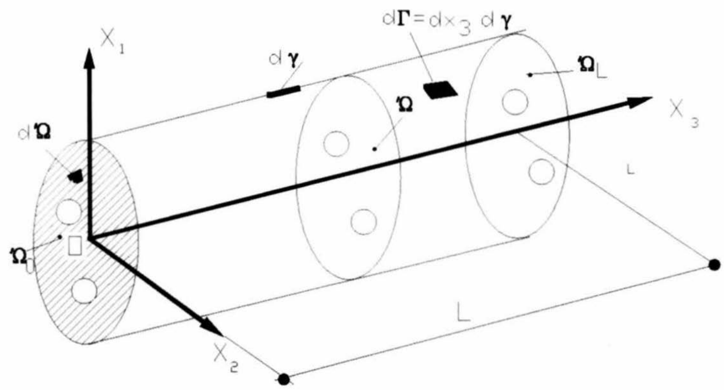


Figure 1: Cylindrical elastic beam.

Partial derivatives of a function $\Phi$ respect to the variable $x_i$ will be written $\Phi_{,i}$. Then derivatives respect to the normal $\mathbf{n}$ are $\Phi_{\mathbf{n}}$. For functions depending only on the variable $x_3$ their successive derivatives will be $\Phi', \Phi'', \Phi''', \ldots$.

The 3-D linear elasticity problem of a cantilever beam can be expressed as follows:

Find the displacement components $u_i$ such that satisfy the boundary value problem

$$\begin{aligned} \sigma_{ij,j} &= f_i && \text{in } V \\ \sigma_{i3} &= p_i && \text{on } \Omega_L \\ \sigma_{i\beta}n_\beta &= g_i && \text{on } \Gamma \\ u_i &= 0 && \text{on } \Omega_0 \end{aligned} \tag{6}$$

where $\mathbf{f} = (f_i)$ are the volume forces, $\mathbf{g} = (g_i)$ and $\mathbf{p} = (p_i)$ are the surface pressures act on $\Gamma$ and $\Omega$ beam boundaries respectively. In case of other boundary conditions to the cantilever ones it may be interesting to emphasized the beam end surface where the forces $\mathbf{p} = p_i$ are acting. Then they will be denoted by $\mathbf{h}_0 = (h_{i0})$ and $\mathbf{h}_1 = (h_{i1})$ for the pressures on $\Omega_0$ and on $\Omega_L$ respectively. Similarly, in case of imposed displacements at $\Omega_0$ and $\Omega_L$ they will written $\mathbf{u}_0 = (u_{0i})$ and $\mathbf{u}_1 = (h_{1i})$.

The Hooke's constitutive law for the beam material is

$$\sigma_{ij} = \frac{E}{2(1+\nu)}(u_{i,j} + u_{j,i}) + \frac{E\nu}{(1+\nu)(1-2\nu)} u_{kk}\delta_{ij} \tag{7}$$

where $\delta_{ij}$ is the Kronecker symbol. The components of the linearized elasticity strain tensor $\mathbf{e} = (e_{ij})$ are expressed in terms of the displacements as follows:

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \tag{8}$$

An alternative formulation to the boundary value problem (6) is given by the following variational problem (Hellinger-Reissner type of mixed formulation):

Find the independent fields $\mathbf{u} = (u_i)$ and $\boldsymbol{\sigma} = (\sigma_{ij})$ such that satisfy for all $\mathbf{u}^* = (u_i^*)$ and $\boldsymbol{\sigma}^* = (\sigma_{ij}^*)$ subjected to the conditions $u_i^* = 0$ on $\Omega_0$ the following equations:

$$\int_V \left(\frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij}\right)\sigma_{ij}^* dV = \int_V e_{ij}\sigma_{ij}^* dV \tag{9}$$

$$\int_V \sigma_{ij} e_{ij}^* dV = \int_V f_i u_i^* dV + \int_\Gamma g_i u_i^* d\Gamma + \int_\Omega p_i u_i^* d\Omega \tag{10}$$

where $e_{ij}^* = \frac{1}{2}(u_{i,j}^* + u_{j,i}^*)$ and $dV = dx_1 dx_2 dx_3$, $d\Gamma = dx_3 d\gamma$ and $d\Omega = dx_1 dx_2$ are the differential beam volume, lateral surface and end surface respectively.

## 3 Asymptotic expansion method

The solution of the boundary value problem defined by (6) or alternatively by (9) and (10) is ill conditioned due to the geometry feature expressed by the condition (2). In order to avoid the inherent numerical difficulties to this fact, the asymptotic expansion method will be applied to find an approximate solution of the previous boundary value problem.

The ideas introduced by Ciarlet and Destuynder [3] for plates have been applied to beams by Bermúdez and Viaño [1]. These authors have found the asymptotic expansion method was very efficient when it is used in the framework of a variational formulation. A complete study of this methodology is given in [6]. In the present paper it will be follow very close the main steps of the analysis there described that will be summarized below.

(1) The beam is considered as a 3-D elastic body and the variational formulae (9) and (10) describe "exactly" its structural behaviour.

(2) The actual beam is imbedded into a family of beams whose cross-sections are homothetic, with ratio $\epsilon$, to a given reference section. The ratio $\epsilon$ is of the same order as the diameter of $\Omega$ and it will be chosen as the small parameter of the problem.

For each beam of the family the applied forces and the material properties are explicitly defined.

(3) By appropriate variables changes the original problem can be transformed into an equivalent one in a fixed reference domain, i.e. independent on $\epsilon$. Then it is possible to apply the asymptotic expansion theory developed by Lions [6] and Goldenweizer [5].

(4) Once the convergence of the power series of $\epsilon$ is justified, it is possible to evaluate any terms in the asymptotic development no matter how high their order is.

(4) Finally the inverse change of variable is carried out and the results are written with respect to the original domain. In this way the successive approximations of the solution for the 3-D original elasticity problem are obtained. According to the expansion terms considered the different classical beam theories are recovered and, sometimes, refined or complemented.

# 4 Application of the asymptotic expansion method to the elastic beam

Applying the methodology of the asymptotic expansion to the 3-D solid beam the following main equations and results are presented.

The variational equations of an imbedded beam according the displacement-stress approach are

$$\int_{V^\epsilon}\left(\frac{1+\nu^\epsilon}{E^\epsilon}\sigma_{ij}^\epsilon-\frac{\nu^\epsilon}{E^\epsilon}\sigma_{kk}^\epsilon\delta_{ij}\right)\sigma_{ij}^{*\epsilon}dV^\epsilon = \int_{V^\epsilon} e_{ij}^\epsilon\sigma_{ij}^{*\epsilon}dV^\epsilon \tag{11}$$

$$\int_{V^\epsilon}\sigma_{ij}^\epsilon e_{ij}^{*\epsilon}dV^\epsilon = \int_{V^\epsilon} f_i^\epsilon u_i^{*\epsilon}dV^\epsilon+\int_{\Gamma^\epsilon} g_i^\epsilon u_i^{*\epsilon}d\Gamma^\epsilon+\int_{\Omega^\epsilon} p_i^\epsilon u_i^{*\epsilon}d\Omega^\epsilon \tag{12}$$

The solid beam $V^\epsilon$ is associated to one of reference $V$ obtained according to the following zoom transformation or scaling $\Pi^\epsilon : V \to V^\epsilon$ defined as

$$\Pi^\epsilon : \mathbf{x}=(x_1,x_2,x_3)\to\mathbf{x}^\epsilon=(x_1,x_2,x_3)=(\epsilon x_1,\epsilon x_2,x_3) \tag{13}$$

and then $V^\epsilon=\Pi^\epsilon(V)$, $\Gamma^\epsilon=\Pi^\epsilon(\Gamma)$, $\mathbf{n}=\mathbf{n}^\epsilon$.

Similarly, the actual and virtual displacements $\mathbf{u}^\epsilon$ and $\mathbf{u}^{*\epsilon}$ are associated to the displacements of the reference problem by the expresions:

$$u_\alpha=\epsilon u_\alpha^\epsilon,\ u_3=u_3^\epsilon,\ u_\alpha^*=\epsilon u_\alpha^{*\epsilon},\ u_3^*=u_3^{*\epsilon}$$

The following hypothesis on data are considered:

$$E^\epsilon=E,\ \nu^\epsilon=\nu \tag{14}$$

$$f_\alpha^\epsilon=\epsilon f_\alpha,\ f_3^\epsilon=f_3,\quad g_\alpha^\epsilon=\epsilon^2 g_\alpha,\ g_3^\epsilon=\epsilon g_3,\ p_\alpha^\epsilon=\epsilon p_\alpha,\ p_3^\epsilon=p_3 \tag{15}$$

i.e. the material properties remain unchanged through the transformation and the virtual work of the system of the applied forces is homogeneous in $\epsilon$, that is

$$\int_{V^\epsilon} f_i^\epsilon u_i^{*\epsilon}dV^\epsilon+\int_{\Gamma^\epsilon} g_i^\epsilon u_i^{*\epsilon}d\Gamma^\epsilon+\int_{\Omega^\epsilon} p_i^\epsilon u_i^{*\epsilon}d\Omega^\epsilon=\epsilon^2\left[\int_V f_i u_i^* dV+\int_\Gamma g_i u_i^* d\Gamma+\int_\Omega p_i u_i^* d\Omega\right]$$

The stress tensors $\boldsymbol{\sigma}=(\sigma_{ij})$ and $\boldsymbol{\sigma}^{*\epsilon}=(\sigma_{ij}^{*\epsilon})$ are associated to the corresponding ones of the reference beam as

$$\begin{aligned}\sigma_{\alpha\beta}&=\epsilon^{-2}\sigma_{\alpha\beta}^\epsilon, & \sigma_{\alpha\beta}^*&=\epsilon^{-2}\sigma_{\alpha\beta}^{*\epsilon}\\ \sigma_{\alpha3}&=\epsilon^{-1}\sigma_{\alpha3}^\epsilon, & \sigma_{\alpha3}^*&=\epsilon^{-1}\sigma_{\alpha3}^{*\epsilon}\\ \sigma_{33}&=\sigma_{33}^\epsilon, & \sigma_{33}^*&=\sigma_{33}^{*\epsilon}\end{aligned} \tag{16}$$

The equations (11) and (12) are reformulated in the following equivalent problem posed now in a fixed domain $V$ i.e. in the domain of the solid beam of reference, by introducing the transformation $\Pi^\epsilon$ given by the expressions (13) to (16):

$$\int_V \frac{1}{E}\sigma_{33}\sigma^*_{33} \; + \; \epsilon^2 \int_V \left\{ \frac{2(1+\nu)}{E}\sigma_{3\alpha}\sigma^*_{3\alpha} - \frac{\nu}{E}\left[\sigma_{33}\sigma^*_{\mu\mu} + \sigma_{\mu\mu}\sigma^*_{33}\right] \right\} dV + $$
$$+ \; \epsilon^4 \int_V \left[ \frac{1+\nu}{E}\sigma_{\alpha\beta} - \frac{nu}{E}\sigma_{\mu\mu}\delta_{\alpha\beta} \right] \sigma^*_{\alpha\beta} dV = \int_V e_{ij}\sigma^*_{ij} dV \tag{17}$$

$$\int_V \sigma_{ij} e^*_{ij} dV \; = \; \int_V f_i u^*_i dV + \int_\Gamma g_i u^*_i d\Gamma + \int_{\Omega_L} p_i u^*_i d\Omega \tag{18}$$

The asymptotic expansion

$$\mathbf{u} = \mathbf{u}^0 + \epsilon^2 \mathbf{u}^2 + \epsilon^4 \mathbf{u}^4 + h.o.t. \tag{19}$$

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^0 + \epsilon^2 \boldsymbol{\sigma}^2 + \epsilon^4 \boldsymbol{\sigma}^4 + h.o.t. \tag{20}$$

are valid provided that the section $\Omega_0$ is weakly clamped (See Ref. 6). Then substitute these formal expressions in (17) and (18) and set the factors of the successive powers of $\epsilon$ to zero, the terms $\mathbf{u}^{2p}$ and $\boldsymbol{\sigma}^{2p}$, $(p = 0, 1, 2, \ldots)$ can be identified.

Associated with the different powers of $\epsilon$ the following successive approximations may be distinguished:

$$\begin{aligned} &Order\ 0: \quad (\mathbf{u}^0, \boldsymbol{\sigma}^0) \\ &Order\ 2: \quad (\mathbf{u}^0, \boldsymbol{\sigma}^0) + \epsilon^2(\mathbf{u}^2, \boldsymbol{\sigma}^2) \\ &Order\ 4: \quad (\mathbf{u}^0, \boldsymbol{\sigma}^0) + \epsilon^2(\mathbf{u}^2, \boldsymbol{\sigma}^2) + \epsilon^4(\mathbf{u}^4, \boldsymbol{\sigma}^4) \end{aligned} \tag{21}$$

Finally, by applying the inverse transformation or "descaling" given by the formulae (13) to (16) the successive approximations of the true displacements $\mathbf{u}^\epsilon$ and stresses boldmath $\sigma^\epsilon$ fields can be obtained i.e.

$$\begin{aligned} &Order\ 0: \quad (\mathbf{u}^{0\epsilon}, \boldsymbol{\sigma}^{0\epsilon}) \\ &Order\ 2: \quad (\mathbf{u}^{0\epsilon}, \boldsymbol{\sigma}^{0\epsilon}) + \epsilon^2(\mathbf{u}^{2\epsilon}, \boldsymbol{\sigma}^{2\epsilon}) \\ &Order\ 4: \quad (\mathbf{u}^{0\epsilon}, \boldsymbol{\sigma}^{0\epsilon}) + \epsilon^2(\mathbf{u}^{2\epsilon}, \boldsymbol{\sigma}^{2\epsilon}) + \epsilon^4(\mathbf{u}^{4\epsilon}, \boldsymbol{\sigma}^{4\epsilon}) \end{aligned} \tag{22}$$

The stress resultants are found from the equilibrium equations on the section $\Omega$ $(p = 0, 1, 2, \ldots)$ and given by the expressions:

$$Q^{2p\epsilon}_\beta = \int_\Omega \sigma^{2p\epsilon}_{3\beta} d\Omega, \quad M^{2p\epsilon}_\beta = \int_\Omega x^\epsilon_\beta \sigma^{2p\epsilon}_{33} d\Omega \tag{23}$$

In this way it can be shown that Navier-Bernoulli beam theory can be reached without any "a priori" assumptions for the first term of the expansion, i.e. $p = 0$. The second order expansion leads to a general stretching-bending-torsion elastic beam theory, justifying and generalizing the basic equations of classical beam theories, namely, Saint Venant uniform torsion theory with Poisson effects, Timoshenko's bending theory and Vlasov's beam theory. In the next section the final results expressed in differential equations form instead of variational equations will be given. For sake of notation simplicity the superscript $\epsilon$ will not be omitted there, although these results refer to the actual elastic beam.

## 5 Asymptotic second order general linear elastic beam theory

The following results obtained for the second order asymptotic expansion are divided in two groups. The first one corresponds to the functions and constants dependent on the cross-section domain $\Omega$. They are solutions of different 2-D laplacian boundary values problem in $\Omega$. The second group of results are solutions of 1-D boundary values problems defined by second or fourth order ordinary differential equations in the length variable $x_3$, $x_3 \in [0, L]$ and specified boundary conditions.

### 5.1 Functions and constants of the beam cross-section

The functions and constants of the cross-section domain of the beam used in the results of the second order asymptotic expansion are:

Area:

$$A = \int_{\Omega} d\Omega = \int \Omega dx_1 dx_2 \tag{24}$$

Second order moment $I_\alpha$ respect to principal axis $Ox_\beta$ with $(\alpha \neq \beta)$:

$$I_\alpha = \int_{\Omega} x_\alpha^2 d\Omega \tag{25}$$

Bimoments of area:

$$H_\alpha = \frac{1}{2}\int_{\Omega} x_\alpha (x_1^2 + x_2^2) d\Omega, \; H_3 = \frac{1}{4}\int_{\Omega} (x_1^2 + x_2^2)^2 d\Omega \tag{26}$$

Functions related to the Poisson's effects in the Saint Venant torsion theory:

$$\mathbf{\Phi} = (\Phi_{\alpha\beta}) = \begin{bmatrix} \frac{1}{2}(x_1^2 - x_2^2) & x_1 x_2 \\ x_1 x_2 & \frac{1}{2}(x_2^2 - x_1^2) \end{bmatrix} \tag{27}$$

Functions $y(x_1, x_2)$ derived by the application of the non symmetric tensor $\varepsilon_{\alpha\beta}$ defined as $\varepsilon = 0$ for $\alpha = \beta$, $\varepsilon = +1$ for even permutation of $\alpha\beta$ and $\varepsilon = -1$ for odd permutation of $\alpha\beta$, i.e

$$y_1 = y(x_1, x_2) = -x_2, \; y_2 = y(x_1, x_2) = x_1 \tag{28}$$

The warping function $w(x_1, x_2)$ is the unique solution of the elliptic boundary value problem:

$$\begin{aligned} -w_{,\alpha\alpha} &= 0 \qquad in\ \Omega \\ w_{,n} &= -y_\alpha n_\alpha \qquad on\ \gamma \\ \int_\Omega w d\Omega &= 0 \end{aligned} \tag{29}$$

The last equation (29) corresponds to an uniqueness condition.

Sectorial moments of area:

$$I_\beta{}^w = 2\int_\Omega x_\beta w d\Omega \tag{30}$$

Warping constant:

$$J_w = \int_\Omega w^2 d\Omega \tag{31}$$

Prandtl function $\Psi(x_1, x_2$ is the unique solution of the elliptic boundary value problem:

$$\begin{aligned} -\Psi_{,\alpha\alpha} &= 2 \qquad in\ \Omega \\ \Psi &= 0 \qquad on\ \gamma_0 \\ \Psi_{,n} &= p_\alpha n_\alpha \qquad on\ \gamma_k \ \ (k = 1, 2, , \ldots, p) \end{aligned} \tag{32}$$

where $p_\alpha = w_{,\alpha} + y_\alpha$.

Torsional constants associated to the Prandtl function

$$I_1^\Psi = -\int_\Omega x_2 \Psi_{,2}\, d\Omega, \quad I_2^\Psi = \int_\Omega x_1 \Psi_{,1}\, d\Omega \tag{33}$$

$$J = -\int_\Omega x_\alpha \Psi_{,\alpha}\, d\Omega = I_1 + I_2 - \int_\Omega (w_{,1}^2 + w_{,2}^2) d\Omega \tag{34}$$

The following shear functions $r_\alpha(x_1, x_2)$ and $s_\alpha(x_1, x_2)$ do not appear explicitly in classical literature:

$$\begin{aligned} -r_{\beta,\alpha\alpha} &= -2x_\beta \ \ in\ \Omega \\ r_{\beta,n} &= 0 \ \ on\ \gamma \\ \int_\Omega r_\beta d\Omega &= 0 \end{aligned} \tag{35}$$

$$\begin{aligned} -s_{\beta,\alpha\alpha} &= 2x_\beta \ \ in\ \Omega \\ s_{\beta,n} &= -\Phi_{\alpha\beta} n_\alpha \ \ on\ \gamma \\ \int_\Omega s_\beta d\Omega &= 0 \end{aligned} \tag{36}$$

The generalized Timoshenko's constants $T_{\alpha\beta}$ are associated with the shear functions and they are defined as combination of the following constants:

$$L^{r}_{\alpha\beta} = \int_{\Omega} x_{\alpha} r_{\beta} d\Omega, \quad L^{s}_{\alpha\beta} = \int_{\Omega} x_{\alpha} s_{\beta} d\Omega$$

$$K^{r}_{\alpha\beta} = \int_{\Omega} \Phi_{\alpha\mu} r_{\beta\mu} d\Omega, \quad K^{s}_{\alpha\beta} = \int_{\Omega} \Phi_{\alpha\mu} s_{\beta\mu} d\Omega \tag{37}$$

The matrix of Timoshenko's constants is $\mathbf{T} = (T_{\alpha\beta})$ where:

$$\begin{aligned} T_{\alpha\beta} &= -\frac{1}{I_{\beta}} \Big\{ (1+\nu) L^{r}_{\alpha\beta} + \nu L_{\alpha\beta}{}^{s} + \frac{\nu}{2(1+\nu)} \left[ (1+\nu) K_{\alpha\beta}{}^{r} + \nu K_{\alpha\beta}{}^{s} + \nu H_3 \delta_{\alpha\beta} \right] \\ & \frac{1}{2(1+\nu)} \left[ (1+\nu) I^{w}_{\alpha} + \nu I^{\Psi}_{\alpha} \right] \left[ (1+\nu) I^{w}_{\beta} + \nu I_{\beta}{}^{\Psi} \right] \Big\} \end{aligned} \tag{38}$$

These constants only depend on the geometry of the cross-section and the Poisson ratio $\nu$ and constitute a generalization of the classical Timoshenko constants [8].

## 5.2 Analysis of the longitudinal beam response

In the following 1-D boundary value problems it will be considered the general case of loading $f_i$, $g_i$ and $p_i$. The boundary conditions on the beam are assumed to be a combination of given forces and specified displacements an the cross sections at $x_3 = 0$ and $x_3 = L$. In order to simplify the notation of the expressions of these 1-D boundary value problems the following functions of the applied loads, representing their resultant values at cross-section level, will be introduced.

-External forces and moments resultants per unit of span length:

$$\begin{aligned} q_{\alpha} &= \int_{\Omega} f_{\alpha} d\Omega + \int_{\gamma} g_{\alpha} d\gamma, \quad q_3 = \int_{\Omega} f_3 d\Omega + \int_{\gamma} g_3 d\gamma \\ m_{\alpha} &= \int_{\Omega} x_{\alpha} f_3 d\Omega + \int_{\gamma} x_{\alpha} g_3 d\gamma, \quad m_3 = \int_{\Omega} (x_2 f_1 - x_1 f_2) d\Omega + \int_{\gamma} (x_2 g_1 - x_1 g_2) d\gamma \end{aligned} \tag{39}$$

The forces $q_{\alpha}$ and $m_{\alpha}$ can be combined into a single resultant force $\overline{q}_{\alpha}$ given by the expression:

$$\overline{q}_{\alpha} = q_{\alpha} + m'_{\alpha} \tag{40}$$

The forces and moments stress resultants at the beam ends $\Omega_0$ and $\Omega_L$ are $(l = 0, 1)$:

$$\overline{Q}_{\alpha l} = \int_{\Omega} h_{\alpha l} d\Omega, \quad \overline{N}_{3l} = \int_{\Omega} h_{3l} d\Omega, \quad \overline{M}_{\alpha l} = \int_{\Omega} x_{\alpha} h_{3l} d\Omega, \quad \overline{M}_{3l} = \int_{\Omega} (x_2 h_{1l} - x_1 h_{2l}) d\Omega, \tag{41}$$

-Non standard functions resultants of the external loads

$$
\begin{aligned}
q_{0\alpha} =&\ \int_\Omega \Phi_{\alpha\beta} f_\beta d\Omega + \int_\gamma \Phi_{\alpha\beta} g_\beta d\gamma, \quad q_{03} = \frac{1}{2}\int_\Omega (x_1^2 + x_2^2) f_3 d\Omega + \frac{1}{2}\int_\gamma (x_1^2 + x_2^2) g_3 d\gamma \\
q_{w3} =&\ \int_\Omega w f_3 d\Omega + \int_\gamma w g_3 d\gamma, \quad m_0 = \int_\Omega x_\alpha f_\alpha d\Omega + \int_\gamma x_\alpha g_\alpha d\gamma \\
q_{r\alpha} =&\ \int_\Omega r_\alpha f_3 d\Omega + \int_\gamma r_\alpha g_3 d\gamma, \quad q_{s\alpha} = \int_\Omega s_\alpha f_3 d\Omega + \int_\gamma s_\alpha g_3 d\gamma
\end{aligned}
\tag{42}
$$

It is convenient to combine these forces to obtain the following ones:

$$
\overline{m}_0 = m_0 + q'_{03}, \quad \overline{m}_3 = m_3 + q'_{w3}, \quad \overline{n}_3 = -\frac{\nu(I_1 + I_2)}{2} q'_3 - \nu \frac{H_\alpha}{I_\alpha} \overline{q}_\alpha + \nu \overline{m}_0 \tag{43}
$$

Now the displacements $u_i(x_1, x_2, x_3)$ may be written in the following way:

$$
u_i(x_1, x_2, x_3) = u_i^0(x_1, x_2, x_3) + u_i^2(x_1, x_2, x_3) \tag{44}
$$

where

$$
u_\alpha^0 = U_\alpha^0 \tag{45}
$$

$$
u_3^0 = U_3^0 - x_\alpha U_\alpha^{0\prime} \tag{46}
$$

$$
u_\alpha^2 = U_\alpha^2 - y_\alpha \Theta_3^2 - \nu \left[ x_\alpha U_3^{0\prime} - \Phi_{\alpha\beta} U_\beta^{0\prime\prime} \right] \tag{47}
$$

$$
\begin{aligned}
u_3^2 =&\ U_3^2 - x_\alpha U_\alpha^{2\prime} - w \Theta_3^{2\prime} + \nu \left[ \frac{1}{2}(x_1^2 + x_2^2) - \frac{1}{2A}(I_1 + I_2) \right] U_3^{0\prime\prime} \\
&+ \left[ (1+\nu) r_\alpha + \nu s_\alpha \right] U_\alpha^{0\prime\prime\prime} + \frac{2(1+\nu)}{E} w^0
\end{aligned}
\tag{48}
$$

The functions $U_i^0 = U_i^0(x_3)$,$U_i^2 = U_i^2(x_3)$ and $\Theta_3^2 = \Theta_3^2(x_3)$ only depend on $x_3$ and they are the solutions of the different boundary values problems in $[0, L]$. In general in these problems it will be considered the possibility of existence of imposed displacements and rotations at the beam cross sections as boundary conditions, i.e. the following values may be are data ($l = 0, 1$): $\overline{U}_{il}$ and $\overline{\Theta}_{il}$. These boundary values problems describe the different beam behaviours according to the asymptotic expansion order considered, and they are listed below:

**Stretching**

*-Order 0*

$$
-EAU_3^{0\prime\prime} = q_3 \ \ in \ (0, L) \tag{49}
$$

Displacement boundary conditions

$$
U_3^0(0) = \overline{U}_{30}, \qquad U_3^0(L) = \overline{U}_{31} \tag{50}
$$

Force boundary conditions

$$
EAU_3^{0\prime}(0) = -\overline{N}_{30}, \qquad EAU_3^0(L) = \overline{N}_{31} \tag{51}
$$

*-Order 2*

$$-EAU_3^{2''} = \overline{n}_3{}' \qquad in \quad (0, L) \tag{52}$$

Displacement boundary conditions $(a = 0, L)$

$$U_3^0(a) = 0 \tag{53}$$

Force boundary conditions $(a = 0, L)$ and the respective values $(\varepsilon = 1, -1)$

$$\varepsilon EAU_3^{2'}(a) = \frac{\nu(I_1 + I_2)}{2} q_3'(a) + \nu \frac{H_\alpha}{I_\alpha} \overline{q}_\alpha(a) - \nu m_0(a) = -\overline{n}_3(a) \tag{54}$$

**Torsion**

*-Order 2*

$$-GJ\Theta_3^{2''} = \overline{m}_3 - \frac{1}{2(1+\nu)} \left[(1+\nu)I_\alpha^w + \nu I_\alpha^\Psi\right] \overline{q}_\alpha \qquad in \quad (0, L) \tag{55}$$

where $G = \frac{E}{2(1+\nu)}$

Displacement boundary conditions $(a = 0, L)$ and the respective values $l = 0, 1$

$$\Theta_3^2(a) = \frac{\nu}{I_1 + I_2} \left[H_2 U_1^{0''}(a) - H_1 U_2^{0''}(a)\right] + \overline{\Theta}_{3l} \tag{56}$$

Force boundary conditions $(a = 0, L)$ and the respective values $(\varepsilon = 1, -1)$

$$\varepsilon GJ\Theta_3^{2'}(a) = G\left[(1+\nu)(I_\alpha^w + \nu I_\alpha^\Psi\right] U_\alpha^{0'''}(a) + q_{w3}(a) - \overline{M}_{3l} \tag{57}$$

**Bending**

*-Order 0* (no sum on $\alpha$)

$$EI_\alpha U_\alpha^{0''''} = \overline{q}_\alpha \qquad in \quad (0, L) \tag{58}$$

Displacement boundary conditions $(a = 0, L)$ and the respective values $l = 0, 1$

$$U_\alpha^0(a) = \overline{U}_{\alpha l}, \;\; U_\alpha^{0'}(a) = \overline{\Theta}_{\alpha l} \tag{59}$$

Force boundary conditions $(a = 0, L)$ and the respective values $(\varepsilon = 1, -1)$

$$\varepsilon EI_\alpha U_\alpha^{0''}(a) = \overline{M}_{\alpha l}, \;\; \varepsilon EI_\alpha U_\alpha^{0'''}(a) = -\overline{Q}_{\alpha l} - m_\alpha(a) \tag{60}$$

*-Order 2* (no sum on $\alpha$)

$$\begin{aligned} EI_\alpha U_\alpha^{2''''} &= -T_{\alpha\beta}\overline{q}_\beta'' - \frac{1}{J}\left[(1+\nu)I_\alpha^w + \nu I_\alpha^\Psi\right] \overline{m}_3'' - \frac{\nu H_\alpha}{A} q_3''' + \\ &+ \left[(1+\nu)q_{r\alpha}''' + \nu q_{s\alpha}'''\right] + \nu q_{0\alpha}'' \qquad in \quad (0, L) \end{aligned} \tag{61}$$

Displacement boundary conditions ($a = 0, L$)

$$U_\alpha^2(a) = -\frac{\nu}{2A}(I_\alpha - I_\beta)U_\alpha^{0''} \quad (\alpha \neq \beta) \tag{62}$$

$$\begin{aligned} U_\alpha^{2'}(a) &= \frac{1}{I_\alpha}\left\{\frac{1}{G}\int_\Omega x_\alpha w^0(a)d\Omega - \frac{1}{2}I_\alpha^w \Theta_3^{2'}(a) - \frac{\nu H_\alpha}{EA}q_3(a) \right. \\ &\left. + \left[(1+\nu)L_{\alpha\beta}^r + L_{\alpha\beta}^s\right] U_\beta^{0'''}(a)\right\} \end{aligned} \tag{63}$$

Force boundary conditions ($a = 0, L$) and the respective values ($\varepsilon = 1, -1$)

$$\begin{aligned} \varepsilon E I_\alpha U_\alpha^{2''}(a) &= -T_{\alpha\beta} q_\beta(a) - \frac{1}{J}\left[(1+\nu)I_\alpha^w + \nu I_\alpha^\Psi\right]\overline{m}_3(a) \\ &- \frac{\nu H_\alpha}{A}q_3'(a) - \left[(1+\nu)q_{r\alpha}'(a) + \nu q_{s\alpha}'(a)\right] + \nu q_{0\alpha}(a) \end{aligned} \tag{64}$$

$$\begin{aligned} \varepsilon E I_\alpha U_\alpha^{2'''}(a) &= -T_{\alpha\beta} q_\beta'(a) - \frac{1}{J}\left[(1+\nu)I_\alpha^w + \nu I_\alpha^\Psi\right]\overline{m}_3'(a) \\ &- \nu\frac{H_\alpha}{A}q_3''(a) - \left[(1+\nu)q_{r\alpha}''(a) + \nu q_{s\alpha}''(a)\right] + \nu q_{0\alpha}'(a) \end{aligned} \tag{65}$$

The additional warping depend on the cross-section $\Omega$ and the acting forces, and it is found from the following 2-D boundary value problem:

$$\begin{aligned} -w^0{}_{,\alpha\alpha} &= \tfrac{1}{A}q_3 - f_3 \qquad in \quad \Omega \\ w_{,n}^0 &= g_3 \qquad on\ \gamma \\ \int_\Omega w^0 d\Omega &= 0 \end{aligned} \tag{66}$$

The above ordinary equations with any combination of displacement and force boundary conditions corresponding to stable beams leads to an unique solution of the displacements. Then it is possible using the technique presented in [7] to obtain the stiffness matrix and the equivalent consistent forces in the framework of this general beam theory

In a similar way the following expressions for the stresses and stress resultants can be found from the displacements $u_i(x_1, x_2, x_3)$ given by (44):

**Stresses**

$$\sigma_{33}^0 = E\left(U_3^0 - x_\alpha U_\alpha^{0''}\right) \tag{67}$$

$$\sigma_{31}^0 = G\left\{-\Psi_{,2}\Theta_3^{2'} + \left[(1+\nu)r_{\beta,1} + \nu(s_{\beta,1} + \Phi_{1\beta})\right]U_\beta^{0'''}\right\} + w_{,1}^0 \tag{68}$$

$$\sigma_{32}^0 = G\left\{-\Psi_{,1}\Theta_3^{2'} + \left[(1+\nu)r_{\beta,2} + \nu(s_{\beta,2} + \Phi_{2\beta})\right]U_\beta^{0'''}\right\} + w_{,2}^0 \tag{69}$$

$$\begin{aligned} \sigma_{\alpha\beta}^0 &= \frac{\nu E}{(1+\nu)(1-\nu)}\left(u_{3,3}^2 - U_3^{2'} + x_\mu U_\mu^{2''}\right)\delta_{\alpha\beta} = \\ &= -w\Theta_3^{2''} + \nu\left[\frac{1}{2}(x_1^2 + x_2^2) - \frac{1}{2A}(I_1 + I_2)\right]U_3^{0'''} + \left[(1+\nu)r_\alpha + \nu s_\alpha\right]U_\alpha^{0''''} + \frac{1}{G}w_{,3}^0 \end{aligned} \tag{70}$$

$$\begin{aligned}
\sigma_{33}^2 &= Eu_{3,3}^2 + \nu\sigma_{\mu\mu}^0 = \\
&= E\left\{U_3^{2'} - x_\alpha U_\alpha'' - w\Theta_3^{2''} - \nu\left[\frac{1}{2}(x_1^2 + x_2^2) - \frac{1}{2A}(I_1 + I_2)\right] U_3^{0'''}\right. \\
&\quad \left. + \left[(1+\nu)r_\alpha + \nu s_\alpha\right] U_\alpha^{0''''}\right\} + 2(1+\nu)w,_3^0 + \nu\sigma_{\mu\mu}^0 \qquad (71)
\end{aligned}$$

Then

$$\sigma_{33} = \sigma_{33}^0 + \sigma_{33}^2 \qquad (72)$$

$$\sigma_{3\alpha} = \sigma_{3\alpha}^0 \qquad (73)$$

$$\sigma_{\alpha\beta} = \sigma_{\alpha\beta}^0 \qquad (74)$$

**Stress resultants**

The stress resultants are according to (23) (no sum on $\alpha$ ):

$$M_\alpha^0 = -EI_\alpha U_\alpha^{0''} \qquad (75)$$

$$Q_\alpha^0 = -EI_\alpha U_\alpha^{0'''} + m_\alpha \qquad (76)$$

$$\begin{aligned}
M_\alpha^2 &= -EI_\alpha u_{\alpha,33}^2 + m_\alpha^2 = \\
&= -EI_\alpha\left[U_\alpha^{2''} - y_\alpha\Theta_3^{2''} - \nu(x_\alpha U_3^{0'''} - \Phi_{\alpha\beta}U_\beta^{0'''})\right] + m_\alpha^2 \qquad (77)
\end{aligned}$$

where

$$\begin{aligned}
m_\alpha^2 &= -T_{\alpha\beta}q_\beta - \frac{1}{J}\left[(1+\nu)I_\alpha + \nu I_\alpha^\Psi\right]\overline{m}_3 \\
&- \frac{\nu H_\alpha}{A}q_3' - \left[(1+\nu)q_{r\alpha}' + \nu q_{s\alpha}'\right] + \nu q_{0\alpha} \qquad (78)
\end{aligned}$$

## 6 Beam cross-section functions and constants determination

It have been shown in the previous section that the constants and functions of the beam cross-section given by the formulae (24) to (37) define completely the structural response of the beam subjected to general loading. This general loading is composed by the loads $f_i, g_i$ and $p_i$, and imposed displacements $u_i$ on a part of the boundary of the beam solid. Therefore, prior to carry out a beam structural analysis the knowledge of these functions and constants of the beam is necessary. In order to compute them in case of an arbitrary beam cross section numerical procedures have to be applied. Among these procedures Finite Element Methods (FEM) and Boundary Element Method (BEM) will be considered. Respective good introductions are the references [10] and [2]. In this respect it should be pointed out that in [4] the shear functions $r_\alpha$ and $s_\alpha$ functions have been computed in order to compare the shear stresses distributions with the values obtained by the standard Strength of Materials formulae.

Due to the simplicity of the boundary value problems involved in the computation of the beam functions and constants the FEM application is straightforward. In fact, a typical case may be described as follows:

Find the function $F = F(x_1, x_2)$, unique solution of the following elliptic boundary value problem

$$\begin{aligned} -F_{,\alpha\alpha} &= G_0 \quad in \quad \Omega \\ F &= G_a \quad on \quad \gamma_a \\ F_{,n} &= G_b \quad on \quad \gamma_b \end{aligned} \tag{79}$$

where the functions $G_0 = G_0(x_1, x_2)$, $G_a = G_s(\gamma_a)$ and $G_b = G_s(\gamma_b)$ are data, and $\gamma_a \in \gamma, \gamma_b \in \gamma$ such that $\gamma_a \cup \gamma_b = \gamma$ and $\gamma_a \cap \gamma_b = \emptyset$ are the length of the cross-section boundary.

As an example, the determination of the warping function $w$, equation (29), corresponds to a Neumann problem, i.e. $G_b = 0, G_a = -y_\alpha$ and $\gamma_a = \emptyset, \gamma_b = \gamma$. The uniqueness of the solution is achieved by introducing the additional condition:

$$\int_\Omega F d\Omega = 0 \tag{80}$$

Similar treatment should be applied to the computation of the shear functions $r_\alpha$ and $s_\alpha$. However, the Prandtl function has to be found in case of a simply connected domain $\Omega$ as a Dirichlet problem, i.e. $G_0 = 2, G_a = 0$ and $\gamma_a = \gamma, \gamma_b = \emptyset$. If the domain $\Omega$ is multiple connected then the boundary value problem (32) is of mixed type.

Then, the application of the FEM , i.e. the introduction the expression $F = \mathbf{NF}$ to the equation (79) leads to the following discrete counterpart

$$\mathbf{KF} = \mathbf{G} \tag{81}$$

where

$$\mathbf{K} = \int_\Omega \mathbf{N}_{,\alpha}^{\mathbf{T}} \mathbf{N}_{,\alpha} \mathbf{d\Omega}, \quad \mathbf{G} = \int_\Omega \mathbf{N}^{\mathbf{T}} \mathbf{G_0} \mathbf{d\Omega} \tag{82}$$

The row vector $\mathbf{N} = (N_1, N_2, \ldots N_m)$ contains $m$ shape or interpolation functions and the column vector $\mathbf{F} = (F_1, F_2, \ldots F_m)$ the unknown values of $F$ at nodes of the Finite Element mesh.

The matrix $\mathbf{K}$ and the vector $\mathbf{G}$ of the system of linear equation (81) should be modified in order to introduce the boundary conditions of (79). In case of Neumann type boundary value problems the singularity of the system of linear equations (81) is eliminated by assuming a given value to one of the components of $\mathbf{F}$, for example, $F_1 = 1$ . Then, in order to obtain the actual value $F_1$ the condition (80) is used in the following way. The actual solution vector $\tilde{\mathbf{F}}$ is found as the sum of the previous one, $\mathbf{F}$ plus a translation given by $F_1 \mathbf{1}$, with $\mathbf{1}$ is the m-th column vector with all their elements equal to the unit. The condition (80 ) can now be written:

$$\int_\Omega \mathbf{N}\tilde{\mathbf{F}} d\Omega = \int_\Omega \mathbf{N}(\tilde{\mathbf{F}} + F_I \mathbf{1}) d\Omega = 0 \tag{83}$$

i.e.

$$F_1 = -\frac{\int_\Omega \mathbf{N}\mathbf{F}d\Omega}{\int_\Omega \mathbf{N}\mathbf{1}d\Omega} \tag{84}$$

and then finally $\tilde{\mathbf{F}} = \mathbf{F} + F_I\mathbf{1}$.

Once the vector $\tilde{\mathbf{F}}$ containing the nodal values of the beam cross-section function is known the beam constants are obtained by an integration over the domain $\Omega$. This can be accomplished directly by Gauss numerical integration formulae over each element.

The finite elements used in the computation of the beam cross-section properties have been $C^0$-triangles with six nodes.

Alternately the BEM has been used for computation of the functions and constants of the beam cross-section. In the BEM the chosen fundamental function for the node $n$ of coordinates $(x_{1n}, x_{2n})$ is

$$f^*(x_1, x_2) = \frac{1}{2\pi}\ln\left(\frac{1}{r}\right) \tag{85}$$

where $r$ is the norm of the vector joining the node $n$ with the point $(x_1, x_2$. The normal derivative along the boundary $\gamma$ will be written as $q^* = f^*_{,n}$.

The boundary $\gamma = \gamma_0 \cup \gamma_1 \cup \ldots \cup \gamma_k$ is divided into $m = m_0 + m_1 + \ldots + m_k$ straight elements. (The number of elements dividing the boundary $\gamma_i$ is $m_i$). The following system of equations can be derived at each boundary node $n$ (no sum on $n$):

$$c^n f_n + \sum_{i=1}^{m}\int_{L_i} f q^* d\gamma = \sum_{i=1}^{m}\int_{L_i} q f^* d\gamma \tag{86}$$

where $L_i$ is the length of the i-th element, $f = F(x_1, x_2)$ and $q = q(x_1, x_2) = f_{,n}$ are the unknown functions to be found dependent on the point $(x_1, x_2) \in \gamma,\ x_1 = x_1(s), x_2 = x_2(s)$ situated along the boundary. They can be linearly interpolated in each element $n$ as follows:

$$f(s) = \frac{1}{2}(1-\eta)f_1 + \frac{1}{2}(1+\eta)f_2 = (N_1, N_2)\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \tag{87}$$
$$q(s) = \frac{1}{2}(1-\eta)q_1 + \frac{1}{2}(1+\eta)q_2 = (N_1, N_2)\begin{pmatrix} q_1 \\ q_2 \end{pmatrix}$$

in which $s$ is the arc length coordinate of the beam boundary, $\eta = \frac{s}{L_i/2}$ and $f_1, f_2, q_1, q_2$ are the unknown values of $f$ and $q$ at nodes 1 and 2 of the element (local numbering) respectively (Figure 2).

Introducing (87) into (86), the equation (86) corresponding to the node $n$ and considering global node numbering becomes (no sum on $n$):

$$c^n f_n + H_{ni}f_i = G_{ni}q_i \quad (i = 1, 2, \ldots, m; \quad (n = 1, 2, \ldots, m) \tag{88}$$

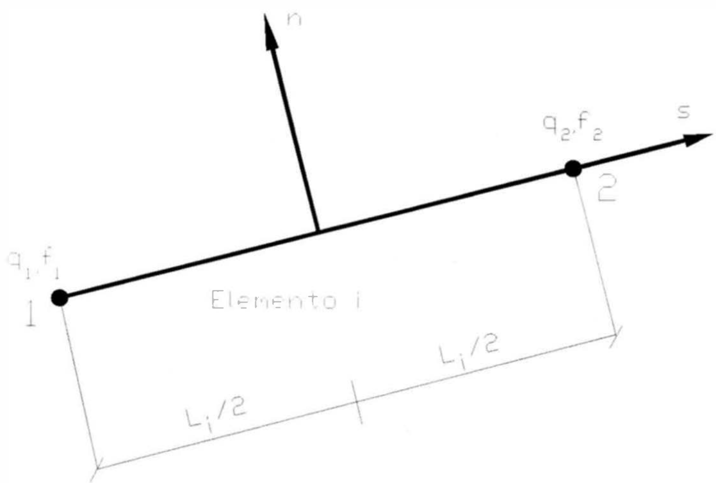


Figure 2: Boundary element

where

$$H_{ni} = \int_{L_i^+} N_1 q^* d\gamma + \int_{L_i^-} N_2 q^* d\gamma, \quad G_{ni} = \int_{L_i^+} N_1 f^* d\gamma + \int_{L_i^-} N_2 f^* d\gamma \tag{89}$$

and $L_i^-$ and $L_i^+$ are the lengths of the elements prior and posterior to the node $i$. The values of the fundamental function and its normal derivative at point $n$ on the boundary $\gamma$ are denoted by $f_n$ and $g_n$ respectively.

The value of $c^n$ can be computed using the condition:

$$c^n + \sum_{i=1}^{m} H_{ni} = 0 \tag{90}$$

The equation (89) may be written in matrix form as

$$\mathbf{HU} = \mathbf{GQ} \tag{91}$$

In the mixed problem (79) part of the elements of the vector $\mathbf{U}$ are known (the ones belonging to the boundary $\gamma_a$) and part of the elements of the vector $\mathbf{Q}$ are also known (the ones situated along the boundary $\gamma_b$). Then the remaining parts of elements of $\mathbf{U}$ and $\mathbf{Q}$ are unknowns and they can be obtained from the system of linear equations (91) modified by algebraic manipulation, i.e. by changing terms from one hand of the equation to the other. This resultant system of linear equations is in general non symmetric and non positive definite, and therefore the standard Gauss elimination procedure can not be applied.

Once the unknown parts of the vectors $\mathbf{U}$ and $\mathbf{Q}$ have been computed, the constants are obtained by means of integral over the domain $\Omega$. Neumann boundary values problems are treated in similar way as in the FEM, however contrary to the FEM in this case it is necessary to compute the values of the functions $f(x_1, x_2)$ and $g(x_1, x_2)$ at internal points of the domain, i.e. $(x_1, x_2) \in \Omega$. Also, the determination of the beam constants demands integration over the cross-section domain. In order to carry out this integration a set equidistant internal points, i.e. a regular grid of points is selected. In this way the integrals over the domain $\Omega$ are easily computed. In the general case of multiple connected domains (k-th connected domains) the internal grid points should be checked if they belong to $\Omega$. In this respect the following test has been used: A point $(x_1^0, x_2^0)$ belongs to $\Omega$ if a ray with origin at the point intersect an odd number of times the boundary $\gamma = \gamma_0 \cap \gamma_1 \cap \ldots \cap \gamma_k$ of $\Omega$. The ray may be defined by the equation:

$$x_1 = x_1^0 + \rho \cos\alpha, \quad x_2 = x_2^0 + \rho \sin\alpha, \quad \rho > 0 \tag{92}$$

## 7 Illustrative results

In order to validate the application of the two procedures, FEM and BEM, just discussed some simple cross-sections for which theoretical values are known in closed form will be considered, namely a circle of unit radius $R = 1$, the square of side $a = 1$ and a half circle of radius $R = 1$. The theoretical values for these sections are collected in Table 1.

| **Beam constants** | | | | | | | |
|---|---|---|---|---|---|---|---|
| Constant | Square | | Circle | | Half circle | | L-shaped |
| | Theor. | Num. | Theor. | Num. | Theor. | Num. | Num. |
| $A$ | 1.000 | 1.000 | 3.142 | 3.142 | 1.570 | 1.561 | 6.000 |
| $I_1$ | 0.083 | 0.083 | 0.785 | 0.785 | 0.392 | 0.388 | 2.500 |
| $I_2$ | 0.083 | 0.083 | 0.785 | 0.785 | 0.108 | 0.108 | 10.000 |
| $H_1$ | 0. | 0. | 0. | 0. | 0. | 0. | 1.789 |
| $H_2$ | 0. | 0. | 0. | 0. | -0.013 | -0.014 | -1.342 |
| $H_3$ | 0.010 | 0.010 | 0.262 | 0.259 | 0.064 | 0.064 | 10.319 |
| $I_1^\omega$ | 0. | 0. | 0. | 0. | 0.066 | 0.065 | 2.493 |
| $I_2^\omega$ | 0. | 0. | 0. | 0. | 0. | 0. | 15.713 |
| $I_1^\psi$ | 0. | 0. | 0. | 0. | 0.007 | 0.007 | 0.084 |
| $I_2^\psi$ | 0. | 0. | 0. | 0. | 0. | 0. | 0.170 |
| $J$ | 0.140 | 0.141 | 1.570 | 1.571 | 0.296 | 0.291 | 1.864 |
| $J_\omega$ | 0. | 0. | 0. | 0. | 0. | 0.009 | 8.107 |
| $T_{11}$ | 0.232 | 0.227 | 0.664 | 0.659 | 0.746 | 0.729 | 2.542 |
| $T_{12}$ | 0. | 0. | 0. | 0. | 0. | -0.060 | 1.475 |
| $T_{21}$ | 0. | 0. | 0. | 0. | 0. | -0.002 | 5.738 |
| $T_{22}$ | 0.232 | 0.227 | 0.664 | 0.659 | 0.161 | 0.164 | 15.444 |

Some illustrative beam functions are also represented in figures 3, 4 and 5.

These results are compared in Table 1 with the ones found by application of the FEM and BEM.

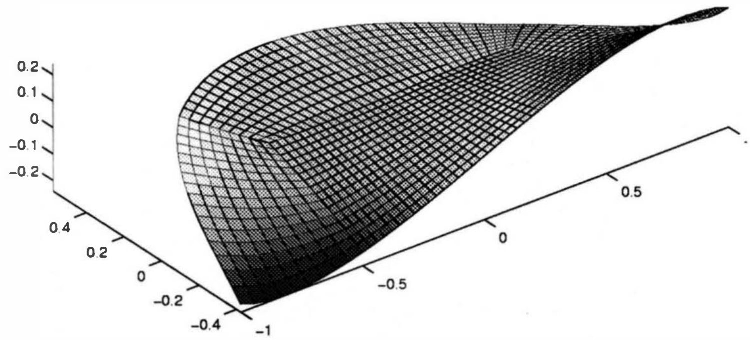


Figure 3: Half circular section. Warping function

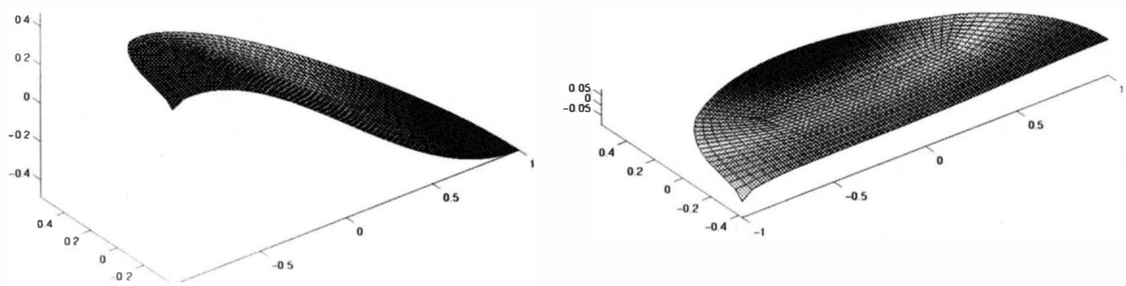

Figure 4: Half circular section. Shear functions $r_1$, $r_2$

In the FEM models the total number of degrees of freedom (dof) used was 1,000 approximately. In the BEM models this number was 100 and the total number of the grid points needed for the integrals evaluation was 400 approximately. A good agreement between the results from both methods has been reached and they also agree reasonably well with the theoretical values, as it can be observed in Table 1. In fact, absolute differences less than $10^{-2}$ for the beam constants and for the maximum differences between ordinates of the beam functions even smaller differences have been found. In order to reach a good agreement between theoretical and numerical results it was observed the importance of an accurate modelling of the section geometry.

Finally, a non symmetric cross-section has been studied, namely, the section "L" which dimensions are shown in figure 6. The values of the constants obtained for this section are also given in table 2.

## 8 Conclusions

A general beam theory has been developed in [9] using an asymptotic expansion technique. The most well known classical beam theories, such as, Navier-Bernoulli, Timoshenko, Vlasov and the Saint Venant torsion are included as special cases, in this general beam theory, without using any additional hypothesis to the ones of the linear elasticity. In this way, it is possible to obtain in the framework of this general beam theory the corresponding stiffness matrices and

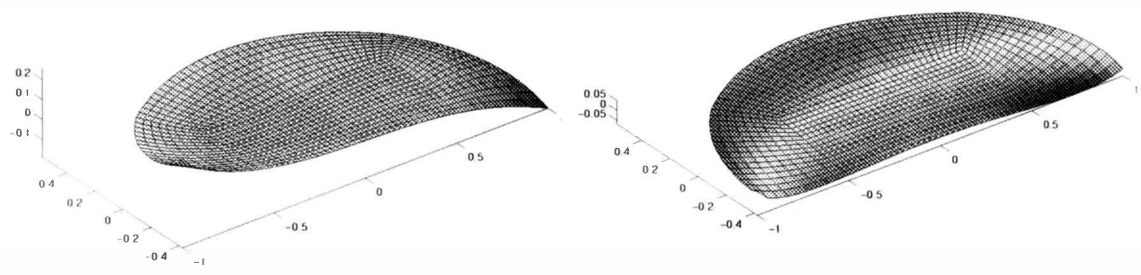

Figure 5: Half circular section. Shear functions $s_1$, $s_2$

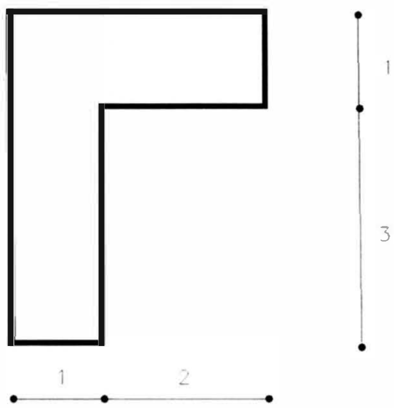


Figure 6: L-shaped section

equivalent nodal forces of the beam and also the stress distribution over the section due to the application of the stress resultants.

The obtained results are given in terms of a set of constants and functions to be found as solutions of laplacian boundary value problems over the beam cross section and standard 1-D boundary value problems on the coordinate $x_3$ i.e. the beam axis along the beam length.

From the examples presented it can be concluded that any of the well known numerical methods FEM and BEM are suitable to determine the functions and constants of the beam. Both methods are computationally efficient and accurate enough for practical structural analysis. These methods can be applied in a systematic way to obtain the characteristics of different beam sections, either massive ones normally used in concrete beams or thin-walled sections typical in the design of the standard rolled steel beams.

# 9 Acknowledgements

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