JOURNAL OF MATHEMATICAL PHYSICS **55**, 082501 (2014)

First-order equivalent to Einstein-Hilbert Lagrangian

M. Castrillón López,[1,a)] J. Muñoz Masqué,[2,b)] and E. Rosado María[3,c)]
[1]*ICMAT (CSIC-UAM-UC3M-UCM), Departamento de Geometría y Topología, Facultad de Matemáticas, UCM, Avda. Complutense s/n, 28040-Madrid, Spain*
[2]*Instituto de Física Aplicada, CSIC, C/ Serrano 144, 28006-Madrid, Spain*
[3]*Departamento de Matemática Aplicada, Escuela Técnica Superior de Arquitectura, UPM, Avda. Juan de Herrera 4, 28040-Madrid, Spain*

(Received 2 September 2013; accepted 6 July 2014; published online 25 July 2014)

A first-order Lagrangian L^∇ variationally equivalent to the second-order Einstein-Hilbert Lagrangian is introduced. Such a Lagrangian depends on a symmetric linear connection, but the dependence is covariant under diffeomorphisms. The variational problem defined by L^∇ is proved to be regular and its Hamiltonian formulation is studied, including its covariant Hamiltonian attached to ∇. © *2014 AIP Publishing LLC*. [http://dx.doi.org/10.1063/1.4890555]

I. INTRODUCTION

Let $p : \mathfrak{M} \to M$ be the bundle of pseudo-Riemannian metrics of a given signature (n^+, n^-), $n^+ + n^- = n = \dim M$, over a connected C^∞ manifold oriented by a volume form $\mathbf{v} \in \Omega^n(M)$. The Einstein-Hilbert (or E-H for short) functional is the second-order Lagrangian density $L_{EH}\mathbf{v}$ on $\mathfrak{M}$ defined along a metric g by $s^g \mathbf{v}_g$, where s^g denotes the scalar curvature of g and $\mathbf{v}_g$ its Riemannian volume form; namely,

$$L_{EH} \circ j^2 g = \sqrt{|\det(g_{ab})|}\, g^{jk} \left\{ \frac{\partial (\Gamma^g)^i_{jk}}{\partial x^i} - \frac{\partial (\Gamma^g)^i_{ik}}{\partial x^j} + \left(\Gamma^g\right)^l_{jk} \left(\Gamma^g\right)^i_{il} - \left(\Gamma^g\right)^l_{ik} \left(\Gamma^g\right)^i_{jl} \right\}, \tag{1}$$

where $(\Gamma^g)^i_{jk}$ are the Christoffel symbols of the Levi-Civita connection ∇^g of the metric g. As is known (e.g., see Secs. 3.3.1 and 3.3.2 of Ref. 1), the first-order Lagrangian L_1 defined along g by $\sqrt{|\det(g_{ab})|}\, g^{jk}((\Gamma^g)^l_{ij}(\Gamma^g)^i_{kl} - (\Gamma^g)^l_{jk}(\Gamma^g)^i_{il})$ differs from L_{EH} by a divergence term, but unfortunately L_1 is *not* an invariantly defined quantity.

Below, we present a completely covariant description of a first-order Lagrangian L^∇ which is variationally equivalent to E-H Lagrangian L_{EH}. Consequently, L^∇ defines the same Euler-Lagrange equations as L_{EH}, namely, Einstein's field equations in the vacuum for arbitrary signature. In particular, this explains why the E-H Lagrangian admits a true first-order Hamiltonian formalism. The difference of our approach with respect to the similar Lagrangian in Ref. 5 is the geometric construction of it, compared with the coordinate expression developed in that article. The purely geometric study of this topic allows one to a better understanding of its structure; specially, from the standpoint of the geometric theory of classical fields.

In addition, although L^∇ depends on an auxiliary symmetric linear connection ∇, this dependence is natural with respect to the action of diffeomorphisms of M on connections and on Lagrangian functions, as proved in Sec. IV. This fact justifies the construction of such a Lagrangian and the interest of its existence.

Furthermore, the Lagrangian L^∇ is seen to be regular and its Hamiltonian formulation is studied, computing explicitly its momenta functions and the covariant Hamiltonian attached to ∇ in the sense of Ref. 11.

[a)]E-mail: mcastri@mat.ucm.es
[b)]E-mail: jaime@iec.csic.es
[c)]E-mail: eugenia.rosado@upm.es

This article is copyrighted as indicated in the article. Reuse of AIP content is subject to the terms at: http://scitation.aip.org/termsconditions. Downloaded to IP: 147.96.20.53 On: Fri, 25 Jul 2014 14:12:23

II. THE EQUIVALENT LAGRANGIAN L^{∇} DEFINED

The difference tensor field between the Levi-Civita connection ∇^g of a metric g and a given symmetric linear connection ∇ on M is the 2-covariant 1-contravariant tensor given by

$$T^{g,\nabla} = \nabla^g - \nabla = \left((\Gamma^g)^h_{ij} - \Gamma^h_{ij}\right) dx^i \otimes dx^j \otimes \frac{\partial}{\partial x^h},$$

where $(\Gamma^g)^i_{jk}$ (resp. Γ^i_{jk}) are the Christoffel symbols of the connection ∇^g (resp. ∇). A Lagrangian function L^{∇} on the bundle of metrics $p : \mathfrak{M} \to M$ is defined as follows:

$$L^{\nabla}\left(j^2_x g\right) \mathbf{v}_x = \left\{s^g(x) + c\left(\left(\mathrm{alt}_{23}\left(\nabla^g T^{g,\nabla}\right)_x\right)^{\sharp}\right)\right\}\left(\mathbf{v}_g\right)_x, \quad \forall j^2_x g \in J^2\mathfrak{M}, \tag{2}$$

where we confine ourselves to consider coordinate systems $(x^1, \dots, x^n)$ on M adapted to $\mathbf{v}$, i.e.,

$$\mathbf{v} = dx^1 \wedge \dots \wedge dx^n, \qquad \mathbf{v}_g = \sqrt{|\det(g_{uv})|}\mathbf{v}, \qquad g = g_{uv} dx^u \otimes dx^v,$$

$\mathrm{alt}_{23} : \otimes^3 T^*M \otimes TM \to \otimes^3 T^*M \otimes TM$ denotes the alternation of the second and third covariant indices, $^{\sharp} : \otimes^3 T^*M \otimes TM \to \otimes^2 T^*M \otimes^2 TM$ is the isomorphism induced by g,

$$w_1 \otimes w_2 \otimes w_3 \otimes X \mapsto w_1 \otimes w_2 \otimes (w_3)^{\sharp} \otimes X, \quad \forall X \in T_x M,\ \forall w_1, w_2, w_3 \in T^*_x M,$$

and, finally, $c : \otimes^2 T^*M \otimes^2 TM \to \mathbb{R}$ denotes the (total) contraction of the first and second covariant indices with the first and second contravariant ones, respectively. We write L^{∇} in order to emphasize the fact that the Lagrangian depends on the auxiliary symmetric linear connection ∇ previously chosen.

If $y_{ij} = y_{ji}$, $i, j = 1, \dots, n$, are the coordinates on the fibres of p induced from a coordinate system $(x^h)^n_{h=1}$ on M, namely, $g_x = y_{ij}(g_x) dx^i \otimes dx^j$ for every metric g_x over $x \in M$, and $(x^h, y_{ij}, y_{ij,k}, y_{ij,kl} = y_{ij,lk})$ denotes the coordinate system induced on $J^2\mathfrak{M}$, then L_{EH} is locally given by

$$L_{EH} = \rho\left(y^{ac}y^{bd} - y^{ab}y^{cd}\right) y_{ab,cd} + L_0, \tag{3}$$

where $(y^{ij}) = (y_{ij})^{-1}$ is the inverse matrix of the symmetric matrix (y_{ij}),

$$\begin{cases} L_0 = \rho y^{ij}\left\{y^{hm}\left(y_{mr,j}G^r_{ih} - y_{mr,h}G^r_{ij}\right) + G^m_{ij}G^h_{hm} - G^m_{ih}G^h_{jm}\right\}, \\ \rho = \sqrt{\left|\det\left(y_{ij}\right)\right|}, \end{cases} \tag{4}$$

and $G^i_{rj} : J^1\mathfrak{M} \to \mathbb{R}$ are defined by $G^i_{rj} = \frac{1}{2} y^{is}\left(y_{rs,j} + y_{js,r} - y_{rj,s}\right)$.

If L'^{∇} is the second-order Lagrangian on $\mathfrak{M}$ determined by the second summand of the right-hand side in the formula (2), namely

$$L'^{\nabla}(j^2_x g) = c\left(\left(\mathrm{alt}_{23}\left(\nabla^g T^{g,\nabla}\right)_x\right)^{\sharp}\right),$$

then (2) can equivalently be rewritten as follows: $L^{\nabla} = L_{EH} + \rho L'^{\nabla}$ and as a calculation shows,

$$\begin{aligned} L'^{\nabla} \circ j^2 g = g^{jr}\Bigg\{ & \frac{\partial\left(T^{g,\nabla}\right)^i_{ri}}{\partial x^j} - \frac{\partial\left(T^{g,\nabla}\right)^i_{rj}}{\partial x^i} \\ & + \left(\Gamma^g\right)^a_{ji}\left(T^{g,\nabla}\right)^i_{ra} - \left(\Gamma^g\right)^a_{jr}\left(T^{g,\nabla}\right)^i_{ai} \\ & + \left(\Gamma^g\right)^a_{ir}\left(T^{g,\nabla}\right)^i_{aj} - \left(\Gamma^g\right)^a_{ai}\left(T^{g,\nabla}\right)^i_{rj}\Bigg\}. \end{aligned} \tag{5}$$

Lemma 2.1. The Lagrangian L^{∇} is of first order.

This article is copyrighted as indicated in the article. Reuse of AIP content is subject to the terms at: http://scitation.aip.org/termsconditions. Downloaded to IP: 147.96.20.53 On: Fri, 25 Jul 2014 14:12:23

Proof. Taking the definition of $T^{g,\nabla}$ and the formulas (5)(1) into account, one obtains

$$\begin{aligned}\sqrt{|\det(g_{uv})|}(L'^{\nabla}\circ j^2g) &= -L_{EH}\circ j^2g \\ &\quad + \sqrt{|\det(g_{uv})|}g^{jr}\left\{\left(\Gamma^g\right)^a_{ji}\left(\Gamma^g\right)^i_{ra}-\left(\Gamma^g\right)^a_{ai}\left(\Gamma^g\right)^i_{rj}\right\} \\ &\quad - \sqrt{|\det(g_{uv})|}g^{jr}\left\{\frac{\partial\Gamma^i_{ri}}{\partial x^j}-\frac{\partial\Gamma^i_{rj}}{\partial x^i}+\left(\Gamma^g\right)^a_{ji}\Gamma^i_{ra}\right. \\ &\quad \left. -\left(\Gamma^g\right)^a_{jr}\Gamma^i_{ai}+\left(\Gamma^g\right)^a_{ir}\Gamma^i_{aj}-\left(\Gamma^g\right)^a_{ai}\Gamma^i_{rj}\right\}.\end{aligned} \tag{6}$$

Hence $(\rho L'^{\nabla}+L_{EH})\circ j^2g$ depends on the values of the metric g and its first derivatives only. □

In fact, the following local expression is readily deduced:

$$L^{\nabla}=\rho y^{jr}\left\{G^a_{ji}T^i_{ra}-G^a_{ai}T^i_{rj}+G^a_{jr}\Gamma^i_{ai}-G^a_{ir}\Gamma^i_{aj}-\frac{\partial\Gamma^i_{ri}}{\partial x^j}+\frac{\partial\Gamma^i_{rj}}{\partial x^i}\right\},$$

$T^i_{jk}: J^1\mathfrak{M}\to\mathbb{R}$ being the functions defined by $T^h_{ij}=G^h_{ij}-\Gamma^h_{ij}$.

Remark 2.1. As L^{∇} has a geometrical definition, the local expression above actually provides a global Lagrangian. Moreover, if ∇ is a flat linear connection and one considers an adapted coordinate system to ∇ (i.e., a coordinate system on which all the Christoffel symbols of ∇ vanish), then the local expression for L^{∇} coincides with the local Lagrangian L_1 defined in the Introduction.

III. L^{∇} AND L_{EH} ARE VARIATIONALLY EQUIVALENT

As a computation shows, the second summand in the definition of L^{∇} can be rewritten in terms of the metric g and the auxiliary connection ∇ only, as follows:

$$\begin{aligned}c\left(\left(\mathrm{alt}_{23}\left(\nabla^gT^{g,\nabla}\right)\right)^{\sharp}\right) &= \left(g^{js}g^{ir}-g^{jr}g^{is}\right)g_{ri,sj} \\ &\quad + \tfrac{1}{2}\left\{\left(2g^{ir}g^{jb}-g^{bi}g^{rj}-g^{br}g^{ij}\right)g^{as}\right. \\ &\quad + \left(g^{ar}g^{ib}+g^{bi}g^{ra}-2g^{ir}g^{ab}\right)g^{js} \\ &\quad - \left(g^{sr}g^{jb}-g^{br}g^{sj}\right)g^{ai} \\ &\quad \left. - \left(g^{ar}g^{sb}-g^{sr}g^{ab}\right)g^{ij}\right\}g_{ab,j}g_{rs,i} \\ &\quad - g^{jr}\left(\tfrac{\partial\Gamma^i_{ri}}{\partial x^j}-\tfrac{\partial\Gamma^i_{rj}}{\partial x^i}\right) \\ &\quad + \tfrac{1}{2}\left\{\left(2g^{js}g^{ar}-g^{jr}g^{as}\right)g_{rj,s}\Gamma^i_{ai}\right. \\ &\quad \left. + \left(g^{jr}g^{ab}-2g^{ar}g^{jb}\right)g_{ab,i}\Gamma^i_{rj}\right\}.\end{aligned}$$

Lemma 3.1. If D_i denotes the total derivative with respect to x^i, then

$$c\left(\left(\mathrm{alt}_{23}\left(\nabla^gT^{g,\nabla}\right)\right)^{\sharp}\right)\mathbf{v}_g=-\left(D_i((L_{EH})^i_{\nabla})\circ j^2g\right)\mathbf{v},$$

where

$$(L_{EH})^i_{\nabla}=\sum_{c\le r}\tfrac{1}{2-\delta_{ib}}\frac{\partial L_{EH}}{\partial y_{cr,ib}}\left(y_{cr,b}-\left(\Gamma^a_{bc}y_{ar}+\Gamma^a_{br}y_{ac}\right)\right). \tag{7}$$

This article is copyrighted as indicated in the article. Reuse of AIP content is subject to the terms at: http://scitation.aip.org/termsconditions. Downloaded to IP: 147.96.20.53 On: Fri, 25 Jul 2014 14:12:23

From this lemma it follows that L^{∇} and L_{EH} are variationally equivalent as, according to the formula (2), one has

$$\left(L^{\nabla}\circ j^2 g\right)\mathbf{v} = \left(L_{EH}\circ j^2 g\right)\mathbf{v} + c\left(\left(\mathrm{alt}_{23}\left(\nabla^g T^{g,\nabla}\right)\right)^{\sharp}\right)\mathbf{v}_g$$
$$= \left\{\left(L_{EH} - D_i\left((L_{EH})^i_{\nabla}\right)\right)\circ j^2 g\right\}\mathbf{v}.$$

Hence $L^{\nabla} = L_{EH} - D_i\left((L_{EH})^i_{\nabla}\right)$ and therefore, L^{∇} and L_{EH} differ in a total divergence.

The proof of Lemma III.1 follows by computing $D_i\left((L_{EH})^i_{\nabla}\right)$ using (3) and (7), taking the identity $D_i\rho = \frac{\rho}{2}y^{rs}y_{rs,i}$ into account, after a simple—but rather long—computation.

IV. DEPENDENCE ON ∇

Below, the dependence of the Lagrangian L^{∇} with respect to the symmetric linear connection ∇, is analysed. First, some geometric preliminaries are introduced.

The image of a linear connection ∇ by a diffeomorphism $\phi : M \to M$ is defined to be $(\phi\cdot\nabla)_X Y = \phi\cdot\left(\nabla_{\phi^{-1}\cdot X}(\phi^{-1}\cdot Y)\right)$, $\forall X, Y \in \mathfrak{X}(M)$. As is well known (e.g., see p. 643 of Ref. 4), the Levi-Civita connection of a metric transforms according to the rule: $\phi^{-1}\cdot\nabla^g = \nabla^{\phi^* g}$. Hence the following formulas hold:

$$\phi^{-1}\cdot T^{g,\nabla} = T^{\phi^* g,\phi^{-1}\cdot\nabla},\quad S^{\phi\cdot\nabla} = (\phi^{-1})^* S^{\nabla} = \phi\cdot S^{\nabla},\quad s^g = s^{\phi^* g},$$

where $S^{\nabla}(X, Y) = \mathrm{trace}(Z \mapsto R^{\nabla}(Z, X)Y)$ is the Ricci tensor of ∇ (e.g., see Sec. VI, p. 248 of Ref. 8). Moreover, the lift of ϕ to the bundle of metrics $p : \mathfrak{M} \to M$ is given by $\bar{\phi}(g_x) = (\phi^{-1})^* g_x$, $\forall g_x \in p^{-1}(x)$ (cf. Ref. 12); hence $p\circ\bar{\phi} = \phi\circ p$, and the mapping $\bar{\phi} : \mathfrak{M} \to \mathfrak{M}$ has an extension to the r-jet bundle $\bar{\phi}^{(r)} : J^r\mathfrak{M} \to J^r\mathfrak{M}$ defined by, $\bar{\phi}^{(r)}\left(j^r_x g\right) = j^r_{\phi(x)}(\bar{\phi}\circ g\circ\phi^{-1})$.

Let $\mathbf{v}_{\mathfrak{M}}$ be the nowhere-vanishing p-horizontal n-form on $\mathfrak{M}$ defined as follows: $(\mathbf{v}_{\mathfrak{M}})_{g_x} = \mathbf{v}_{g_x}$, $\forall g_x \in \mathfrak{M}$, where, as above, $\mathbf{v}_{g_x}$ denotes the Riemannian volume form attached to g_x. Hence $\mathbf{v}_{\mathfrak{M}} = \rho\mathbf{v}$, where ρ is as in (4). Every rth order Lagrangian density Λ on $\mathfrak{M}$ can thus be written as $\Lambda = L\mathbf{v}_{\mathfrak{M}}$ for a certain Lagrangian function $L \in C^{\infty}(J^r\mathfrak{M})$ and Λ is invariant under diffeomorphisms, i.e., $(\bar{\phi}^{(r)})^*\Lambda = \Lambda$, $\forall\phi \in \mathrm{Diff}\,M$, if and only if L is, i.e., $L\circ\bar{\phi}^{(r)} = L$, as $(\bar{\phi}^{(r)})^*\Lambda = (L\circ\bar{\phi}^{(r)})(\bar{\phi}^*\mathbf{v}_{\mathfrak{M}})$ and, according to Proposition 7 of Ref. 13, $\mathbf{v}_{\mathfrak{M}}$ is invariant under diffeomorphisms, i.e., $\bar{\phi}^*\mathbf{v}_{\mathfrak{M}} = \mathbf{v}_{\mathfrak{M}}$.

The E-H Lagrangian density $L_{EH}\mathbf{v}$ is known to be invariant under diffeomorphisms, i.e., $(\bar{\phi}^{(2)})^*(L_{EH}\mathbf{v}) = L_{EH}\mathbf{v}$, $\forall\phi \in \mathrm{Diff}\,M$. In fact, there exists a classical result by Weyl (Appendix II of Ref. 15, also see Refs. 6 and 9), according to which the only $\mathrm{Diff}\,M$-invariant Lagrangians on $J^2\mathfrak{M}$ depending linearly on the second-order coordinates $y_{ab,ij}$ are of the form $\lambda L_{EH} + \mu$, for scalars λ, μ.

Therefore, transforming the equation $L^{\nabla}\mathbf{v} = L_{EH}\mathbf{v} + L'^{\nabla}\mathbf{v}_{\mathfrak{M}}$ by a diffeomorphism ϕ, one obtains $(\bar{\phi}^{(1)})^*\left(L^{\nabla}\mathbf{v}\right) = L_{EH}\mathbf{v} + (L'^{\nabla}\circ\bar{\phi}^{(2)})\mathbf{v}_{\mathfrak{M}}$, and one is led to compute $L'^{\nabla}\circ\bar{\phi}^{(2)}$, which, by using the formulas above, is proved to transform according to the following rule:

$$L'^{\nabla}\circ\bar{\phi}^{(2)} = L'^{\phi^{-1}\cdot\nabla}. \tag{8}$$

V. HAMILTONIAN FORMALISM

A. Regularity of L^{∇}

Proposition 5.1. For $\dim M = n \geq 3$, *the Lagrangian* L^{∇} *is regular, namely, the following square matrix of size* $\frac{1}{2}n^2(n+1)$ *is non-singular*:

$$\left(\frac{\partial p^{uv,w}}{\partial y_{ab,c}}\right)^{u\leq v,w}_{a\leq b,c} = \left(\frac{\partial^2 H^{\nabla}}{\partial y_{ab,c}\partial y_{uv,w}}\right)^{u\leq v,w}_{a\leq b,c}, \tag{9}$$

This article is copyrighted as indicated in the article. Reuse of AIP content is subject to the terms at: http://scitation.aip.org/termsconditions. Downloaded to IP: 147.96.20.53 On: Fri, 25 Jul 2014 14:12:23

where

$$p^{ij,k}=\frac{\partial L^{\nabla}}{\partial y_{ij,k}},\quad H^{\nabla}=\sum_{i\leq j}\frac{\partial L^{\nabla}}{\partial y_{ij,k}}y_{ij,k}-L^{\nabla}. \tag{10}$$

Proof. From the very definition of H^{∇} it follows:

$$\frac{\partial H^{\nabla}}{\partial y_{ab,c}}=\sum_{i\leq j}\frac{\partial^2 L^{\nabla}}{\partial y_{ab,c}\partial y_{ij,k}}y_{ij,k},$$

and the formula (9) above. Moreover, we claim that the functions $p^{uv,w}$ depend linearly on the variables $y_{ab,c}$. In fact, as a calculation shows,

$$\begin{aligned}\frac{\partial p^{uv,w}}{\partial y_{ab,c}}&=\frac{\partial^2 L^{\nabla}}{\partial y_{ab,c}\partial y_{uv,w}}\\&=\rho y^{jr}\frac{\partial^2}{\partial y_{ab,c}\partial y_{uv,w}}\left(G^l_{ji}G^i_{rl}-G^l_{li}G^i_{rj}\right)\\&=\tfrac{1}{(1+\delta_{ab})(1+\delta_{uv})}\rho\left\{y^{bw}\left(y^{au}y^{cv}+y^{av}y^{cu}\right)+y^{aw}\left(y^{bu}y^{cv}+y^{bv}y^{cu}\right)\right.\\&-y^{ab}\left(y^{cu}y^{vw}+y^{cv}y^{uw}\right)-y^{uv}\left(y^{aw}y^{bc}+y^{ac}y^{bw}\right)\\&\left.-\left(y^{ua}y^{vb}+y^{ub}y^{va}\right)y^{wc}+2y^{ab}y^{uv}y^{wc}\right\}.\end{aligned}$$

Therefore, in order to prove that the matrix (9) is non-singular, it suffices to prove that the variables $y_{ab,c}$ can be written in terms of the functions $p^{uv,w}$. To do this, we first compute

$$\begin{aligned}\sum_{u,v,w}\tfrac{1+\delta_{uv}}{\rho}p^{uv,w}y_{ur}y_{vs}y_{wq}&=y_{qr,s}+y_{qs,r}-y_{rs,q}\\&-\tfrac{1}{2}\sum_{a,b}y^{ab}\left(y_{sq}y_{ab,r}+y_{rq}y_{ab,s}\right)\\&+\sum_{a,b}y^{ab}y_{rs}\left(y_{ab,q}-y_{qa,b}\right).\end{aligned}$$

Evaluating the previous formula at g_{x_0}, by using adapted coordinates (i.e., $y_{ij}(g_{x_0})=\varepsilon_i\delta_{ij}$, $\varepsilon_i=\pm1$), and letting $\Upsilon_{rsq}(j^1_{x_0}g)=\frac{1+\delta_{rs}}{\rho}p^{rs,q}(j^1_{x_0}g)\varepsilon_r\varepsilon_s\varepsilon_q$, it follows:

$$\begin{aligned}\Upsilon_{rsq}(j^1_{x_0}g)&=y_{qr,s}(j^1_{x_0}g)+y_{qs,r}(j^1_{x_0}g)-y_{rs,q}(j^1_{x_0}g)\\&-\tfrac{1}{2}\sum\nolimits_a\varepsilon_a\varepsilon_q\left(\delta_{sq}y_{aa,r}(j^1_{x_0}g)+\delta_{rq}y_{aa,s}(j^1_{x_0}g)\right)\\&+\sum\nolimits_a\varepsilon_a\varepsilon_r\delta_{rs}\left(y_{aa,q}(j^1_{x_0}g)-y_{qa,a}(j^1_{x_0}g)\right).\end{aligned}$$

If $q\neq r\neq s\neq q$, then $\Upsilon_{rsq}(j^1_{x_0}g)=y_{qr,s}(j^1_{x_0}g)+y_{qs,r}(j^1_{x_0}g)-y_{rs,q}(j^1_{x_0}g)$. Hence

$$y_{qr,s}(j^1_{x_0}g)=\tfrac{1}{2}\left(\Upsilon_{rsq}(j^1_{x_0}g)+\Upsilon_{qsr}(j^1_{x_0}g)\right). \tag{11}$$

If $q=r$, $r\neq s$, then

$$\Upsilon_{rsr}(j^1_{x_0}g)=y_{rr,s}(j^1_{x_0}g)-\tfrac{1}{2}\sum\nolimits_a\varepsilon_a\varepsilon_r y_{aa,s}(j^1_{x_0}g). \tag{12}$$

If $r=s$, $q\neq r$, then

$$\begin{aligned}\Upsilon_{rrq}(j^1_{x_0}g)&=2y_{qr,r}(j^1_{x_0}g)-y_{rr,q}(j^1_{x_0}g)\\&+\sum\nolimits_a\varepsilon_a\varepsilon_r\left(y_{aa,q}(j^1_{x_0}g)-y_{qa,a}(j^1_{x_0}g)\right).\end{aligned} \tag{13}$$

The formula (12) can be rewritten as

$$2\varepsilon_r\Upsilon_{rsr}(j^1_{x_0}g)=\varepsilon_r y_{rr,s}(j^1_{x_0}g)-\sum\nolimits_{a\neq r}\varepsilon_a y_{aa,s}(j^1_{x_0}g).$$

This article is copyrighted as indicated in the article. Reuse of AIP content is subject to the terms at: http://scitation.aip.org/termsconditions. Downloaded to IP: 147.96.20.53 On: Fri, 25 Jul 2014 14:12:23

Summing up over the index r, $2\sum_r \varepsilon_r \Upsilon_{rsr}(j^1_{x_0}g) = (2-n)\sum_r \varepsilon_r y_{rr,s}(j^1_{x_0}g)$, and replacing this formula into (12) it follows:

$$\Upsilon_{rsr}(j^1_{x_0}g) = y_{rr,s}(j^1_{x_0}g) - \tfrac{1}{2-n}\varepsilon_r \sum_a \varepsilon_a \Upsilon_{asa}(j^1_{x_0}g).$$

Therefore

$$y_{rr,s}(j^1_{x_0}g) = \Upsilon_{rsr}(j^1_{x_0}g) + \tfrac{\varepsilon_r}{2-n}\sum_a \varepsilon_a \Upsilon_{asa}(j^1_{x_0}g). \tag{14}$$

Replacing (14) into (13), we eventually obtain

$$\sum_a \varepsilon_a y_{qa,a}(j^1_{x_0}g) = \tfrac{1}{n-2}\sum_a \varepsilon_a \Upsilon_{aaq}(j^1_{x_0}g) - 2\tfrac{n-1}{(n-2)^2}\sum_a \varepsilon_a \Upsilon_{aqa}(j^1_{x_0}g), \tag{15}$$

and replacing $y_{rr,q}(j^1_{x_0}g)$, $\sum_a \varepsilon_a y_{aa,q}(j^1_{x_0}g)$, and $\sum_a \varepsilon_a y_{qa,a}(j^1_{x_0}g)$ into (13) it follows:

$$\begin{aligned}\Upsilon_{rrq}(j^1_{x_0}g) = {} & 2y_{qr,r}(j^1_{x_0}g) - \Upsilon_{rqr}(j^1_{x_0}g) + \tfrac{n\varepsilon_r}{(n-2)^2}\sum_a \varepsilon_a \Upsilon_{aqa}(j^1_{x_0}g) \\ & - \tfrac{\varepsilon_r}{n-2}\sum_a \varepsilon_a \Upsilon_{aaq}(j^1_{x_0}g).\end{aligned}$$

Hence

$$\begin{aligned}y_{qr,r}(j^1_{x_0}g) = {} & \tfrac{1}{2}\tfrac{n-1}{n-2}\Upsilon_{rrq}(j^1_{x_0}g) + \tfrac{1}{2}\left(1 - \tfrac{n}{(n-2)^2}\right)\Upsilon_{rqr}(j^1_{x_0}g) \\ & - \tfrac{n\varepsilon_r}{2(n-2)^2}\sum_{a\neq r} \varepsilon_a \Upsilon_{aqa}(j^1_{x_0}g) \\ & + \tfrac{\varepsilon_r}{2(n-2)}\sum_{a\neq r} \varepsilon_a \Upsilon_{aaq}(j^1_{x_0}g).\end{aligned} \tag{16}$$

If $q = r = s$, then $\Upsilon_{rrr}(j^1_{x_0}g) = -\sum_{a\neq r} \varepsilon_a \varepsilon_r y_{ra,a}(j^1_{x_0}g)$. From (15) we obtain $\sum_{a\neq r} \varepsilon_a y_{ra,a}(j^1_{x_0}g)$ and then

$$\begin{aligned}\sum_{a\neq r} \varepsilon_a y_{ra,a}(j^1_{x_0}g) = {} & -\varepsilon_r y_{rr,r}(j^1_{x_0}g) + \tfrac{1}{n-2}\sum_a \varepsilon_a \Upsilon_{aar}(j^1_{x_0}g) \\ & - 2\tfrac{n-1}{(n-2)^2}\sum_a \varepsilon_a \Upsilon_{ara}(j^1_{x_0}g),\end{aligned}$$

and replacing it into the previous equation,

$$\Upsilon_{rrr}(j^1_{x_0}g) = y_{rr,r}(j^1_{x_0}g) - \tfrac{\varepsilon_r}{n-2}\sum_a \varepsilon_a \Upsilon_{aar}(j^1_{x_0}g) + 2\tfrac{(n-1)\varepsilon_r}{(n-2)^2}\sum_a \varepsilon_a \Upsilon_{ara}(j^1_{x_0}g).$$

Hence

$$\begin{aligned}y_{rr,r}(j^1_{x_0}g) = {} & \Upsilon_{rrr}(j^1_{x_0}g) + \varepsilon_r \tfrac{1}{n-2}\sum_a \varepsilon_a \Upsilon_{aar}(j^1_{x_0}g) \\ & - 2\varepsilon_r \tfrac{n-1}{(n-2)^2}\sum_a \varepsilon_a \Upsilon_{ara}(j^1_{x_0}g).\end{aligned} \tag{17}$$

The formulas (11), (14), (16) and (17) end the proof. □

B. Hamilton-Cartan equations

The Poincaré-Cartan form for the density $L^\nabla \mathbf{v}$ is the n-form on $J^1\mathfrak{M}$ given by

$$\Theta_{L^\nabla \mathbf{v}} = \sum_{i\leq j}(-1)^{k-1}p^{ij,k}dy_{ij}\wedge \mathbf{v}_k - H^\nabla \mathbf{v},$$

the momenta $p^{ij,k}$ and the Hamiltonian function H^∇ being defined as in (10), and the Hamilton-Cartan equations can geometrically be written as

$$\left(j^1 g\right)^*(i_Y d\Theta_{L^\nabla \mathbf{v}}) = 0, \tag{18}$$

for every p^1-vertical vector field $Y \in J^1\mathfrak{M}$, which are known to be equivalent to Euler-Lagrange equations, where $p^1 : J^1\mathfrak{M} \to M$ is the natural projection.

This article is copyrighted as indicated in the article. Reuse of AIP content is subject to the terms at: http://scitation.aip.org/termsconditions. Downloaded to IP: 147.96.20.53 On: Fri, 25 Jul 2014 14:12:23

According to Proposition 5.1, $(x^i, y_{jk}, p^{uv,w})$, $j \leq k$, $u \leq v$, is a coordinate system on $J^1\mathfrak{M}$. Letting $Y = \partial/\partial y_{ab}$ and $Y = \partial/\partial p^{uv,w}$ in (18), it follows, respectively:

$$\sum_k \frac{\partial \left(p^{ab,k} \circ j^1 g\right)}{\partial x^k} = -\frac{\partial H^\nabla}{\partial y_{ab}} \circ j^1 g,$$

$$\frac{\partial \left(y_{uv} \circ j^1 g\right)}{\partial x^w} = \frac{\partial H^\nabla}{\partial p^{uv,w}} \circ j^1 g,$$

which are the Hamilton-Cartan equations in the canonical formalism.

C. Covariant Hamiltonian

An Ehresmann (or nonlinear) connection on a fibred manifold $p : E \to M$ is a differential 1-form γ on E taking values in the vertical sub-bundle $V(p)$ such that $\gamma(X) = X$ for every $X \in V(p)$, e.g., see Refs. 10, 11, and 14. Given γ, one has $T(E) = V(p) \oplus \ker\gamma$, $\ker\gamma$ being the horizontal sub-bundle attached to γ.

According to Ref. 11, the covariant Hamiltonian $\mathcal{H}^\gamma$ associated to a Lagrangian density Λ on J^1E with respect to γ is the Lagrangian density defined by setting $\mathcal{H}^\gamma = \left((p_0^1)^*\gamma - \theta\right) \wedge \omega_\Lambda - \Lambda$, where $p^1 : J^1E \to M$, $p_0^1 : J^1E \to J^0E = E$ are the natural projections, and ω_Λ is the Legendre form attached to Λ, i.e., the $V^*(p)$-valued p^1-horizontal $(n-1)$-form on J^1E given by

$$\omega_\Lambda = (-1)^{i-1}\frac{\partial L}{\partial y_i^\alpha} dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n \otimes dy^\alpha, \quad \Lambda = L\mathbf{v},$$

and $\theta = \theta^\alpha \otimes \partial/\partial y^\alpha$, $\theta^\alpha = dy^\alpha - y_i^\alpha dx^i$, is the $V(p)$-valued contact 1-form on J^1E. Locally, $\mathcal{H}^\gamma = \left(\left(\gamma_i^\alpha + y_i^\alpha\right)\frac{\partial L}{\partial y_i^\alpha} - L\right)\mathbf{v}$.

Let $\pi : F(M) \to M$ be the bundle of linear frames and let $q : F(M) \to \mathfrak{M}$ be the projection given by $q(X_1, \ldots, X_n) = g_x = \varepsilon_h w^h \otimes w^h$, where $(w^1, \ldots, w^n)$ is the dual coframe of $(X_1, \ldots, X_n) \in F_x(M)$, i.e., g_x is the metric for which $(X_1, \ldots, X_n)$ is a g_x-orthonormal basis and $\varepsilon_h = 1$ for $1 \leq h \leq n^+$, $\varepsilon_h = -1$ for $n^+ + 1 \leq h \leq n$. The projection q is a principal G-bundle with $G = O(n^+, n^-)$. Given a symmetric linear connection Γ with associated covariant derivative ∇, and a tangent vector $X \in T_xM$, for every $u \in \pi^{-1}(x)$ there exists a unique Γ-horizontal tangent vector $X_u^{h_\Gamma} \in T_u(FM)$ such that, $\pi_* X_u^{h_\Gamma} = X$. Given a metric $g_x \in q^{-1}(x)$, let $u \in \pi^{-1}(x)$ be a linear frame such that $q(u) = g_x$. The projection $q_*(X_u^{h_{\Gamma_x}})$ does not depend on the linear frame u chosen over g_x; we refer the reader to Lemma 3.3 of Ref. 12 for a proof of this fact. In this way a section $\sigma^\nabla : p^*TM \to T\mathfrak{M}$ of the projection $p_* : T\mathfrak{M} \to p^*TM$ is defined by setting $\sigma^\nabla(g_x, X) = q_*(X_u^{h_{\Gamma_x}})$. The retract $\gamma^\nabla : T\mathfrak{M} \to V(p)$ associated to σ^∇, namely, $\gamma^\nabla(Y) = Y - \sigma^\nabla(p_*Y), \forall Y \in T_{g_x}\mathfrak{M}$, determines an Ehresmann connection on the bundle of metrics and the Lagrangian density $\Lambda^\nabla = L^\nabla\mathbf{v}$ admits a "canonical" covariant Hamiltonian $\mathcal{H}^{\gamma^\nabla}$. Locally,

$$\gamma^\nabla\left(g_x, \partial/\partial x^j\right) = -\sum_{k\leq l}\left\{\Gamma_{jk}^a(x) y_{al}\left(g_x\right) + \Gamma_{jl}^a(x) y_{ak}\left(g_x\right)\right\}(\partial/\partial y_{kl})_{g_x}.$$

Hence, $\gamma_{kl,j} = -(\Gamma_{jk}^a y_{al} + \Gamma_{jl}^a y_{ak})$, and

$$\mathcal{H}^{\gamma^\nabla} = \left(\sum_{k\leq l}\left(y_{kl,j} - \left(\Gamma_{jk}^a y_{al} + \Gamma_{jl}^a y_{ak}\right)\right)\frac{\partial L^\nabla}{\partial y_{kl,j}} - L^\nabla\right)\mathbf{v}.$$

From a direct computation the following result is deduced:

If $\mathcal{H}^{\gamma^\nabla} = H^{\gamma^\nabla}\mathbf{v}$, then

$$H^{\gamma^\nabla}(j_x^1 g) = L^\nabla(j_x^1 g) - 2\rho(g_x)s^{g,\nabla}(x), \quad \forall j_x^1 g \in J_x^1\mathfrak{M},$$

This article is copyrighted as indicated in the article. Reuse of AIP content is subject to the terms at: http://scitation.aip.org/termsconditions. Downloaded to IP: 147.96.20.53 On: Fri, 25 Jul 2014 14:12:23

where $s^{g,\nabla}$ is the scalar curvature of the symmetric linear connection ∇ with respect to the metric g, namely

$$s^{g,\nabla} = g^{jk}\left\{\tfrac{\partial\Gamma^i_{jk}}{\partial x^i} - \tfrac{\partial\Gamma^i_{ik}}{\partial x^j} + \Gamma^l_{jk}\Gamma^i_{il} - \Gamma^l_{ik}\Gamma^i_{jl}\right\}.$$

The Hamilton-Cartan equations for a covariant Hamiltonian H^γ attached to a connection γ are

$$\sum_k \frac{\partial\left(p^{ab,k}\circ j^1 g\right)}{\partial x^k} - \sum_{u\leq v}\left(\frac{\partial\gamma_{uv,w}}{\partial y_{ab}}\circ g\right)\left(p^{uv,w}\circ j^1 g\right) = -\frac{\partial H^\gamma}{\partial y_{ab}}\circ j^1 g,$$

$$\frac{\partial\left(y_{uv}\circ j^1 g\right)}{\partial x^w} + \gamma_{uv,w}\circ j^1 g = \frac{\partial H^\gamma}{\partial p^{uv,w}}\circ j^1 g,$$

(for example, see Ref. 2). Note that for $\gamma = 0$ (that is, the trivial connection induced by the coordinate system) these equations coincide with the local expression of the Hamilton-Cartan equations for H^∇ given in Sec. V B.

VI. CONCLUSIONS

We have defined a first-order Lagrangian L^∇ on the bundle of metrics which is variationally equivalent to the second-order classical Einstein-Hilbert Lagrangian.

This Lagrangian depends on an auxiliary symmetric linear connection, but this dependence is covariant under the action of the group of diffeomorphisms.

We have also proved that the variational problem defined by L^∇ is regular and its Hamiltonian formulation has been studied, including the covariant Hamiltonian attached to ∇.

Moreover, we should finally mention the completely different behaviour of L^∇ with respect to the Palatini Lagrangian.

Let $q : \mathfrak{C} \to M$ be the bundle of symmetric linear connections on M. The Palatini variational principle consists in coupling a metric g and a symmetric linear connection ∇ as independent fields, thus defining a first-order Lagrangian density $L_P\mathbf{v}$ on the product bundle $\mathfrak{M}\times_M \mathfrak{C}$ as follows:

$$(L_P\mathbf{v})\left(g_x, j^1_x\nabla\right) = s^{g,\nabla}(x)(\mathbf{v}_g)_x,$$

and varying g and ∇ independently. The Palatini method can also be applied to other different settings; e.g., see Refs. 7 and 3, but below we confine ourselves to consider the classical setting for the Palatini method. As is known, the Euler-Lagrange equations of L_P are the vanishing of the Ricci tensor of g (Einstein's in the vacuum) and the condition $\nabla = \nabla^g$ expressing that ∇ is the Levi-Civita connection of the metric.

In our case, we can similarly define a first-order Lagrangian $\mathfrak{M}\times_M\mathfrak{C}$ by setting $L(j^1 g, j^1\nabla) = L_{HE}(j^2 g) + c\left((\mathrm{alt}_{23}(\nabla^g T^{g,\nabla})^\sharp\right)(\rho\circ g)$. Assuming M is compact, then the action associated with L is given as follows:

$$\mathcal{S}(g,\nabla) = \int_M L^\nabla(j^1 g, j^1\nabla)\mathbf{v},$$

and by considering (1) an arbitrary 1-parameter variation g_t of g and (2) the 1-parameter variation $\nabla_t = \nabla + tA$ attached to $A\in\Gamma(S^2T^*M\otimes TM)$ of ∇, we obtain (1) Einstein's equation and (2) $0 = \int_M c\left(\mathrm{alt}_{23}(\nabla^g A)^\sharp\right)\mathbf{v}_g$, $\forall A\in\Gamma(S^2T^*M\otimes TM)$, which lead us to a contradiction.

ACKNOWLEDGMENTS

The authors thank the referee for his/her suggestions and remarks which have improved the presentation of this work. This work was supported by Ministerio de Ciencia e Innovación of Spain under Grant No. MTM2011–22528.

[1] M. Carmeli, *Classical Fields: General Relativity and Gauge Theory* (John Wiley & Sons, Inc., New York, 1982).
[2] M. Castrillón López and J. Marsden, "Some remarks on Lagrangian and Poisson reduction for field theories," J. Geom. Phys. **48**(1), 52–83 (2003).

This article is copyrighted as indicated in the article. Reuse of AIP content is subject to the terms at: http://scitation.aip.org/termsconditions. Downloaded to IP: 147.96.20.53 On: Fri, 25 Jul 2014 14:12:23

[3] N. Dadhich and J. M. Pons, "On the equivalence of the Einstein-Hilbert and the Einstein-Palatini formulations of general relativity for an arbitrary connection," Gen. Relativ. Gravitation **44**(9), 2337–2352 (2012).
[4] D. B. A. Epstein, "Natural tensors on Riemannian manifolds," J. Differ. Geom. **10**, 631–645 (1975).
[5] M. Ferraris and M. Francaviglia, "Covariant first-order Lagrangians, energy-density and superpotentials in general relativity," Gen. Relativ. Gravitation **22**(9), 965–985 (1990).
[6] P. Von der Heyde, "A generalized Lovelock theorem for the gravitational field with torsion," Phys. Lett. A **51**(3), 381–382 (1975).
[7] J. Katz and G. I. Livshits, "Affine gravity, Palatini formalism and charges," Gen. Relativ. Gravitation **43**(12), 3313–3336 (2011).
[8] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry* (John Wiley & Sons, Inc., New York, 1963), Vol. I.
[9] D. Lovelock, "The Einstein tensor and its generalizations," J. Math. Phys. **12**, 498–501 (1971).
[10] L. Mangiarotti and G. Sardanashvily, *Connections in Classical and Quantum Field Theory* (World Scientific Publishing Co., Inc., River Edge, NJ, 2000).
[11] J. Marsden and S. Shkoller, "Multisymplectic geometry, covariant Hamiltonians, and water waves," Math. Proc. Cambridge Philos. Soc. **125**, 553–575 (1999).
[12] J. Muñoz Masqué and M. Eugenia Rosado María, "Diffeomorphism-invariant covariant Hamiltonians of a pseudo-Riemannian metric and a linear connection," Adv. Theor. Math. Phys. **16**(3), 851–886 (2012).
[13] J. Muñoz Masqué and A. Valdés Morales, "The number of functionally independent invariants of a pseudo-Riemannian metric," J. Phys. A: Math. Gen. **27**, 7843–7855 (1994).
[14] G. A. Sardanashvily, *Gauge Theory in Jet Manifolds*, Hadronic Press Monographs in Applied Mathematics (Hadronic Press, Inc., Palm Harbor, FL, USA, 1993).
[15] H. Weyl, *Space-Time-Matter*, translated by H. L. Brose (Dover Publications, Inc., 1952).

This article is copyrighted as indicated in the article. Reuse of AIP content is subject to the terms at: http://scitation.aip.org/termsconditions. Downloaded to IP: 147.96.20.53 On: Fri, 25 Jul 2014 14:12:23