Finite Element Model of the Innovation Diffusion: An Application to Photovoltaic Systems

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Abstract:

This paper presents a Finite Element Model, which has been used for forecasting the diffusion of innovations in time and space. Unlike conventional models used in diffusion literature, the model considers the spatial heterogeneity. The implementation steps of the model are explained by applying it to the case of diffusion of photovoltaic systems in a local region in southern Germany. The applied model is based on a parabolic partial differential equation that describes the diffusion ratio of photovoltaic systems in a given region over time. The results of the application show that the Finite Element Model constitutes a powerful tool to better understand the diffusion of an innovation as a simultaneous space-time process. For future research, model limitations and possible extensions are also discussed.

Keywords: Innovation, Adoption, Spatiotemporal, prediction, solar photovoltaics

1. Introduction

Photovoltaic systems (PV) have a vital importance to succeed the transition toward sustainable energy production. In 2013, the global cumulative installed capacity of PV increased 35% in compare to the year before albeit a slowdown in European countries (EPIA, 2014). The diffusion of ecological innovations such as PV is hard to forecast. The diffusion patterns of such innovations vary not only from country to country (Beise and Rennings, 2005) but also from a spatial region to another (see e.g. Karakaya et al., 2014). Can we model the diffusion of PV when spatial heterogeneity exists? It is a hard task, not only for the case of PV, or for ecological innovations, but also for any kind of innovations in general (see Kiesling et al., 2011; Meade and Islam, 2006).

Wrestling with similar questions in different contexts, the emergence of big data has opened new perspectives both in the natural and social sciences. There is a myriad of examples that point to this. The forecasting of disease arrival times is simply possible (Brockmann and Helbing, 2013). The effect of social network incentives on people´s cooperation is already predictable (Mani et al., 2013). Using big data, predictive models of building successful teams and making good decisions has been also emerged (see Pentland, 2012, 2013). It is not only the big data, that make such studies possible, but also the diverse models which recently became available with the increasing computational power.

A variety of models has also been extensively studied in the literature on diffusion of innovations. The pioneering scholars on this phenomenon described the diffusion as an one-dimensional time-dependent S-curve (see Rogers, 1962), studying the factors influencing the diffusion through regression analysis (Griliches, 1957; Mansfield, 1961). Several studies provided a variety of differential equations formulating such one-dimensional S-curves (e.g. Bain, 1963; Harvey, 1984; Meade & Islam, 1998; Sharif & Islam, 1980). Some other studies introduced two-dimensional models, considering both time and space (Haynes et al., 1977; Hägerstrand, 1967; Morrill, 1970). However, the two-dimensional models brought computational complexity and had been rarely applied for a long time. With the emergence of agent-based models in 1990s and increasing computational capacity, some recent studies attempted to tackle both the space and time dimensions on modeling diffusion of innovations (e.g. Berger, 2001; Dunn and Gallego, 2010; Guseo and Guidolin, 2014; Schwarz and Ernst, 2009).

Unlike the conventional models of PV diffusion (e.g. Guidolin and Mortarino, 2010; Luque, 2001; Masini and Frankl, 2002; Mesak and Coleman, 1992), space dimension has recently been considered in some studies on diffusion of PV (e.g. Higgins et al., 2014; Kwan, 2012). These studies have used a downscaled unit of analysis at local or regional level rather than national level. Although this approach explained the differences of one dimensional diffusion paths at different locations, it still lacked of explaining the dynamics of spatial heterogeneity on two dimensional diffusion paths, i.e. how innovations propagate in a spatiotemporal domain that has heterogonous variables.

Addressing this problem; this study implements a Finite Element Model that forecasts the diffusion of PV in time and space. Finite Element Models are numerical tools for approximating the solutions of large scale problems that are based on partial differential equations. It reduces the spatiotemporal continuum problem, which consists of an infinite number of unknowns, to one with a finite number of unknowns (Lewis et al., 2004). It originated from the need for solving complex elasticity and structural analysis problems in engineering sciences in 1960s. The model has since been extended and applied to the broad field of continuum mechanics for engineering disciplines (such as civil, aeronautical, biomechanical, chemical and automotive) (Huebner et al., 2001). Recently, the model has been borrowed by a number of social science studies.

The rest of the paper is organized as following. Section 2 explains the theoretical background and presents a literature review. Section 3 describes the formulation of Finite Element Model applied in this study. Section 4 presents the results and discussions. Finally, Section 5 is devoted to conclusions, presenting the limitations and future work.

2. Theoretical Background and Literature Review

2.1. Modeling Diffusion of PV

The diffusion of PV has been modelled from various theoretical perspectives through a variety of methods. The study of Mesak and Coleman (1992) appears as one of the first contributions of this phenomenon. They extended the model of Bass (1969) in order to forecast the PV diffusion for the residential sector in Kuwait. Their study addressed the link between the government subsidies and the PV diffusion. Assuming that diffusion of PV is based on price evolution, Luque (2001) combined the learning curve approach (Yelle, 1979) with a demand elasticity model. As a result, he forecasted both the market and price evolution of PV. Learning curve approach was also used by Masini and Frankl (2002). They simulated the diffusion of PV systems in 5 southern Europe countries under different scenarios. Watanabe and Asgari (2004) attempted to link the dynamic behavior of learning coefficient with that of innovation diffusion. They developed a mathematical formulation of the logistic growth function which incorporates the innovation diffusion and its carrying capacity. The model was applied to the Japanese case of PV from1986 to 2000.

Kobos et al. (2006) extended the debate by combining the frameworks of learning by doing and learning by searching. Based on the estimation of the cost curves and learning elasticity through regression analysis, they examined experience curve analysis of wind energy and PV under several scenarios. Based on the well-known Bass model (Bass, 1969) and the extension of it (Bass et al., 1994), Guidolin and Mortarino (2010) forecasted the diffusion patterns of PV in several countries. Their paper showed the important differences among studied countries, presenting the different stages of the diffusion process. However, the comparison between the forecasted values of this study and what happened in reality indicates a big mismatch, e.g. in the case of Germany.

Since 2011, the number of studies addressing the modeling diffusion of PV has rapidly increased. This is probably, among others, due to the increasing available empirical data. For example, Popp et al. (2011) studied the relationship between global technology stock influence the diffusion of renewable energy technologies such as PV. They explained the investment per capita in capacity of installed renewable energy through a statistical fixed effects model. Although many scholars used a downscaled unit of analysis at regional or local level (Gooding et al., 2013; Higgins et al., 2014; Kwan, 2012), how spatial differences can affect the two dimensional diffusion paths have not been addressed yet.

2.2. Finite Element Models

The research, that applied the Finite Element Model in social science related fields, has flourished in the 1990s[1]. These studies can be categorized under three groups (see Table 2 in Appendix for the categories and the contribution of some selected papers). The first category is the Economic Geography. This stream is dominated by the studies from transportation

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[1] E.g. the studies have appeared in a variety of journals from different fields. Some examples are the Annals of Regional Science (Wong and Sun, 2001; Wong et al., 2006), an international journal of urban, regional and environmental studies; Computational Economics (Cao-Alvira, 2010, 2011; Shinohara and Okuda, 2010), a journal dedicated to the interface of computer science, economics and management science; Transportation Research Part B- Methodological (Ho et al., 2006; Jiang et al., 2011; Wong, 1998; Yang et al., 1994), a journal presenting methodological aspects applied in transportation systems; and Management Science (Ben-Ameur et al., 2002), a journal that publishes on the practice of management.

research. The second is the Finance, with a particular emphasis on American and European options pricing. Although the first studies on implementing Finite Element Model in option pricing appeared in the early 2000s (e.g. Allegretto et al., 2001; Tomas and Yalamanchili, 2001), there have been already a large number of papers published in the literature of finance. Lastly, the third category consisted studies from various topics, including economic growth models (e.g. Cao-Alvira, 2011) and cash management optimization (e.g. Baccarin, 2009). In all three categories, most of the studies focus on methodological and modelling aspects, such as optimization, improvement and extension of applied models with Finite Element Method (e.g. Angermann and Wang, 2007 in option pricing; Ho et al., 2006 in transportation systems in a city).

Finite Element Models usually start with a continuum problem and a numerical model, an example of which is illustrated in Figure 1. Then, one part of the model carries out domain discretization, i.e. meshing, and the other deals with discretization of the equation in respect to both spatial and temporal dimensions. In the end, the model is solved with a numerical software tool, which typically contains three parts: preprocessing, the main processing and post processing. The results are typically tested both for accuracy and convergence.

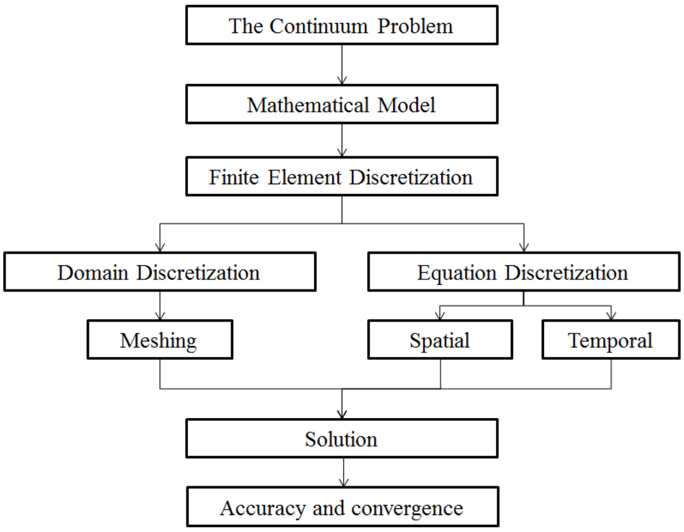

Figure 1- Typical implementation steps for Finite Element Model (adopted from Lewis et al., 2004)

The continuum problem and mathematical model can vary, based on the studied phenomena (see Table 3 in Appendix for some examples). There are different continuum problems considered. For example, Ho and Wong (2007) study a continuum problem of the macro role for the interacting housing and travel patterns in both land-use and transportation system; Wong et al., (2006) considers a continuum traffic equilibrium problem in a city with several competing facilities; and Shinohara and Okuda (2010) study continuum innovation diffusion via an analogy with natural physical phenomena. Some studies deal with a minimization problem while others caries out a maximization problem. In general, the prediction of customer behavior is the focal point of most of the mathematical models that are applied in these studies.

The domains, which are subject to discretization, are usually chosen as a whole region of a city (e.g. Wong et al., 2004) or a country (e.g. Shinohara and Okuda, 2010). The selected domains are typically meshed with a set of three-node linear triangular finite elements. The studied variables, such as diffusion rate or traffic density, can be then mapped on these meshed domains. This enables a deeper understanding on the studied phenomena, such as pinpointing the sources of diffusion of innovations or the congested roads.

After finite element discretization, the solutions of are calculated by a software, which also offers accuracy and convergence analysis. Accuracy in Finite Element Model is usually tested by numerical solutions of restricted problems with known results; whereas convergence can be tested by observing whether the numerical results are changing with increasing number of elements or not (Walz et al., 1968). In the literature, the convergence of the method have been validated with different approaches (e.g. see Ho and Wong, 2007; Wong and Sun, 2001).

3. Formulation of the Applied Model

The model considers continuum geography where the innovations are diffused as if heat diffusion. Innovation diffusion rate *(D),* the proportion of actual number of adoptions to the total number of maximum possible adoptions in a given area, is the object of the model. E.g. if innovation diffusion rate in a given area is 70% at a given time, it means that 70% of the potential adopters have already adopted the innovation by this given time. If it is 100%, it means that the diffusion in the given area has already saturated and there is no expected adoption left. Innovation diffusion rate *(D),* as defined in this study, is therefore similar to the temperature in heat diffusion equation. It depends both on time and space. Generated from an innovation source, the innovations flow in space in all directions. For example, Figure 2 shows an illustrative case, where innovations, i.e. PV, flow from the innovation source, e.g. a company, to the adopters.

Figure 2- The illustration of flow in innovations in space

The innovation flow, or in other words, innovation flux *(q)* can be formulated as similar to the fundamental rule for flow of heat energy equation.

$$q = -c\nabla D$$

where c can be called as innovation diffusivity[2]. This rule indicates that innovation flux *(q)* is proportional to the negative innovation rate across the space. With the transformation of this equation, the governing equation on the diffusion rate (D) can be derived as the following.

$$\frac{\partial D}{\partial t} = c(x, y)\left(\frac{\partial^2 D}{\partial x^2} + \frac{\partial^2 D}{\partial y^2}\right)$$

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[2] Innovation diffusivity value can also be interpreted as a combination of the coefficient of innovation and the coefficient of imitation in Bass model (Bass, 1969)

where t is time. The x and y are the coordinates of the location in space. The derivation of such formulation, which is based on of the model of Haynes et al, (1977), can also be seen in the study of Shinohara and Okuda (2010). It is also similar to the formulation of Morrill (1970), that describes the innovation diffusion as if a wave propagation. Such mathematical models also ssume that the innovation will diffuse from a source to the adopters.

For the application of the Finite Element Model, the semi-real case of a solar company, called HET, is considered. The HET´s target market includes 10 zip codes in southern Germany (see Figure 3). It is assumed that the innovation source is the HET in this selected area. This means that adoption depends on both the distance to the HET and the innovation diffusivity of a spatial point. Householders are assumed to be the potential adopters that are homogenously spread in the geography.

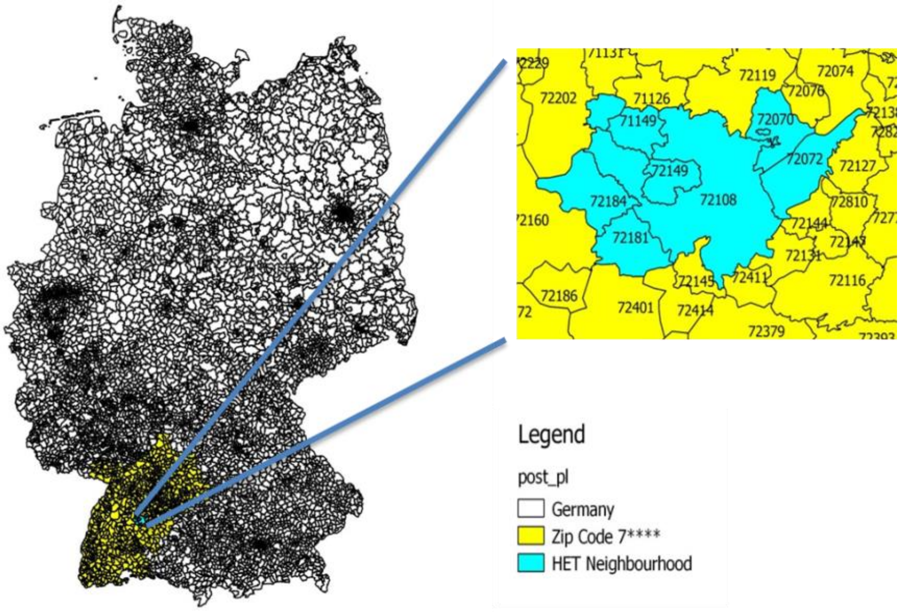

Figure 3- The modelled geography from Southern Germany

As shown in Table 1, a variety of innovation diffusivity numbers are considered in order to address a possible spatial heterogeneity problem. These values are chosen on a purpose, making the zip code area 72149 an extreme case with very low innovation diffusivity value. As an alternative, these values could have been obtained from the history data of diffusion curves.

Table 1- The values for innovation diffusivity

Zip code	**71126**	**71149**	**71559**	**72070**	**72072**	**72108**	**72119**	**72149**	**72181**	**72184**
$c(x,y)$	72	0.5	35	58	138	150	23	80	46	112

To solve the mathematical equation, both the domain and equation are meshed by ABAQUS software (Abaqus, 2013). This computational mesh on approximated geographic geometry is shown in Figure 4. The HET is located between zip code areas 72108 and 72119.

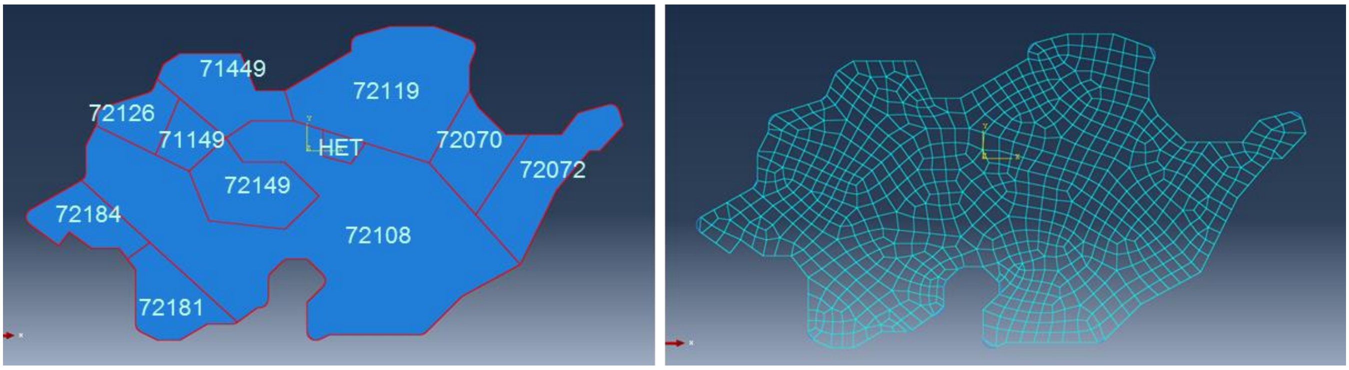

Figure 4- The computational mesh on approximated geographic geometry

The type of the elements used for computational mesh is 4-node linear quadrilateral. The total number of elements is 752. The solution of the model is based on a 13-year calculation. As boundary condition, the location of the HET is assumed to have 100% diffusion rate during the whole time period.

4. Results and Discussions

The governing equation on the diffusion rate *(D)* has been solved at 43 incremental time steps through the software. This calculation has lasted for 35 seconds[3]. Based on this solution, it is possible to obtain the forecasted values of diffusion rate *(D)* for any spatial point in a given time. For example, the diffusion rate in each spatial location after 13 years of diffusion is plotted as a counter map in Figure 5. The red color accounts for 100% diffusion rate, while blue color indicates for around 9% diffusion rate. The HET is located on the red area and that is why this area has 100% diffusion ratio. The zip code area 72149, which has the lowest innovation diffusivity value, has also accounts for the lowest diffusion ratio after 13 years. As expected, not only the distances between potential adopter and innovation source but also innovation diffusivity values seem to be decisive for the heterogeneous diffusion rate values that are spread in the given geography.

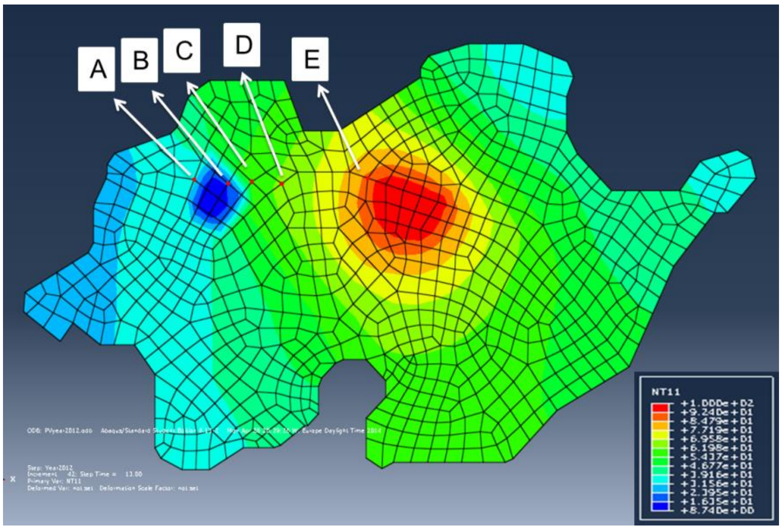

Figure 5- Diffusion rate after 13 years

In order to see the differences on diffusion patterns on different spatial points, the 5 locations, namely as A,B,C,D and E (as shown in Figure 5), are additionally analyzed. The

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[3] Such calculation is almost impossible to conduct without a computational software

resultant curves present the evolutions of diffusion rate (D) on given points as illustrated in Figure 6. The curves have a typical s-shape as studied by literature (Bass, 1969; Rogers, 1962), showing that the selected locations are at a variety of stages of diffusion process. If a location is nearer to the HET, it is most likely to have a rapid diffusion. However, the diffusion is also bound to the innovation diffusivity. For example, the location B has a slower diffusion than the location A, although the location B is closer to the HET than the location A is. This can be explained due to the fact that location B has lower innovation diffusivity value than the location A does.

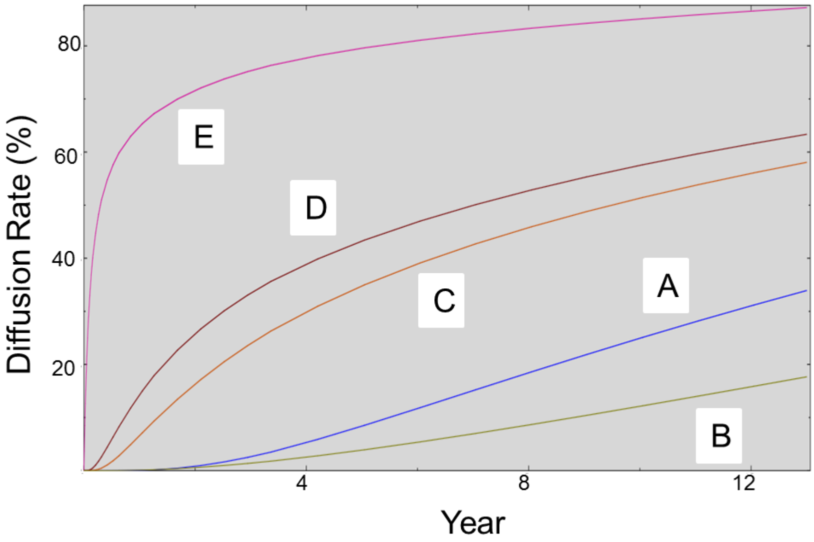

Figure 6- The patterns of s-curves at 5 locations

The interrelation among the s-curves (as presented in Figure 6) are similar to those of what Hägerstrand (1967) and Morrill (1970) proposed. The patterns resemble the wave propagation. With the words of Morrill (1970), spatial heterogeneity causes the diffusion to spread gradually in space and time. Such diffusion pattern can be also recognized in the lead market model of Beise (2004) which argues that some regions may have a rapid diffusion with a leading role. The differences on diffusion patterns of innovations both in space and time are commonly associated with the characteristics of the potential adopters, such as socioeconomic factors, and the environmental context such as geographical and political settings (Wejnert, 2002). Therefore, it can be assumed that the innovation diffusivity $c(x,y)$ contains the combined information of the adopters and the environmental context. This means, for example, the low innovation diffusivity value of zip code area 71148 may represent the unfavorable conditions of this region, e.g. lack of householders or low income level etc.

The Finite Element Model also gives detailed information on the innovation flux *(q)*, which is not possible to obtain neither from the one dimensional conventional models (e.g. Bass et al., 1994; Bass, 1969) nor from those of that study the innovation diffusion at local level (e.g. Gooding et al., 2013; Higgins et al., 2014). Figure 7 presents the evolution of innovation flux for a time period of 10 years. As expected, the innovations flow from the HET to the other regions. However, the innovation flux is not homogenous in all directions. For example, the flux is higher into the direction of zip code area 72108 than the one into the zip code are 72119. This is expected due to the fact that zip code area 72108 has higher innovation diffusivity than that of 72119 has. One can also observe that the maximum value of innovation flux decreases with the time. This can be explained with the diffusion rate differences among regions. In the beginning, e.g. after 3 months, the high diffusion rate differences result in higher flux. However, in the later phases, e.g. after 10 years, the diffusion rate differences among regions decrease, resulting in lower innovation flux. In addition, the result of propagated evolution in innovation flux can be observable in the evolution of diffusion ratio (see Figure 8). Diffusion ratio increases heterogeneously on the space as driven by the heterogonous innovation flux.

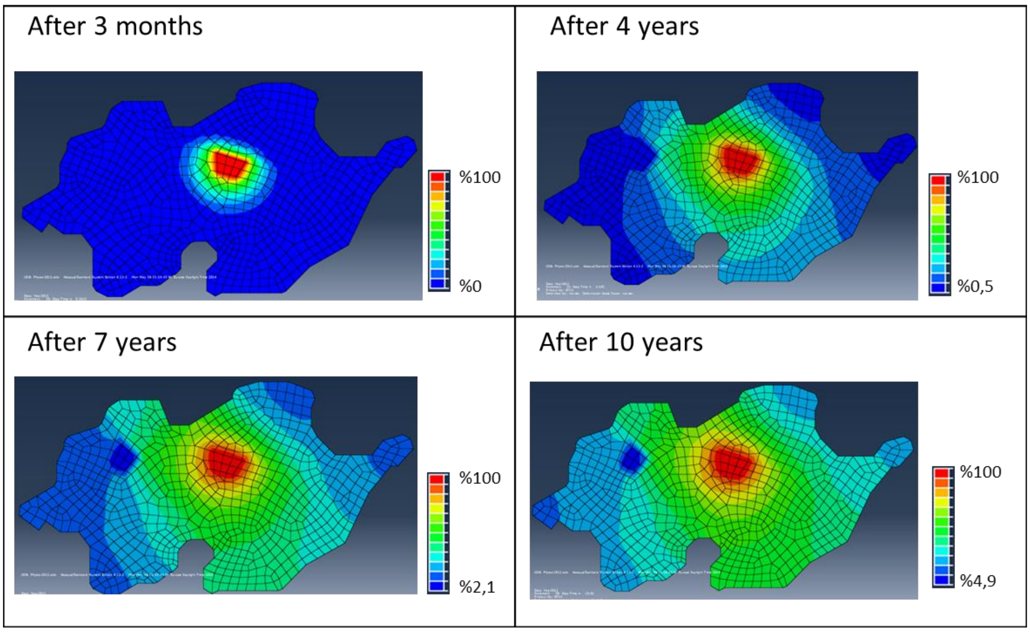

Figure 7- Innovation flux after 3 months, 4 years, 7 years and 10 years

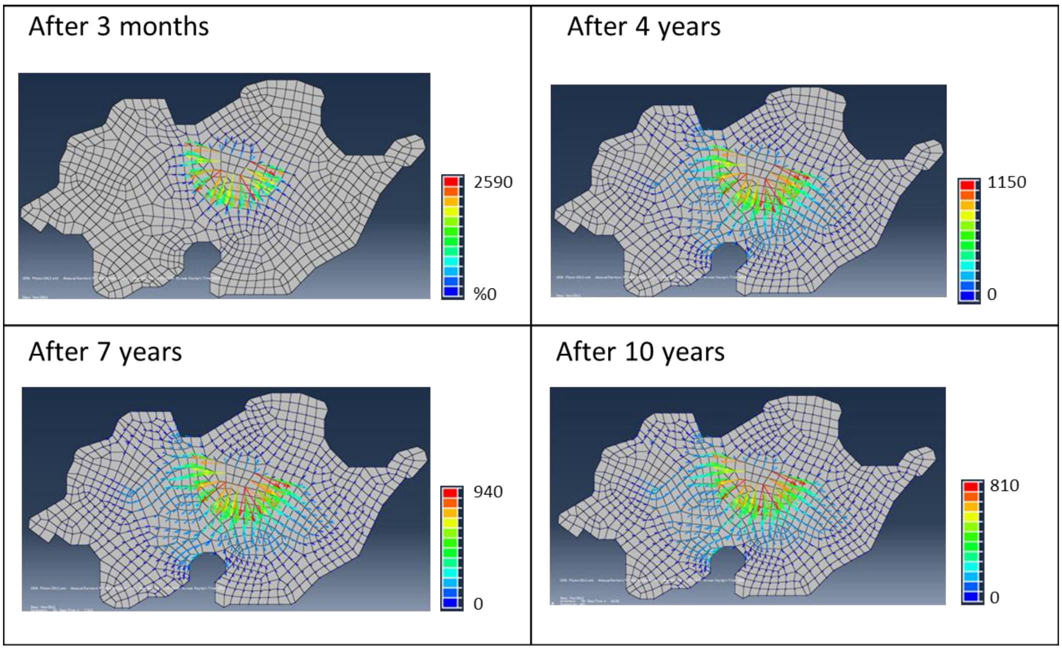

Figure 8- Diffusion ratio after 3 months, 4 years, 7 years and 10 years

The findings of the Finite Element Model on the case of PV diffusion have important implications for scholars who seek to study the diffusion of innovations as a space-time process. In agreement with Shinohara (2012), the Finite Element Model applied in this paper presents a comprehensive explanation on the spatial patterns of the diffusion. In addition, the model serves as a powerful method to solve the partial differential equations which have infinite number of unknowns. This means that the Finite Element Models can be also easily applicable to the other mathematical formulations of innovation diffusion that are based on partial differential equations (e.g. Bass, 1969; Morrill, 1970).

5. Conclusions and Future Work

Although science has advanced on understanding and modelling social phenomena (e.g. Ajzen, 1991; Pentland and Liu, 1999; Pentland, 2014; Skinner, 1953), forecasting is a challenge. However, the availability of big data and advanced computation power has yet opened new perspectives. This paper uses such model, the 50 years-old Finite Element Model, which has been lately started to be used in social sciences.

This study implements the Finite Element Model to a semi-real case on the diffusion of PV in a particular area in southern Germany. The model is capable of generating the spatiotemporal patterns of diffusion of PV systems. The mathematical equation describing this phenomenon is based on of the model of Haynes et al. (1977). The applied model responds to the difficulty of modelling the influence of spatial heterogeneity on diffusion. It also gives detailed information on the dynamics of the diffusion in spatiotemporal domain, e.g. in terms of spatial propagation of innovation flux. Such insight from the spatiotemporal diffusion is not possible to obtain from many other alternative models that have been studied in the literature for decades (e.g. Bass et al., 1994; Bass, 1969; Guidolin and Mortarino, 2010; Sharif and Islam, 1980). In this sense, the model could be an alternative to the emerging agent based models that also address spatial heterogeneity (e.g. Guseo and Guidolin, 2014; Schwarz and Ernst, 2009). This study can be also considered as a context-only extension of the study of Shinohara and Okuda (2010), which applied a Finite Element Model to the case of diffusion of hybrid cars in Japan.

However, this study presents several limitations. Firstly, the geometry of the geography is approximate. Secondly, the innovation coefficient values are not empirically obtained. As future work, I plan to use the real geographic data and obtain the empirical innovation diffusivity values. This will let to test the power of the model in real life situations.

Nevertheless, this study also serves as an entry point for the researchers seeking to implement the emerging Finite Element Model in social sciences. The model presents a comprehensive potential on modelling and forecasting purposes, especially if the spatiotemporal dimension needs to be considered.

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Appendix

Table 2. The categorization and contribution of some selected studies on Finite Element Models [4]

Field	Contribution
Economic Geography	A proposal of a user equilibrium model for the spatial trip assignment in a city (Yang et al., 1994)
	Extension of the continuum approximation of network flow for a city (Wong, 1998; Wong et al., 1998)
	A new approach for calculating the market shares and areas captured by competitive facilities (Wong and Yang, 1999)
	Presenting a continuous equilibrium approach for calculating the market shares of competitive facilities (Yang and Wong, 2000)
	Optimization, improvement and sensitivity analysis for solving the equilibrium problem in a city, e.g. the route-choice behavior of customers, with several competing facilities (Wong and Sun, 2001; Wong et al., 2004, 2006)
	A finite element model of traffic flow networks (Mohr, 2005)
	An application of the continuum traffic equilibrium model for the cordon-based congestion pricing problem (Ho et al., 2005)
	Presenting or extending a model for a continuum traffic equilibrium problem with multiple user classes (Ho et al., 2006, 2013)
	Modeling diffusion of innovations (Shinohara and Okuda, 2010)
	Dynamic continuum model for pedestrian flow in regard to travel costs and densities (Xia et al., 2009)
	A formulation of a predictive continuum dynamic user-optional model (Jiang et al., 2011)
	Proposing a model for housing allocation (and transportation emission) (Ho and Wong, 2007; Yin et al., 2013)
Financial Option Pricing	Demonstrating capabilities (including convergence) of Finite Element Method in Black Scholes equilibrium pricing model (Andalaft-Chacur et al., 2011; Angermann and Wang, 2007; Golbabai et al., 2013; Huang et al., 2009; Markolefas, 2008; Tomas and Yalamanchili, 2001)
	Presenting a partial differential equation model for pricing a ratchet cap (Suárez-Taboada and Vázquez, 2012)
	Presenting finite element method (along with finite volume) for American put options (Allegretto et al., 2003)
	Pricing options on a foreign currency in a stochastic interest rate economy with different methods (Choi and Marcozzi, 2003)
	Proposing a method or new approach for pricing American or European options (Allegretto et al., 2001; Ballestra and Sgarra, 2010; Christara and Dang, 2011; Marcozzi, 2008; Rambeerich et al., 2009, 2013; Reisinger and Witte, 2012)
	Derivation of formulations of the Kolmogorov equations arising in asset pricing (Reich et al., 2009)
	Presenting an inverse finite element method (Zhu and Chen, 2013)
	Solving two-factor convertible bonds valuation (Barone-Adesi, 2003)
	A pricing method for the American-Style Asian Options (Ben-Ameur et al., 2002)
	Introduction of a new method (that is comparable with Finite Element Method) (Cont et al., 2011)
Other Interrelated Fields	Obtaining an accurate numerical solution for homothetic utility function in Grossman-Weiss monetary model (Li, 1998)
	Proposing a method for the global approximation of the equilibrium of a cash-in-advance model economy (Cao-Alvira, 2011)
	Parameterizing true policy functions that solve for the equilibrium of an optimal growth model (Cao-Alvira, 2010)
	Presenting a method to solve a multidimensional portfolio cash management problem (Baccarin, 2009)
	Application of Finite Element Method for stochastic growth model (McGrattan, 1996)
	Solving the portfolio optimization problem (Muthuraman and Kumar, 2006)
	Numerical solution of a distributional control problem (Calvo Calzada and Goetz, 2001)
	Demonstration of the feasibility and utility of developing economic cost mode (Tufts et al., 2010)

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[4] These studies are selected through a 3-stage methodology. First, the studies, that contain "Finite Element" term in their abstracts, titles or keywords, are detected and retrieved from Social Sciences Citation Index (SSCI) of Web of Science Core Collection. By using "Finite Element" term, it is aimed to capture all studies that used different variations of term: e.g. "Finite Element Analysis" or "Finite Element Modeling". This resulted in 364 publications from the time span between 1975 and 2013. Secondly, based on the abstracts and titles, the studies that do not belong to any of social science fields are excluded from the database. Most of these exclusions are from the interdisciplinary journals that are indexed both in Science Citations Index and SSCI. This resulted in inclusion of 46 publications for literature review.

Table 3. The continuum problem and governing equations of some example studies

The continuum problem	Governing Equation
The prediction of the spatial pattern of trip assignment in a city, given that each commuter seeks to minimize travel time (Yang et al., 1994)	$\min_{\boldsymbol{f},\boldsymbol{v}} \theta(\boldsymbol{f},\boldsymbol{v}) = \sum_{a\in A}\int_0^{f_a} t_a(z)dz + \iint_D \left\{\int_0^{\lvert\boldsymbol{v}\rvert} c(\boldsymbol{z},\boldsymbol{x})dz\right\} dx_1 dx_2$ $\boldsymbol{v}$ **:**traffic flow $\boldsymbol{f}$ **:**traffic flow on a link
Commuters' movements in the city with a central business district (Wong et al., 1998)	$\min_{\boldsymbol{f}} z(\boldsymbol{f}) = \iint_\Omega a\lvert\boldsymbol{f}\rvert + \frac{b}{\gamma+1}\lvert\boldsymbol{f}\rvert^{\gamma+1} - \int_0^q W(\zeta)d\zeta d\Omega$ $\boldsymbol{f}$ **:**traffic flow in the city
A multi-commodity traffic flow with several highly central business districts (Wong, 1998)	$\min_{\boldsymbol{f},\boldsymbol{q}} \theta(\boldsymbol{f},\boldsymbol{q}) = \iint_\Omega a\sum_{m=1}^{M}\lvert\boldsymbol{f}_m\rvert + \frac{b}{2}\left(\sum_{m=1}^{M}\lvert\boldsymbol{f}_m\rvert\right)^2 - \sum_{m=1}^{M} D_m^{-1}(\xi)d\xi d\Omega$ $\boldsymbol{q}$ **:**demand for each commodity as a function of travel cost $\boldsymbol{f}$ **:**traffic flow in the city
Customer spatial choice equilibrium, assuming that each consumer choose a facility based both congested travel time and attributes of facility (Wong and Yang, 1999)	$\min_{\boldsymbol{f}} z(\boldsymbol{f}) = \sum_{n=1}^{N}\overline{u_n}\, Q_n + \iint_\Omega \left(a\lvert\boldsymbol{f}\rvert + \frac{b}{\gamma+1}\lvert\boldsymbol{f}\rvert^{\gamma+1}\right) d\Omega$ $\boldsymbol{f}$ **:**traffic flow in the city
Estimation of the market areas and market shares captured by each competitive facility (Yang and Wong, 2000)	$\min_{\boldsymbol{f}} z(\boldsymbol{f}) = \sum_{n=1}^{N}\int_0^{Q_n} C_n(\xi)d\xi + \iint_\Omega \left(a\lvert\boldsymbol{f}\rvert + \frac{b}{\gamma+1}\lvert\boldsymbol{f}\rvert^{\gamma+1} - \int_0^q D^{-1}(\zeta)d\zeta\right) d\Omega$ $\boldsymbol{f}$ **:**flow state in the city
Estimation of the market areas and market shares captured by given competitive facilities (Wong and Sun, 2001; Wong et al., 2004)	$\min_{\boldsymbol{f},\boldsymbol{q}} \theta(\boldsymbol{f},\boldsymbol{q}) = \iint_\Omega a\sum_{i=1}^{N}\lvert\boldsymbol{f}_i\rvert + \frac{b}{2}\left(\sum_{i=1}^{N}\lvert\boldsymbol{f}_i\rvert\right)^2 + \frac{1}{\zeta}\sum_{i=1}^{N}(q_i \ln q_i - q_i)d\Omega$ $\boldsymbol{q}$ **:**demand for each commodity $\boldsymbol{f}$ **:** flow state in the city
A constrained optimization problem of the housing allocation, maximizing the total utility of the system (Ho and Wong, 2007)	$\max_{\boldsymbol{h}} z(h) = \iint_\Omega \sum_m q_m^*(x,y)U_m(q^*(x,y),u_m^*(x,y))d\Omega$ $\boldsymbol{h}$ **:** housing provision at a location
Spatial innovation diffusion modelling (Shinohara and Okuda, 2010)	$\frac{\partial N}{\partial t} = \alpha\left(\frac{\partial^2 N}{\partial x^2} + \frac{\partial^2 N}{\partial y^2}\right)$ *N*: The innovation adopter ration in a given local area
Optimization of toll rates that maximize the total utility of the system (Ho et al., 2013)	$\max_{\boldsymbol{f},\boldsymbol{q}} \widetilde{U}(\boldsymbol{f},\boldsymbol{q}) = \iint_\Omega \sum_m\sum_{\tilde{n}} q_{m\tilde{n}}U_{m\tilde{n}} + \sum_m \frac{1}{\zeta_m}q_m \ln q_m - \sum_m\sum_{\tilde{n}}\frac{1}{\zeta_m}q_{m\tilde{n}}\ln q_{m\tilde{n}} - \sum_m\sum_n c_m\lvert\boldsymbol{f}_{\boldsymbol{mn}}\rvert\, d\Omega$ $\boldsymbol{q}$ **:**demand for each commodity $\boldsymbol{f}$ **:** flow state in the city in the city

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