NUMERICAL METHODS IN NONLINEAR ANALYSIS OF SHELL STRUCTURES

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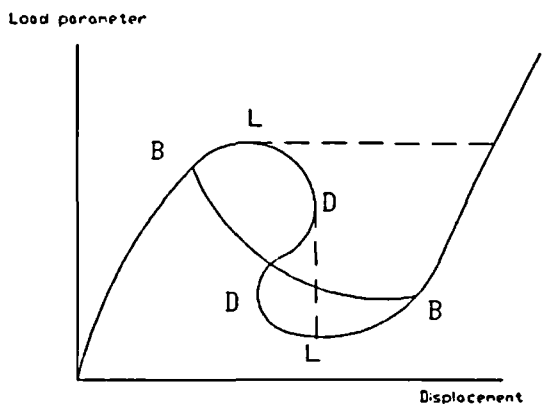
1.INTRODUCTION

The design of shell and spatial structures represents an important challenge even with the use of the modern computer technology. If we concentrate in the concrete shell structures many problems must be faced, such as the conceptual and structural disposition, optimal shape design, analysis, construction methods, details etc. and all these problems are interconnected among them. As an example the shape optimization requires the use of several disciplines like structural analysis, sensitivity analysis, optimization strategies and geometrical design concepts [1]. Similar comments can be applied to other space structures such as steel trusses with single or double shape and tension structures.



B.- Bifurcation point.
D.- Displacement limit point,
L.- Force limit point.

Fig.1.- Static load-displacement path of a structure

In relation to the analysis the Finite Element Method appears to be the most extended and versatile technique used in the practice. In the application of this method several issues arise. First the derivation of the pertinent shell theory or alternatively the degenerated 3-D solid approach should be chosen. According to the previous election the suitable FE model has to be adopted i.e. the displacement, stress or mixed formulated element. The good behavior of the shell structures under dead loads that are carried out towards the supports by mainly compressive stresses is impaired by the high imperfection sensitivity usually exhibited by these structures. This last effect is important particularly if large deformation and material nonlinearities of the shell may interact unfavorably, as can be the case for thin reinforced shells. In this respect the study of the stability of the shell represents a compulsory step in the analysis. Therefore there are currently very active fields of research such as the different descriptions of consistent nonlinear shell models given by Simo, Fox and Rifai, Mantzenmiller and Büchter and Ramm among others, the consistent formulation of efficient tangent stiffness as the one presented by Ortiz and Schweizerhof and Wringgers, with application to concrete shells exhibiting creep behavior given by Scordelis and coworkers; and finally the development of numerical techniques needed to trace the nonlinear response of the structure.

The objective of this paper is concentrated in the last research aspect i.e. in the presentation of a state-of-the-art on the existing solution techniques for nonlinear analysis of structures. In this presentation the following excellent reviews on this subject [2,3] and [4] will be mainly used.

For many real shell analysis it is necessary to trespass the critical points because the sensitivity of the shell to the geometrical imperfections is very much connected to the nature of the structural response at the neighborhood of the critical points. A very detailed discussion on this subject is

presented by Medwadowski in [5] where the classification of the critical instability points (bifurcation points, force limit points displacement limit point with the related phenomena of snap-trough and snap-back) is given and related to the sensitivity of the shell to the geometrical imperfections (figure 1). Therefore only in few cases the isolated value of the first critical point may be sufficient for design purposes and even then this information can be unreliable if the numerical analysis employed can not efficiently handle situations near the limit point in which the stiffness matrix is near singular. In these cases the instability of the structure can be misinterpreted by the instability of the numerical algorithm used in the analysis.

2.CLASSIFICATION OF THE NONLINEAR SOLUTION PROCEDURES.

The classification of the numerical techniques of solving nonlinear systems of equations is a rather difficult task. This fact is because there are several points of view for the classification and also some procedures are currently under modification in order to improve their performance and take some benefits from others. In this way it is not possible to define a clear cut between two in principle different procedures.

A first classification corresponds to the objective of the analysis. Some procedures are concerned with the solution of the problem for a given load level (Single solution algorithms). Others more often are related to the tracing of the structural response for a continuous set of load or displacement values (Tracing path algorithms). Obviously the latter includes the first group of algorithms, but due to essential property of the nonlinearity there is not an unique solution. Therefore it is necessary to define in every single solution algorithm a path point situated near to the searched solution.

Normally in the tracing path algorithms it is assumed that the structure is stable at initial (unloading) state, but there exist several important situations for which this assumption is not valid as ii is the case of cable structures with moderate prestress. It is possible to extend the tracing path algorithms to cope with these cases. The work of Hangai and coworkers [6] is relevant in this respect.

Specific difficulties have to be handled carefully by the tracing path algorithms. These difficulties arise in several important situations as when a bifurcation point is trespasing from a stable branch to one unstable ones, or when several branches exist at the bifurcation point, or when is necessary to treat and recognize force or displacement limit points. In this last case the existence of a quasi-singular tangent stiffness and the limitation of the load or displacement level demands different tracing path strategies.

Another classification point of view corresponds to the nature of the mathematical procedure used in finding of the solution point. There are iterative algorithms and incremental algorithms. Usually both types of algorithms are used simultaneously in a solution procedure, but the iterative ones are connected with the searching of the solution along the neighborhood of the searched solution, by successively improving the approximation of a trial initial solution.

Finally another typical classification of nonlinear solution procedures refers to the level of the mathematical formulation of the nonlinear structural problem.

A first level of setting up the structural problem is the use of a energy functional- usually the total potential energy- that has to become stationary at the solution point.

The second level is represented by the set of simultaneous equilibrium equations. These equations can be derived either as a consequence of the principle of the virtual work or as the first order conditions (first variation) of the above mentioned energy functional.

Finally the third level corresponds to the application of the principle of the virtual work at incremental level or alternatively to equal zero the second variation of the energy functional.

The nonlinear solution algorithms that are applied to the two first levels of formulation are known as implicit methods and the ones

using the third level of formulation are called explicit methods. In the implicit methods it is not necessary to build the tangent stiffness matrix of the problem contrary to the explicit ones. The main computational difference between the two types of methods is that in the explicit methods a linear problem needs to be solved in each iteration and in the implicit methods this is not the case. However this difference is currently becoming more fuzzy because explicit methods trend is to avoid or at least to diminish this drawback and the implicit methods are trying to become more accurate in each iteration by solving some type of linear problem as it will be shown below.

3. IMPLICIT METHODS.

3.1. DEFINITIONS.

It is assumed that the shell or the spatial structure is discretized in finite elements and its static and nonlinear behavior is described by the system of simultaneous equations:

$$r(u) = K(u)u - p = 0 \qquad [1]$$

where u is the displacement vector and p is the equivalent forces vector applied at nodes.

The nonlinear behavior of the structure is caused by the nonlinear constitutive equations of the material (plasticity, elasto-plasticity etc) and/or the modification of the initial geometry of the structure. This nonlinear behavior is contained in the stiffness matrix $K=K(u)$ and also in the force vector $p=p(u)$ in the case that the loads depend on the geometry of structure (non conservative structures).

Usually in the Finite Element Method the stiffness matrix is computed from the internal stresses σ (or stress-resultants) as follows

$$K(u)u = \int_V B^T \sigma \, dV \qquad [2]$$

The tangent stiffness matrix K_t is defined by the expression:

$$K_t = \frac{\partial}{\partial u}[K(u)u] \qquad [3]$$

plus a contribution from the external forces $\frac{\partial p}{\partial u}$ if they depend on the deformed geometry of the structure. This contribution is usually omitted in order to keep the symmetry of the tangent stiffness matrix. In the following this simplification will assumed to be valid.

The equation [1] can be derived by minimizing the total potential energy of the structure $V(u)$ i.e.,setting equal zero the gradient of this functional $V(u)$ that is given by the formula:

$$\frac{\partial V(u)}{\partial u} = K(u)u - p = r(u) \qquad [4]$$

3.2. DIRECT SUBSTITUTION METHOD (DS)

It is the simplest method of iteration in a point. Starting from an approximation of the displacement solution u_0 the following successive values $u_1, u_2, u_3, \ldots u_i$ are obtained according to the algorithm:

$$K(u_i)u_{i+1} - p = 0 \qquad [5]$$

i.e

$$u_{i+1} = K^{-1}(u_i)p \qquad [6]$$

The set defined in [6] is convergent to a solution near to the starting point u_0 if the Lipschitz condition is fulfilled:

$$\|K^{-1}(u_i)p - K^{-1}(u_j)\| < M \|u_i - u_j\| \qquad [7]$$

where $0 \le M \le 1$ for all values $u_i, u_j \in B$ and B a neighborhood of the solution.

The condition [7] is fulfilled if the moduli of all the eigenvalues of the jacobian matrix $J(u)$ are smaller than the unity for all point u belonging to B where the jacobian is defined as follows:

$$J(u) = \frac{\partial [K^{-1}(u)p]}{\partial u} \qquad [8]$$

The DS algorithm can be considered as belonging to a general algorithms family expressed by the formula[7]:

$$u_{i+1} = -\frac{r(u_i)}{q_i} + \left(\frac{e_{i-1}}{q_i} + 1\right)u_i - \frac{e_{i-1}}{q_i}u_{i-1} \qquad [9]$$

where $r(u_i) = r_i = K(u_i)u_i - p$, , q_i y e_{i-1} are

specific coefficients for the particular algorithm under consideration.

For the values $q_i = -1$; $e_i = 0$ para *todo* i the equation [9] becomes to the one described by the DS algorithm, i.e.

$$u_{i+1} = r(u_i) + u_i = r_i + u_i \qquad [10]$$

In the following sections several interesting particular cases of the algorithms family [9] are described.

3.3.CONJUGATE GRADIENT METHODS.(CG).

The CG methods have been initially developed in order to minimize quadratic functions. As introduction to these methods this initial application is summarized here.

The total potential energy of a linear structure discretized by the FEM is:

$$V(u) = V_0 - u^T p + \frac{1}{2} u^T K u \qquad [11]$$

where the stiffness (tangent) matrix $K = K_t$ is now constant.

The direction of the gradient of this function is

$$\frac{dV}{du} = -p + K_t u = r(u) \qquad [12]$$

and the displacements in the gradient direction are obtained in the iteration i as:

$$u_{i+1} = u_i + \alpha_i \left(\frac{dV}{du}\right)_{u=u_i} = u_i + \alpha_i r_i \qquad [13]$$

The value of α_i can be computed from the condition of minimum of $V(u_{i+1}) = V(u_i + \alpha_i r_i)$, i.e.

$$\frac{dV}{d\alpha_i} = \left(\frac{dV}{du}\right)_{u=u_{i+1}} . r_i = r_{i+1}^T r_i = 0 \qquad [14]$$

or equivalently

The expressions [13] y [14] define the steepest gradient method. This method is convergent [8] if the ratio between the

$$\alpha_i = -\frac{r_{i+1}^T u_i}{r_i^T r_i} \qquad [15]$$

maximum and minimum eigenvalues of the stiffness matrix is less than 2.

It is possible to define an algorithm that is convergent within a fixed given number of iterations by changing the equation [13] by the more general following one:

$$u_{i+1} = u_i + \alpha_i s_i \qquad [16]$$

where the unknown vector s_i , that represents the optimal direction, is expressed as a linear combination of the gradient r_i and of the optimal direction vector of the previous iteration s_{i-1}, i.e:

$$s_i = r_i + \beta_i s_{i-1} \qquad [17]$$

The coefficients α_i and β_i can be computed similarly as for the case of the steepest descent methods and then it results:

$$r_{i+1}^T s_i = 0 \qquad [18]$$

$$s_{i-1}^T K_t s_i = 0 \qquad [19]$$

The two above equations lead to the following formula:

$$\beta_i = \frac{s_{i-1}^T K_t r_i}{s_{i-1}^T K_t s_{i-1}} \qquad [20]$$

The application of the equation [12] to the iterations i and i-1 allows us to write:

$$K_t (u_i - u_{i-1}) = r_i - r_{i-1} \qquad [21]$$

i.e

$$\alpha_{i-1} K_t s_{i-1} = r_i - r_{i-1}$$

and the expression [20] becomes in the following one:

$$\beta_i = \frac{r_i^T (r_i - r_{i-1})}{s_{i-1}^T (r_i - r_{i-1})} \qquad [22]$$

The value of α_i can be obtained by iteration

from the expression [19] and the computed value of β_i. If it is possible to satisfy exactly the equation [19] then the line search along the direction s_i is called exact.
It is possible to simplify the computation of β_i according to [22] if the two following hypothesis are fulfilled:

(1) The search along s_i and s_{i-1} are exact.
(2) The function V is quadratic.

Then the Fletcher-Reves formula[9] given by [23] can be applied:

$$\beta_i = \frac{r_i^T r_i}{r_{i-1}^T r_{i-1}} \qquad [23]$$

If the hypothesis (2) does not hold then it is convenient to use the so called Polak-Ribiere formula[10]:

$$\beta_i = \frac{r_i^T (r_i - r_{i-1})}{r_{i-1}^T r_{i-1}} \qquad [24]$$

It is observed that the CG method corresponds to the algorithm of the family given by the equation [9] for the parameters values $q_i = \frac{1}{\alpha_i}$; $e_{i-1} = \frac{\beta_{i-1}}{\alpha_{i-1}}$ for $i \geq 1$ and $e_{i-1} = 0$ for $i = 0$.

3.4.CONJUGATE NEWTON METHODS AND SCALED METHODS.

The convergence of the CG methods are very much dependent on the ellipticity of the problem, expressed by the positive definite (pd) character of the stiffness matrix K_t. A measure of this pd property is the conditioning number of the matrix. Then the stiffness matrix is becoming singular near a limit point and therefore a scaling procedure is needed in order to improve the CG method convergence. A scaling algorithm can be applied as follows

Let it be K_{ta} an approximation of the stiffness matrix K_{ti} and L and L^T the Choleski factors, i.e.

$$K_{ta} = L\, L^T$$

The following transformations are carried out:

$$\bar{u}_i = L^t u_i \; ; \; \bar{r}_i = L^{-1} r_i \; ; \; \bar{p} = L^{-1} p \qquad [25]$$

and the new stiffness matrix becomes:

$$\bar{K}_{ti} = L^{-1} K_{ti} L^{-T}$$

that is nearly an unit matrix, that means, their eigenvalues are approximately equal.

The application of a CG method to the transformed problem

$$\bar{r} = \bar{K}_t \bar{\mu} - \bar{p} = 0$$

gives

$$\bar{s}_i = -\bar{r}_i + \beta_i \bar{s}_{i-1}$$

i.e.

$$s_i = -K_{ta}^{-1} r_i + \beta_i s_{i-1}$$

where the coefficient β_i is computed from the expression [22] transformed into the following one:

$$\beta_i = \frac{r_i (g_i - g_{i-1})}{s_{i-1}^T (g_i - g_{i-1})} \qquad [26]$$

and $g_i = K_{ta}^{-1} r_i$ corresponds to the search direction obtained according to the modified Newton method that will be described later. Obviously the transformations expressed in [25] have not to be carried explicitly by the computer.

The approximate stiffness matrix k_{ta} proposed in linear problems is the diagonal matrix defined as $K_{ta} = \text{diag}[K_t]$ or one banded matrix with all their elements within a fixed given band equal to the original stiffness matrix. For nonlinear problems it has been developed a conjugate Newton method that produces a direction vector satisfying the equation [19] where K_t is the current stiffness matrix at iteration i and the information contained in [18] is also used.

The efficiency of the CG methods is very much problem dependent. If the problem is

quasi-linear and the line search is exact then the formula [23] can be applied. In the case of very strong nonlinear problems to be solved the use of the expression [23] is recommended instead. But if the line search is not exact the original formula [22] is the one to be applied.

3.5.METHOD OF THE DYNAMIC RELAXATION.(DR).

These methods have been introduced in the sixties by Otter et alia [11] and they are very robust iterative algorithms to solve strong nonlinear problems. The solution of these problems is considered to be the steady part of a fictitious dynamic problem defined by the equations:

$$M\ddot{u}+C\dot{u}+K_t u-p=0$$

where M and C are the mass and damping matrices to be selected in order to reach numerical efficiency. The DR methods can be included into the family of algorithms expressed by [9] with the following values for the parameters:

$$q_i=\frac{(\omega_{max}+\omega_{min})^2}{4}\ ;\ e_{i-1}=\frac{(\omega_{max}-\omega_{min})^2}{4}\quad \text{for } i\geq 1$$

$$q_i=\frac{(\omega_{max}+\omega_{min})^2}{4}\ ;\ e_{i-1}=0\quad \text{for } i=0$$

where ω_{max} and ω_{min} are the maximum and minimum frequencies of the pseudodynamic problem. In reference [12] some criteria to select the parameters of the DR method in order to optimize its efficiency are given.

Respect to the explicit methods it can be concluded that they are very efficient for large and strongly nonlinear problems, because the computation of the residuals r_i is not very costly. However the number of iterations can be very large if convergence is going to be achieved but this drawback can be avoided if paralell and vectorized procedures are used. In this context it seems to be more efficient the DR method than the CG method particularly if the problem to be solved is highly nonlinear one.

4.IMPLICIT METHODS.

4.1.INTRODUCTION.

In general the implicit methods are less efficient than the explicit ones, particularly for very extreme situations involving highly nonlinear problems with a large number of equations, but the total number of iterations required to reach convergence are in these methods much smaller than in the implicit methods. Therefore the scope of the implicit methods are restricted mainly to medium size problems with no very strong nonlinearities and including complex finite elements, such as hyperelements. In these cases the residual computation can be very costly in the explicit methods with regard to the total computation cost.

The most frequently used implicit algorithms correspond to the Newton type algoritm. They will be described in the subsequent sections starting with the review of the original Newton method.

4.2.NEWTON-RAPHSON METHOD. (NR).

The solution to the nonlinear problem [1] near to the point u_0,p can be obtained by approximating the residual function $r(u)$ by other defined by the following linear application:

$$r_L(u)=r(u_0)+\left.\frac{\partial r(u)}{\partial u}\right|_{u=u_0}(u-u_0)$$

i.e

$$r_L(u)=r_0+K_{t0}(u-u_0) \qquad [27]$$

where $r_0=r(u_0)$ and $K_{t0}=K_t(u_0)$ is the tangent stiffness matrix at point $u=u_0$.
If the initial point u_0 is a good appoximation of $r(u)=0$ then a better value for the solution is obtained by setting to zero the approximation [27] of the residual function, i.e.:

$$u_1=u_0-K_{t0}^{-1}r_0$$

Using an iterative procedure the value of approximated solution at iteration i is:

$$u_{i+1} = u_i - K_{ti}^{-1} r_i \qquad [28]$$

where u_i is the value of the displacement vector obtained in the previous iteration and

$$K_{ti} = K_t(u_i) \quad ; \quad r_i = r(u_i)$$

In order to study the possible failures of the method it is convenient to consider the total potential energy $V(u)$ that becomes minimum at the same point as the solution of $r(u) = 0$.

This function $V(u)$ can be expanded at point u_i, what represents an approximation to the solution to be found, and then it results:

$$V(u_i + s) = V(u_i) + s^T r_i + \frac{1}{2} s^T K_{ti} s + O(s^3) \qquad [29]$$

A local quadratic approximation of the function $V(u)$ at the neigborhood of u_i is

$$V_L(s) = V(u_i) + s^T r_i + \frac{1}{2} s^T K_{ti} s \qquad [30]$$

The value of $s = s_i$ is selected such that [30] becomes minimum. The gradient of [30] is $r_i + s^T K_{ti}$ and then the following equation results

$$K_{ti} s_i = -r_i \qquad [31]$$

This equation [31] is identical to the equation [28] because the new point is now $u_{i+1} = u_i + s_i$.

If the stiffnes matrix K_{ti} is positive definite (pd) the following equation is satisfied:

$$s_i^T K_{ti} s_i = -s_i^T r_i < 0 \qquad [32]$$

and

$$V(u_i + s_i) - V(u_i) = -\frac{1}{2} s_i^T K_{ti} s_i + O(s^3) < 0 \qquad [33]$$

The equation [33] means that the new point u_{i+1} is a better approximation to the solution than the old point u_i,. assuming that the stiffness matrix is pd and the old point is itself a good approximation to the solution.

If the above mentioned assumptions are fulfilled the set of values $u_1, u_2, u_3, \ldots, u_i$ generated by the NR algorithm converges quadratically to the solution u^*, namely for a given norm:

$$\| u_{i+1} - u^* \| \leq c \| u_i - u^* \| \qquad [34]$$

where c is a constant depending on the stiffnes matrix and its derivatives.

From the previous discussion it may occur that $V(u_{i+1}) > V(u_i)$ and the NR method fails due to one of the following reasons:

(1).- K_{ti}^{-1} exists and is pd but the vector defined in the line search s_i is so large that occurs $V(u_{i+1}) > V(u_i)$, i.e. the vector u_i is not a good approximation of the solution.

(2).- K_{ti}^{-1} exists but is not pd, then it may occur that

$$s_i \cdot \left. \frac{\partial V}{\partial u} \right|_i > 0 \quad \text{i.e.} \quad r_i^T s_i = -s_i^T K_{ti} s_i \qquad [35]$$

and the new point is not a better approximation to the solution than the old one, i.e. in this case

$$V(u_i + s_i) > V(u_i) \qquad [36]$$

(3).- K_{ti} is singular and then the direction and modulus of s_i can not be defined. Therefore if the NR algorithm needs to be continued it is necessary to introduce an alternative procedure to handle this situation, in order to construct s_i

To avoid these possible failures the following alternative remedials can be applied.

(1).-If the starting point is a bad estimation of the solution it is convenient to use a line search in order to ameliorate it. The following problem must then be solved:

$$\min_{\alpha} \; V(u_i + \alpha s_i) \;\rightarrow\; s_i^T . r(u_i + \alpha s_i) = 0 \qquad [37]$$

(2).- K_{ti} is not a pd matrix. There are several posibilities to deal with this situation that will be commented below.

(a) Change the usual NR algorithm for a steepest descent gradient algorithm.

(b) If the failure is due to the change in the direction of the line search (ascendent direction) it can be treated by the reversal of the search direction.

(c) Add to the stiffness matrix a diagonal matrix D with positive terms such that the resultant matrix becomes pd. In this case the iterative algorithm is

$$u_{i+1}=u_i-(K_{ti}-\mu D)^{-1}r_i \qquad [38]$$

where μ is a scalar such that $0<\mu<1$.

This technique has been applied by Popov and it can be interpreted as the introduction of the a set of ficticious elastic springs in all degrees of freedom (dof) of the structure in order to it becames stable.

4.3. MODIFIED NEWTON RAPHSON METHOD. (mNR).

If the total number of dof of the structure is very large, the application of NR method represents a very heavy cost consisting of the computation of the stiffness matrix K_{ti}, the solution of the system of equations $K_{ti}s_i=r_i$ and the posible improvement represented by the line search $s_i^T.r(u_i+\alpha s_i)=0$ at each iteration. The two first cost items can be reduced if the stiffness matrix is kept constant during a certain number of iterations, i.e., the following algorithm known as the modified Newton Raphson is used:

$$u_{i+1}=u_i-K_m^{-1}r_i \quad (i>n) \qquad [39]$$

The update of the stiffness matrix can be carried out according to a given strategy or if some convergence criterium is satisfied.

The number of iterations required in this mNR method is larger than the one needed in the NR method for a given level of accuracy in the solution. However the computational effort in each iteration is much smaller in this mNR method than in the original NR algorithm, because the stiffness matrix is constant troughout several iterations and its inverse (or even better its Choleski decomposition) can be stored in the computer memory.

4.4. APPROXIMATE LINE SEARCH.

The line search is used as it was commented in order to improve the bad approximation of the first point to the solution of the structural problem. However this line search represents a major computational task, and it consists of the determination of the value of constant α such that the following equation is satisfied within a high accuracy:

$$s_i^T.r(u_i+\alpha s_i)=0 \qquad [40]$$

Therefore this line search should be carefully used and in the strictly necesary cases. Some authors recommend the substitution of the equation [40] by the following less restrictive one:

$$|s_i^T.r(u_i+\alpha s_i)|<\eta|s_i r_i| \qquad [41]$$

where η is an appropiate limit that should be adjust according the degree of the nonlinearity of the problem and the type of iterative procedure used. Usual values are $0.25<\eta<0.50$. If the equation [41] is satisfied for $\alpha=1$ then the line search is not needed.

4.5.QUASI NEWTON METHODS. (QN).

As it has been already pointed out one of the biggest inconvenience of the NR methods corresponds to the computation of the stiffness matrix K_t and its inverse K_t^{-1} at each iteration point u_i. The mNR method represents an answer in order to diminish the computational effort demanded by the obtention of these matrices. Another efficient possibility is the group of algorithms known as the Quasi-Newton methods.

In order to show the ideas behind the QN methods this preliminary discussion will be concerned with 1 dof structure. The local tangent whose slope corresponds to the stiffness matrix K_t is replaced by the straight line joining the two consecutive points in the iterative procedure. Later it will be shown that the extension of this concept to the structures with several dof is not a straightforward task. In this regard the first order expansion of the residual fuction $r(u)$ at the neighborhood of u_{i+1} is:

$$r(u_i)=r(u_{i+1})+K_{ti}(u_i-u_{i+1})+\Delta \qquad [42]$$

where $\Delta\to 0$ as $u_i\to u_{i+1}$.

The two following new functions are defined

$$q_i = r(u_{i+1}) - r(u_i) = r_{i+1} - r_i$$
$$s_i = u_{i+1} - u_i \qquad [43]$$

and then if Δ is neglected the equation [42] becomes:

$$q_i = K_{t,i+1} s_i \qquad [44]$$

i.e.

$$s_i = K_{t,i+1}^{-1} q_i = F_{t,i+1} q_i \qquad [45]$$

The expresion [45] is known as the Newton equation. It is exact if the functional of the problem is quadratic, otherwise it represents a good approximation in the neigboorhood of the solution assuming a strictly convex functional.

The matrix $F_{t,i+1}$ can be easily computed by adding a corrective matrix C_i to the matrix $F_{t,i}$ of the previous iteration:

$$F_{t,i+1} = F_{t,i} + C_i \qquad [46]$$

This corrective matrix is in general a function of $F_{t,i}$, s_i and q_i. It should be selected in such a way that the sucessive matrices $F_{t,i}$ keep their symmetry and pd properties. There exist several formulae to define these matrices C_i in [46]. Each of them defines a variant of the QN methods and they can be classified according to the rank of these updating matrices C_i.

(a) Rank-1 updates.

The most simple expression of C_i is:

$$C_i = z_i u_i^T \qquad [47]$$

where the arbitrary vector u_i satisfies $u_i^T q_i \neq 0$ and z_i is chosen so that the equation [45] is valid, i.e.

$$F_{t,i+1} q_i = F_{t,i} q_i + z_i u_i^T q_i = s_i \rightarrow z_i = \frac{s_i - F_{t,i} q_i}{u_i^T q_i} \qquad [48]$$

Then the general expression for the rank-1 update formulae is

$$F_{t,i+1} = F_{t,i} + \frac{(s_i - F_{t,i} q_i) u_i^T}{u_i^T q_i} \qquad [49]$$

Several possibilities for the selection of the arbitrary vector u_i have been used. Broyden proposes the following one:

$$u_i^T = q_i^T F_{t,i}^B$$

i.e.

$$F_{t,i+1}^B = F_{t,i}^B + \frac{(s_i - F_{t,i}^B q_i) q_i^T F_{t,i}^B}{q_i^T F_{t,i}^B q_i} \qquad [50]$$

The formula [50] does not keep the symmetry of rhe matrix $F_{t,i}^B$ and in order to avoid this possible inconvenience Davidon has suggested the following update matrix

$$u_i = s_i - F_{t,i}^D q_i$$

i.e.

$$F_{t,i+1}^D = F_{t,i}^D + \frac{(s_i - F_{t,i}^D q_i)(s_i - F_{t,i}^D q_i)^T}{(s_i - F_{t,i}^D q_i)^T q_i} \qquad [51]$$

Both formulae [50] and [51] do not need a line search when they are applied to a quadratic functional. This property hints the possibility to use these formulae to general nonlinear cases with an approximated line search instead of an exact one. This fact is very attractive from a computational point of view. However these rank-1 updates can not keep the pd property of the matrix $F_{t,i}$. Therefore in the solution procedure it is necessary to check if the set of equations becomes non pd and then to adequately handle this possibility.

(b) Rank-2 updates

The rank-2 updates allow us to keep the pd property of the matrices $F_{t,i}$. The Davidon-Fletcher-Powell (DFP) formulae can be expressed as follows:

$$F_{t,i+1}^{DFP} = F_{t,i}^{DFP} + \beta s_i s_i^T + \gamma \left[F_{t,i}^{DFP} q_i\right]\left[F_{t,i}^{DFP} q_i\right]^T \qquad [52]$$

where the coefficients β and γ are obtained so that the condition [45] is fulfilled. The DFP formulae correspond to the values:

$$\beta = \frac{1}{s_i^T q_i} \quad ; \quad \gamma = -\frac{1}{q_i^T F_{t,i}^{DFP} q_i}$$

i.e.

$$F_{t,i+1}^{DFP} = F_{t,i}^{DFP} + \frac{s_i s_i^T}{s_i^T q_i} - \frac{\left[F_{t,i}^{DFP} q_i\right]\left[F_{t,i}^{DFP} q_i\right]^T}{q_i^T F_{t,i}^{DFP} q_i} \quad [53]$$

The approximation of the stiffness matrix in the above DFP approximation is given by the expression:

$$K_{t,i+1}^{DFP} = \left(I - \frac{q_i s_i^T}{q_i^T s_i}\right) K_{t,i}^{DFP} \left(I - \frac{s_i q_i^T}{q_i^T s_i}\right) + \frac{q_i q_i^T}{q_i^T s_i} \quad [54]$$

where

$$K_{t,i+1} = \left[F_{t,i+1}^{DFP}\right]^{-1}$$

It can be shown that if the stiffness (or flexibility) matrix of the previous iteration is pd then it is also pd the update corresponding matrix assuming a search line has been carried out. Moreover if the search line is exact then the DFP method coincides with the Conjugate Direction method. In this case if the starting or initial stiffness (or flexibility) matrix is the unity matrix then the method becomes the Conjugate Gradient method.

It is possible to obtain a couple of complementary expressions to the ones described by [53] and [54] if the conditions to be fulfilled by the coefficients β and γ are the [44] instead of the [45]. In this case the following expressions known as the Broyden-Flectcher-Goldfarb-Shanno formulae are found and deducted by permutation of the variables s_i and q_i in the DBF equations [53] and [54]:

$$K_{t,i+1}^{BFGS} = K_{t,i}^{BFGS} + \frac{q_i q_i^T}{s_i^T q_i} - \frac{\left[K_{t,i}^{BFGS} s_i\right]\left[K_{t,i}^{BFGS} s_i\right]^T}{s_i^T K_{t,i}^{BFGS} s_i} \quad [55]$$

$$F_{t,i+1}^{BFGS} = \left(I - \frac{s_i q_i^T}{q_i^T s_i}\right) F_{t,i}^{BFGS} \left(I - \frac{q_i s_i^T}{q_i^T s_i}\right) + \frac{s_i s_i^T}{q_i^T s_i} \quad [56]$$

It is widely recognized [13] that equations [55] and [56] represent the most efficient procedures in the present state of the art in order to solve non linear structural problems with pd tangent stiffness matrix.

In the application of the QN methods a very severe inconvenience is that the bandwidth of the matrix K_t (or F_t) is not kept trough out the iterations. That means the demand for larger resources of the memory in this case than in the conventional NR methods. Several attempts have been developed in order to keep constant the bandwidth of the matrices but most of them mean an extra computational work that reduces the advantages of the standard QN methods. Perhaps the idea of Mathies and Strang is one of the most efficient ones in relation to the line search used in conjunction to the QN algorithms, because they recognizes explicitly the band property of the involved matrices. The main points of this QN method improvement are as follows.

An alternative expression to the one given by [56] is

$$F_{t,i+1}^{BFGS} = \left(I + w_i v_i^T\right) F_{t,i}^{BFGS} \left(I + v_i w_i^T\right) \quad [57]$$

where

$$w_i = \frac{1}{s_i^T q_i} s_i \; ; \; v_i = a_i^{\frac{1}{2}} K_{t,i} s_i - q_i \; ; \; a_i = \frac{s_i^T q_i}{s_i^T K_{t,i} s_i} \quad [58]$$

The value of a_i is positive assuming that $K_{t,i}$ and $K_{t,i+1}$ are pd matrices as it is the usual hypothesis in the BFGS method. In the contrary case the original expression [56] should be used.

The application of the method in the first iteration step gives:

$$s_1 = K_{t,2} r_1 = F_{t,2} r_1 = \left(I + w_1 v_1^T\right) F_{t,1} \left(I + v_1 w_1^T\right) r_1 \quad [59]$$

The equation [59] can be solved without actually construct the matrix $K_{t,2}^{-1}$ or $F_{t,2}$ by using the vectors:

$$b_1 = r_1 + (w_1 r_1) v_1 \quad ; \quad c_1 = F_{t,1} b_1$$

and then

$$s_1 = c_1 + (v_1^T c_1) w_1 \quad [60]$$

Once the vector s_1 is computed an exact or

an approximated (with tolerance η) line search permits to obtain the value α_1.

Similarly the next iteration step leads to the equation

$$s_2=\left(I+w_2v_2^T\right)\left(I+w_1v_1^T\right)F_{t,1}\left(I+v_1w_1^T\right)\left(I+v_2w_2^T\right)r_2$$

that can be computed without actually construct the matrices $F_{t,1}$ or $F_{t,2}$. In this way this procedure is applied to the successive steps.

The main inconvenience of this approach lies in the need to accumulate more and more vectors as the method proceeds. However it is possible to disregard the first vectors and to keep only the last ones, according to the amount of memory available in the computer.

The above technique used to avoid the increase of the bandwidth can be applied to the 1-rank updates. In these cases the alternative expression to [49] is the following one:

$$F_{t,i}=F_{t,1}+\sum_{j=1}^{i-1}\beta_jv_jv_j^T \qquad [61]$$

For example in the Davidon updating with a search line it is obtained:

$$\begin{aligned} v_j&=\alpha_js_j-F_{t,j}q_j \\ \beta_j&=\left[\left(\alpha_js_j-F_{t,j}q_j\right)^Tq_j\right]^{-1} \end{aligned} \qquad [62]$$

In the i-th iteration the previous vectors v_j and coefficients β_j $(j<i)$ are kept in the computer memory. The search direction for the following iteration is according to [61]:

$$s_i=\left(F_{t,1}+\sum_{j=1}^{i-1}\left(\beta_jv_jv_j^T\right)\right)r_i \qquad [63]$$

and the new vector for the Davidon updating is then

$$v_i=\alpha_is_i-F_{t,1}r_i-\sum_{j=0}^{i-1}\beta_jv_jv_j^Tr_j \qquad [64]$$

It is important to check the condition number of the matrices $F_{t,i}$ during the application of the QN methods. If this number at i-th iteration is larger than a given tolerance (usually 10^5) then it is necessary to compute directly the corresponding stiffness (or flexibility matrix), or to state (in case of the stiffness matrices): 2

$$K_{t,i+1}=K_{t,i} \quad \text{or} \quad K_{t,i+1}=K_{t,1}$$

Finally a comparison between the NR and mNR methods and the QN methods can be made from the computational point of view. The NR method needs in order to obtain the matrix $K_{t,i}$ and its triagularization $O(n^3)$ arithmetic operations but the mNR only needs in each iteration step except the first one $O(n^2)$ where n is the total number of dof. In the QN methods the same order of operations as for mNR methods is needed, but the coefficient is smaller in the QN methods because for them only one backsubstitution is required for each iteration.

4.6. NEWTON-SECANT ALGORITHMS. MEMORYLESS QN METHODS.

This method has been developed by Crisfield in [14] and it presents some similarities with the conjugate gradient method with the formulation represented by the formula [15] but without the requirement of the orthogonality condition [20]. This condition is replaced now by the secant equation [46] of the QN method.

This method is known sometimes as the method of the rapid iterations in the NR technique or the secant-Newton method (SN method). The basic idea in this method is to assume the following approximation for any of the different updating formulae of the QN method:

$$F_{t,k}=K_{ta}^{-1}=F_{ta} \qquad [65]$$

where K_{ta} is an approximative tangent stiffness matrix. In this memoryless method th different updates can be written as follows:

$$s_{i+1}=AF_{ta}r_{i+1}+B\alpha_is_i+CF_{ta}r_i \qquad [66]$$

where the parameters A,B and C depend on the particular update formula and α_i is the length obtained in the line search.

As an example in the Davidon updating it is obtained:

$$A=-(1+B)\ ;\ B=\frac{r_{i+1}^T(\alpha_i s_i-F_{ia}r_i)}{q_i^T(\alpha_i s_i-F_{ia}r_i)}\ ;\ C=B$$

and in the case of the BFGS update it is found:

$$A=1-C\ ;$$

$$B=-C-\frac{(F_{ia}r_{i+1}-F_{ia}r_i)r_{i+1}}{s_i^T q_{i+1}}+C\frac{(F_{ia}r_{i+1}-F_{ia}r_i)q_{i+1}}{s_i^T q_{i+1}}\ ;$$

$$C=\frac{s_i^T r_{i+1}}{s_i^T q_{i+1}} \qquad [67]$$

The formula [66] represents an expression that iteratively modify three vectors. It is possible to obtain analogous formulae for three vectors if the approximative tangent stiffness matrix satisfies the relationship:

$$\alpha_i r_i=-K_{ia}^{-1}q_i \qquad [68]$$

For the BFGS case [68] leads to the following expression:

$$s_i=\bar{A}F_{ia}r_i+\bar{B}s_{i-1} \qquad [69]$$

where $\bar{A}=A$: $\bar{B}=-C-A\frac{s_1^T q_i}{s_{i-1}^T q_i}$ and A and C given by [67].

In the particular case $F_{ia}\equiv I$ (unity matrix) the above method coincides with the conjugate gradient method when it is applied to quadratic functionals with exact line search.

Finally it is important to point out that the SN methods include in their process an approximate line search and this fact can avoid the need for further more precise line searchs.

5.- ITERATIVE AND INCREMENTAL METHODS.

The algorithms described in the previous sections solve the problem of finding the nearest solution $u=u^*$ to the value $u=u_0$ of the nonlinear equation [1]. The developed procedures are iterative, i.e. in successive steps is found the searched solution. In each step i it is known the approximative solution $u=u_i$ and a better aproximation $u=u_{i+1}$ is found if the process converges.

There exists another possibility of solving nonlinear equations known as incremental procedure. The methods belonging to this procedure have in common that the solution is found by sucessive increasing the value of the vector load given by the right hand side of the equation [1]. This means that the load vector varies from an initial value (usually a zero value) up to the final value for which the solution is to be found. Then if it is represented by λ the proportional factor the following sucessive equations are solved:

$$K(u)u-\lambda p=0 \qquad [70]$$

where sucessive values are given to λ starting from 0 up to the final value 1 and with specified incremental values $d\lambda$.

Usually the incremental formulation is expressed as follows:

$$K_t du=dp=p\,d\lambda \qquad [71]$$

and to integrate [71] by one of the classical integration procedures as the Euler method or better by a Runge-Kutta method. These methods are step by step algorithms and they start from an initial solution, usually $\lambda=0$, $u=0$.

The incremental methods of solution of the equation:

$$p\,d\lambda=K_t(u)\,du \qquad [72]$$

allow us to advance from a value $u=u_j$ corresponding to a load level λ^j and to obtain the new solution u^{j+1} for the load level $\lambda^{j+1}=\lambda^j+d\lambda^{j+1}$. In this computation it is necessary to introduce some hypothesis respect to the law of variation of the matrix $K_t(u)$ in the load interval λ^j,λ^{j+1} and the displacement interval u^j,u^{j+1}. In general the hypothesis used assumes a series expansion of the matrix in the interval and the solution of [72] is approximated and therefore the values λ^{j+1},u^{j+1} at the end of the interval are obtained with an error. This error is

accumulated and propagated as the algorithm is applied to the sucessive intervals. Then the final solution to be obtained can be significativly different to the exact one, particularly if the intervals are not small enough or too many are needed in order to reach the final value of the load. Naturally there are several tecniques to diminish the level of this error propagation. One often used introduces at the end of each computational step ficticious forces that equilibrate (i.e. satisfy [72]) the starting values u^j, λ^j for the following load interval and in this way the level of the error may be limited.

It is obvious from the above considerations that typically pure incremental methods are rarely used in the practice. It is usual to applied incremental methods but introducing some iterative techniques inside of each load interval $d\lambda^{j+1}$ in order to diminish or eliminate, for a given level, the error produced in the computation. These procedures are known as incremental-iterative methods and they will be described in the next section as the typical tool used to trace the structural response of a structure under a proportional increasing set of loads.

6.-TRACE OF THE STRUCTURAL RESPONSE.

6.1.- INTRODUCTION.

As it has been commented in section 3.1 it is necessary sometimes to know the behavior of a structure in the neighborhood of its critical points. The knowledge of this behavior and particularly of the postcritical branch of the structural response gives some light about the sensitivity of the structure to imperfections. Ascendent and descendent bifurcation branchs, load and displacement limit points can indicate the importance of the imperfection in the value of the real collapse load as it is shown in the recent review on the subject [16] . In the IASS Shell Recommendations [16] it is commented that the critical load of a real shell can be as low as the 50 per cent of the ideal critical load. Therefore the trace of the structure near critical points is not only an academic exercise but also has practical implications. Normally the designer wants to know the load level that produces the first critical point. However if the algorithm used in this computation is not suitable to handle the tracing of the structural response near the critical points the computed collapse load may be different to the real one. That is because the computed collapse load can be also be obtained as a consequence of the lack of convergence of the numerical algorithm used or its implementation in the computer.

6.2.- NUMERICAL DIFFICULTIES AT THE NEIGHBORHOOD OF A CRITICAL POINT.

In the already commented iterative procedures the load vector remains constant, i.e. in the iterative-incremental methods the load parameter λ is kept constant during the iteration until convergence in the load step is reached. Trespassing the limit points represents a computationally difficult problem due to the fact the tangent stiffness matrix is almost singular. Several techniques have been proposed in order to handle this situation. Bergan [17] uses an indicator called current stiffness parameter that it is activated at the neighborhood of a limit point and consequently supress the equilibrium iterations. Other possibility consists of the introduction of ficticious stiffness in the steps near the critical point in order to keep the stiffness matrix of the structure positive definite. This rather heuristic procedure used in [18] has the disavantage of the need to proceed by trial and error in order to select a suitable set of dof where to introduce the extra springs. Another strategy very much extended in the practical computations is represented by the consideration of increasing incremental loads until the limit point is reached. Then the point is trespassing by the increase of one typical displacement or a set of displacements and then the load is evaluated. This procedure can work well in some cases but it is not suitable in the situation of critical displacement limit points. Besides, in this method introduced in [19], the choice of the representative displacement to be increased is not an obvious task and normally the symmetry of the stiffness matrix is lost. In further research some of these drawbacks have been tried to be removed but with the cost of extra computation.

In the next section an unified iterative-incremental approach of tracing the complete structural response is presented. In order to reduce the scope of the exposition it is assumed that the iterative procedure to be used in each load step is the mNR. The tangent stiffness matrix is updated at the beginning of each load increment and remains constant until the number of iterations in this load step permits to reach convergence. Naturally other iterative methods are suitable to be used, but some of them can be sensitive to the singularity of the stiffness matrix at or near the critical point and others may demand more computational effort than the mNR. However there is not a definitive answer respect to the more convenient procedure to solve a general non linear problem, i.e. in some cases some methods works better than others and contrary.

6.3.- GENERAL FORMULATION OF THE ITERATIVE-INCREMENTAL SOLUTION TECHNIQUE.

In an iterative-incremental method each load step j from the factor λ^j to $\lambda^{j+1} = \lambda^j + \Delta\lambda^j$ consists of the application of the load increment and the equilibrium restoration by means of sucessive iterations inside the step load. In the following the usual notation will be used. The superscript j denotes the load step. The subscript i express the iteration cycle in the load step. The iterations start at $i = 1$ when the load is incremented to the level j. Then the equilibrium iterations start from $i = 2$.

To describe an algorithm iterative-incremental it is necesary to define the two following strategies.

2.-Strategy for the iterations.

1.-Strategy for the automatic load incrementation.

Finally it is also important to know the convergence criterion used in the algorithm.

In the following sections some comments will be given about the posible alternative strategies.

6.4.- ITERATIVE STRATEGIES.

This strategy tries to define the sucessive iteration cycles ($i \geq 2$) that allow to reach the equilibrium in the minimum number of iterations. Using the notation of the figure 2, the strategy must define the additional restriction between the unknown displacements vector u_i^j and the incremental load parameter Δ_i^j, because near the critical point it is usual to iterate no only on the displacements but also on the incremental load parameter.

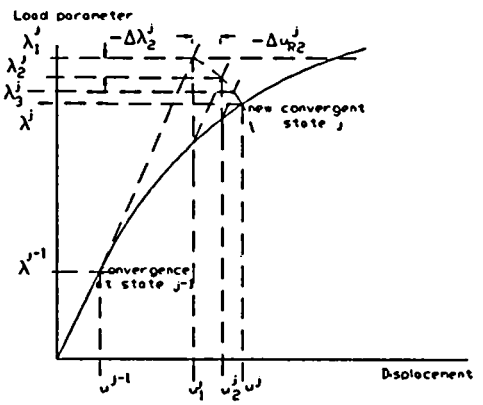


Figure 2.-Notation. Iterative-incremental methods with nMR iterations and variable load parameter.

It is assumed at the beginning of the load step j that the solution values obtained in the previous step j-1 (λ^{j-1}, u^{j-1}) satisfy the total equilibrium. Two cases will be discussed.

1.-First iteration cycle. (i = 1).

In the new load step the computation starts by determining the tangent stiffness matrix K_1^j from the displacements and known stresses obtained at the end of the previous load step. The direction of the tangent displacement vector u_1^j in this load step is found from the solution of the following system of linear equations:

$$K_1^j u_1^j = p_1^j \qquad [73]$$

where p_1^j is the reference load vector ussually defined in the input data of the structural analysis.

Using one of the strategies described later the initial increment load parameter Δ_1^j is obtained and the incremented displacement vector is found from proportionality of the solution of [73], i.e.

$$\Delta u_1^j = \Delta \lambda_1^j u_I^j \qquad [74]$$

The load level and the total displacements of the previous load step are now updated by the equations:

$$u_1^j = u^{j-1} + \Delta u_1^j \qquad [75]$$

$$\lambda_1^j = \lambda^{j-1} + \Delta \lambda_1^j \qquad [76]$$

In this situation the values given by [75] and [76] do not satisfy the total equilibrium and therefore some iterations are needed in order to restore it. The following part is dedicated to this topic.

2.-Equilibrium iterations (i>1).

The load parameter λ_i^j must vary simultaneously with the unknown displacement vector u_i^j if a limit point is to be traspassed. In this case it is possible to design a general technique in order to find the incremental change of the displacement vector for each iteration ($i \geq 2$) in the load step j as the solution of

$$K_I^j u_i^j = \Delta \lambda_i^j p_I^j - r_{i-1}^j \qquad [77]$$

where the residual forces are

$$r_{i-1} = p_{int,i-1}^j - p_{ext,i-1}^j \qquad [78]$$

and they represent the resultant of the unbalanced forces acting on the structure at the end of the previous iteration (i-1 th). The internal forces $p_{int,i-1}^j$ are computed, as is usual in the standard Finite Element Method, as the summation of the contribution of the generalized stresses existing on each element volume i.e. by the expression:

$$p_{int,i-1}^j = \int_V B_{i-1}^{jT} \sigma_{i-1}^j dV \qquad [79]$$

where in the strain-displacement matrix B_{i-1}^j non linear terms may be included and σ_{i-1}^j represents the vector of generalized stresses.

The internal forces $p_{ext,i-1}^j$ are computed by scaling the equation [73] and then it results:

$$p_{ext,i-1}^j = \lambda_{i-1}^j p_I^j \qquad [80]$$

The solution of the equation [77] can be obtained as a linear combination of the solution for each term of the right hand side, i.e.

$$\Delta u_i^j = \Delta \lambda_i^j u_I^j + u_{R,i}^j \qquad [81]$$

where u_I^j is given by [73] and the displacement vector u_R satisfies the equation:

$$K_I^j u_{R,i}^j = -r_{i-1}^j \qquad [82]$$

The variation of the load parameter $\Delta \lambda_i^j$ is obtained according to the different strategies that will be summarized at the end of this section. Therefore it is possible to obtain the displacement increment Δu_i^j for the current iteration from the equation [81] and then by the use of the expressions

$$u_i^j = u_{i-1}^j + \Delta u_i^j \qquad [83]$$

$$\lambda_i^j = \lambda_{i-1}^j + \Delta \lambda_i^j \qquad [84]$$

The iterations should be continued until convergence is reached according to a criterion based on a displacement or a forces norm. However, in a computer program the total number of iterations should be limited to a maximum value in such a way that if this maximum is reached or it is observed divergence the load step should be reinitiated with a load step smaller than the previously assumed, normally a half of it.

In the following the current most used iterative strategies are summarized, i.e. the different relationships between the incremental load factor $\Delta \lambda_i^j$ ($i \geq 2$) and the residual displacements $\Delta u_{R,i}^j$

(a) Constant load iteration.

$$\Delta \lambda_i^j = 0 \qquad [85]$$

(b) Constant displacement iteration

The key displacement is the n component of the displacement vector and it can be

expressed as linear combination of the residual displacement vector

$$\Delta u_n = b_n^T \Delta u_{R,i}^j \qquad [86]$$

where all the elements of the vector b are zero except the n-th.
By consideration of the equation [81] the following result[20] is reached

$$\Delta \lambda_i^j = -\frac{b_n \Delta u_{R,i}^j}{b_n^T u_i^j} \qquad [87]$$

(c) Constant arc length iteration.
In order to limit the initial load factor the following restriction have been introduced [21 22]

$$\Delta u_1^{jT} . \Delta u_1^j + (\Delta \lambda_1^j) . p_1^{jT} = (l_j)^2 \qquad [88]$$

where l_j is the generalized arc length of the tangent at the equilibrated step load j-1 in the space loads-displacements.
Crisfield has presented an alternative to [88] more convenient for the convergence and expressed as follows:

$$\Delta u_{ai}^{jT} . \Delta u_{ai}^j = (l_j)^2 \qquad [89]$$

where the vector Δu_{ai}^j collects the accumulated displacements in the load step j, i.e.
In equation [88], the iterations are in a plane

$$\Delta u_{ai}^j = u_i^j - u^{j-1} \qquad [90]$$

and with [89] the iterations occur on a sphere with center the last equilibrated load step (λ^{j-1}, u^{j-1}) and radius l^j.

The equation [89] leads to the following one:

$$A(\Delta \lambda_i^j)^2 + B \Delta \lambda_i^j + C = 0 \qquad [91]$$

where

$$A = u_I^{jT} u_I^j \;\; ; \;\; B = \left[\Delta u_{a,i-1}^j + \Delta u_{R,i}^j\right]^T u_I^j$$
$$C = \left[\Delta u_{a,i-1}^j + \Delta u_{R,i}^j\right]^T \left[\Delta u_{a,i-1}^j + \Delta u_{R,i}^j\right] - (l^j)^2 \qquad [92]$$

The choice between the two roots $\Delta \lambda_{i1}^j, \Delta \lambda_{i2}^j$ of [92] must be done in order to avoid the backwards solution. To this end the "angle" between the increment displacement vector at the beginning of the current iteration and the one after the iteration should be positive. The angles of the two roots of [92] are given by the expression:

$$\theta_k = \left[\Delta u_{a,i-1}^j + \Delta \lambda_{ik}^j u_I^j + \Delta u_{R,i}^j\right]^T \Delta u_{a,i-1}^j \qquad [93]$$

The root to be selected should give positive value for the angle θ_k. In the case of two positive angles the root to be selected should be the nearest to the solution of the [91]. If the two roots are imaginary then the initial increment load should be reduced.

(d) Constant external work iteration.
The increment of the external work is

$$\Delta W^j = \Delta \lambda_i^j p_I^{jT} \Delta u_i^j = 0 \qquad [94]$$

and from [94] the following equation results:

$$\Delta \lambda_i^j = -\frac{p_I^{jT} \Delta u_{R,i}^j}{p_I^{jT} u_I^j} \qquad [95]$$

(e) Minimum norm of unbalanced displacements iteration

The condition is

$$\frac{\partial}{\partial \Delta \lambda_i^j}\left[\Delta u_i^{jT} \Delta u_i^j\right] = 0 \qquad [96]$$

that leads to the restriction

$$\Delta \lambda_i^j = -\frac{u_I^{jT} \Delta u_{R,i}^j}{u_I^{jT} u_I^j} \qquad [97]$$

(f) Minimum norm of unbalanced loads iteration.
The external loads should be closest as possible to the internal loads, i.e., the condition in the iteration is written as:

$$\frac{\partial d}{\partial \lambda_i^j} = 0 \qquad [98]$$

where

$$d = \left(p_{int,i-1}^j - \lambda_i^j p_I^j\right)^T \left(p_{int,i-1}^j - \lambda_i^j p_I^j\right)$$

and $p_{int,i-1}^j$ is given by [79].

The equation [98] is transformed into the following one:

$$\Delta\lambda_i^j = -\frac{p_i^{jT} p_{i\alpha,i-1}^j}{p_i^{jT} p_i^j} \qquad [99]$$

Bergan has suggested the use of different weighting factors for the rotational and translational dof of the structure.

(g) General strategy. Constant weighted response iteration.
The strategies summarized in (a) to (f) are particular cases of the general formulation given in [23] and extended more recently in the reference [24]. The idea is to introduce the length of the weighted response vector defined by the expression:

$$(L_i^j)^2 = u_i^{jT} G^j u_i^j + (\lambda_i^j)^2 p_i^{jT} H^j p_i^j \qquad [100]$$

where G^j and H^j are diagonal matrices with stiffness and flexibility dimensions respectively. A common practice is to use the following diagonal elements:

$$g_{mm} = k_{mm} \;;\; h_{mm} = \frac{1}{k_{mm}}$$

where k_{mm} is the m-th diagonal element of the tangent stiffness matrix K_i^j of the structure.

Then it can be defined L_i^j for the iteration i and similarly to the constant arc length (c) the incremental length ΔL_i^j can be expressed as follows:

$$(\Delta L_i^j)^2 = \Delta u_{a,i}^{jT} G^j \Delta u_{a,i}^j + (\Delta\lambda_i^j)^2 p_i^{jT} H^j p_i^j \qquad [101]$$

where $\Delta u_{a,i}^j$ is defined by [91] and $\Delta\lambda_{a,i}^j = \lambda_i^j - \lambda^{j-1}$.

The general strategy is represented by the following condition at iteration i:

$$\Delta L_i^j - \Delta L_{i-1}^j = 0 \qquad [102]$$

The equation [102] can be transformed into the following one:

$$A(\Delta\lambda_i^j)^2 + B(\Delta\lambda_i^j) + C = 0 \qquad [103]$$

where

$$A = u_i^{jT} G^j u_i^j + p_i^{jT} H^j p_i^j$$

$$B = 2\left[\Delta u_{a,i-1}^j + \Delta u_{R,i}^j\right]^T G^j u_i^j + 2\Delta\lambda_{a,i-1}^j p_i^{jT} H^j p_i^j$$

$$C = \left[2\Delta u_{a,i-1}^j + \Delta u_{R,i}^j\right]^T G^j \Delta u_{R,i}^j$$

The correct choice of the root of the second degree equation [102] follows the same pattern as in (c). There exists also a very efficient approximation of the value of the correct root which is the one given by the following formula:

$$\Delta\lambda_i^j = -\frac{\Delta u_{R,i}^{jT} G^j \Delta u_{a,i-1}^j}{u_i^{jT} G^j u_{a,i-1}^j + \Delta\lambda_{a,i-1}^j p_i^{jT} H^j p_i^j} \qquad [103]$$

6.5.- INCREMENTAL STRATEGIES.

A strategy respect to the increment load factor should consider the following two aspects:

(1) The size of the increment load factor $\Delta\lambda_1^j$ at the begining of the load step i.

(2) The sign of the increment load $\Delta\lambda_1^j$ that has to be reversed near a load limit point.

Besides the two above aspects in the computation of the solution of a nonlinear problem following data should given.

- $\Delta\lambda_1^1$ i.e. the increment load factor to be used in the first load step and at the initial iteration. Typically a value of the 20 to 40 per cent of the foreseen maximum total load.

- J_d or the desired number of iteration in a load step to reach convergence. Typical values are between 3 and 5.

- γ an exponential number to be defined later.

- ξ_c or convergence tolerance as it will be defined in the next section 6.6.

In the following the different strategies for load incrementation will be commented.

6.5.1.- STRATEGIES FOR THE SIZE OF THE INCREMENTAL LOAD.

A small size of $\Delta\lambda_i^1$ leads to an inefficient and costly computation of the equilibrated solution (λ_i^j, u_i^j) due the need to carry out too many iterations in the load step. On the contrary a

large initial value $\Delta\lambda_1^1$ can produce a very slow convergence or even no convergence at all.

Three groups of strategies for the size of the increment load will be commented below.

The first group uses the following expression

$$\Delta\lambda_1^j = \Delta\lambda_1^{j-1}\left(\frac{J_d}{J_{j-1}}\right)^\gamma \qquad [103]$$

where J_{j-1} is the number of iterations needs to reach convergence in the previous load step and $0.5 \le \gamma \le 1.0$. In this respect Crisfield suggests the value 1 and Ramm [25] with 0.5 obtains smooth results.

If the constant displacement is used as an iterative strategy then the formula [103] is replaced by the following one:

$$\Delta u_n^j = \Delta u_n^{j-1}\left(\frac{J_d}{J_{j-1}}\right)^\gamma \;;\; (0 \le \gamma \le 1) \qquad [104]$$

and if [104] is expressed in terms of the increment load parameter it becomes:

$$\Delta\lambda_1^j = \frac{\Delta u_n^j}{b_n^T u_1^j} \qquad [105]$$

Similarly if the iterative constant arc length is used then at the j-th step load it is obtained:

$$l^j = l^{j-1}\left(\frac{J_d}{J_{j-1}}\right)^\gamma \qquad [106]$$

or equivalently

$$\Delta\lambda_1^j = \pm\frac{l^j}{\sqrt{u_1^{jT} u_1^j}} \qquad [107]$$

Finally if the iterative strategy related to the constant external work is considered the following formula can be used:

$$\Delta W^j = \Delta W^{j-1}\left(\frac{J_d}{J_{j-1}}\right)^\gamma \qquad [108]$$

$$\Delta\lambda_1^j = \frac{\Delta W^j}{p_1^{jT} u_1^j} \qquad [109]$$

The second group of incremental strategies introduces the scalar known as the stiffening parameter of the structure S_λ defined by the expression:

$$S_\lambda = \frac{\left[\frac{du}{d\lambda}\right]_0^T \bar{p}}{\left[\frac{du}{d\lambda}\right]^T \bar{p}} \qquad [110]$$

where $\left[\frac{du}{d\lambda}\right]_0$ and $\left[\frac{du}{d\lambda}\right]$ are the derivatives of the displacement vector respect to the load parameter at the initial and the current step load respectively. The vector $\bar{p}$ corresponds to the reference load vector. In the discrete case [110] becomes

$$S_\lambda^j = \frac{\left[u_1^1\right]_0^T p_1^j}{\left[u_1^j\right]^T p_1^j} \qquad [111]$$

The values of the parameter S_λ smaller than the unity means a structural softening and greater than the unity a hardening. Bergan has proposed [26] the following expression for the automatic load incrementation at the load step j:

$$\Delta\lambda_1^j = \pm\Delta\lambda_1^{j-1}\left|\frac{\Delta S_\lambda}{\Delta S_\lambda^j}\right| \le \Delta\lambda_{max} \qquad [112]$$

where $\Delta S_\lambda^j = S_\lambda^j - S_\lambda^{j-1}$ and ΔS_λ is a prescribed constant given as datum or estimated during the first load steps by the expression:

$$\Delta S_\lambda = S_\lambda^2 - S_\lambda^1$$

Obviously in the regions of quasi-linear structural behavior the value of ΔS_λ^j becomes very small and therefore the increments of load $\Delta\lambda_1^j$ are very large. This is the reason to introduce an upper limit in [112].

Sometimes the following alternative expression to [112] and more simple to apply is used:

$$\Delta\lambda_1^j = \pm\Delta\lambda_1^1 |S_1^1|^{\gamma} / |S_1^j|^{\gamma} \leq \Delta\lambda_{max} \quad ; \quad (\gamma = 1) \quad [113]$$

If the computation is carried out in the neighborhood of a displacement limit point then S_1^j can be very large and therefore $\Delta\lambda_1^j$ must be also limited by $\Delta\lambda_{max}$.

In the expressions [112] and [113] it was assumed a nearly constant truncation error i.e. that the number of iterations needed to reach convergence in each load step remains practically the same.

Finally in the third group of strategies it is assumed that the optimal increment of loading must produce a constant truncation error in every load step. This means a number practically constant of iterations 2in each load step. In order to achieve this situation the norm of the displacement vector $||u||$ and the load parameter λ is approximated by a parabola. In this case it is possible to compute the load step using the condition that the error τ (difference between the exact value of the parabola and its approximation by the tangent) is constant (figure 3). Then it is obtained the value

$$\Delta\lambda_1^j = \pm\Delta\lambda_1^{j-1} \sqrt{\frac{\tau}{||u_1^j|| \cdot |\lambda^{j-1} - \lambda^{j-2}| - ||u^{j-1} - u^{j-2}||}} \quad [114]$$

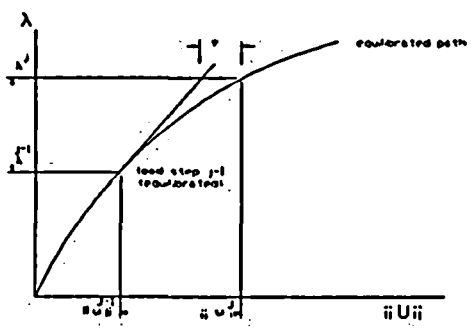


Figure 3.-Parabolic approximation of the response.

6.5.2.- STRATEGIES FOR THE SIGN OF THE INCREMENTAL LOAD.

In the expressions of the size of the increment load vector $\Delta\lambda_1^j$ to be used the sign is not determined. Near displacement and load limit points it is important to choose the right sign in order to trace the structural response and avoid going back. Crisfield and Ramm have suggested the following rule

$$sign(\Delta\lambda_1^j) = sign(\Delta\lambda_1^{j-1}) \frac{sign|K_1^j|}{sign|K_1^{j-1}|} \quad [115]$$

The tangent stiffness matrix determinants $|K^{j-1}|$ and $|K^j|$ corresponding to the load steps j-1 and j respectively are computed as the product of the diagonal terms of the upper triangular matrices obtained in the Gauss decomposition, that it is used as a standard solution technique of the linear system of equations [73].

Alternatively Bergan and coworkers proposed as a more reliable sign check the following one:

$$sign(\Delta\lambda_1^j) = sign(\Delta\lambda_1^{j-1}) \frac{sign\, W^j}{sign\, W^{j-1}} \quad [116]$$

where ΔW^j has been defined in [94]

The above formula [116] is similar to the following one:

$$sign(\Delta\lambda_1^j) = sign(\Delta\lambda_1^{j-1}) \frac{sign\, S_1^j}{sign\, S_1^{j-1}} \quad [117]$$

The reliability of the above expressions when they are applied to complex structures such as shell and space structures is still an open research topic under current intensive study.

6.6.- CRITERIA OF CONVERGENCE.

There exist several convergence criteria and they indicate when the current solution is so close to the true or equilibrating solution that the iteration process can be terminated. The convergence criteria are usually based on displacements, residual forces or energy. The natural with the FEM (displacement formulation) is the displacement based criteria but they can sometimes be misleading and

satisfy by a slow convergence rate. Residual force criteria are more reliable because they are checking the global equilibrium within a specified tolerance. The value of the tolerance is very important because a small value for the tolerance coefficient demands an excessive and costly number iterations and large values can produce inaccurate results.

The norms used to measure the vector $x=(x_i)$ can be of one of the following classes, where N is the vector dimension:

The absolute norm: $\|x\|_1 = \frac{1}{N}\sum_{n=1}^{N}|x_n|$

The euclidean norm: $\|x\|_2 = \sqrt{\frac{1}{N}\sum_{n=1}^{N}|x_n|^2}$

The maximum norm: $\|x\|_\infty = \max_n |x_n|$

The residual forces vector r_i^j is defined as the difference vector between the external and the internal forces vectors for the i-th iteration of the load step j. These vectors have already been introduced in previous sections and the residual components corresponding to the n-th dof are denoted by $r_{i,n}^j$ and similarly for the external forces by $p_{i,n}^j = \lambda_i^j p_{I,n}^j$.

The convergence checks can be related to the vectors containing the forces belonging to a particular type of dof or to all types of the dof. In both cases they can be expressed using one of the above defined norms as:

$$\frac{\|r_{i,n}^j\|}{\|p_{i,n}^j\|} \le \xi_n \quad ; \quad \frac{\|r_i^j\|}{\|p_i^j\|} \le \xi \qquad [118]$$

and in the case of considering all the dof as in the second formula there exists one difficulty due to the inconsistency of using mixed units and then the above expressions [118] should be modified by dividing all components of one type of dof by a reference value such the maximum value or the square root of the diagonal of the tangent stiffness matrix. In this way the value of the norm becomes unit independent.

It has been observed that the above criteria related to residual forces are rather difficult to accomplish even if the displacements convergence is very good. This situation appears very often when analyzing concrete structures with total distribution of the unbalanced tensile forces released due to the cracking. The areas of the unbalanced forces are sometimes very small and have not too much influence in the overall structural behavior. The convergence criteria based in displacements are written as follows:

$$\frac{\|\Delta u_{i,n}^j\|}{\|u_{ref,n}^j\|} \le \xi_n \quad ; \quad \frac{\|\Delta u_i^j\|}{\|u_{ref}^j\|} \le \xi \qquad [119]$$

where $u_{ref,n}$ and u_{ref} are the reference vectors with the type n of dof and all types of dof respectively. Reference vectors can be the displacements in the initial iteration or the current incremental displacements. As it has been commented, the use of mixed units demands the consideration of modified norms.

Finally the energy based convergence criteria use both displacements and forces. In these criteria the amount of work in the i-th iteration is compared with the value for the first iteration as it is indicated below:

$$\frac{r_i^{jT}\Delta u_i^j}{r_1^{jT}\Delta u_1^j} \le \xi \qquad [120]$$

These criteria do not present any difficulties if mixed units are used. However they are not very reliable criteria particularly when line search is used in the iterative procedure.

7.- ANALYSIS OF KINEMATIC INDETERMINATE STRUCTURES.

Finally there exist specific problems related to the nonlinear analysis of special structures such as cable and membrane structures with weak prestressing or during their construction. In these structures numerical difficulties owing to their kinematic indeterminacy can appear. To analyze them by any iterative solution it is necessary to known an initial point (equilibrated solution) near to the solution to be found under a given loading. In stable unstressed structures the initial point is obviously the origin (displacements and forces null) however in the kinematic indeterminate structures this is not the situation because the original

geometry of the structures has to be obtained.

In order to treat this class of structural problems it is normally necessary to the use a large displacement and large rotation theory with the large strains and large tensions with stiffening effects and nonlinear strain-displacement relations, and therefore the nMR may be not suitable to be used in the iterations carried in a particular load step. The tangent stiffness matrix should be updated several times in each load step.

The problem just described is very much related with the shape finding analysis and therefore only some comments will be given here. First it is possible to use the methods described in the previous sections but some numerical instabilities may occur due to the critical value produced by the zero load condition that should be treated by an iterative-incremental procedure. Another possibility is to consider that the transition from an initial unstable configuration to the final stable one is taken place only by unstrained states i.e. only rigid body movements will occur and then the application of the ideas developed by Hangai and coworkers [27] could be of interest. In the application of the methodology shown here some difficulties may be encountered in obtaining convergence in a very unstable regions of the structure. On the contrary the method developed by Hangai et alia has to handle rectangular full matrices and obtain the generalized inverse or Moore-Penrose inverse matrices. These facts require large computational time and computer memory, but the convergence stability is assured trough out all the computations.

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