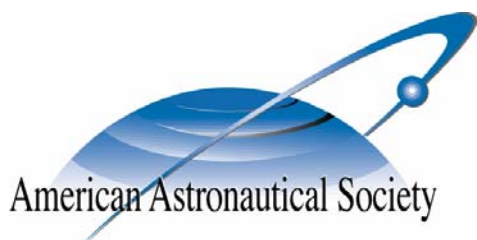Paper AAS 05-167

A SPECIAL PERTURBATION METHOD IN ORBITAL DYNAMICS

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A Special Perturbation Method in Orbital Dynamics

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Abstract

Bead models are used in dynamical simulation of tethers. These models discretize a cable using beads distributed along its length. The time evolution is obtained numerically. Typically the number of particles ranges between 5 and 50, depending on the required accuracy. Sometimes the simulation is extended over long periods (several years). The complex interactions between the cable and its spatial environment require to optimize the propagators —both in runtime and precision— that constitute the central core of the process. The *special perturbation method* treated on this article conjugates simpleness of computer implementation, speediness and precision, and is capable to propagate the orbit of whichever material particle. The paper describes the evolution of some orbital elements, which are constants in a non-perturbed problem, but which evolve in the time scale imposed by the perturbation. It can be used with any kind of orbit and it is free of singularities related to small inclination and/or small eccentricity. The use of Euler parameters makes it robust.

1 Introduction

This article has been developed within a three-year project called **"Stability and Dynamical Simulation of Tethers"**, whose main objective has been to understand the dynamical instabilities that poses an electrodynamic tether when operating in slanted orbits. The control of these instabilities is essential in order to exploit the advantageous characteristics of tethers to get a reliable, economic, and safe propulsion of space vehicles —particularly the bare tether.

Theoretical studies have been done in this project to increase the knowledge of the complex dynamics of this kind of tethers. Simultaneously, a general purpose dynamical simulator of tethers will be developed to study electrodynamic tethers. Finally, the predesign of two bare tethers will be done. The first one is conceived to compensate the decrease of height of the International Space Station due to aerodynamical drug at three different heights (300, 400 and 500 Km). In the second one, the re-entry of a satellite initially at a height of 1000 Km in a reasonable time will be forced. In both cases the emphasis is put in demonstrating the controllability of the tether.

Basically, there are tree kinds of models to simulate the dynamics of tethers: 1) models based on finite elements, 2) the *continuous models* —which use form functions to different modes—, and 3) *bead models* or *lumped-mass models*. In the last family, the mass of the cable is discretized in a collection of material particles whose time evolution is required. First and second kinds of models are preferred by some authors that argue in favour of then a greater precision to describe the lateral dynamics of the tether, but there is no consensus on this open question.

Nevertheless, the bead models keep a direct connection to physical effects involved in the simulation and they are preferable when the dynamics is not well known. Moreover, they are advantageous at predesign phase and at the preliminary mission analysis, because of their ease of programming and —if the number of beads is small— their quickness. We can improve the precision of analysis increasing the number of beads used to discretize the tether.

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Tethers emerge in space in the 70's and since then a lot of dynamical simulations have been carried on. Most of then span weeks, perhaps months, but simulations with flexible tethers that extend over years have never been done. However, electrodynamic tethers show their better properties in long missions, where large masses are present. The needed simulation runtime is longer in these missions. The International Space Station (ISS) would be a good instance of them.

The computational cost of tether simulations is due to different factors. First, the complex interaction tether-environment imposes the use of sophisticated models. For instance, the description of forces operating in electrodynamic tethers requires to model: 1) the ionosphere, 2) the terrestrial magnetic field, and 3) the electromagnetic interaction between the surrounding plasma and the conducting cable. This last item implies to determine the time evolution of the cable temperature, that affects decisively the electrical resistance and therefore the current collected in the system. This last variable is an essential datum to describe electromagnetic forces.

Some small perturbations that don't affect shorter simulations must be included with longer simulation times, where the accumulative effect can became important. Finally, the lack of previous results compels to a precision increase. The consequence is an increase of the number of beads used to discretize the system. In summary, it is important to optimize the methods and the routines used to propagate the orbits of the particles that compose the tether. Next, a propagator is going to be presented to be used as a central core of the simulator.

2 Equations for the particle

Let $Ex_1y_1z_1$ be the inertial geocentric system of reference with its origin E at the mass center of the Earth. The problem of determining the time evolution of the position vector $\vec{x} = \vec{x}(t)$ of a particle M with mass m in this reference is a 6-order differential problem. The equations governing its motion are:

$$m\ddot{\vec{x}} = -\frac{m\mu}{|\vec{x}|^3}\vec{x} + \vec{F}_p \tag{1}$$

where $\vec{x} = \overrightarrow{EM}$ is the position vector of the particle and $\vec{F}_p$ is the **perturbation force**, which is defined as the vector resultant of all the forces except the first term of the gravitacional potencial. At first, the *natural* variables for the numerical integration of the system (1) would be x_1, y_1, z_1, $\dot{x}_1$, $\dot{y}_1$ y $\dot{z}_1$, that is, the cartesian coordinates of M at the geocentric reference and its time derivatives. Altogether they define the dynamical state of M. The solution of this system of equations has the form

$$\vec{x} = \vec{x}(t; \vec{x}_0, \dot{\vec{x}}_0)$$

Without loss of generality, the position and the velocity of M at $t = 0$ (initial time) are taken as integration constants.

An **orbital reference** $\mathcal{R} = \{O; \vec{i}, \vec{j}, \vec{k}\}$ can be associated to the particle motion. Its origin O coincides with the particle M and its unit vectors ($\vec{i}, \vec{j}, \vec{k}$) are defined as follows

- $\vec{i}$, in the same direction and way of the position vector $\vec{x}$

- $\vec{k}$, contained in the plane defined by E, $\vec{x}$ and the velocity $\vec{v} = \dot{\vec{x}}$ of the particle, and with its way such that $\vec{k} \cdot \vec{v} \geq 0$

- $\vec{j}$, is selected in a manner that the reference would be *right-handed*

Thus, if the particle evolution is known the orbital reference evolution is known as well, and it is given by the following relations

$$\vec{i} = \frac{\vec{x}}{|\vec{x}|} = \frac{\vec{x}}{R}, \quad \vec{j} = \frac{\vec{v} \times \vec{x}}{|\vec{v} \times \vec{x}|} = -\frac{\vec{h}}{h}, \quad \vec{k} = \vec{i} \times \vec{j} \tag{2}$$
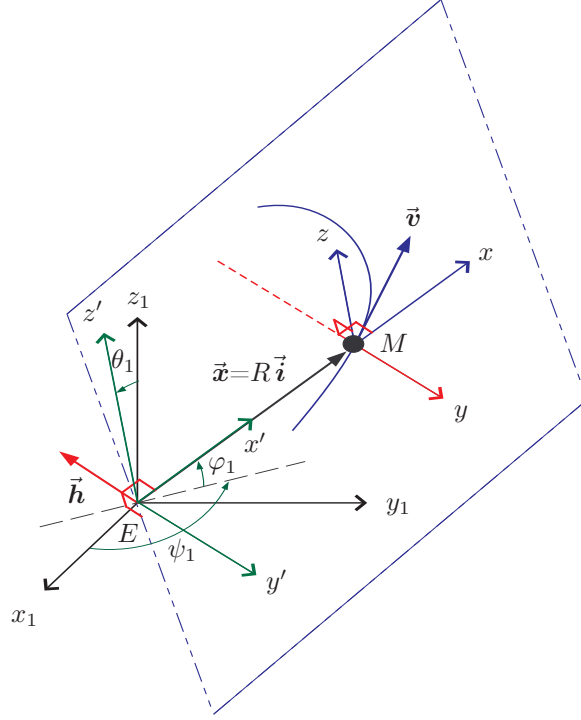
Figure 1: Orbital reference

where $R(t)$ is the distance $R = |\vec{x}|$ and $\vec{h} = \vec{x} \times \vec{v}$ is the specific angular momentum of the particle. Outside of the particle, the time evolution of a free solid S in a reference system is, generally, a 12-order differential problem, because we have to determine the time evolution of one of its points (6 state variables) and additionally the time evolution of its attitude (6 state variables more). *Though, the time evolution of the orbital reference $\mathcal{R}$ afore defined is governed by a 6-order system, equivalent to the system* (1) *that govern the time evolution of the particle M.* This is a basic point of the numerical integration method that is proposed in these pages.

In fact, let $\vec{\omega} = p\,\vec{i} + q\,\vec{j} + r\,\vec{k}$ be the angular velocity of the orbital reference $\mathcal{R}$, and $(\psi_1, \theta_1, \varphi_1)$, for a moment, the Euler angles that provide its attitude with respect to inertial geocentric reference $Ex_1y_1z_1$.

To determine the time evolution of $\mathcal{R}$ implies obtaining the time evolution of R, (p, q, r) and $(\psi_1, \theta_1, \varphi_1)$. Seemingly, the problem seems to be of 7th order but indeed it is of 6th order, because of the definition of the orbital reference, where the r component of the angular velocity $\vec{\omega}$ *is always null*. This fundamental property is deduced from the time derivatives of $(\vec{i}, \vec{j})$ unit vectors, that can be expressed in two different ways. The first one is obtained from the time derivative of equations (2), and the second one is obtained from the elemental properties of the solid angular velocity.

$$\frac{d\vec{i}}{dt} = \frac{\vec{v}}{R} - \frac{1}{R^2}\dot{R}\,\vec{x} = \vec{\omega} \times \vec{i} = r\,\vec{j} - q\,\vec{k}$$

$$\frac{d\vec{j}}{dt} = \frac{\dot{h}}{h^2}\,\vec{h} - \frac{1}{h}\frac{d\vec{h}}{dt} = \vec{\omega} \times \vec{j} = -r\,\vec{i} + p\,\vec{k}$$

The components of the angular velocity of $\mathcal{R}$ are deduced by identifying the corresponding values

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of both expressions

$$p = -\frac{1}{h}(\vec{k} \cdot \frac{d\vec{h}}{dt}), \; q = -\frac{1}{R}(\vec{v} \cdot \vec{k}), \; r = \frac{1}{R}(\vec{v} \cdot \vec{j}) = 0$$

Otherwise, considering the angular moment equation

$$\frac{d\vec{h}}{dt} = \frac{1}{m} \, \vec{x} \times \vec{F}_p$$

which is easily deduced from equation (1), the components of $\vec{\omega}$ take the form

$$p = -\frac{R}{mh}(\vec{F}_p \cdot \vec{j}), \quad q = -\frac{h}{R^2}, \quad r = 0$$

The r component is null independently of the perturbation force affecting the particle, as it was aforemention in advance.

Note that if the perturbation disappears, ($\vec{F}_p = \vec{0}$), the particle has a keplerian motion, $\vec{h}$ is a constant vector —proper of central motions— and the area rule $R^2\dot{\theta} = h$ is satisfied. The angular velocity components are in this case:

$$p = 0, \quad q = -\dot{\theta}, \quad r = 0$$

where θ represent the true anomaly.

The equations governing the time evolution of the orbital reference will be established and, therefore, the motion of M. For this, the time derivatives of the position vector $\vec{x} = R\vec{i}$ will be calculated

$$\frac{d\vec{x}}{dt} = \dot{R}\vec{i} + R\vec{\omega} \times \vec{i}$$

$$\frac{d^2\vec{x}}{dt^2} = \ddot{R}\vec{i} + R\{\dot{\vec{\omega}} \times \vec{i} + \vec{\omega} \times (\vec{\omega} \times \vec{i})\} + 2\dot{R}\vec{\omega} \times \vec{i}$$

Equation (1) takes the form

$$\ddot{R}\vec{i} + 2\dot{R}\vec{\omega} \times \vec{i} + R\{\dot{\vec{\omega}} \times \vec{i} + \vec{\omega} \times (\vec{\omega} \times \vec{i})\} = -\frac{\mu}{R^2}\vec{i} + \frac{\vec{F}_p}{m}$$

and projecting on the three axis of $\mathcal{R}$, we obtain the following equations

$$\ddot{R} - Rq^2 + \frac{\mu}{R^2} = +\frac{1}{m}(\vec{F}_p \cdot \vec{i}) \tag{3}$$

$$Rpq = +\frac{1}{m}(\vec{F}_p \cdot \vec{j}) \tag{4}$$

$$\frac{d}{dt}(R^2q) = -\frac{R}{m}(\vec{F}_p \cdot \vec{k}) \tag{5}$$

which by integration provide $R(t), p(t), q(t)$. Considering them, we can obtain the evolution of $(\vec{i}, \vec{j}, \vec{k})$ through the relations

$$\frac{d\vec{i}}{dt} = \vec{\omega} \times \vec{i} = -q\vec{k} \tag{6}$$

$$\frac{d\vec{j}}{dt} = \vec{\omega} \times \vec{j} = +p\vec{k} \tag{7}$$

$$\frac{d\vec{k}}{dt} = \vec{\omega} \times \vec{k} = +q\vec{i} - p\vec{j} \tag{8}$$

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Before proceeding a simplification of the equations will be done. Non-dimensional variables will be introduced and the system state will be expressed with other, more suitable coordinates to obtain this objective. Finally, Euler parameters will be introduced to simplify the equations (6-8).

Three fundamental characteristic magnitudes (length, time and mass) will be introduced to have non-dimensional equations; other magnitudes are derived from the fundamental magnitudes using their definitions. The characteristic values used are:

Characteristic length	R_0
Characteristic time	$\omega_0^{-1} = \sqrt{\dfrac{R_0^3}{\mu}}$
Characteristic mass	m
Characteristic velocity	$R_0\omega_0$
Characteristic acceleration	$R_0\omega_0^2$
Characteristic force	$mR_0\omega_0^2$

where R_0 is the initial distance from the satellite to E —Earth's center of mass— and ω_0 is the orbital frequency of the circular orbit at distance R_0 from E.

$$\tau = \omega_0 t, \quad R = R_0 r, \quad q = \omega_0 Q, \quad p = \omega_0 P \tag{9}$$

From now on r will represent a non-dimensional distance and not the third component of $\vec{\omega}$ angular velocity. P and Q are the only two non-dimensional non-vanished components of $\vec{\omega}$.

2.1 State variables

The equations (3-5) will be rewritten using (r, u, ψ) as state variables; r has been defined in (9); u and ψ are defined by:

$$u = \frac{dr}{d\tau}, \qquad \psi = -r^2 Q$$

When equations are expressed in these state variables, they take the form:

$$\frac{dr}{d\tau} = u \tag{10}$$

$$\frac{du}{d\tau} = \frac{\psi^2}{r^3} - \frac{1}{r^2} + (\vec{f}_p \cdot \vec{i}) \tag{11}$$

$$\frac{d\psi}{d\tau} = r(\vec{f}_p \cdot \vec{k}) \tag{12}$$

where

$$\vec{f}_p = \frac{1}{mR_0\omega_0^2} \vec{F}_p$$

is the non-dimensional value of the perturbation force acting on the particle.

Note that ψ coincides with the non-dimensional value constant in the law of areas in the non-perturbed problem ($\vec{F}_p = \vec{0}$), that is to say, it turns to be a constant magnitude.

Non-dimensional components of angular velocity (P, Q) can be expressed as function of state variables by the relations:

$$P = -\frac{r}{\psi}(\vec{f}_p \cdot \vec{j}) \tag{13}$$

$$Q = -\frac{\psi}{r^2} \tag{14}$$

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2.2 Euler parameters

When p and q are known, equations (6-8) give us the time evolution of $(\vec{i}, \vec{j}, \vec{k})$, unit vectors of the orbital frame $\mathcal{R}$. There are two fundamental frames in the proposed problem: 1) the inertial geocentric frame $Ex_1y_1z_1$, or $\mathcal{R}_1$ frame, whose unit vectors are denoted by $(\vec{i}_1, \vec{j}_1, \vec{k}_1)$, and 2) the orbital frame $\mathcal{R}$, whose unit vectors are denoted by $(\vec{i}, \vec{j}, \vec{k})$. The relation between the two sets of unit vectors is given by the equation[1]

$$[\vec{i} \ \vec{j} \ \vec{k}] = [\vec{i}_1 \ \vec{j}_1 \ \vec{k}_1] \ \mathcal{Q}(t)$$

where $\mathcal{Q}$ is an **orthogonal matrix** of third order, which evolves in time as the particle is moving. Only three of the nine elements of the matrix are independent. It is not easy to describe the matrix in terms of three of its elements, by the necessity of selecting then one of each row and one of each column. Usually the nine elements of $\mathcal{Q}$ are expressed in terms of three independent coordinates, for example, Euler angles. From the viewpoint of numerical calculus this election has a disadvantage: There is always a singular direction in which the matrix represented by three generalized coordinates is not bijective. If the calculus passes over or is near the singularity the numerical simulation can fail or can be erroneous because of accumulated errors.

The minimum number of parameters needed to remove the singularity problem in the representation of $\mathcal{Q}$ is **four**. This is the reason why Euler parameters ($\vec{\varepsilon} = (\varepsilon_1, \varepsilon_2, \varepsilon_3), \eta$) will be used in this paper. The $\mathcal{Q}$ matrix as function of these parameters takes the form:

$$\begin{bmatrix} 1 - 2(\varepsilon_2^2 + \varepsilon_3^2) & 2\varepsilon_1\varepsilon_2 - 2\eta\varepsilon_3 & 2\varepsilon_1\varepsilon_3 + 2\eta\varepsilon_2 \\ 2\varepsilon_1\varepsilon_2 + 2\eta\varepsilon_3 & 1 - 2(\varepsilon_1^2 + \varepsilon_3^2) & 2\varepsilon_2\varepsilon_3 - 2\eta\varepsilon_1 \\ 2\varepsilon_1\varepsilon_3 - 2\eta\varepsilon_2 & 2\varepsilon_3\varepsilon_2 + 2\eta\varepsilon_1 & 1 - 2(\varepsilon_1^2 + \varepsilon_2^2) \end{bmatrix}$$

Following Euler's theorem, the transformation of the base $(\vec{i}_1, \vec{j}_1, \vec{k}_1)$ into the base $(\vec{i}, \vec{j}, \vec{k})$ by means of $\mathcal{Q}$ is geometrically obtained by a rotation of value ϕ around a spatial direction defined by a unit vector $\vec{a}$. In this context, Euler parameters define the rotation elements by relations:

$$\vec{\varepsilon} = \vec{a}\sin\frac{\phi}{2}, \qquad \eta = \cos\frac{\phi}{2}$$

The time evolution of Euler parameters is given by the equations:

$$\frac{d\vec{\varepsilon}}{dt} = \frac{1}{2}(\vec{\varepsilon} \times \vec{\omega} + \eta\,\vec{\omega}), \qquad \frac{d\eta}{dt} = -\frac{1}{2}\vec{\varepsilon} \cdot \vec{\omega}$$

where all vectors have to be expressed in the orbital frame $(\vec{i}, \vec{j}, \vec{k})$. The equations (6-8) in terms of non-dimensional variables take the form:

$$\left.\begin{aligned} \frac{d\varepsilon_1}{d\tau} &= +\frac{1}{2}P\eta - \frac{1}{2}Q\varepsilon_3 \\ \frac{d\varepsilon_3}{d\tau} &= -\frac{1}{2}P\varepsilon_2 + \frac{1}{2}Q\varepsilon_1 \\ \frac{d\varepsilon_2}{d\tau} &= +\frac{1}{2}P\varepsilon_3 + \frac{1}{2}Q\eta \\ \frac{d\eta}{d\tau} &= -\frac{1}{2}P\varepsilon_1 - \frac{1}{2}Q\varepsilon_2 \end{aligned}\right\} \tag{15}$$

where non-dimensional components (P, Q) of angular velocity $\vec{\omega}$ are given by relations (13-14) and depend only on the perturbation $\vec{f}_p$ acting on the particle.

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[1]The notation uses some *matrices*, such as $[\vec{i}_1 \ \vec{j}_1 \ \vec{k}_1]$, which are no true matrices (matrix elements are scalars but not vectors). In algebraic operations these *pseudo-matrices* obey the same algebra rules, *mutatis mutandi*, than true matrices. This notation is easy to use and causes no error if everyone is forewarned.

As is well known, the following additional condition must be satisfied by the four Euler parameters.

$$\varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2 + \eta^2 = 1 \tag{16}$$

therefore, *only three* of the four equations (15) are independent. Nevertheless, relation (16) will not be used to reduce the system order, but to check the quality of numerical solutions obtained by integration of equations (15).

2.3 Scheme of method generation

Equations (10-14) and (15) are the starting point of the process that leads to the integration method of this paper. The scheme followed to deduce equations of the method is summarized in three basic points:

• A *fictitious time* σ is introduced by mean of a change of independent variable, similar to the Sundmann transformation (see [2]), first step in classical regularization process of the two-body problem (see [1]). It is defined by:

$$\frac{d\sigma}{d\tau} = -Q(\sigma) \ \Rightarrow \ \tau = -\int_{\sigma_0}^{\sigma} \frac{d\sigma}{Q(\sigma)} = \int_{\sigma_0}^{\sigma} \frac{r^2 d\sigma}{\psi} \tag{17}$$

• Then, the analytical solution of *non-perturbed problem* is obtained

$$\vec{X} = \vec{X}(\sigma; \vec{E}) \tag{18}$$

Here $\vec{X}$ represents the state of the system and $\vec{E}$ represents the six integration constants (orbital elements in broad sense) that appear in the non-perturbed problem solution.

• The variation of constants method will be used to solve *perturbed problem*.

The perturbed problem solution is expressed in the form of (18), where orbital elements are already not constants from now on, but unknown functions of σ, $\vec{E} = \vec{E}(\sigma)$. This is equivalent to taking (18) as a change of variables that permits to express the state of the system $\vec{X}$ as function of orbital elements $\vec{E}$. Such a change permits to rewrite equations of the perturbed problem in terms of $\vec{E}$:

$$\frac{d\vec{E}}{d\sigma} = \vec{G}(\sigma, \vec{E}, \vec{f}_p), \qquad \vec{G}(\sigma, \vec{E}, \vec{0}) = \vec{0}$$

Thereby, we obtained equations whose second members cancel if the perturbation disappears.

The basic scheme will be developed below, firstly with equations (10-14) and finally with equations (15).

3 First set of equations

The first set of equations (10-14) will be rewritten taking σ as independent variable, which is defined in (17). At the same time, for reasons that will be exposed below, we will take a new state variable

$$z = \frac{1}{r}$$

instead of r. The next set of equations is obtained:

$$\frac{d\tau}{d\sigma} = \frac{r^2}{\psi} \qquad\qquad \rightarrow \frac{d\tau}{d\sigma} = \frac{1}{z^2\psi} \tag{19}$$

$$\frac{dr}{d\sigma} = u\frac{r^2}{\psi} \qquad\qquad \rightarrow \frac{dz}{d\sigma} = -\frac{u}{\psi} \tag{20}$$

$$\frac{du}{d\sigma} = \frac{\psi}{r} - \frac{1}{\psi} + \frac{r^2}{\psi}(\vec{f}_p \cdot \vec{i}) \rightarrow \frac{du}{d\sigma} = z\psi - \frac{1}{\psi} + \frac{1}{z^2\psi}(\vec{f}_p \cdot \vec{i}) \tag{21}$$

$$\frac{d\psi}{d\sigma} = \frac{r^3}{\psi}(\vec{f}_p \cdot \vec{k}) \qquad\qquad \rightarrow \frac{d\psi}{d\sigma} = \frac{1}{z^3\psi}(\vec{f}_p \cdot \vec{k}) \tag{22}$$

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and the following relations will be added besides

$$P = -\frac{1}{z\psi}(\vec{f}_p \cdot \vec{j}) \tag{23}$$

$$Q = -\psi z^2 \tag{24}$$

that give us the components of angular velocity $\vec{\omega}$ as functions of the state of the system.

3.1 Non-perturbed problem

In the non-perturbed problem we have $\vec{f}_p = \vec{0}$. Equations (20-22) are simplified and the solution can be analytically obtained. This result is not surprising, since it is known that the problem is integrable. We can obtain the next relations

$$z = \frac{1}{\psi^2} + A\cos\sigma + B\sin\sigma \tag{25}$$

$$u = \psi(A\sin\sigma - B\cos\sigma) \tag{26}$$

$$\psi = \psi_0 \tag{27}$$

where ψ_0, A and B are integration constants depending on the initial conditions.

It can be noticed that σ has a precise geometrical meaning in the non-perturbed problem: it coincides with the *true anomaly* of the keplerian orbit followed by the particle. In fact, if the law of areas is compared with equation (19)

$$\text{Law of areas:} \quad r^2\frac{d\theta}{dt} = C \ \leftrightarrow \ r^2\frac{d\sigma}{d\tau} = \psi_0 \quad (19)$$

the previous result is deduced easily: $\sigma \equiv \theta$. The initial condition to be imposed to σ emerges immediately from this: $\sigma(\tau = 0) = \theta_0$, where θ_0 is the initial true anomaly of the particle. This initial condition is the same in both problems, perturbed and non-perturbed, since it doesn't depend on presence or absence of perturbation.

The introduction of the $z = 1/r$ variable is due to a well known fact. The second formula of Binet for central motions

$$\gamma = -\frac{C^2}{r^2}\left\{\frac{d^2}{d\theta^2}(\frac{1}{r}) + \frac{1}{r}\right\}$$

shows that the equation which give us the trajectory in the two-body problem is linear in z. This linearization es one of the basic goals of all the processes of equation regularization.

The same equation (19) provides, by means of an additional integration, the relation between the non-dimensional time τ and the true anomaly σ. Nevertheless, this relation is not interesting in this context. Moreover, depending on the type of keplerian orbit (elliptic, parabolic or hyperbolic) it could take three different forms.

3.2 Perturbed problem

Guided by the solution (25-27) of the *non-perturbed problem*, a solution of the *perturbed problem* is sought in the form

$$z = \frac{1}{\psi^2(\sigma)} + A(\sigma)\cos\sigma + B(\sigma)\sin\sigma$$

$$u = \psi(\sigma)(A(\sigma)\sin\sigma - B(\sigma)\cos\sigma)$$

$$\psi = \psi(\sigma)$$

where $\psi(\sigma)$, $A(\sigma)$ and $B(\sigma)$ are now unknown functions of σ to be determined as part of the solution. It is equivalent to considering equations (25-27) as a change of variable that permits to express the dynamical state of the system in terms of the generalized coordinates ψ, A and B.

If this relations are introduced in equations (19-22), the equations governing the evolution of $\psi(\sigma)$, $A(\sigma)$ y $B(\sigma)$ are obtained. Thereby we get the system

$$\frac{d\psi}{d\sigma} = \frac{1}{z^3\psi}(\vec{f}_p \cdot \vec{k}) \tag{28}$$

$$\frac{dA}{d\sigma} = +\sin\sigma \cdot \frac{1}{\psi^2 z^2}(\vec{f}_p \cdot \vec{i}) + \frac{1}{\psi}\frac{d\psi}{d\sigma}\{\cos\sigma(\frac{1}{\psi^2} + z) - A\} \tag{29}$$

$$\frac{dB}{d\sigma} = -\cos\sigma \cdot \frac{1}{\psi^2 z^2}(\vec{f}_p \cdot \vec{i}) + \frac{1}{\psi}\frac{d\psi}{d\sigma}\{\sin\sigma(\frac{1}{\psi^2} + z) - B\} \tag{30}$$

$$\frac{d\tau}{d\sigma} = \frac{1}{z^2\psi} \tag{31}$$

whose second members include the perturbation force $\vec{f}_p$, that is a function of the state of the system. If $\vec{f}_p$ is null, we recover the non-perturbed problem, where ψ, A, and B will keep constant. To calculate the second member of these expressions the following relation must be considered

$$z = \frac{1}{\psi^2(\sigma)} + A(\sigma)\cos\sigma + B(\sigma)\sin\sigma \tag{32}$$

$$\frac{dz}{d\sigma} = -A(\sigma)\sin\sigma + B(\sigma)\cos\sigma \tag{33}$$

that are satisfied at any time.

4 Second set of equations

Equations (15), if they are rewritten taking σ as the independent variable, are of the form

$$\left.\begin{array}{l} \dfrac{d\varepsilon_1}{d\sigma} = -\dfrac{1}{2}\lambda(\sigma)\eta + \dfrac{1}{2}\varepsilon_3 \\[2mm] \dfrac{d\varepsilon_3}{d\sigma} = +\dfrac{1}{2}\lambda(\sigma)\varepsilon_2 - \dfrac{1}{2}\varepsilon_1 \\[2mm] \dfrac{d\varepsilon_2}{d\sigma} = -\dfrac{1}{2}\lambda(\sigma)\varepsilon_3 - \dfrac{1}{2}\eta \\[2mm] \dfrac{d\eta}{d\sigma} = +\dfrac{1}{2}\lambda(\sigma)\varepsilon_1 + \dfrac{1}{2}\varepsilon_2 \end{array}\right\} \tag{34}$$

where the function $\lambda(\sigma)$, defined by the relation

$$\lambda(\sigma) = \frac{P}{Q} = \frac{1}{\psi^2 z^3} \cdot (\vec{f}_p \cdot \vec{j}) \tag{35}$$

is the quotient between the two non-zero components of angular velocity $\vec{\omega}$ of the $\mathcal{R}$ frame. $\lambda(\sigma)$ can be singular if $z = 0$ or $\psi = 0$. The case $z = 0$ appears when the trajectory reaches a point of the infinite ($r = \infty$); the case $\psi = 0$ appear when the angular momentum of the particle with respect to the origin E is null, that is to say, the particle velocity is contained into the local vertical. None of these situations are presented in tether dynamics, nor in most of the problems of orbital dynamics. Nevertheless, the hyperbolic case will lead to a possible singularity when the particle approaches the asymptote, with λ taken increasing positive values; in this case an asymptotic solution can be obtained in the limit $\lambda \to \infty$, but for brevity's sake it is not developed in these pages.

4.1 Non-perturbed problem

In the non-perturbed problem $\lambda(\sigma) \equiv 0$, since $\vec{f}_p = \vec{0}$. Equations (34) can be simplified and take the form

$$\frac{d\varepsilon_1}{d\sigma} = +\frac{1}{2}\varepsilon_3, \qquad \frac{d\varepsilon_2}{d\sigma} = -\frac{1}{2}\eta$$

$$\frac{d\varepsilon_3}{d\sigma} = -\frac{1}{2}\varepsilon_1, \qquad \frac{d\eta}{d\sigma} = +\frac{1}{2}\varepsilon_2$$

This system of equations have analytical solution, that can be expressed —changing the order of Euler parameters lightly— as

$$\left\{ \begin{array}{c} \varepsilon_1 \\ \varepsilon_3 \\ \varepsilon_2 \\ \eta \end{array} \right\} = \mathcal{M}(\sigma - \sigma_0) \left\{ \begin{array}{c} \varepsilon_1^0 \\ \varepsilon_3^0 \\ \varepsilon_2^0 \\ \eta^0 \end{array} \right\} \tag{36}$$

where the matrix $\mathcal{M}(\zeta)$, given by

$$\begin{bmatrix} \cos(\zeta/2) & \sin(\zeta/2) & 0 & 0 \\ -\sin(\zeta/2) & \cos(\zeta/2) & 0 & 0 \\ 0 & 0 & \cos(\zeta/2) & -\sin(\zeta/2) \\ 0 & 0 & \sin(\zeta/2) & \cos(\zeta/2) \end{bmatrix}$$

is a orthogonal 4×4-matrix; Note that $\mathcal{M}(0) \equiv I$, where I is the identity 4×4-matrix. σ_0 represents the initial value of σ in (36) and it coincides with the initial value of the true anomaly of the particle if it will keep its keplerian orbit from the initial time without perturbations. Values ε_1^0, ε_3^0, ε_2^0 and η^0 are integration constants that define the attitude of the orbital frame $\mathcal{R}$ at the initial time. Note that the position of orbital frame $\mathcal{R}$ changes with time beginning from an initial position. The solution (36) relates the attitude of $\mathcal{R}$ in a generic time with the attitude of $\mathcal{R}_0$, namely, the reference $\mathcal{R}$ at the initial time ($t = 0$).

4.2 Perturbed problem

To approach the perturbed problem, it is appropriate to write equations (34) in matricial form

$$\left\{ \begin{array}{c} d\varepsilon_1/d\sigma \\ d\varepsilon_3/d\sigma \\ d\varepsilon_2/d\sigma \\ d\eta/d\sigma \end{array} \right\} = S(\sigma) \left\{ \begin{array}{c} \varepsilon_1 \\ \varepsilon_3 \\ \varepsilon_2 \\ \eta \end{array} \right\} \tag{37}$$

where matrix $S(\sigma)$ is

$$S(\sigma) = \frac{1}{2} \begin{bmatrix} 0 & 1 & 0 & -\lambda(\sigma) \\ -1 & 0 & \lambda(\sigma) & 0 \\ 0 & -\lambda(\sigma) & 0 & -1 \\ \lambda(\sigma) & 0 & 1 & 0 \end{bmatrix}$$

In the perturbed problem we seek a solution to equations (37) of type

$$\left\{ \begin{array}{c} \varepsilon_1 \\ \varepsilon_3 \\ \varepsilon_2 \\ \eta \end{array} \right\} = \mathcal{M}(\sigma - \sigma_0) \left\{ \begin{array}{c} \varepsilon_1^0(\sigma) \\ \varepsilon_3^0(\sigma) \\ \varepsilon_2^0(\sigma) \\ \eta^0(\sigma) \end{array} \right\} \tag{38}$$

in which $\varepsilon_1^0(\sigma)$, $\varepsilon_3^0(\sigma)$, $\varepsilon_2^0(\sigma)$ and $\eta^0(\sigma)$ are functions of σ that must be determined as part of the solution. This is equivalent to considering equations (36) as a change of variables that permits to express the dynamical state of the system in terms of generalized coordinates ε_1^0, ε_3^0, ε_2^0 and η^0.

Introducing expressions (38) into equations (37), after algebraic manipulation, these take the form

$$\left\{\begin{array}{c} d\varepsilon_1^0(\sigma)/d\sigma \\ d\varepsilon_3^0(\sigma)/d\sigma \\ d\varepsilon_2^0(\sigma)/d\sigma \\ d\eta^0(\sigma)/d\sigma \end{array}\right\} = F(\sigma) \left\{\begin{array}{c} \varepsilon_1^0(\sigma) \\ \varepsilon_3^0(\sigma) \\ \varepsilon_2^0(\sigma) \\ \eta^0(\sigma) \end{array}\right\}$$

where the matrix

$$F(\sigma) = \left(\mathcal{M}^T(\sigma - \sigma_0)S(\sigma)\mathcal{M}(\sigma - \sigma_0) + \right.$$
$$\left. -\mathcal{M}^T(\sigma - \sigma_0)\frac{d\mathcal{M}}{d\sigma}(\sigma - \sigma_0)\right)$$

can be written, after introducing the value $\Omega = \sigma - \sigma_0$, as

$$F(\sigma) = \frac{\lambda(\sigma)}{2} \left[\begin{array}{cccc} 0 & 0 & -\sin\Omega & -\cos\Omega \\ 0 & 0 & \cos\Omega & -\sin\Omega \\ \sin\Omega & -\cos\Omega & 0 & 0 \\ \cos\Omega & \sin\Omega & 0 & 0 \end{array}\right]$$

After some development, equations take the form

$$\frac{d\varepsilon_1^0}{d\sigma} = -\frac{\lambda(\sigma)}{2}\{\sin(\sigma - \sigma_0)\varepsilon_2^0 + \cos(\sigma - \sigma_0)\eta^0\} \tag{39}$$

$$\frac{d\varepsilon_2^0}{d\sigma} = +\frac{\lambda(\sigma)}{2}\{\sin(\sigma - \sigma_0)\varepsilon_1^0 - \cos(\sigma - \sigma_0)\varepsilon_3^0\} \tag{40}$$

$$\frac{d\varepsilon_3^0}{d\sigma} = +\frac{\lambda(\sigma)}{2}\{\cos(\sigma - \sigma_0)\varepsilon_2^0 - \sin(\sigma - \sigma_0)\eta^0\} \tag{41}$$

$$\frac{d\eta^0}{d\sigma} = +\frac{\lambda(\sigma)}{2}\{\cos(\sigma - \sigma_0)\varepsilon_1^0 + \sin(\sigma - \sigma_0)\varepsilon_3^0\} \tag{42}$$

and can be integrated with suitable initial conditions

Note again that if perturbations disappear, $\lambda \equiv 0$, generalized coordinates ε_1^0, ε_3^0, ε_2^0 and η^0 will keep constant, that is to say, the non-perturbed problem result is recovered. The value of λ depends only on the component of perturbation perpendicular to the orbital plane (see (35)), and if it is *small* the value of λ will be *small*, this is, $\lambda << 1$. In that case, the evolution of ε_1^0, ε_3^0, ε_2^0 and η^0 is produced in a time scale which is as long as the perturbation is small.

Note that now there are *three* frames in the problem: the inertial geocentric frame $Ex_1y_1z_1$ that is fixed, the orbital frame $\mathcal{R}_0$, whose attitude is defined by generalized coordinates ε_1^0, ε_3^0, ε_2^0 and η^0, and finally, the orbital frame $\mathcal{R}$ located at the real position of the particle, whose attitude is defined by Euler parameters ε_1, ε_3, ε_2 and η.

The $\mathcal{R}_0$ reference is fixed in the $Ex_1y_1z_1$ reference in the non-perturbed problem, but in the perturbed problem its position changes with time (slower as the perturbation acting on the particle is smaller). In both cases the attitude of the orbital reference $\mathcal{R}$ located at the real position of the particle is given by equations (38). However, in the non-perturbed problem $\mathcal{R}$ evolves in an unique time scale; by example, if the orbit followed by the particle M is elliptic this unique time scale is defined by the orbital period. On the contrary, $\mathcal{R}$ evolves in two time scales —different in principle— in the perturbed problem. One of them, defined by the osculatrix orbit (quick scale), an the other one defined by the perturbation (slow scale); when the perturbation is small, the second is much greater than the former. The relation between the two orbital references, $\mathcal{R}_0$ and $\mathcal{R}$, is much clearer: if in a given time the perturbation is eliminated, the particle will follow a keplerian orbit from this time forward: the osculatrix orbit; in this orbit, $\mathcal{R}_0$ would be the orbital frame in the initial time and $\mathcal{R}$ the orbital frame at a generic time.

It is important to understand this propagator structure, since in order to calculate the perturbation situated at the second members of equations, the orbital frame located al the particle needs to be

used, as integrating equations provide only the orbital frame $\mathcal{R}_0$. The pass from one frame to the other is obtained by means of equations (38).

All in all, the proposed method integrates equations (28-31) and (39-42) beginning from suitable initial conditions. Relations (32-33), (35), and (38) —satisfied at any time— must be considered in the process of integration and in the computation of right sides of equations.

4.3 Change of variables

The definitive method formulation is reached after doing a last change of variables, defined by the relations

$$q_1 = \psi A, \qquad q_2 = \psi B, \qquad q_3 = \frac{1}{\psi} \tag{43}$$

The change is inspired by equations structure, where the derivatives of ψA and ψB appear in a natural form. Moreover, the total energy of the system in the non-perturbed problem takes —as the reader can check— the simplified form

$$E = \frac{1}{2}mv^2 - \frac{m\mu}{R} = \frac{m\mu}{2R_0}\{q_1^2 + q_2^2 - q_3^2\}$$

The last change is not essential. Its introduction in the method was initially guided by an erroneous argument.

The change of variables can be conceived as a transformation —in the space of phases of the non-perturbed problem— of the original variables R, p_R, θ and p_θ into the variables q_1, q_2, q_3 and θ, given by equations

$$
\begin{aligned}
\frac{1}{R} &= q_3(q_3 + q_1\cos\theta + q_2\sin\theta\}\frac{1}{R_0} \\
\theta &= \theta \\
p_R &= m\sqrt{\frac{\mu}{R_0}}(q_1\sin\theta - q_2\cos\theta\} \\
p_\theta &= m\sqrt{\mu R_0}\frac{1}{q_3}
\end{aligned}
$$

Initially it was suspected that this was a canonical transformation that would preserve the hamiltonian structure of equations, and would give an advantageous algorithm. A later detailed calculation shows the error on this supposition. The previous transformation **is not canonical** and doesn't bring any advantage at all in the theoretical formulation. Nevertheless, a faster (38) calculation was obtained in several simulation tests developed to check the goodness of the method using the change of variables of (43). Although the differences are not important, it was finally adopted because there is no reason against using it.

4.4 Summary of equations

All in all, the method proposed on this pages is based on the following set of equations

$$\frac{d\tau}{d\sigma} = +\frac{1}{q_3 s^2} \tag{44}$$

$$\frac{dq_1}{d\sigma} = +\frac{\sin\sigma}{q_3 s^2}\,(\vec{f}_p \cdot \vec{i}) + \cos\sigma\frac{s+q_3}{q_3 s^3}\,(\vec{f}_p \cdot \vec{k}) \tag{45}$$

$$\frac{dq_2}{d\sigma} = -\frac{\cos\sigma}{q_3 s^2}\,(\vec{f}_p \cdot \vec{i}) + \sin\sigma\frac{s+q_3}{q_3 s^3}\,(\vec{f}_p \cdot \vec{k}) \tag{46}$$

$$\frac{dq_3}{d\sigma} = -\frac{1}{s^3}(\vec{f}_p \cdot \vec{k}) \tag{47}$$

$$\frac{d\varepsilon_1^0}{d\sigma} = -\frac{\lambda(\sigma)}{2}\{\sin(\sigma-\sigma_0)\varepsilon_2^0 + \cos(\sigma-\sigma_0)\eta^0\} \tag{48}$$

$$\frac{d\varepsilon_2^0}{d\sigma} = +\frac{\lambda(\sigma)}{2}\{\sin(\sigma-\sigma_0)\varepsilon_1^0 - \cos(\sigma-\sigma_0)\varepsilon_3^0\} \tag{49}$$

$$\frac{d\varepsilon_3^0}{d\sigma} = +\frac{\lambda(\sigma)}{2}\{\cos(\sigma-\sigma_0)\varepsilon_2^0 - \sin(\sigma-\sigma_0)\eta^0\} \tag{50}$$

$$\frac{d\eta^0}{d\sigma} = +\frac{\lambda(\sigma)}{2}\{\cos(\sigma-\sigma_0)\varepsilon_1^0 + \sin(\sigma-\sigma_0)\varepsilon_3^0\} \tag{51}$$

that can be integrated simultaneously with the relations needed to calculate the second members of equations

$$\lambda(\sigma) = \frac{1}{q_3 s^3}\,(\vec{f}_p \cdot \vec{j})$$

$$s = q_3 + q_1\cos\sigma + q_2\sin\sigma$$

$$z = \frac{1}{r} = q_3 \cdot s$$

$$\frac{dr}{d\tau} = -\psi\frac{dz}{d\sigma} = q_1\sin\sigma - q_2\cos\sigma$$

$$\left\{\begin{array}{c} \varepsilon_1 \\ \varepsilon_3 \\ \varepsilon_2 \\ \eta \end{array}\right\} = \mathcal{M}(\sigma-\sigma_0)\left\{\begin{array}{c} \varepsilon_1^0(\sigma) \\ \varepsilon_3^0(\sigma) \\ \varepsilon_2^0(\sigma) \\ \eta^0(\sigma) \end{array}\right\}$$

$$1 = (\varepsilon_1^0)^2 + (\varepsilon_2^0)^2 + (\varepsilon_3^0)^2 + (\eta^0)^2$$

5 Advantages of the method

The proposed method presents advantages of diverse nature. Test results will be shown to check the goodness of the method. Nevertheless, some of the advantages can be deduced directly from the theoretical formulation developed and it is convenient to underline then before showing the figures. The advantages are

- Unique formulation for the three types of orbits: elliptic, parabolic and hyperbolic. So, the singularity that appears in the proximity of parabolic motion when using different formulation for elliptic and hyperbolic orbits can be avoided.

- It uses orbital elements as generalized coordinates, as is the case of Lagrange's Planetary equations. Hereby, the truncation error is null in the non-perturbed problem and is scaled by the perturbation itself in the perturbed one.

- The error propagation is not exponential, as is the case of Cowell's or Encke's methods (see [2]).

- The method doesn't have singularities for small inclination and small eccentricities, unlike the Lagrange's planetary equations. The orbital plane attitude is determined by the Euler parameters and not the Euler angles, that always present a singular direction for null inclination.

- The use of Euler parameters gives easy auto-correction as well as robustness. When the sum of the first members of equation (16) differs from 1 more than a certain limit, Euler parameters can be normalized dividing by the module of the associated quaternion.

- Easy programming, since they use the components of perturbation forces in the orbital frame. This makes easy the use of models proper of Orbital Dynamics.

- A precise and quick simulator is obtained if it is integrated by a variable step routine with an effective step control, as Runge-Kutta-Fehlberg or Dormand-Prince types.

- It is not necessary to resolve Kepler's equation in the elliptic case, nor the equivalent for hyperbolic and parabolic cases, since time is an integration variable determined by the method itself.

Finally note that the method integrates a system of eight differential equations to resolve a six-order problem and this is a characteristic shared with other regularization methods. Although increasing the order of the system in two unities can seem troublesome, there is no disadvantage, as can be shown by regularization methods and the results of the next section.

6 Method checking

The method has been used to reproduce results obtained in the example 2b of the book by Stiefel & Scheifele (see [1], pag. 122). It deals with a satellite that is in an inclined ($i = 30°$) elliptic orbit, of great eccentricity ($e = 0.95$) and only subjected to two perturbation forces: 1) the gravitational perturbation due to Earth oblateness and 2) the lunar perturbation.
Parameters associated to Earth gravitation —needed to develop the simulation— are

$$J_2 = 1.08265 \cdot 10^{-3}$$
$$R_E = 6371.22 \text{ Km}$$
$$\mu = 398601.0 \text{ Km}^3\text{s}^{-2}$$

Lunar perturbation is modeled with a force of value

$$\vec{F}_{PL} = -m\mu_L\{\frac{\vec{R} - \vec{\rho}}{|\vec{R} - \vec{\rho}|^3} + \frac{\vec{\rho}}{\rho^3}\}$$
$$\text{where } \mu_L = 4902.66 \text{ Km}^3\text{s}^{-2}$$

where $\vec{R}$ and $\vec{\rho}$ are the position vectors of satellite and Moon respectively, in the inertial geocentric frame. The Moon position is given by the following ephemeris

$$\vec{\rho} = \rho\{\sin \Omega_L t\, \vec{i}_1 - \frac{\sqrt{3}}{2} \cos \Omega_L t\, \vec{j}_1 - \frac{1}{2} \cos \Omega_L t\, \vec{k}_1\}$$

where ρ and Ω_L are constants of values

$$\rho = 384400 \text{ Km}$$
$$\Omega_L = 2.665315780887 \cdot 10^{-6} \text{ s}^{-1}$$

and correspond to a inclined circular orbit.

Beginning from the initial conditions

$$(x_1, y_1, z_1) = (0.0, \ -5888.9727, \ -3400.0) \text{ Km}$$
$$(\dot{x}_1, \dot{y}_1, \dot{z}_1) = (10.691338, \ 0.0, \ 0.0) \text{ Km s}^{-1}$$

corresponding to the perigee, at a distance $R = 6800$ Km of the Earth's center of mass, the satellite's position is determined after 288.12768941 mean solar days, time to describe 50 revolutions. The most precise calculus of final position (x_{1f}, y_{1f}, z_{1f}) given in the reference [1] and using a numerical integration scheme of 498 steps per revolution is:

$$x_{1f} = -24219.0503 \text{ Km}$$
$$y_{1f} = 227962.1064 \text{ Km} \tag{52}$$
$$z_{1f} = 129753.4424 \text{ Km}$$

The most precise calculus of final position achieved in our group leads to

$$x_{1f} = -24219.0501159 \text{ Km}$$
$$y_{1f} = 227962.1063730 \text{ Km}$$
$$z_{1f} = 129753.4424001 \text{ Km}$$

The same problem is used in the reference [2] to compare different integration methods. Ad hoc, the solution (52) given in [1] is adopted as authentic and the method error is defined as the distance (in Km) between the final position given by a method and the authentic one. All the cases use variable step-size Runge-Kutta algorithms. In the five first columns of table 1, taken from the reference [2], results given by different calculus method are shown. The sixth and last column corresponds to the results obtained using the exposed method.

Method	Stiefel & Scheifele [1]	Sperling & Burdet [3]	Kustaanheimo & Stiefel [4]	Cowell [5]	GDT
x(Km)	-24219.050	-24218.818	-24219.002	-24182.152	-24219.279
y(Km)	227962.106	227961.915	227962.429	227943.989	227962.207
z(Km)	129753.442	129753.343	129753.822	129744.270	129753.492
Steps/rev	500	62	62	240	62
Error		0.318	0.501	42.5	0.250

Table 1: Comparison of integration methods (taken from [2])

Note that Cowell's method requires working with a much greater number of steps per revolution (240) and, in spite of it, it is the less precise method, because the exponential error propagation (see [2]). Sperling-Burdet's method, considered as the most efficient method in the book by V. Bond [2], gives an error of 318 m using 62 steps per revolution. As it is shown in the last column of table 1 the method developed by our group clearly competes with Sperling-Burdet's method as precision is referred, as it provides a smaller error, of 250 m, using 62 steps per revolution.

In order to do a more detailed comparison between the Sperling-Burdet's method and our method, we have coded both orbit propagators and we have computed the former problem with them to obtain the position vector at final time for different step error tolerances. Both computations have been done

- in the same computer (Intel Xeon 3056 MHz microprocessor, 2 Gb RAM),

- with the same compiler (Intel C++ 8.1.022),

- with the same integrating algorithm (Runge-Kutta-Fehlberg 7(8) of variable step-size),

- in the same computer conditions (processor load, etc.)

The variable step size is controlled by using four different parameters in the integrating algorithm. The problem has been resolved using different sets of values of these parameters to show the method's features better.

To minimize the effect of uncontrolled factors in the computation time we have done the former task several times (exactly 30) and we have obtained the mean value of runtime.

The results are shown in the figure 6, where we plot the mean computation runtime in ordinates and the common logarithm of the norm of the error vector $(-\log(|\Delta \bar{x}|)$ in abscissas. It's is easy to see that this last quantity is a quality parameter of the solution, that is approximately equivalent to the number of exact decimal digits of the solution plus one.
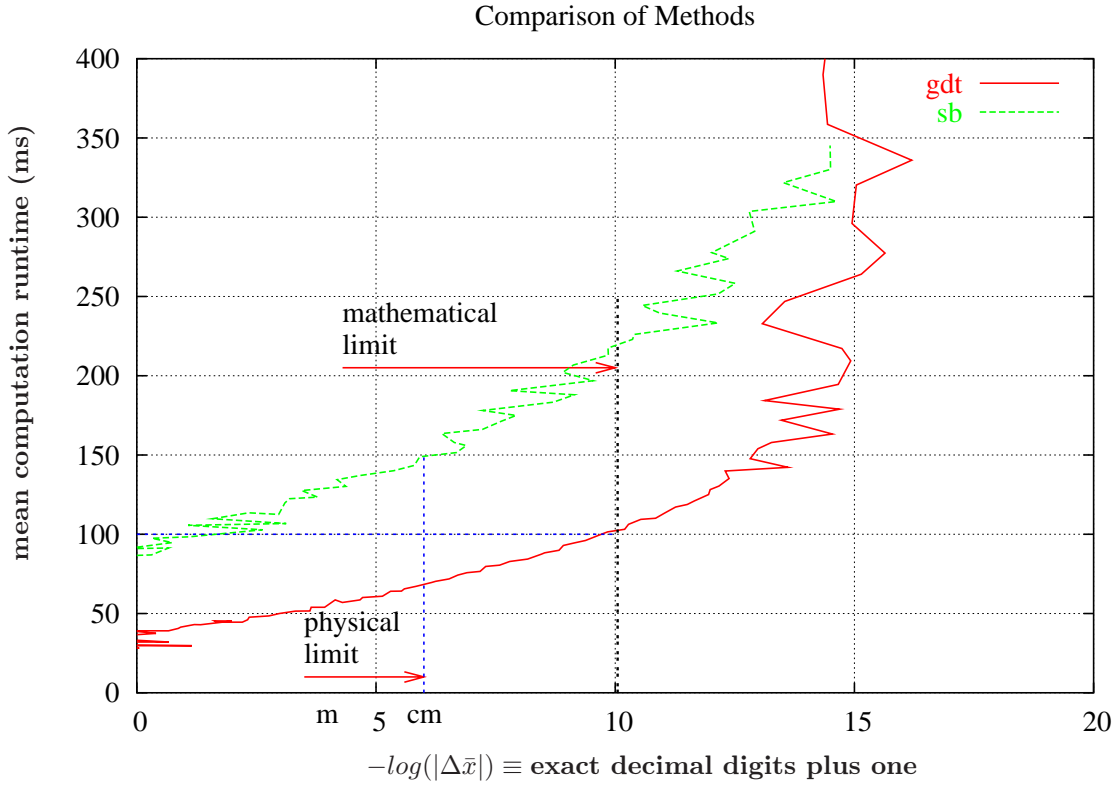

Figure 2: Mean computation runtime vs. quality parameter

The plot shows that our method seems to be quicker for the same precision, or equivalently, it seems to be more accurate for identical computational time.

We think this method is better because an essential reason: in Sperling-Burdet's method —and other similar methods based on regularization techniques as can be Kustaanheimo-Stiefel's method— to obtain the second members of equation to be integrated requires to process perturbation forces by numerical treatments of some length. However, forces hardly require manipulation in the method exposed in this article, because their components in the orbital frame $\mathcal{R}$ only appear in the second members and are obtained by simple scalar products. Moreover, the simplicity of programming, joined to the clearness and the simplicity of equations governing the evolution of Euler parameters, strengthen our belief in the method's advantages.

6.1 Similar formulations

An analysis of the available literature on the subject shows the existence of formulations similar to the one exposed in this article, although in different context. Particularly, almost identic formulations are used in the Department of Astronomy of the Cairo University, collected in references [7, 9]. Likewise, similar schemes have been formulated in the Space Mechanics Group of the Zaragoza University, being the nearest the one contained in the reference [11].

Whichever the case may be, the method is clearly advantageous for us in several aspects when it is compared to other traditional methods. The reasons why it is so are no perfectly clear and are no simple to describe. The reader interested in deepening the subject can encounter an extensive analysis about regularization and linearization aspects in the reference [10], advanced techniques to make new perturbation methods in the last decades.

7 Conclusions

The formulation of a special perturbation method have been exposed in detail. An analysis of equation structure shows some of the advantages intrinsic to the method, as are

- Unified formulation for the three kinds of orbits: elliptic, parabolic and hyperbolic
- Null truncation error in the *non-perturbed* problem
- Truncation error scaled by perturbation in the *perturbed* problem
- It is no singular for small inclination nor small eccentricity
- There is no exponential error propagation
- Robustness and ease of programming

It has been displayed that the method exhibits indubitable advantages concerning precision, when it is compared to classical methods (the Cowell's or Encke's methods), as well as concerning computation runtime compared to more sophisticated methods (the KS or the SB methods).

All these reasons make this perturbation method suitable to be used in tether dynamics, where it is required to follow the time evolution of a great number of particles. But the method is also useful for general orbital motion.

8 Acknowledgments

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