Some consistent finite element formulations of 1-D beam models: a comparative study

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A consistent Finite Element formulation was developed for four classical 1-D beam models. This formulation is based upon the solution of the homogeneous differential equation (or equations) associated with each model. Results such as the shape functions, stiffness matrices and consistent force vectors for the constant section beam were found. Some of these results were compared with the corresponding ones obtained by the standard Finite Element Method (i.e. using polynomial expansions for the field variables).

Some of the difficulties reported in the literature concerning some of these models may be avoided by this technique and some numerical sensitivity analysis on this subject are presented.

Key words: finite element formulation, differential equations, polynomial functions.

1 INTRODUCTION

Piecewise polynomial functions were extensively used to express the expansion of the field variables in the Finite Element Method.[1] However, this approach, in the structural analysis, presents some difficulties even in simple 1-D elements when shear strain is included.[2]

An alternative to the use of polynomial functions is to consider the solution of the boundary value field problem. Some inconsistencies described in the previous reference about the formulations developed in the references[3–6] can be in this way overcome. In some of those formulations, overstiff solutions for long slender beams were obtained (shear locking); in some cases more than two nodes per element were used, or, in other cases, only two nodes were required, but with more than two degrees of freedom at each node; in all those formulations a significant number of elements was required to accurately represent the behaviour of short beams under complex distributions of the load.

Here, the previously mentioned alternative is applied to the following four beam models and in some of them the results are compared with the standard FEM solutions:

1. Navier–Bernoulli beam model
2. Timoshenko beam model, i.e., shear strain included
3. Beam–column model
4. Beam–column model considering shear strain.

The notation used in Sections 2–7 is given in Appendix A. The abscissa ξ of the beam element is defined along an element length $L = 1$, i.e. $\xi \in [0,1]$. Other variables related to the beam of unitary length are denoted by an overline. However, for sake of clarity in the figures the overline was dropped. The properties (bending and shear stiffnesses, $\overline{EI}$ and $\overline{GA}$ respectively) are functions of the abscissa ξ.

This study can be easily extended to a beam of arbitrary length L according to Appendix A.

2 NAVIER–BERNOULLI BEAM MODEL

This model, for the case of non-existence of body forces is described by the following boundary value problem:

$$\frac{d^2}{d\xi^2}\left(\overline{EI}\frac{d^2\bar{w}}{d\xi^2}\right) = 0 \qquad \xi \in (0,1) \tag{1}$$

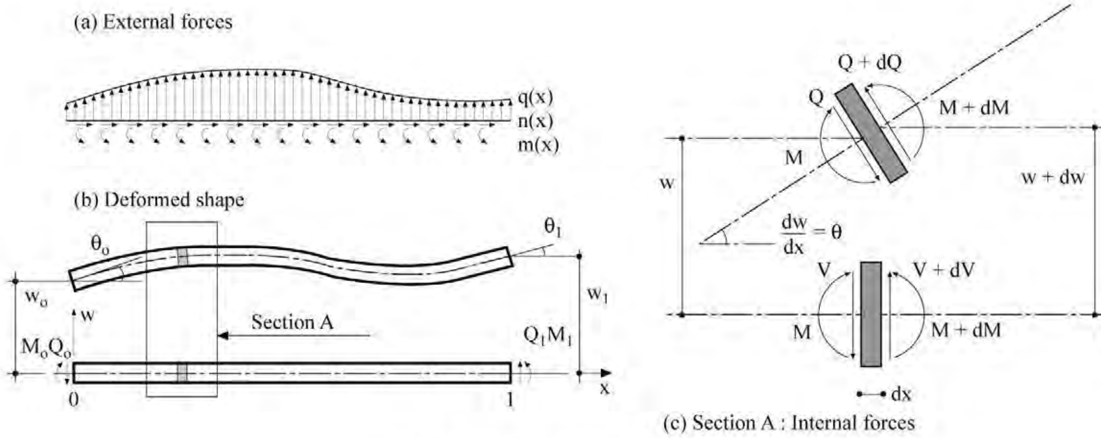

Fig. 1. Navier–Bernoulli beam.

and the boundary conditions at ξ_i for $i = 0,1$

$$\bar{w} = \bar{w}_i \quad \text{or} \quad -\frac{d}{d\xi}\left(\overline{EI}\frac{d^2\bar{w}}{d\xi^2}\right)_i = \bar{Q}_i \tag{2}$$

$$\frac{d\bar{w}}{d\xi} = \bar{\theta}_i \quad \text{or} \quad \left(\overline{EI}\frac{d^2\bar{w}}{d\xi^2}\right)_i = \bar{M}_i$$

where $\xi_0 = 0$ and $\xi_1 = 1$, and $\overline{w}$ and $\bar{\theta}$ are the unknown deflection and slope at the abscissa ξ of the beam. $\overline{w}_i$, $\bar{\theta}_i$, $\overline{Q}_i$ and $\overline{M}_i$ are data (see Fig. 1).

The solution of this problem is

$$\bar{w}(\xi) = A_0 + A_1\xi + A_2\mathcal{F}_1(\xi) + A_3\mathcal{F}_2(\xi) = \underline{\mathcal{P}}(\xi)\underline{A} \tag{3}$$

where

$$\underline{\mathcal{P}}(\xi) = \left(1 \;\; \xi \;\; \mathcal{F}_1(\xi) \;\; \mathcal{F}_2(\xi)\right) \tag{4}$$

$$\underline{A}^T = (A_0, A_1, A_2, A_3)$$

$$\mathcal{F}_1(\xi) = \int_0^{\xi}(\xi - u)\frac{du}{\overline{EI}(u)}, \quad \mathcal{F}_2(\xi) = \int_0^{\xi}(\xi - u)\frac{u\,du}{\overline{EI}(u)}$$

The arbitrary constants A_i can be obtained in terms of the essential boundary conditions, i.e.

$$\underline{d} = G_d\underline{A} \tag{5}$$

where

$$\underline{d}^T = (\bar{w}_0, \bar{\theta}_0, \bar{w}_1, \bar{\theta}_1) \tag{6}$$

$$\Delta = I_0 I_2 - I_1^2$$

$$\bar{G}_d = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & (I_0 - I_1) & (I_1 - I_2) \\ 0 & 1 & I_0 & I_1 \end{pmatrix}$$

and

$$I_0 = \int_0^1 \frac{d\xi}{\overline{EI}(\xi)}, \quad I_1 = \int_0^1 \xi\frac{d\xi}{\overline{EI}(\xi)}, \quad I_2 = \int_0^1 \xi^2\frac{d\xi}{\overline{EI}(\xi)} \tag{7}$$

The deflections can be expressed in terms of the shape functions:

$$\bar{w}(\xi) = \underline{\mathcal{P}}(\xi)\bar{G}_d^{-1}\bar{d} = \bar{N}^w(\xi)\bar{d} \tag{8}$$

where

$$\underline{N}^w(\xi) = (N_1^w, N_2^w, N_3^w, N_4^w) \tag{9}$$

$$\underline{G}_d^{-1} = \frac{1}{\Delta}\begin{pmatrix} \Delta & 0 & 0 & 0 \\ 0 & \Delta & 0 & 0 \\ -I_1 & -I_2 & I_1 & -(I_1 - I_2) \\ I_0 & I_1 & -I_0 & (I_0 - I_1) \end{pmatrix} \tag{10}$$

and

$$N_1^w(\xi) = 1 - \frac{I_1}{\Delta}\mathcal{F}_1 + \frac{I_0}{\Delta}\mathcal{F}_2 \tag{11}$$

$$N_2^w(\xi) = \xi - \frac{I_2}{\Delta}\mathcal{F}_1 + \frac{I_1}{\Delta}\mathcal{F}_2$$

$$N_3^w(\xi) = \frac{I_1}{\Delta}\mathcal{F}_1 - \frac{I_0}{\Delta}\mathcal{F}_2$$

$$N_4^w(\xi) = \frac{(I_2 - I_1)}{\Delta}\mathcal{F}_1 + \frac{(I_0 - I_1)}{\Delta}\mathcal{F}_2$$

The shape functions for the slope can be found from the previous functions:

$$\bar{\theta}(\xi) = \frac{d\bar{w}}{d\xi} = \underline{N}^{\theta}\underline{d} \tag{12}$$

$$\underline{N}^{\theta} = (N_1^{\theta}, N_2^{\theta}, N_3^{\theta}, N_4^{\theta})$$

$$N_i^{\theta} = \frac{dN_i^w}{d\xi} \quad (i = 1, 2, 3, 4)$$

From the consideration of natural boundary conditions the arbitrary constants A_i can be related to the boundary forces in the following way

$$\underline{p} = \underline{G_p}\underline{A} \tag{13}$$

where

$$\underline{p} = (-\bar{Q}_0, -\bar{M}_0, \bar{Q}_1, \bar{M}_1) \tag{14}$$

and

$$G_p = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \tag{15}$$

Then, the stiffness matrix is:

$$\underline{p} = \bar{G}_p \bar{G}_d^{-1} \underline{d} = \underline{K}\underline{d} \tag{16}$$

with

$$\underline{K} = \frac{1}{\Delta} \begin{pmatrix} I_0 & I_1 & -I_0 & (I_0 - I_1) \\ I_1 & I_2 & -I_1 & (I_1 - I_2) \\ -I_0 & -I_1 & I_0 & -(I_0 - I_1) \\ (I_0 - I_1) & (I_1 - I_2) & (I_1 - I_0) & (I_0 - 2I_1 + I_2) \end{pmatrix} \tag{17}$$

If there were body forces applied in the beam, the right hand side of eqn (1) should be replaced with a term containing these forces. The solution of the new boundary value problem permits to obtain the consistent equivalent forces.

The results for the case of the beam with constant section were obtained and they are summarised in Appendix B as a special case of the Timoshenko beam described in the following section (formulae (B.1.1) to (B.1.4)).

3 TIMOSHENKO BEAM MODEL

The boundary value problem describing this model is

$$\frac{d}{d\xi}(\overline{GA}\bar{\gamma}) = 0 \tag{18}$$

$$\frac{d}{d\xi}\left(\overline{EI}\frac{d\bar{\theta}}{d\xi}\right) + \overline{GA}\bar{\gamma} = 0$$

$$\xi \in (0,1) \quad \bar{\gamma} = \frac{d\bar{w}}{d\xi} - \bar{\theta} \quad G = \frac{E}{2(1+\nu)}$$

and the boundary conditions at ξ_i for i = 0,1

$$\bar{w} = \bar{w}_i \ \text{ or } \ \overline{GA}\left(\frac{d\bar{w}}{d\xi} - \bar{\theta}\right)_i = \bar{Q}_i \tag{19}$$

$$\bar{\theta} = \bar{\theta}_i \ \text{ or } \ \left(\overline{EI}\frac{d\bar{\theta}}{d\xi}\right)_i = \bar{M}_i$$

where $\bar{w}$ and $\bar{\theta}$ are the unknown deflection and rotation of the normal section at the abscissa ξ of the beam and A is the shear area. $\bar{w}_i, \bar{\theta}_i, \bar{Q}_i$ and $\bar{M}_i$ are data and $\xi_0 = 0$ and $\xi_1 = 1$ (see Fig. 2). Note that the rotation $\bar{\theta}$ is no longer the derivative of the deflection $\bar{w}$.

Similarly to the previous section, the shape functions, stiffness matrix, and consistent equivalent loads can be obtained. The results for the case of constant section are summarised in Appendix B (formulae (B.1.1) to (B.1.4)). It is interesting to point out that now, the shape function coefficients depend on the material properties given by the dimensionless coefficient μ, where

$$\mu = \frac{\overline{EI}}{\overline{GA}} \tag{20}$$

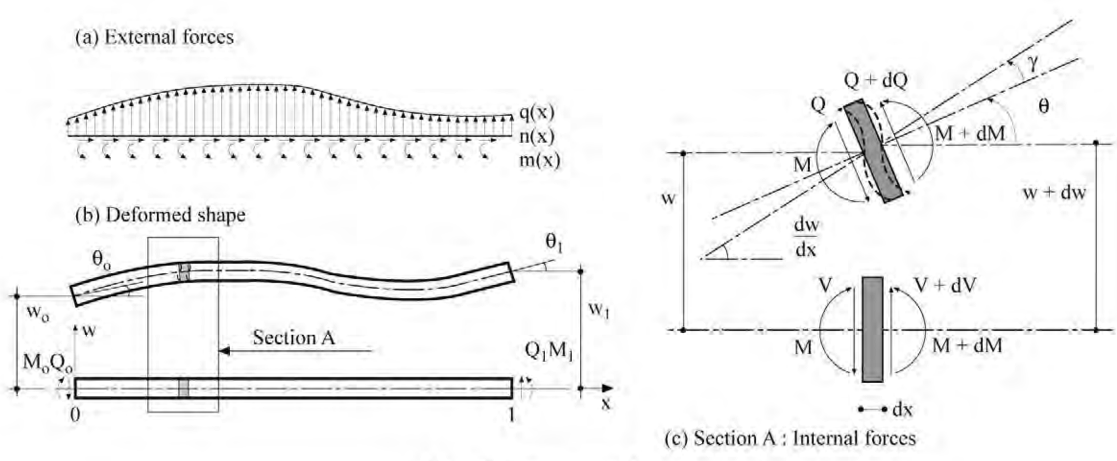

Fig. 2. Timoshenko beam.

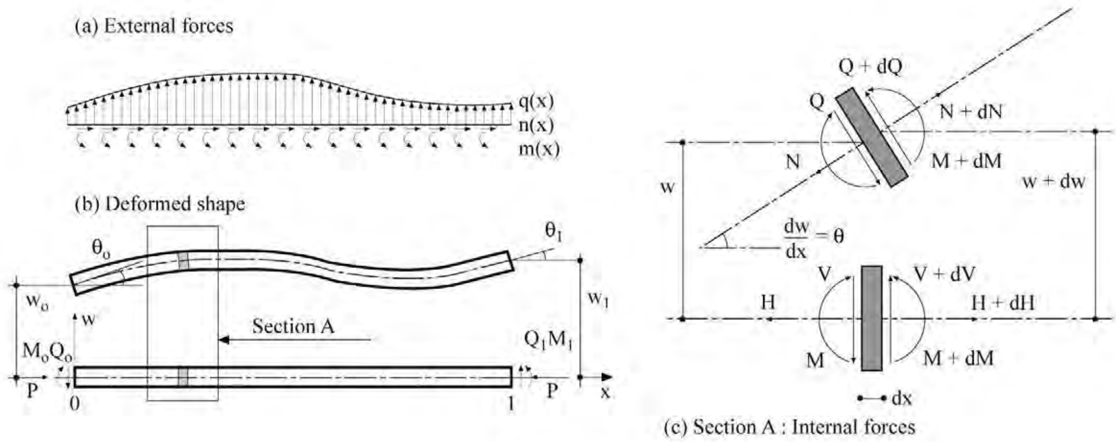

Fig. 3. Beam–column.

4 BEAM–COLUMN MODEL

It is assumed here that the axial forces of the beam–column are applied only at the beam ends. The boundary value problem is defined (see for example reference[7]) by:

$$\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}\left(\frac{\overline{EI}\mathrm{d}^2\bar{w}}{\mathrm{d}\xi^2}\right)+\bar{P}\left(\frac{\mathrm{d}^2\bar{w}}{\mathrm{d}\xi^2}\right)=0 \quad \xi\in(0,1) \tag{21}$$

and the boundary conditions at ξ_i for i = 0,1

$$\bar{w}=\bar{w}_i \;\text{ or }\; -\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\overline{EI}\frac{\mathrm{d}^2\bar{w}}{\mathrm{d}\xi^2}\right)_i-\bar{P}\left(\frac{\mathrm{d}\bar{w}}{\mathrm{d}\xi}\right)_i=\bar{Q}_i \tag{22}$$

$$\frac{\mathrm{d}\bar{w}}{\mathrm{d}\xi}=\bar{\theta}_i \;\text{ or }\; \left(\overline{EI}\frac{\mathrm{d}^2\bar{w}}{\mathrm{d}\xi^2}\right)_i=\bar{M}_i$$

where $\bar{P}$ is the axial compressive force applied at both ends of the beam and $\bar{w}$ and $\bar{\theta}$ are the unknown deflection and slope at the abscissa ξ of the beam. $\bar{w}_i$, $\bar{\theta}_i$, $\bar{Q}_i$ and $\bar{M}_i$ are data and $\xi_0 = 0$ and $\xi_1 = 1$ (see Fig. 3).

The shape functions, stiffness matrix, and equivalent force vectors for the case of constant section were obtained; they are presented in Appendix B as a special case of the general beam–column model with shear strain to be described later ((B.2.1) to (B.2.4)).

5 BEAM–COLUMN MODEL WITH SHEAR STRAIN

This model is a Timoshenko beam in which axial forces are applied at the beam ends similarly to the previous case. Then the problem is defined by the equations:

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\overline{GA}\bar{\gamma}-\bar{P}\bar{\theta}\right)=0 \tag{23}$$

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\overline{EI}\frac{\mathrm{d}\bar{\theta}}{\mathrm{d}\xi}\right)+\left(\overline{GA}+\bar{P}\right)\bar{\gamma}=0$$

where

$$\xi\in(0,1) \quad \bar{\gamma}=\frac{\mathrm{d}\bar{w}}{\mathrm{d}\xi}-\bar{\theta} \quad G=\frac{E}{2(1+\nu)}$$

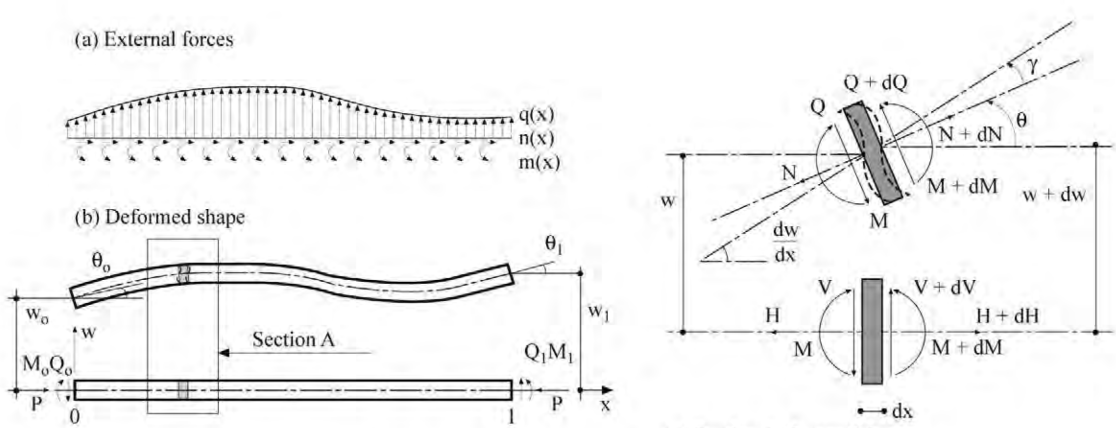

Fig. 4. Beam–column with shear strain.

and the boundary conditions at ξ_i for i = 0,1

$$\bar{w}=\bar{w}_i \quad \text{or} \quad \overline{GA}\left(\frac{d\bar{w}}{d\xi}-\bar{\theta}\right)_i - \bar{P}\bar{\theta}_i=\bar{Q}_i \tag{24}$$

$$\bar{\theta}=\bar{\theta}_i \quad \text{or} \quad \left(\overline{EI}\frac{d\bar{\theta}}{d\xi}\right)_i=\bar{M}_i$$

where A is the shear area, $\bar{P}$ is the axial compressive force applied at both ends of the beam, and $\bar{w}$ and $\bar{\theta}$ are the unknown deflection and rotation of the normal section at the abscissa ξ of the beam. $\bar{w}_i$, $\bar{\theta}_i$, $\bar{Q}_i$ and $\bar{M}_i$ are data and $\xi_0 = 0$ and $\xi_1 = 1$ (see Fig. 4). In Appendix B the shape functions, stiffness matrix and equivalent forces vector for the constant section situation are presented (formulae (B.2.1) to (B.2.4)).

It can be observed that in all the previous models the shape functions satisfy the following well known properties in the limit i.e.:

1. Interpolation
2. Rigid body movements
3. Constant strain properties

6 FINITE ELEMENT APPROXIMATIONS

6.1 Introduction

An alternative to the earlier described approach is to transform the corresponding boundary value equations into weak forms that are suitable for application of the FEM. In this case the following results are reached:

Stiffness matrix for Navier–Bernoulli and Timoshenko beam models:[8]

$$\underline{K}=\int_0^1 \underline{B}^T\underline{D}\underline{B}d\xi \tag{25a}$$

Stiffness matrix for beam–column model

$$\underline{K}=\int_0^1 \underline{B}^T\underline{D}\underline{B}d\xi-\bar{P}\int_0^1 \underline{N}^{\theta^T}\underline{N}^{\theta}d\xi \tag{25b}$$

Stiffness matrix for beam–column model with shear strain

$$\underline{K}=\int_0^1 \underline{B}^T\underline{D}\underline{B}d\xi-\bar{P}\int_0^1\left[\left(\frac{d\underline{N}^w}{d\xi}\right)^T\underline{N}^{\theta} + \underline{N}^{\theta^T}\left(\frac{d\underline{N}^w}{d\xi}\right)-\underline{N}^{\theta^T}\underline{N}^{\theta}\right]d\xi \tag{25c}$$

Consistent force vector

$$\underline{f}_{eq}=\int_0^1 \underline{N}^{w^T}\bar{q}\,d\xi+\int_0^1 \underline{N}^{\theta^T}\bar{m}d\xi \tag{26}$$

where the matrices $\underline{B}$ and $\underline{D}$ are dependent on the model under consideration and $\bar{q}$ and $\bar{m}$ are the distributed vertical forces and moments along the beam (see Fig. 1).

In the case of the Navier–Bernoulli beam and in the beam–column model, the expressions for matrices $\underline{B}$ and $\underline{D}$ can be written as follows:

$$\underline{B}=\left(\frac{d^2\underline{N}^w}{d\xi^2}\right) \quad \underline{D}=\overline{EI} \tag{27}$$

For the Timoshenko beam and the beam–column model with shear strain the expressions for the earlier mentioned matrices become:

$$\underline{B}=\begin{pmatrix}\left(\frac{d\underline{N}^{\theta}}{d\xi}\right)\\ \left(\frac{d\underline{N}^w}{d\xi}-\underline{N}^{\theta}\right)\end{pmatrix} \quad \underline{D}=\begin{pmatrix}\overline{EI} & 0\\ 0 & \overline{GA}\end{pmatrix} \tag{28}$$

If the shape functions $\underline{N}$ correspond to the ones obtained in the previous sections for the beam of constant section and are used in eqns (25) and (26), then the results for $\underline{K}$ and f_{eq} are identical to those presented in Appendix B.

However these shape functions can be introduced in the formula even for the cases of beams with variable section and the result obtained then represents a good approximation of the exact solution, which in many cases cannot be analytically found.

In order to estimate the accuracy of this type of approximation, the stiffness matrix of some constant section beam models will be computed using eqns. (25) with shape functions corresponding to different and simpler beam models. Namely, two cases will be analysed:

(a) Beam–column model using Hermitian polynomials as shape functions.
(b) Beam–column model with shear strain using the shape functions of the Timoshenko beam model.

6.2 Beam–column model

The expression eqn (25a) for the stiffness matrix becomes now:

$$\underline{K}=\overline{EI}\int_0^1\left\{\left(\frac{d^2\underline{N}^w}{d\xi^2}\right)^T\frac{d^2\underline{N}^w}{d\xi^2}-\alpha^2\underline{N}^{\theta^T}\underline{N}^{\theta}\right\}d\xi$$
$$=\underline{K}_0-\alpha^2\underline{K}_G \tag{29}$$

where $\underline{N}^w$ and $\underline{N}^{\theta}$ are the shape functions vectors defined in (B.1.1), for the special case of the Navier Bernoulli model, and α is given by the following expression:

$$\alpha=\sqrt{\frac{\bar{P}}{\overline{EI}}} \tag{30}$$

$\underline{K}_0$ represents the stiffness matrix of Navier–Bernoulli beam (B.1.2) (for the particular case of Navier Bernoulli model), and the expression for $\underline{K}_G$ is as follows:

$$\underline{K}_G=\frac{\overline{EI}}{30}\begin{pmatrix}36 & 3 & -36 & 3\\ 3 & 4 & -3 & -1\\ -36 & -3 & 36 & -3\\ 3 & -1 & -3 & 4\end{pmatrix}. \tag{31}$$

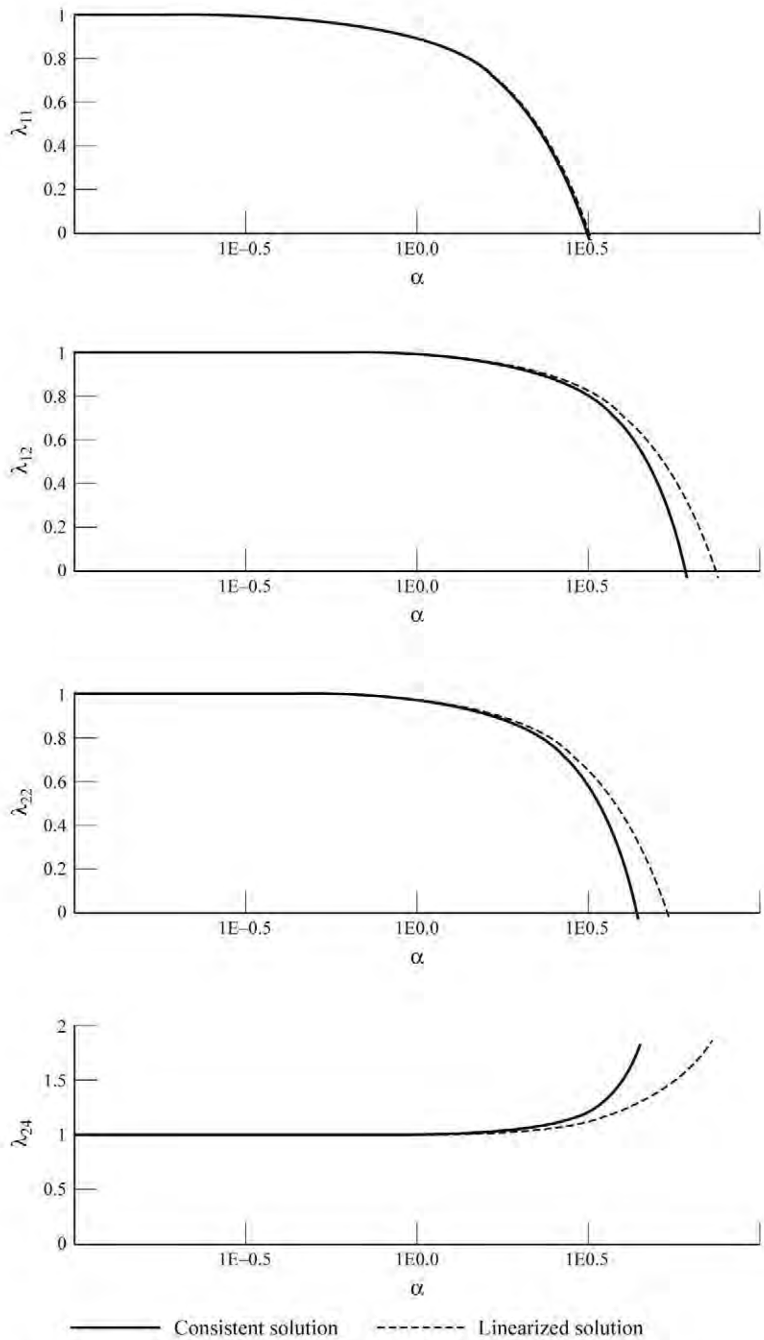

Fig. 5. Coefficients of the stiffness matrices of the beam column in function of the load.

It is interesting to compare the coefficients of this approximate stiffness matrix with those of the 'exact' or consistent one, derived in Section 4. This comparison will be carried out as a function of the dimensionless coefficient α. In order to normalize the stiffness matrix coefficients for different values of α they are divided by the corresponding ones for $\alpha = 0$, i.e. the following quotients λ_{ij} are introduced:

$$\lambda_{ij} = \frac{k_{ij}(\alpha)}{k_{ij}(0)} = \frac{k_{ij}^3}{k_{ij}^1} \tag{32}$$

with $\underline{K} = \{k_{ij}\}$, and, k_{ij}^1 and k_{ij}^3, which represent the coefficients k_{ij} of the stiffness matrices of models 1 (Navier–Bernoulli beam) and 3 (Beam–column), respectively, which can be obtained as particular cases in the tables in Appendix B.

In Fig. 5, two sets of values of λ_{ij} are represented, in a semilogarithmic scale. The first one corresponds to the coefficients $k_{ij}(\alpha)$, given in (B.2.2) for the particular case of the beam column without shear strain, which can also be calculated consistently using eqn (25) with the shape functions (B.2.1) for that particular case, and the second one corresponds to those coefficients but calculated according to the approximate formula eqn (29).

6.3 Beam–column model with shear strain

Similarly to the previous case, the stiffness matrix for a beam element becomes:

$$\underline{K} = \overline{EI}\int_0^1\left\{\left(\frac{\mathrm{d}\underline{N}^\theta}{\mathrm{d}\xi}\right)^{\mathrm{T}}\frac{\mathrm{d}\underline{N}^\theta}{\mathrm{d}\xi} + \frac{1}{\mu}\left(\frac{\mathrm{d}\underline{N}^{\mathrm{w}}}{\mathrm{d}\xi}-\underline{N}^\theta\right)^{\mathrm{T}}\left(\frac{\mathrm{d}\underline{N}^{\mathrm{w}}}{\mathrm{d}\xi}-\underline{N}^\theta\right) - \alpha^2\left[\left(\frac{\mathrm{d}\underline{N}^{\mathrm{w}}}{\mathrm{d}\xi}\right)^{\mathrm{T}}\underline{N}^\theta + \underline{N}^{\theta\mathrm{T}}\left(\frac{\mathrm{d}\underline{N}^{\mathrm{w}}}{\mathrm{d}\xi}\right) - \underline{N}^{\theta\mathrm{T}}\underline{N}^\theta\right]\right\}\mathrm{d}\xi = \underline{K}_0 - \alpha^2\underline{K}_{\mathrm{G}} \quad (33)$$

with

$$\alpha = \sqrt{\frac{\overline{P}}{\overline{EI}}},\quad \mu = \frac{\overline{EI}}{\overline{GA}} \quad (34)$$

where $\underline{N}^{\mathrm{w}}$ and $\underline{N}^\theta$ are the shape functions vectors defined in (B.1.1), $\underline{K}_0$ corresponds here to the stiffness matrix of Timoshenko beam (B.1.2), and the $\underline{K}_{\mathrm{G}}$ can be expressed as follows:

$$\underline{K}_{\mathrm{G}} = \overline{EI}\begin{pmatrix} c_1 & c_2 & -c_1 & c_2 \\ c_2 & c_3 & -c_2 & c_4 \\ -c_1 & -c_2 & c_1 & -c_2 \\ c_2 & c_4 & -c_2 & c_3 \end{pmatrix} \quad (35)$$

where

$$c_1 = \frac{36}{30\Delta_0^2}(1+20\mu) \quad (36)$$

$$c_2 = \frac{1}{\Delta_0^2}\left(\frac{1}{10}-72\mu^2\right)$$

$$c_3 = \frac{1}{\Delta_0^2}\left(2\mu[1-12\mu]+\frac{2}{15}\right)$$

$$c_4 = -\frac{1}{\Delta_0^2}\left(2\mu[24\mu+1]+\frac{1}{30}\right)$$

$$\Delta_0 = 1+12\mu\,.$$

The coefficients of the stiffness matrix $\underline{K}$ obtained in (33) will be next compared to their respective ones obtained in Section 5, in function of the dimensionless parameters α and μ. As in the beam column model without shear strain, it is interesting to introduce the following coefficients λ_{ij}, where

$$\lambda_{ij} = \frac{k_{ij}(\alpha,\mu)}{k_{ij}(0,\mu)} = \frac{k^4_{ij}(\alpha,\mu)}{k^2_{ij}(0,\mu)} \quad (37)$$

with $\underline{K} = \{k_{ij}\}$, and k_{ij} can be k^2_{ij} or k^4_{ij}. These represent the coefficients k_{ij} of the stiffness matrices of models 2 (Timoshenko beam) and 4 (Beam–column with shear strain), respectively.

In Fig. 6, six sets of values of λ_{ij} are represented in a semilogarithmic scale: three different values of the dimensionless parameter μ are chosen and for each of them, the two sets of values of λ_{ij} are plotted in function of α. Similarly to paragraph 6.2, from these two sets, the first one corresponds to the coefficients $k_{ij}(\alpha,\mu)$, given in (B.2.2), that can be calculated consistently using eqn (25) with the shape functions in (B.2.1), and the second one corresponds to those coefficients but calculated according to the approximate eqn (33).

6.4 Numerical stability

Typically, the use of the Timoshenko and the beam column with shear strain models is limited to a range of beams with a dimensionless parameter μ greater than a critical value. This critical value is dependent on the computer word length, and in general it is not possible, with these two models, to simulate the limit cases represented by the Navier–Bernoulli and the beam column models, because numerical difficulties arise in the computation of the structural response.

The use of the previously presented consistent shape functions in the F.E. formulation practically avoids these numerical problems produced by the shear overstiffness, i.e., the well-known locking phenomenon.

In Fig. 7 the dimensionless deflection:

$$w^* = \frac{w}{\left(\dfrac{VL^3}{3EI}\right)}$$

of a cantilever beam with $\nu = 0$, subjected to a concentrated load V at its tip is represented for different values of the axial load P given as a fraction of the Euler critical load of the cantilever. These results were obtained using two different computer word lengths, namely, single precision real values (i.e. occupying 4 bytes) and double precision real values (occupying 8 bytes). It can be observed that with single precision, the results are deteriorated beyond $\mu = 10^{-10}$, but for double precision this critical value of μ is not reached even for small values such as 10^{-20} in all analysed cases of the axial load P.

6.5 Comments on the results

The approximation reached by the use of the typical Finite Element methods, i.e. by using polynomial expansions for the field variables, was studied in several instances. It is possible to obtain exact or very approximate solutions when the exact or consistent shape functions are polynomials. That means that the values obtained with the Navier Bernoulli model coincide with the ones from the FEM. The same with some remarks can be applied to the Timoshenko model as was pointed out in[2]. Here the

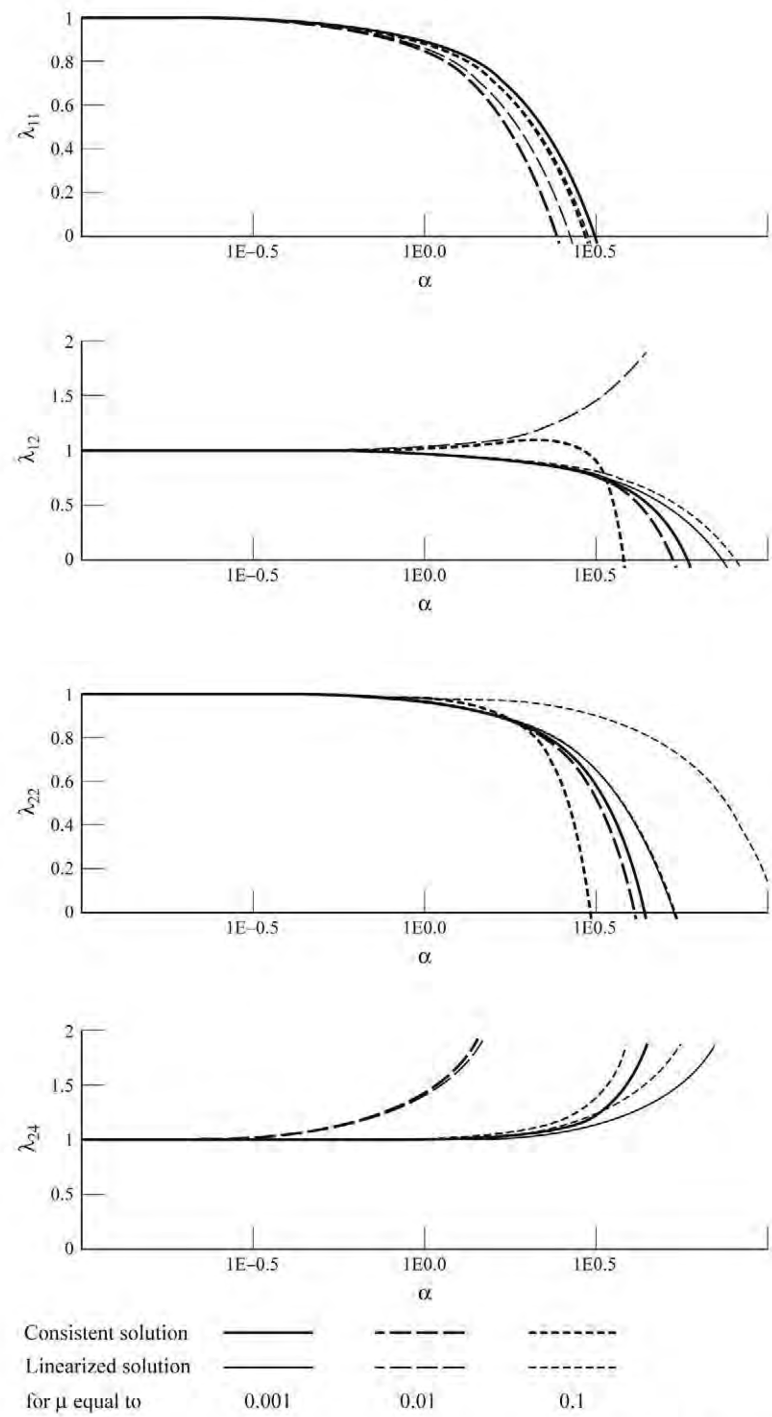

Fig. 6. Coefficients of the stiffness matrices of the beam column with shear strain in function of the load.

results of the two last models are compared, namely beam–column and beam–column with shear strain.

In the case of the beam column, from Fig. 5 it can be observed that in general the stiffness matrix obtained for the linearized solution is stiffer than the consistent one for values of α greater than $10^{0.5}$. However, the linearized coefficient k_{11} has a negligible difference from the consistent one for the range of normal values of α. The largest differences between coefficients of both stiffness matrices occur for coefficients k_{22} and k_{24} for large values of α.

Similarly, in Fig. 6, the coefficients of the consistent and linearized stiffness matrices of the beam column with shear strain are compared for three different shear strain levels. In general, for μ less than 0·001 the comments given previously for the beam column without shear strain are valid. For very large values of the shear strain the differences between coefficients k_{11} moderately

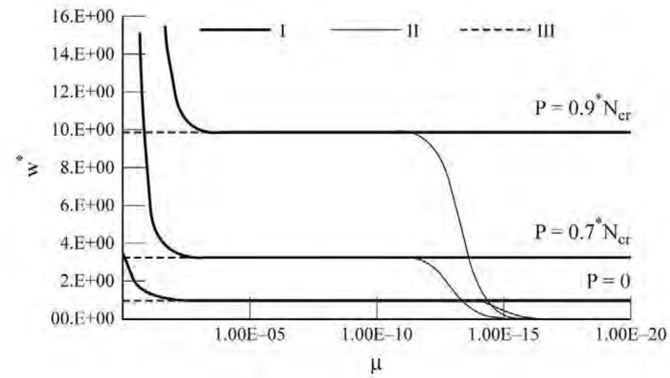

Fig. 7. Cantilever beam of length L subjected to an axial load P and a lateral load V at its tip (material properties EI and GA):
$w^* = \frac{w}{\left(\frac{VL^3}{3EI}\right)} = \frac{\bar{w}}{\left(\frac{V}{3\overline{EI}}\right)}$, $\mu = \frac{EI}{GAL^2} = \frac{\overline{EI}}{GA}$, $N_{cr} = \frac{\pi^2 EI}{4L^2} = \frac{\pi^2 \overline{EI}}{4}$ where w represents the vertical deflection under the load at the tip.
In addition: I: w* computed in double precision. II: w* computed in single precision. III: w* computed in double precision neglecting the effect of shear strain.

increase, but these differences become very large for k_{22} and k_{24}.

7 CONCLUSIONS

The use of shape functions derived from the solutions of the homogeneous differential equations governing a given beam model in the FEM (these functions are not necessarily polynomials), leads to results that are much more accurate than those obtained with the standard polynomial functions of the FEM when one element is used.

The consistent formulation avoids some of the inconsistencies reported in the literature concerning the use of finite elements to model beams with shear strain.

The consistent shape functions determined may also be used to obtain the vectors and matrices of elements with longitudinal variation of the cross sections using the Finite Element technique. The results found in this way, although approximate, would be more accurate than those obtained with the standard polynomial shape functions.

However, the determination of the consistent shape functions demands an important programming and computing effort for each one of the different beam models considered, unlike standard polynomial functions which are much more versatile.

APPENDIX A

The four consistent beam models considered were presented in the main text for a beam of unity length. The abscissa, ξ, is equal to:

$$\xi = \frac{x}{L}$$

where x is the abscissa of the general beam of length L.

The other variables used in the beam of unitary length are denoted in the paper by an overline. They are equal to the corresponding variables of the general beam of length L, except for the following ones:

$$\bar{w} = \frac{w}{L}, \quad \overline{EI} = \frac{EI}{L^2}, \quad \overline{M} = \frac{M}{L}, \quad \bar{q} = qL, \quad \bar{n} = nL \tag{A1}$$

where w, EI, M, q and n are the deflection, flexural stiffness, end moment and distributed vertical and horizontal forces, respectively of the beam of length L (E is Young's modulus and I is the moment of inertia).

These expressions permit to transform the expressions of the unitary length beam into a general form for a beam of length L.

APPENDIX B

In the following tables, the shape functions, stiffness matrices and consistent load vectors for two load cases relevant to the four consistent beam models are presented.

The shape functions correspond to a beam of unity length and satisfy the following equations:

$$w(\xi) = \underline{N}^w(\xi)\underline{d} \tag{B1.5}$$

and

$$\theta(\xi) = \underline{N}^\theta \underline{d} \tag{B1.6}$$

where

$$\underline{d}^T = (w_0, \theta_0, w_1, \theta_1)\,. \tag{B1.7}$$

The extension of these results to the general beam of length L follow the same expressions as given in Appendix A.

Table 1. Timoshenko beam. The Navier Bernoulli beam formulae are found from the Timoshenko beam by assuming $\mu = 0$ and $\Delta_0 = 1$.

Timoshenko beam	
Shape functions	
Deflection $\overline{w}$ shape functions	
$N_1^w(\xi) = \frac{1}{\Delta_0}((1+12\mu)-12\mu\xi-3\xi^2+2\xi^3)$ $N_2^w(\xi) = \frac{1}{\Delta_0}((1+6\mu)\xi-(2+6\mu)\xi^2+\xi^3)$ $N_3^w(\xi) = \frac{1}{\Delta_0}(12\mu\xi+3\xi^2-2\xi^3)$ $N_4^w(\xi) = \frac{1}{\Delta_0}((-6\mu)\xi+(6\mu-1)\xi^2+\xi^3)$	**[B.1.1.a]**
Rotation $\overline{\theta}$ shape functions	
$N_1^\theta(\xi) = \frac{1}{\Delta_0}(-6\xi+6\xi^2)$ $N_2^\theta(\xi) = \frac{1}{\Delta_0}((1+12\mu)-2(2+6\mu)\xi+3\xi^2)$ $N_3^\theta(\xi) = \frac{1}{\Delta_0}(6\xi-6\xi^2)$ $N_4^\theta(\xi) = \frac{1}{\Delta_0}(2(6\mu-1)\xi+3\xi^2)$	**[B.1.1.b]**
Stiffness matrix	
$\underline{K} = \frac{\overline{EI}}{\Delta_0}\begin{pmatrix} 12 & 6 & -12 & 6 \\ 6 & 4(1+3\mu) & -6 & 2(1-6\mu) \\ -12 & -6 & 12 & -6 \\ 6 & 2(1-6\mu) & -6 & 4(1+3\mu) \end{pmatrix}$	**[B.1.2]**
Consistent force vectors	
Distributed vertical force $\overline{q}$	
$\overline{q}\left(-\left(\frac{1}{2}\right) ; -\left(\frac{1}{12}\right) ; -\left(\frac{1}{2}\right) ; \left(\frac{1}{12}\right)\right)$	**[B.1.3]**
Distributed moment $\overline{m}$	
$\frac{\overline{m}}{\Delta_0}(1 ; -6\mu ; -1 ; -6\mu)$	**[B.1.4]**

Stiffness matrices and load vectors correspond also to a beam of unity length. They satisfy the equation:

$$\underline{p} = \underline{Kd} \tag{B1.8}$$

Vector $\underline{p}$ is an end load vector (see Fig. 1). The consistent load vectors presented in this appendix are a function of a distributed load or moment.

The following notation was used in the Table 1 and Table 2: E is Young's modulus and, I the moment of inertia.

For the Timoshenko beam model, specific notation was used:

$$\Delta_0 = 1 + 12\mu \quad \mu = \frac{\overline{EI}}{GA} = \frac{EI}{GAL^2} \tag{B2.5}$$

where G is the shear modulus and A the shear area. For the ordinary Navier Bernoulli beam $\mu = 0$ and $\Delta_0 = 1$.

For the beam–column model the following notation is applied:

$$\alpha = \sqrt{\frac{\overline{P}}{\overline{EI}}} = \sqrt{\frac{P}{EI}}L \quad c = \cos\alpha$$

$$s = \sin\alpha \quad \Delta_1 = 2(1-c) - s\alpha \tag{B2.6}$$

where $P = \overline{P}$ is the applied axial load, and if shear strain is included then the new variables are introduced:

$$\bar{\alpha} = \alpha\sqrt{1+\alpha^2\mu} \quad \bar{c} = \cos\bar{\alpha} \quad \bar{s} = \sin\bar{\alpha}$$

$$\Delta_2 = 2(1-\bar{c})(1+\alpha^2\mu) - \bar{s}\bar{\alpha} \tag{B2.7}$$

Table 2. Beam–column with shear strain. The beam–column without shear strain expressions are found by assuming $\mu = 0$.

Beam column with shear strain

Shape functions

Deflection $\overline{w}$ shape functions

$$N_1^w(\xi) = \frac{1}{\Delta_2}\left([(1+\alpha^2\mu)(1-\bar{c}) - \bar{s}\,\bar{\alpha}] + \bar{s}\,\bar{\alpha}\xi + [(1+\alpha^2\mu)(1-\bar{c})]\cos(\bar{\alpha}\xi) - [\bar{s}(1+\alpha^2\mu)]\sin(\bar{\alpha}\xi)\right)$$

$$N_2^w(\xi) = \frac{\bar{\alpha}}{\alpha^2\Delta_2}\left([\bar{c}\,\bar{\alpha} - \bar{s}(1+\alpha^2\mu)] + (1-\bar{c})\bar{\alpha}\xi - [\bar{c}\,\bar{\alpha} - \bar{s}(1+\alpha^2\mu)]\cos(\bar{\alpha}\xi) + [(1-\bar{c})(1+\alpha^2\mu) - \bar{s}\,\bar{\alpha}]\sin(\bar{\alpha}\xi)\right)$$

$$N_3^w(\xi) = \frac{1}{\Delta_2}\left([(1+\alpha^2\mu)(1-\bar{c})] - \bar{s}\,\bar{\alpha}\xi - [(1+\alpha^2\mu)(1-\bar{c})]\cos(\bar{\alpha}\xi) + [\bar{s}(1+\alpha^2\mu)]\sin(\bar{\alpha}\xi)\right)$$

$$N_4^w(\xi) = \frac{\bar{\alpha}}{\alpha^2\Delta_2}\left([\bar{s}(1+\alpha^2\mu) - \bar{\alpha}] + (1-\bar{c})\bar{\alpha}\xi - [\bar{s}(1+\alpha^2\mu) - \bar{\alpha}]\cos(\bar{\alpha}\xi) - [(1+\alpha^2\mu)(1-\bar{c})]\sin(\bar{\alpha}\xi)\right)$$

[B.2.1.a]

Rotation $\overline{\theta}$ shape functions

$$N_1^\theta(\xi) = \frac{\alpha^2}{\bar{\alpha}\Delta_2}\left(\bar{s}(1+\alpha^2\mu) - [(1+\alpha^2\mu)(1-\bar{c})]\sin(\bar{\alpha}\xi) - [\bar{s}(1+\alpha^2\mu)]\cos(\bar{\alpha}\xi)\right)$$

$$N_2^\theta(\xi) = \frac{1}{\Delta_2}\left((1-\bar{c})(1+\alpha^2\mu) + [\bar{c}\,\bar{\alpha} - \bar{s}(1+\alpha^2\mu)]\sin(\bar{\alpha}\xi) + [(1-\bar{c})(1+\alpha^2\mu) - \bar{s}\,\bar{\alpha}]\cos(\bar{\alpha}\xi)\right)$$

$$N_3^\theta(\xi) = \frac{\alpha^2}{\bar{\alpha}\Delta_2}\left(-\bar{s}(1+\alpha^2\mu) + [(1+\alpha^2\mu)(1-\bar{c})]\sin(\bar{\alpha}\xi) + [\bar{s}(1+\alpha^2\mu)]\cos(\bar{\alpha}\xi)\right)$$

$$N_4^\theta(\xi) = \frac{1}{\Delta_2}\left((1-\bar{c})(1+\alpha^2\mu) - [\bar{\alpha} - \bar{s}(1+\alpha^2\mu)]\sin(\bar{\alpha}\xi) - [(1-\bar{c})(1+\alpha^2\mu)]\cos(\bar{\alpha}\xi)\right)$$

[B.2.1.b]

Stiffness matrix

$$\underline{K} = \overline{EI}\,\frac{\alpha^2}{\Delta_2}\begin{pmatrix} \bar{\alpha}\,\bar{s} & (1+\alpha^2\mu)(1-\bar{c}) & -\bar{\alpha}\,\bar{s} & (1+\alpha^2\mu)(1-\bar{c}) \\ (1+\alpha^2\mu)(1-\bar{c}) & \frac{\bar{\alpha}}{\alpha^2}(\bar{s}(1+\alpha^2\mu) - \bar{c}\,\bar{\alpha}) & -(1+\alpha^2\mu)(1-\bar{c}) & \frac{\bar{\alpha}}{\alpha^2}(\bar{\alpha} - \bar{s}(1+\alpha^2\mu)) \\ -\bar{\alpha}\,\bar{s} & -(1+\alpha^2\mu)(1-\bar{c}) & \bar{\alpha}\,\bar{s} & -(1+\alpha^2\mu)(1-\bar{c}) \\ (1+\alpha^2\mu)(1-\bar{c}) & \frac{\bar{\alpha}}{\alpha^2}(\bar{\alpha} - \bar{s}(1+\alpha^2\mu)) & -(1+\alpha^2\mu)(1-\bar{c}) & \frac{\bar{\alpha}}{\alpha^2}(\bar{s}(1+\alpha^2\mu) - \bar{c}\,\bar{\alpha}) \end{pmatrix}$$

[B.2.2]

Consistent force vectors

Distributed vertical force $\overline{q}$

$$\frac{\overline{q}}{2\alpha^2\Delta_2}\begin{pmatrix} -\alpha^2\Delta_2 \\ -[\bar{\alpha}^2(1+\bar{c}) + 2\Delta_2 - 2\bar{s}\,\bar{\alpha}(1+\alpha^2\mu)] \\ -\alpha^2\Delta_2 \\ [\bar{\alpha}^2(1+\bar{c}) + 2\Delta_2 - 2\bar{s}\,\bar{\alpha}(1+\alpha^2\mu)] \end{pmatrix}$$

[B.2.3]

Distributed moment $\overline{m}$

$$-\frac{\overline{m}}{\Delta_2}\begin{pmatrix} \bar{s}\,\bar{\alpha} - 2(1-\bar{c}) \\ (1-\bar{c})\alpha^2\mu \\ 2(1-\bar{c}) - \bar{s}\,\bar{\alpha} \\ (1-\bar{c})\alpha^2\mu \end{pmatrix}$$

[B.2.4]

where α and μ have already been defined. It is found that for the case of the beam column without shear strain:

$$\Delta_2 = \Delta_1,\ \bar{\alpha} = \alpha,\ \bar{c} = c \text{ and } \bar{s} = s \qquad \text{(B2.8)}$$

REFERENCES

1. Turner, M. J., Clough, R. W., Martin, H. and Topp, L. J., Stiffness and deflection analysis of complex structures. *J. Aeron. Sci.*, 1956, **23**, 805–823.
2. Friedman, Z. and Kosmatka, J. B., An improved two-node Timoshenko beam finite element. *Computers and Structures*, 1993, **47**(3), 473–481.
3. Nickell, R. E. and Secor, G. A., Convergence of consistently derived Timoshenko beam finite elements. *Int. J. Numer. Meth. Engng.*, 1972, **5**, 243–253.
4. Thomas, D. L., Wilson, J. M. and Wilson, R. R., Timoshenko beam finite elements. *J. Sound Vibr.*, 1973, **31**, 315–330.
5. Hughes, T. R. J., Taylor, R. L. and Kanoknukulchoii, W., A simple and efficient plate element for bending. *Int. J. Numer. Meth. Engng.*, 1977, **11**, 1529–1943.

6. Tessler, A. and Dong, S. B., On a hierarchy of conforming Timoshenko beam elements. *Comput. Struct.*, 1981, **14**, 335–344.
7. Samartín, A., Teoría elemental de la viga-columna con aplicación al estudio de la estabilidad elástica de estructuras. Publicación AE-84-4, E.T.S.I.C.C.P. de Santander, Universidad de Santander, 1984.
8. Zienckiewicz, O. C. and Taylor, R. L., *The Finite Element Method*, 4th edn. McGraw-Hill, New York, 1991.