

STABILITY OF SLENDER, AXISYMMETRIC LIQUID BRIDGES BETWEEN UNEQUAL DISKS

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1. Introduction

It is well known that, in the absence of gravity, a cylindrical liquid bridge bounded by two coaxial parallel end disks having the same radius, becomes unstable when the length of the liquid column exceeds the perimeter of the disks. If the length of the liquid bridge is smaller than the critical value, a bifurcation to unstable equilibrium shapes appears. The character of the bifurcation, as well as the influence of perturbations like small volume changes, disk rotation, and microgravitational effects, are being extensively studied [1-4]. The case

of unequal disks (fig. 1) has been also considered, equilibrium shapes have been analyzed in ref. [5], and numerical results concerning the variation, with disks ratio of minimum volume stability limit are graphically presented in ref. [1]. Some of these results are checked here through a perturbation analysis, the results obtained allowing the extension of the dynamical study performed in ref. [6] to the case of liquid bridges between unequal disks.

2. Bifurcation to equilibrium shapes

In the following, all lengths are made dimensionless with $R = (R_1 + R_2)/2$ and pressures with σ/R , σ being the surface tension. The equation governing the equilibrium interface shapes of the liquid bridge is obtained by expressing the equilibrium between surface tension and local pressure forces at the free surface of the liquid

$$-\frac{1}{2}(2S + S_z^2 - SS_{zz})(S + \frac{1}{4}S_z^2)^{3/2} + P = 0, \quad (1)$$

where P is a constant related to the origin of pressures and $S(z) = F^2(z)$ stands for the cross-sectional area of each slice of the liquid bridge. Two additional nondimensional parameters are introduced, the slenderness of the liquid bridge, $\Lambda = L/(2R)$, and the disks radius ratio $w = R_1/R_2$.

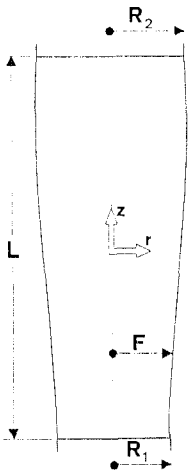


Fig. 1. Geometry and coordinate system for the liquid bridge.

Boundary conditions are

$$S(\pm A) = 1 \pm H, \quad (2)$$

where $H = (1 - w^2)/(1 + w^2)$, and the volume of liquid must be equal to that of a cylindrical liquid bridge having the same slenderness

$$\int_{-A}^A S(z) dz = 2A. \quad (3)$$

For convenience, let us introduce asymptotic expansions as in ref. [6]

$$\Lambda = \pi(1 + \epsilon^{1/2}\lambda_1 + \epsilon\lambda_2 + \epsilon^{3/2}\lambda_3 + \dots), \quad (4)$$

$$S = 1 + \epsilon^{1/2}s_1 + \epsilon s_2 + \epsilon^{3/2}s_3 + \dots, \quad (5)$$

$$P = 1 + \epsilon^{1/2}p_1 + \epsilon p_2 + \epsilon^{3/2}p_3 + \dots \quad (6)$$

and, to normalize boundary conditions, a new spatial coordinate is used

$$x = z/(1 + \epsilon^{1/2}\lambda_1 + \epsilon\lambda_2 + \epsilon^{3/2}\lambda_3 + \dots). \quad (7)$$

Before pursuing further it must be pointed out that, in order to reach the bifurcation to unstable equilibrium shapes, it is required to anticipate certain properties of the solution, that is, the order of magnitude of the parameter $H = \epsilon^n h$ appearing in boundary conditions. If $n = 1/2$ or 1, unequal-disks boundary condition will appear in the $\epsilon^{1/2}$ or ϵ problem, respectively; the first case gives $h = 0$, whereas in the second one linearized interface shapes are obtained. Therefore, H must be of $\epsilon^{3/2}$ order and thence, after introducing expressions (4)–(7) in eqs. (1), (2) and (3), and equating coefficients of equal power in ϵ , the following recursive sequence of problems results:

$\epsilon^{1/2}$ order

$$s_{1,x} + s_1 + 2p_1 = 0, \quad (8)$$

$$s_1(\pm\pi) = 0, \quad \int_{-\pi}^{\pi} s_1 dx = 0; \quad (9)$$

ϵ order

$$s_{2,x} + s_2 - \frac{1}{4}(s_{1,x}^2 + 2s_1 s_{1,x} + 3s_1^2) - 2\lambda_1 s_{1,x} + 2p_2 = 0, \quad (10)$$

$$s_2(\pm\pi) = 0, \quad \int_{-\pi}^{\pi} s_2 dx = 0; \quad (11)$$

$\epsilon^{3/2}$ order

$$s_{3,x} + s_3 - \frac{1}{2}(s_{1,x} s_{2,x} + s_1 s_{2,x} + s_{1,x} s_2 + 3s_1 s_2) - \frac{1}{8}[3s_{1,x}^2(s_{1,x} - s_1) - 3s_1^2 s_{1,x} - 5s_1^3] + 2(2\lambda_1^2 - \lambda_2)s_{1,x} + \frac{1}{2}\lambda_1(s_{1,x}^2 + 2s_1 s_{1,x} - 4s_{2,x}) + 2p_3 = 0, \quad (12)$$

$$s_3(\pm\pi) = \pm h, \quad \int_{-\pi}^{\pi} s_3 dx = 0. \quad (13)$$

The solution of (8) with conditions (9) is $p_1 = 0$, $s_1 = A_1 \sin x$, where A_1 is a unknown constant, and the solution of (10)–(11), after replacing s_1 by its expression, is $p_2 = A_1^2/8$, $\lambda_1 = 0$, $s_2 = A_2 \sin x$. The same procedure must be applied to the $\epsilon^{3/2}$ order problem, and s_3 results

$$s_3 = A_3 \sin x + A_4 \cos x + \frac{1}{2}A_1 A_2 - 2p_3 + \frac{1}{64}A_1^3 \sin 3x + \left(\frac{3}{16}A_1^3 + \lambda_2 A_1\right) x \cos x, \quad (14)$$

with $p_3 = A_1 A_2/4$ to meet volume requirements, whereas the fulfilment of boundary conditions yields $A_4 = 0$ and

$$-\left(\frac{3}{16}A_1^3 + \lambda_2 A_1\right)\pi = h, \quad (15)$$

which determines the amplitude of the interface deformation A_1 . The number of real roots of (15) depends on the sign of the discriminant, as it is well known from cubic equation solution, and bifurcation appears when this discriminant vanishes, that is, when

$$\lambda_2 = -(3/2)^{4/3}(h/2\pi)^{2/3}, \quad (16)$$

and the following stability limit results

$$\Lambda = \pi \left[1 - (3/2)^{4/3} (H/2\pi)^{2/3} \right]. \quad (17)$$

A comparison of the approximation of (17) with numerical results obtained by Martínez [1] is given in fig. 2.

It must be remarked that (17) represents a stability limit similar to that obtained by Vega and Perales [3] for cylindrical liquid bridges when subject to microgravitational forces acting parallel to the liquid bridge axis

$$\Lambda = \pi \left[1 - (3/2)^{4/3} \text{Bo}^{2/3} \right], \quad (18)$$

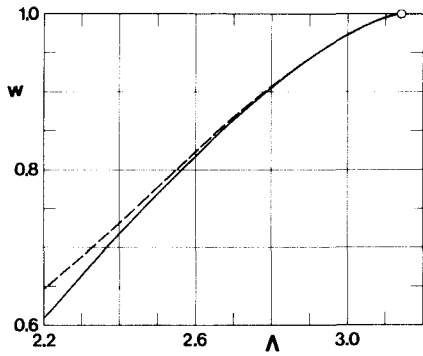


Fig. 2. Stability limit of axisymmetric liquid bridges between unequal disks: (---) calculated by Martínez [1]; (—) as given by eq. (17). The results correspond to configurations having the same volume as cylindrical liquid bridges of the same slenderness.

where Bo stands for the gravitational Bond number, $Bo = \rho g R^2 / \sigma$, ρ being the liquid density and g the acceleration due to microgravity ($\mathbf{g} = -g\mathbf{u}_z$).

In the case of liquid bridges between unequal disks, when microgravitational effects are taken into account, the term $-Bo z$ must be added to the left hand side of (1). According to (18), the Bond number may be expanded as $Bo = \epsilon^{3/2} b$, and, in consequence, microgravitational effects will only appear as a new addend, $2bx$, in eqs. (12) and (14) of the $\epsilon^{3/2}$ problem. Therefore, when the Bond number is considered, the fulfilment of boundary conditions leads to

$$-\left(\frac{3}{16}A_1^3 + \lambda_2 A_1\right)\pi + 2b\pi = h \quad (19)$$

instead of (15), and the following stability limit is obtained

$$\Lambda = \pi \left[1 - (3/2)^{4/3} (Bo - H/2\pi)^{2/3} \right]. \quad (20)$$

3. Conclusions

The first aspect to be pointed out concerns the results obtained in ref. [6]; the dynamical analysis there performed may be extended to the case of liquid bridges between unequal disks if, in expression (3.22) and followings in that paper, $b - h/(2\pi)$ is read instead of b . On the other hand, from the static point of view (20) indicates that, for a given Bond number, the critical slenderness increases if $H > 0$ (larger disk at the top), whereas the contrary occurs if $H < 0$ (small disk at the top). This behaviour agrees with experimental evidence; actually, gravity tends to bulge out the liquid column at the bottom region and to neck in at the upper one; therefore, placing the small disk at the top accentuates this necking effect whereas the large disk at the top causes the opposite effect.

Acknowledgement

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