

# Changing assessment methods: New rules, new roles

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## A B S T R A C T

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Over the past 20 years, the use of Computer Algebra Systems (CAS) has helped with the teaching of mathematics in engineering schools. However the traditional use of CAS only in math labs has led to a narrow view by the student: the CAS is an additional work, not included in the learning process. The didactic guidelines of the European Higher Education Area (EHEA) propose a new teaching-learning model based on competencies. We suggest the use of the CAS be adapted to the new rules. In this paper, we present a model for the integrated use of the CAS, and we describe and analyze two experiments carried out in the academic year 2011–2012. Our analysis suggests that the use of CAS in all learning and assessment activities has the potential to positively influence the development of competencies.

## 1. Introduction

The teaching of mathematics has drastically changed in recent years and the current teaching practices differ from what was being done 30 years ago. This change has taken place not only owing to the impact of new technologies in the classroom. There has also been a contribution from the changing legislative framework and the implementation of the educational model adapted to the European Higher Education Area (EHEA) <http://www.ehea.info>.

The didactic guidelines of the EHEA propose a new teaching and learning model based on competencies, with new roles for students, who then become the protagonists of their learning. The teachers must guide the students' work and evaluate not only their knowledge and skills but also, above all, their competencies.

The present paper aims to address the problem of knowing the best way of using a Computer Algebra System (CAS) in math teaching in engineering degrees within the framework of competence-based learning as well as the possible contribution of such a system to the process of evaluation of competencies.

Over the past two decades, the use of CAS has become common in mathematics courses within engineering studies and the literature reflects many experiences gained through their use (see [Marshall et al., 2012](#)). [Artigue \(2002\)](#) established the theoretical framework of the *instrumental genesis*. According to this theory, the use of CAS involves a process during which the object or artifact is turned into a mathematical instrument. This framework was completed by [Drijvers et al. \(2009\)](#) with the *instrumental orchestration*, for analyzing teaching practices.

The literature also reports experiences in the use of CAS in assessments (see, for example, [Thomson et al., 2009](#)). [MacAogáin \(2002\)](#) proposes a model aimed at measuring how CAS affect the solving of the different tasks in an exam. [Brown \(2001\)](#) concluded that *the introduction of CAS into examinations has the potential to allow the student to move away from an examination where the examiner controls the solution strategy to one in which the student control the solution strategy*.

The use of technology is gaining ground in curricula and assessment activities ([Meagher, 2000](#)) and it is now being allowed for use in authorities examinations, sometimes without modifying the tasks being evaluated. [Brown \(2010\)](#) analyzed three authorities' examinations and found that the testing of mechanical skills predominated both prior and after the introduction of the graphics calculator.

In recent years the use of CAS has been included in teaching-learning models based on competencies (see [Niss and Højgaard, 2011](#)). [García et al. \(2011a\)](#) described how the use of CAS in problem-solving promotes the development of generic competencies.

Here we propose a model of integration of CAS in the teaching of mathematics to engineering students within a learning framework based on competencies that involves their use as an element of constructive learning. This model is the result of our long experience. The theoretical background is based on the instrumental genesis theory ([Artigue, 2002](#)) and the toolbox approach ([García et al., 2009](#)). We propose the full integration of CAS in all the learning and assessment activities, including examinations. Two experiences, in two different environments, have been carried out, analyzed and discussed.

The paper is organized as follows. First we address general issues, offering a brief overview of the use of CAS and positing the theoretical framework of the model proposed. Then, we describe and analyze the two different experiences obtained after implementing the model. Finally, we offer some conclusions.

## **2. Learning mathematics and computer algebra systems**

### *2.1. History: twenty years teaching mathematics in a CAS environment*

A CAS can be defined as a software with numerical, graphic and symbolic capacities. Its origins can be found in the sixties, with systems such as REDUCE, MuMATH and MACSYMA (see [Davenport et al., 1988](#); [Moses, 2012](#)). In this period, the use of CAS was basically reduced to performing mathematical calculations for scientific research. Towards the end of the eighties, CAS such as DERIVE, *Mathematica*, MAPLE, etc., began to be used as teaching tools (see [Amrhein et al., 1997](#)). Slowly, versions that were easier to use and with better performances led CAS to become an important teaching instrument. Textbooks also began to include examples and exercises using CAS (see [Villa, 2011](#)). Around the same time, an interesting debate involving the curricula and CAS teaching methods took place.

Many studies have addressed the teaching of mathematics, above all at the secondary school level, in a CAS environment (see [Lagrange et al., 2003](#)). The studies also contain references to the use of CAS as functional and pedagogical tools in university teaching, above all in mathematics courses for engineering students. In [Marshall et al. \(2012\)](#) the authors conducted a comparative study, between

a review of the literature and an international survey, of the integration of CAS in the teaching of mathematics at university level, analyzing the factors involved, their strengths and their weaknesses. The authors concluded that the most frequent way of integrating CAS is by means of laboratory sessions in which the students perform tasks under the supervision of an instructor and they use the CAS as a functional and pedagogical tool. They also detected some barriers to full integration, such as the cost of commercial software licenses.

Currently, on-line CAS, such as WIRIS (<http://www.wiris.com>), are becoming increasingly popular. Moreover there are CAS that can be accessed from mobile phones, such as Wolfram Alpha (<http://www.wolframalpha.com>), which students are now beginning to use as an everyday tool. Drijvers and Trouche (2010) suggest that we should be aware that hand-held technology is no longer an isolated artifact but integrated in and articulated within a network of resources, particularly on-line resources.

To end this historical overview we comment on two problematic issues:

- Open or commercial software?: There is a debate concerning the use of free open-code or commercial software. Those who defend open software, such as Stallman (1999) and Joyner (2008), not only offer economic reasons but also, and above all, educational and ideological reasons. The open source free software offers: freedom to use it anywhere and for any purpose; freedom to study and adapt it to our needs and to distribute it to students in the comfort of their home. However, the maintenance of commercial software is better guaranteed and, additionally, the long tradition of certain systems of private licensing means that they have become so widely implemented that they are difficult to substitute.
- Assessments with CAS. There is a general agreement about the effectiveness of CAS in the learning process but their inclusion in assessment activities has aroused considerable controversy. The comparative study performed by Marshall et al. (2012) showed that more than 80% of teachers, respondents from an international survey, required the use of CAS in homework and assignment problems, but there are few instructors who allow the use of CAS in assessment activities. In addition, usually the assessments with CAS are reduced to solving some exercises in a mathematics laboratory.

Automatic assessment tools like STACK (<http://www.stack.bham.ac.uk>), supported by CAS, have also been developed with the aim to facilitate the on-line assessment (see Sangwin, 2004).

## 2.2. Here comes the future: EHEA

The historical process of the creation of a transnational higher education area points to a necessary idea stemming from the political conception of the European Union. The reasons supporting the need to create an EHEA can be summarized in terms of the following needs of Europe:

- (1) To increase the number of University degrees in Europe.  
To illustrate this need, it can be said that in 2002 the percentage of the active population with higher education in Europe was 21%, a figure clearly lower than the 43% seen in Canada or the 38% found in the USA.
- (2) To facilitate the free circulation of students and degrees in Europe.  
A common credit system has been set up based on ECTS, the European Credit Transfer System, with consensus that one ECTS stand for around 25 to 30 working hours, together with a system of comparable and compatible degrees, and a guarantee of common quality.
- (3) To make Europe a more attractive place to study and research.  
A challenge of EHEA is to increase the number of foreign students coming to Europe to study for a PhD.
- (4) To promote a change in the educational paradigm.  
The EHEA places the student at the center of the learning process, that is the best way to allow them to become autonomous and learn *how* to learn. Accordingly they will be better prepared to tackle issues inherent to a knowledge-based society.
- (5) To link higher education to life-long learning.

The university period is only the beginning of students' learning activity before the professional careers. Therefore the key to training is the development of competencies.

The road to the creation of the EHEA has been long and comprises several ministerial declarations (from Bologna (1999) to Budapest (2012), <http://www.ehea.info>) that have helped to uphold the process.

Currently, university teaching is becoming adapted to the new normative. The teaching framework has been modified in the past few years: teaching scenarios, tutorials, assessments, curricula and teaching methods. Again, to emphasize the inexorable advance in technology is permitting the wide use of open courses and learning content management systems, like Moodle (Modular Object-Oriented Dynamic Learning Environment, <http://www.moodle.org>).

However, it is also necessary to comment on the main difficulties that are arising along this road. In our opinion, the main problems that arise when teaching mathematics in engineering studies are as follows:

- The mathematics subjects in many engineering degrees have been reduced and standardized.
- The low level of mathematical knowledge for freshman students.
- The spectacular change in teaching materials, methods and resources has not been reflected in assessment techniques.

### 2.3. *The role of assessments*

Assessment is a core component for effective learning. The change from a teacher-centered instruction towards a learner-centered instruction and competencies based learning implies the development of new assessment methods. If teaching and learning are based on acquiring competencies, then assessments must determine the acquisition of these competencies. There is a strong relationship between learning and assessment: *What is assessed strongly influences what is learned.*

Arguments for introducing CAS in assessment activities are well established (Meagher, 2000; Thomson et al., 2009). Some authors have described the impact of the introduction of CAS in examinations (MacAogáin, 2002; Brown, 2001), and they concluded that using CAS gives students the opportunity to be more responsible for their own learning.

Moreover, assessment must be more than a summative assessment. A formative assessment with feedback is an important strategy which can help students to take control of their own learning and develop critical thinking. Virtual learning environments such as Moodle provide many opportunities for high quality feedback and formative assessment (see Limniou and Smith, 2012). CAS may play an important role in any model of formative assessment in mathematics courses among engineering students. For example, CAS is a very useful tool for problem-based learning in mathematics.

Despite this many instructors continue to feel that a good exam in mathematics should consist of a collection of problems to be solved with pencil and paper. However, a good model of formative assessment about mathematical competencies, consistent with the literature on student-centered learning (see Baartman et al., 2006; Niss and Højgaard, 2011), should include:

- Team work for solving problems and doing projects, because collaborative learning has a higher efficiency than individualistic learning method (Hsiung, 2010).
- Quizzes with feedback on-line.
- Solving written exercises or problems related to the real world, using aids and tools.
- Exams with free use of mathematical software allowing evaluation of mathematical competencies.

### 2.4. *A proposal for the integrated use of CAS*

We have more than 20 years of experience in the use of CAS in mathematical subjects for engineering students. Over the years we have used DERIVE, MAPLE, *Mathematica*, MAXIMA and MATLAB and we have seen that the developed activities are more important than the chosen CAS. We have usually

used CAS as tools for graphic support in instructors' explanations and in laboratory sessions in which students work following a list of previously established tasks.

For about 10 years we have been including, at least in some exams, certain exercises and problems to be solved with CAS. The marks in these exercises are quite similar to those of the other parts of the exam.

Recently we have observed that students do not see CAS as a learning tool that allows them to build upon their knowledge of mathematics but as an additional task with a low percentage in the final grade.

In recent years we have attempted to implement a more integrated use of CAS in the learning process by proposing small projects to be completed in team work (see [García et al., 2011b](#)) and the toolbox theory ([García et al., 2009](#)), encouraging students to use CAS to tackle different mathematical routines for use in different environments since, as stated by [Aguilera et al. \(2007\)](#), the use of the programming language of a CAS promotes creativity in mathematics and a better assimilation of concepts and algorithms and promotes the process of instrumental genesis ([Artigue, 2002](#)).

In order to get students to build good toolboxes it is necessary to allow their use in all types of environment. We have designed a teaching-learning model in which CAS is no longer a separate part restricted to the laboratory but is completely integrated in all teaching and assessment activities, including examinations.

The students should develop a working style that will allow them to advance in the building of their knowledge, not only within the framework of a given subject but also in successive academic years and in their later professional careers. Accordingly, the model is based on the following requirements:

- To integrate the use of CAS in all the learning and assessment activities.
- To encourage the students to build their own toolboxes.
- To promote the use of CAS, outside the sphere of the subject, for solving engineering problems.

For implementing this model it is necessary to select carefully the CAS to be used, according with its functionality, usability, accessibility (for home use) and professional diffusion (mainly in the mathematical subject related with technical topics).

It would seem easier to propose this model for use in advanced courses, but since most of our teaching is concentrated in the first years of university degrees, we decided to perform two experiments simultaneously. The first was in Linear Algebra in the first academic year, establishing an experimental group and a control group. The second was in an advanced course on mathematics addressing signal processing. In this case, the low number of students registered in the course did not allow the establishment of a control group, although it did favor fluid communication between teachers and students.

To evaluate the effectiveness of the proposal we designed a framework similar to that proposed by [Pierce and Stacey \(2004\)](#). We have analyzed the experiences using:

- Observation and evaluation of student's work.
- Surveys about mathematical and technical aspects.
- Surveys about personal aspects including the perception of the students as regards the development of generic competencies.

### 3. Two experiences

In this section we will describe two different experiences carried out involving two different topics: Linear Algebra and Signal Processing. The purpose of both experiences was to develop on students generic competencies (self learning, problem solving, team work, use of technology) together with specific mathematical competencies of each subject. This has been accomplished by promoting cooperative and autonomous work on students as well as the oriented use of mathematical software.

We explain in a more detailed form the results of the first experience, because it could be used in a general context.

### 3.1. Linear algebra for mechanical engineering

Linear Algebra for Mechanical Engineering (LAME) at the Polytechnic University in Madrid (UPM) is taught in the first semester of the first academic year, with 6 ECTS, equivalent to 156 hours of student work.

Instruction and learning are based on competencies. For the LAME subject we defined:

- Generic competencies: Self Learning, Problem Solving, Use of Technology, Team Work.
- Specific competency: On a successful completion of the course, the student should be able to use the appropriate mathematical tools provided by LAME and the appropriate software to solve problems in technological subjects.

In order to perform the experiment, during the 2011–2012 academic year, the subject was organized within two groups of students: the control group (47 students) and the experimental group (49 students). In both groups, the subject was taught over 15 weeks. Five hours per week are devoted to face-to-face activities. As a complement, the Moodle platform was used to announce the different activities and callings for exams and to distribute didactic material: students' guide, worksheets, bibliography, files for autonomous student's work etc.

The control group followed the subject, with theoretical and practical lectures and three practical sessions at the laboratory, using the CAS DERIVE.

In the experimental group, the subject was taught with full integration, in both teaching and assessment, of MAXIMA (<http://maxima.sourceforge.net/>). This CAS was chosen because it is a free and open source code, and the students have used it in face-to-face teaching and home-work.

Each week there was one session of theory lasting two hours, another two-hour session devoted to the solution of problems, and one hour working with MAXIMA. During this time, the teacher answers questions related to student's autonomous work, giving some feedback together with ideas and suggestions for solving problems with MAXIMA.

The basic teaching material used by the students is a textbook, with a CD containing utility files, including several MAXIMA files.

The use of MAXIMA was open, it was based on the tutorials of the textbook's CD. To enhance the self learning competency the students worked, in autonomous way, with the following documents:

- Doc 1: Tutorial on matrices.
- Doc 2: Tutorial on linear systems.
- Doc 3: Vector spaces, problems solved.
- Doc 4: Matrix analysis, problems solved.
- Doc 5: Determinants, problems solved.
- Doc 6: Linear systems, problems solved.
- Some more specific tutorials and solved problems, concerning eigenvalues, eigenvectors and inner products, were also been provided.

An extract of two different files regarding the Jordan canonical form and the solution of a real problem using linear systems can be found in Appendices A and B, respectively.

The students' home-work involved a collection of problems to be solved with MAXIMA. The students worked in teams of two or three people. After teacher revisions of each work, each group can improve it according to the feedback.

A formative assessment model based on different learning activities was used. In all the activities the students have the opportunity to comment errors with the teacher.

Table 1 summarizes the assessment activities.

Table 2 shows the academic results obtained for the students of each group.

The last examination was identical for both groups. The marks are quite similar, as Table 3 shows. There is no significative differences for specific mathematical competencies in both groups. However the evolution of the students' work allows to conclude that generic competencies (self learning, team work, problem solving and use of technology) are better in the experimental group.

**Table 1**

Assessment activities.

Control Group (CG)	Experimental Group (EG)
80% Three traditional written exams with "paper and pencil"	80% Three traditional written exams with free use of MAXIMA
10% DERIVE lab sessions	10% Team work, with free use of MAXIMA
10% Quizzes	10% Face-to-face problem solving with MAXIMA

**Table 2**

Academic results.

	CG	EG
Students	47	48
Does not complete the course activities	1	13
Successfully complete the course	40	33
Do not pass the course	6	3
Efficiency rate	85%	67.3%
Success rate	87%	91%

**Table 3**

Number of students in each grade range.

Marks obtained	0–3	3–5	5–8	8–10
Number of students. Experimental group	6	7	17	6
Number of students. Control group	5	16	16	9

**Table 4**

Evaluation of Doc 1.

Doc 1: Tutorial of matrices	SD	D	A	SA
The document is easy to follow	0%	5.26%	43.37%	43.37%
The MAXIMA defined functions are useful	0%	15.79%	52.63%	31.58%
The document helps to understand the concepts	0%	26.32%	31.58%	42.11%
The examples are accessible	0%	10.53%	15.79%	73.68%
The document is useful for solving problems	0%	5.26%	36.84%	57.89%
The document is useful in examinations	5.26%	0%	31.58%	63.16%

Following a model designed to assess the efficacy of the experience we gave out questionnaires collecting the opinions of the students about the different aspects of the experiment performed:

- Regarding the files made available to the students.
- Regarding the solution of problems in autonomous work with MAXIMA.
- Regarding the overall experience and improvement in the acquisition of competencies.

We also asked the students to provide an estimate of the time spent in the autonomous work in order to check the estimation made by the instructor.

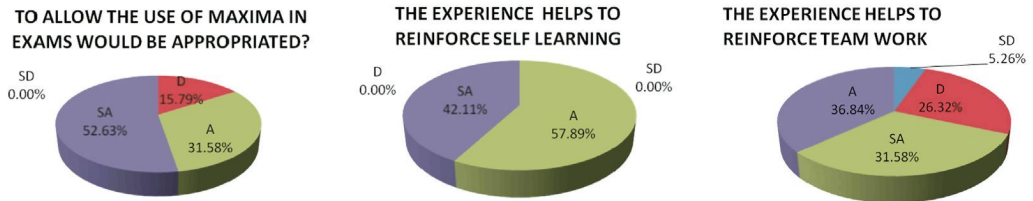
The questionnaires were answered by the students of the experimental group. One formulary was completed for each team (2–3 students). Nineteen questionnaires were retrieved; the results are commented below.

First, the students assessed Doc 1 to Doc 5. The results obtained were similar for the five documents analyzed and the answers were fairly favorably. The answers relating to Doc 1: Tutorial on matrices are collected in Table 4. All the students used this document and the percentages of those who answered *SD* (*strongly disagree*), *D* (*disagree*), *A* (*agree*) or *SA* (*strongly agree*) to each of the items are shown.

In the second part of the questionnaire we asked the students about the use of MAXIMA in autonomous work involving the solution of problems in team work.

**Table 5**  
Evaluation of block 1.

	SD	D	A	SA
The amount and variety of problems are adequate	10.53%	5.26%	52.63%	31.58%
To solve the problems with MAXIMA has been easy	5.25%	15.79%	36.84%	42.11%
To distinguish the problems that cannot be solved with MAXIMA has been easy	10.53%	36.84%	31.58%	21.05%



**Fig. 1.** Evaluation: Personal aspects.

The proposed problems have been divided into two blocks:

- *Block 1:* Problems concerning vector spaces, matrix analysis, determinants and linear systems.
- *Block 2:* Problems of linear applications, euclidean vector spaces and orthogonal diagonalization. Orthogonal transformations.

For block 1, the aspects to be evaluated and the corresponding answers are shown in Table 5. The aspects evaluated in the block 2 are the same and the answers are similar.

It should be noticed that, in the autonomous work, the difficulty the students had was understanding the limitations of the CAS employed: That is, the detection of problems that MAXIMA was unable to solve for them.

In the third section, the values to be evaluated were as follows: whether it is appropriate to allow the use of MAXIMA in exams; whether the experience helped students to reinforce the self learning and to enhance the team work.

The results are shown in Fig. 1.

The students' answers regarding the number of hours they devoted to their work, the mean time per student was compatible with what had been foreseen, with the exception of two atypical values: one is excessively low and corresponded to students who had decided not to study the subject and another excessively high one, corresponding to a group that evaluated the experience as being highly positive.

Finally, we offer some of the comments made by the students:

- (1) *"The inclusion of an informatics program for problem solving in Algebra is a good idea. It allows us to perform basic calculations but we must have knowledge of the contents of the subject. So in fact it is really only a help, not a solution. I also believe that the choice of MAXIMA is also a good one since it is a multiplatform program. The use of MAXIMA in exams is also a good thing and I hope that it will continue to be used in Algebra and that it will eventually be extended to other subjects".*
- (2) *"Allowing the use of MAXIMA in exams seems to be a good choice since this saves a lot of time and minimizes the risk of errors when performing intermediate calculations to solve a problem (determinants, matrix products, ...)".*

### 3.2. Mathematical methods for signal processing

Mathematical Methods for Signal Processing (MMSP) is an optional subject for Computer Engineering students at Polytechnic University of Madrid (UPM), which is offered as a continuing education in a part-time course with a blended learning methodology. There are very few students enrolled (only six students in the academic year 2011–2012), who are technical graduates with professional



**Table 6**  
Learning planning of MMSP and estimated time.

Learning activities	Face-to-face	Autonomous work
Attending lectures	9 h	
Displaying on-line presentations		6 h
Individual study		12 h
Tutorials	3 h	
On-line quizzes (two attempts with feedback)		3 h
Solving exercises with MATLAB	10 h	
Doing a MATLAB Personal-Toolbox		10 h
Small projects (team-work)	2 h	20 h
Exams	3 h	

**Table 7**  
Allocation of marks.

Assessment instruments	Mark
Two exams, with free use of MATLAB and the Personal-Toolbox	50
Three on-line quizzes	10
Two short team projects using MATLAB	40
Total	100

experience. They are “workers who study”, maintaining their full-time job. Most of them have not studied mathematics in recent years.

The generic competencies defined for the subject MMSP were the same as the LAME (Self Learning, Problem Solving, Use of Technology, Team Work).

As a specific competency: Students of MMSP must use mathematical skills and software for developing a strong grounding in the fundamentals of digital signal processing, including:

- Understand different digital signal and systems.
- Determine if a system is a Linear Time-Invariant System (LTI).
- Use  $z$  transforms and discrete time Fourier transforms to analyze an LTI.
- Use  $z$  transform in solving difference equations.
- Understand the relationship between poles, zeros, and stability.
- Determine the spectrum of a signal using the Discrete Fourier Transform (DFT).
- Implement DFT using Fast Fourier Transform (FFT) algorithm.
- Use FFT for efficient implementation of linear convolutions and correlations.

The Mathematical Software used is MATLAB. It is the most widely used software for Signal Processing and Industrial Applications. With MATLAB students can define tools to be used in other subjects (Signal Processing, Control Systems, Robotic...).

We have used the Toolbox approach García et al. (2009), encouraging students to create their own well-organized “Personal-Toolbox” for solving mathematical problems using MATLAB to later transfer knowledge and skills to career problems. This toolbox is a collection of m-files, for the usual topics in the subject. In addition, students completed their work by writing a brief “user’s manual” for their tools. Each student can use his Personal-Toolbox in all the learning and assessment activities. The Toolbox includes functions programmed for the usual topics and for solving typical exercises.

As an example, a student’s toolbox is shown in Fig. 2. On the left the picture shows the functions and scripts in different folders. A script where FFT is used for filtering a signal is open.

For this subject, 3 ECTS (78 hours of students’ work) have been allocated. These hours are distributed between face-to-face activities and autonomous work with on-line teacher–student interaction (see planning in Table 6).

The Learning Management System Moodle was used as an e-learning tool. Work materials and on-line assessment tools were delivered through this platform. The Moodle tools used were: communications forum, online tutorials, upload files and quizzes with feedback.

The assessment model is shown in Table 7.

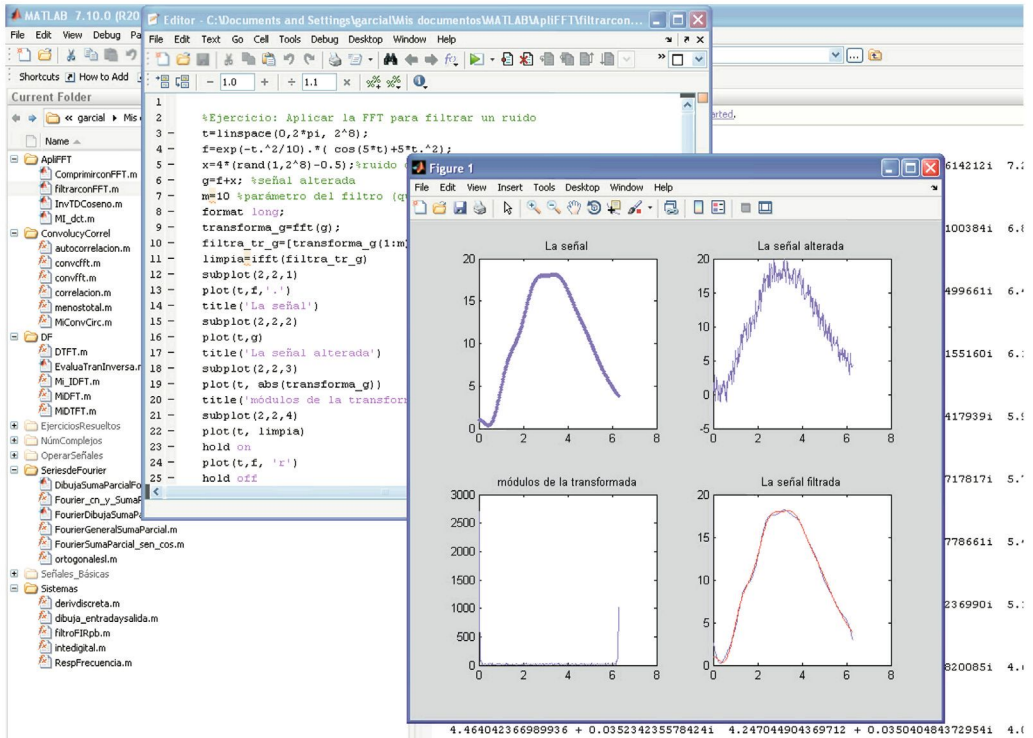


Fig. 2. A student toolbox.

Surveys, interviews and observation of student's work were used to collect data and analyze the experience. Next, we comment the main results in the major aspects that contribute to how well a student use the CAS for learning and doing mathematics.

Mathematical aspects: the result is satisfactory. The CAS has helped to solve the problems due to the lack of mathematical training of the students and allowed them to fix the bases for the subject that comes after: Signal Processing. The instructor in this subject has evaluated positively the mathematical competencies of students and their MATLAB toolboxes.

Technical aspects: the students have not had any problems with the MATLAB syntax since they already had some experience in programming. In some cases, they had difficulties in interpreting the results but these were mainly due to mathematics *per se* and not to the software. We also observed that although they were familiar with their own toolbox, they are no able to use fluently the toolboxes provided by other students. In general, students found difficulties for accurately writing a user manual.

Personal aspects: In the questionnaires evaluated from 0 to 6, the students showed a positive attitude (mean value, 4.1) and were satisfied with the model of assessment (5.5). They considered that their generic competencies had improved (4.4), especially those of Self-Learning and Team Work.

Work time: Following the instructions in [European Commission \(2009\)](#), we contrasted the estimated time established for each activity with the time it took students to carry out their work. Students stated that the course demanded too much work time. They have worked less than 12 hours on their individual study activities but more than 20 hours on projects.

#### 4. Conclusions

The technology can be used as a powerful tool in all teaching and learning activities. The use of CAS in assessment activities is a crucial part to mathematical learning on a framework based on

competencies, since these tools foster self-efficacy and promote a way of working closer to the real work.

The use of CAS in exams allows the evaluation of different aspects of learning. Certain calculation skills lose relevance and is easier to evaluate the level of mathematical competencies.

The two experiences analyzed here, in which CAS (MAXIMA and MATLAB) were integrated in all the learning and assessment activities, proved to be positive according to the academic results obtained and students' perceptions.

We can conclude that it is possible to improve some generic competencies while the students have also acquired the provided specific mathematical competencies. The students had a great acceptance of the experience.

We have observed that using mathematical software in all learning and assessment activities, students created an appropriate work style and a set of tools that allowed them to cope with greater ease the mathematical problems of engineering.

Evidently, the ways in which CAS were used were different in both experiences. The students of LAME incorporated the software in their usual problem-solving activities, mainly using functions within the system or routines programmed by the instructor. By contrast, the students of MMSP programmed their own functions. In any case, the competencies in the use of technology, team work and autonomous learning at the different levels were very much favored by the work performed with the CAS.

From a student's viewpoint, active learning helps to improve their competencies. The feeling about material and learning strategy has been good.

From the point of view of the instructors, the experiences analyzed show that the change in methodology is positive and with that further work should be possible to become adapted to the new rules and new roles demanded by the current situation.

## Acknowledgement

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## Appendix A. The Jordan canonical form

The Jordan canonical form of a matrix  $m$  can be found using the command `jordan(m)`, after uploading the package "diag".

```
(%i1) load("diag");  
(%o1) C:/PROGRA 1/MAXIMA 1.0/share/maxima/5.24.0/share/contrib/diag.mac
```

EXAMPLES:

```
(%i2) a: matrix([1,3,2,0], [0,1,0,0], [0,2,-2,0], [6,6,-3,4]);  
(%o2) 
$$\begin{pmatrix} 1 & 3 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & -2 & 0 \\ 6 & 6 & -3 & 4 \end{pmatrix}$$
  
(%i3) jordan(a);  
(%o3) [[-2, 1], [1, 2], [4, 1]]  
  
(%i4) b:matrix([2,0,0,0,0,0,0,0], [1,2,0,0,0,0,0,0], [-4,1,2,0,0,0,0,0],  
[2,0,0,2,0,0,0,0], [-7,2,0,0,2,0,0,0], [9,0,-2,0,1,2,0,0],  
[-34,7,1,-2,-1,1,2,0], [145,-17,-16,3,9,-2,0,3]);
```

$$(\%04) \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ -4 & 1 & 2 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ -7 & 2 & 0 & 0 & 2 & 0 & 0 & 0 \\ 9 & 0 & -2 & 0 & 1 & 2 & 0 & 0 \\ -34 & 7 & 1 & -2 & -1 & 1 & 2 & 0 \\ 145 & -17 & -16 & 3 & 9 & -2 & 0 & 3 \end{pmatrix}$$

(%i5) jordan(b);

(%o5) [[2, 3, 3, 1], [3, 1]]

The output is unexpected. Each bracket shows an eigenvalue and the dimension of the associated submatrices. In the matrix a:

- the eigenvalue  $-2$  has associated a submatrix of order 1,
- the eigenvalue  $1$  has associated a submatrix of order 2, and
- the eigenvalue  $4$  has associated a submatrix of order 1.

In the matrix b:

- the eigenvalue  $2$  has associated 3 submatrices of orders 3, 3 and 1, and
- the eigenvalue  $3$  has associated a submatrix of order 1.

To get the Jordan canonical form, we use the instruction `dispJordan(1)`. The input 1 is the output of the jordan instruction.

(%i6) s: jordan(a);

(%o6) [[-2, 1], [1, 2], [4, 1]]

(%i7) dispJordan(s);

$$(\%07) \begin{pmatrix} -2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}$$

(%i8) t: jordan(b);

(%o8) [[2, 3, 3, 1], [3, 1]]

(%i9) dispJordan(t);

$$(\%09) \begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \end{pmatrix}$$

## Appendix B. Application of linear systems

Fig. 3 shows the traffic in a road network.

The directions for driving are shown by the corresponding arrows. Only in the streets AB and CD is allowed the driving in two directions. Find the traffic in each street in the following cases:

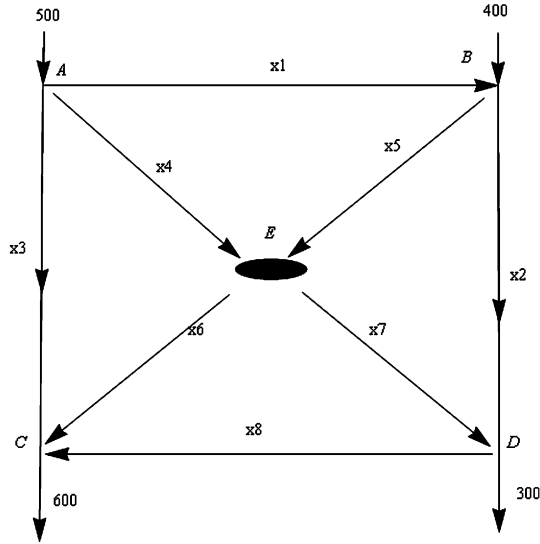


Fig. 3. Traffic network.

- (i) It is forbidden to drive in the knot E.
- (ii) It is forbidden to drive in the street AB and the knot D.

**Remark 1.** The unknowns  $x_1$  and  $x_8$  are positive if the traffic direction is according with the picture. (A negative value for  $x_j$  indicates traffic in the opposite direction.)

We establish the linear system, taking into account that in every intersection the sum incoming traffic must match the output:

$$\begin{aligned}
 500 &= x_1 + x_4 + x_3, \\
 400 &= x_2 + x_5 - x_1, \\
 0 &= x_4 + x_5 - x_6 - x_7, \\
 600 &= x_3 + x_6 + x_8, \\
 300 &= x_2 + x_7 - x_8
 \end{aligned}$$

We use `linsolve` for solving the system:

```
(%i1) linsolve([500 = x1 + x4 + x3, 400 = x2 + x5 - x1,
              0 = x4 + x5 - x6 - x7,
              600 = x3 + x6 + x8, 300 = x2 + x7 - x8],
              [x1, x2, x3, x4, x5, x6, x7, x8]);
```

*solve: dependent equations eliminated: (5)*

```
(%o1) [x1 = %r4 - %r2 + %r1 - 100, x2 = -%r2 + %r1 + 300, x3 = -%r3 - %r1 + 600, x4 = -%r4 + %r3 + %r2, x5 = %r4, x6 = %r3, x7 = %r2, x8 = %r1]
```

There are infinite solutions. We change the 4 parameters by letters:

```
(%i2) [x1=d-b+a-100, x2=-b+a+300, x3=-c-a+600, x4=-d+c+b, x5=d,
      x6=c, x7=b, x8=a];
```

```
(%o2) [x1 = d - b + a - 100, x2 = -b + a + 300, x3 = -c - a + 600, x4 = -d + c + b, x5 = d, x6 = c, x7 = b, x8 = a]
```

(i) If in the knot E the traffic is 0 we have the new equations (corresponding with the conditions):  
 $x_4=x_5=x_6=x_7=0$

```
(%i3) linsolve([500 = x1 + x4 + x3, 400 = x2 + x5 - x1,
0 = x4 + x5 - x6 - x7,
600 = x3 + x6 + x8, 300 = x2 + x7 - x8, x4=0, x5=0,
x6=0, x7=0], [x1, x2, x3, x4, x5, x6, x7, x8]);
```

*solve: dependent equations eliminated: (94)*

```
(%o3) [x1=%r5 - 100, x2=%r5 + 300, x3=600 - %r5, x4=0, x5=0, x6=0, x7=0, x8=%r5]
```

One parameter appears in the solution, associated to the unknown  $x_8$ .

If  $x_2$  and  $x_3$  are positives, then the possible values for  $x_8$  are in the interval  $[-300, 600]$ .

If  $x_8=-300$  then, changing the parameter by the letter  $a$ , we obtain

```
(%i4) [x1=a-100, x2=a+300, x3=600-a, x4=0, x5=0, x6=0, x7=0, x8=a], a=-300;
```

```
(%o4) [x1=-400, x2=0, x3=900, x4=0, x5=0, x6=0, x7=0, x8=-300]
```

If  $x_8=-300$  then  $x_1=-400$ ,  $x_2=0$  and  $x_3=900$ .

We are doing the same with  $x_8=600$ .

```
(%i5) [x1=a-100, x2=a+300, x3=600-a, x4=0, x5=0, x6=0, x7=0, x8=a], a=600;
```

```
(%o5) [x1=500, x2=900, x3=0, x4=0, x5=0, x6=0, x7=0, x8=600]
```

If  $x_8=600$  then  $x_1=500$ ,  $x_2=900$  and  $x_3=0$ .

Then we can conclude that  $x_1 \in [-400, 500]$ ,  $x_2 \in [0, 900]$  and  $x_3 \in [0, 900]$ .

(ii) We do the following substitutions in the initial system:  $x_1=0$  (traffic forbidden in AB)  
 $x_2=x_7=x_8=0$  (traffic forbidden in D). We get

```
(%i7) linsolve([500 = x1 + x4 + x3, 400 = x2 + x5 - x1,
0 = x4 + x5 - x6 - x7,
600 = x3 + x6 + x8, 300 = x2 + x7 - x8, x1=0, x2=0, x7=0, x8=0],
[x1, x2, x3, x4, x5, x6, x7, x8]);
```

```
(%o7) []
```

The system is not compatible. The result is trivial taking into account that the equation  $300=x_2+x_7-x_8$  corresponding to intersection D is not compatible with the conditions  $x_2=x_7=x_8=0$ .

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