Proceedings of the International Association for Shell and Spatial Structures (IASS) Symposium 2009, Valencia
Evolution and Trends in Design, Analysis and Construction of Shell and Spatial Structures
28 September - 2 October 2009, Universidad Politécnica de Valencia, Spain
Alberto DOMINGO and Carlos LAZARO (eds.)

Optimal boundary geometry in an elasticity problem: a systematic adjoint approach.

J. Garcia-Palacios*, C. Castro[a] and A. Samartín[b]

* Professor Titular. Dept. of Civil Engineering: Hydraulic and Energetic.
ETSICCYP. UPM. Ciudad Universitaria s/n. 28040 Madrid
jgpalacios@caminos.upm.es

[a] Professor Titular. Dept. of Civil Engineering: Mathematics and Computer Science. UPM
[b] Catedrático. Dept. of Civil Engineering: Continuum Mechanics and Theory of Structures. UPM

Abstract

In different problems of Elasticity the definition of the optimal geometry of the boundary, according to a given objective function, is an issue of great interest. Finding the shape of a hole in the middle of a plate subjected to an arbitrary loading such that the stresses along the hole minimizes some functional or the optimal middle curved concrete vault for a tunnel along which a uniform minimum compression are two typical examples. In these two examples the objective functional depends on the geometry of the boundary that can be either a curve (in case of 2D problems) or a surface boundary (in 3D problems). Typically, optimization is achieved by means of an iterative process which requires the computation of gradients of the objective function with respect to design variables.

Gradients can by computed in a variety of ways, although adjoint methods either continuous or discrete ones are the more efficient ones when they are applied in different technical branches. In this paper the adjoint continuous method is introduced in a systematic way to this type of problems and an illustrative simple example, namely the finding of an optimal shape tunnel vault immersed in a linearly elastic terrain, is presented.

Keywords: Structural Optimization, Boundary Geometry, Continuous Adjoint Approach, Finite Elements, Numerical techniques

1. Introduction

Structural Optimal Design is a very fast growing topic, particularly after the advent of high speed computation. An increasing number of general purpose computer programs on structural analysis based on Finite Elements discretization includes user's optimization capabilities [4]. However despite the important results achieved in the optimization field, the theory and applications are currently facing new and difficult challenges.

The notion of improving the performance of a structure implies some freedom to change the structure itself and also a measure to compare the different structural designs. Then, as it is well known, in a standard structural optimization design problem the following elements are involved: (1) Design or control variables, typically described by a vector $\mathbf{x}$; (2) An objective or merit function $f(\mathbf{x})$ and (3) Some constraints for the design variables, i.e. relationships among the design variables in order the design be feasible.

In order to solve the problem of the structural optimal design a large variety of classical, analytical and numerical tools exist. An extensive bibliography represented by general texts as [1], [8], [5] exists in which different optimization methods are described. They can be classified in several groups according to the structure of the involved elements in the optimization problem. Typically, an objective function is an application of design variables to a scalar value, although cases of multi-objective function exists, but they will not be treated here. Then, from this point of view, three main optimization groups can be distinguished. The first group corresponds to the classical problem of maximizing or minimizing an objective function $f(\mathbf{x})$, that relates the unknowns $\mathbf{x} \in R^n$ to a scalar value $f(\mathbf{x})$. Differential calculus have been used for solving this group of problems during many years. Naturally unconstraint and constraint conditions among the design can be incorporated, applying in this last case Lagrange multipliers technique. In the second group variational methods are used when problems in which the objective function represents a functional, i.e. an application of a function space to a scalar numbers. For example the functional of the functions space $\mathbf{y}$ and its derivatives up to order k as given by the expression $f(\mathbf{y}) = \int_\Omega I(\mathbf{x}, \mathbf{y}', \mathbf{y}'', \dots, \mathbf{y}^k) d\mathbf{x}$. The third group of of optimization problems has appear more recently than the former ones. In this group the design variable are a domain Ω or its boundary $\partial\Omega$. The functional is now of the following class $f(\mathbf{y}, \Omega) = \int_\Omega I(\mathbf{x}, \mathbf{y}', \mathbf{y}'', \dots, \mathbf{y}^k) d\mathbf{x}$ and the design variables are $\mathbf{y}$ and Ω as well. This optimization problem will be discussed here and some illustrative examples will be shown.

Several approaches have been applied to solve optimization problems of the third group since the pioneering work on Shape optimal design by Zienkiewicz and Campbell [10]. Typically, a topological optimization problem is reduced to other optimization problem belonging to the second group by a direct way. This way consist in to represent the shape of the structural domain or of the structural boundary by a set of specified control variables [6], then the problem is reduced to follow these specified control variables. Two approaches can be applied. The first is the Lagrangian approach in which the control variables or key points are a list of sequentially numbered points. Between consecutive points interpolation functions are used and therefore the whole structural domain, shape or boundary, is represented. In the other approach, known as Eulerian approach, the unknown domain design is embedded into a larger and fictitious domain. The fictitious boundary is discretized by FE and the optimal domain is found by a successive elimination of the different FE. Different optimal search techniques as genetic algorithms [2] or moving mesh techniques can be used. In both approaches steepest descent method in conjunction to line search can be applied.

In the procedure here presented the adjoint problem of elastic structural problem is used in order to avoid the computation of the gradients of the different degrees of freedom (dof) representing the domain. In fact, from the solution of the adjoint problem the gradient function along the boundary domain can be computed. That means the whole boundary curve is modified to a new boundary curve in such a way that the value of the objective function along the new boundary become smaller than the value obtained along the previous boundary. Therefore, if the boundary is represented by a set of dof then only a single gradient step defines the gradient steps of all dof and the computation effort is dramatically reduced.

This adjoint approach is in fact a quite general method and it can be applied to a great number of optimal design problems and cost functionals. Here we focus on a particular example where the cost functional is chosen in order to obtain a uniform compression along the profile.

2. Problem statement

It is considered a two dimensional plane strain elastic problem defined by a domain $\Omega \subset R^2$ subjected to a set of distributed external forces $\mathbf{f} = (f_1, f_2)$ and along the part Γ_1 of its boundary $\partial\Omega \equiv \Gamma$, where $\Gamma_1 \bigcup \Gamma_2 = \Gamma$ and $\Gamma_1 \bigcap \Gamma_2 = \emptyset$, the displacements $\mathbf{u} = (u_1, u_2)$ are specified and along the complementary part of the boundary Γ_2 the pressure forces $\mathbf{p} = (p_1, p_2)$ are also known. It is supposed a part $S \in \Gamma$ of the boundary can be modified in order that a functional along S of the stresses obtained from the plane strain problem is minimized. This type of situation can occur when the geometry of a hole of an elastic plate subjected to boundary forces has to be designed such the sum of the absolute values of the tangential stresses along the contour of the hole is minimum. Other example can be to find the tunnel shape such that the sum of deviation of the tangential stresses, respect some average stress, along its contour is minimum.

For a given geometry S, the stresses along S are obtained by solving the two dimensional elasticity system:

$$\sigma_{\alpha\beta,\beta} + f_\alpha = 0 \quad \mathbf{x} \in \Omega, \quad \alpha, \beta = 1, 2 \tag{1}$$

$$\sigma_{33} = \nu(\sigma_{11} + \sigma_{22}) \quad \mathbf{x} \in \Omega \tag{2}$$

$$\varepsilon_{\alpha\beta} = \frac{1+\nu}{E}(\sigma_{\alpha\beta} - \sigma^0_{\alpha\beta}) - \frac{\nu}{E}(\sigma_{kk} - \sigma^0_{kk})\delta_{\alpha\beta} \tag{3}$$

$$\varepsilon_{13} = \varepsilon_{23} = \varepsilon_{33} = 0 \tag{4}$$

where $\mathbf{x} = (x_1, x_2, 0) \in \Omega$ is a generic point of the elastic body, $\sigma_{\alpha\beta}$ and $\varepsilon_{\alpha\beta}$ are the components of the stress and the strain tensors respectively. The elastic constants of the isotropic material are the Young modulus E and Poisson ratio ν. Partial derivative is denoted by a comma (,). The expression of strain tensor components are given as function of the displacements as follows:

$$\varepsilon_{\alpha\beta} = \frac{1}{2}\left(u_{\alpha,\beta} + u_{\beta,\alpha}\right) \tag{5}$$

The following boundary conditions are assumed for this illustrative example

$$u_\alpha = \bar{u}_\alpha, \quad \mathbf{x} \in \Gamma_1, \quad \alpha = 1, 2 \tag{6}$$

$$\sigma_{\alpha\beta} n_\beta = \bar{p}_\alpha, \quad \mathbf{x} \in \Gamma_2, \quad \alpha, \beta = 1, 2 \tag{7}$$

where $\bar{u}_\alpha$ and $\bar{p}_\alpha$ are specified displacement and pressure values, $\mathbf{n} = (n_1, n_2)$ is the the outward normal unit vector to the boundary Γ. The unitary tangent vector will be denoted by $\mathbf{t} = (t_1, t_2)$ obtained by rotating $\mathbf{n}$ clockwise.

The optimization problem to be solved is the following:
Find S^{min} such that

$$J(S^{min}) = \min_S J(S).$$

where the functional J is defined as

$$J(S) = \frac{1}{2}\oint_S (\sigma_t - \sigma_{tm})^2 \, ds = \frac{1}{2}\oint_S (\sigma_{\alpha\beta} t_\alpha t_\beta - \sigma_{tm})^2 \, ds \qquad (8)$$

where

$$\sigma_t(s) = \sigma_{\alpha\beta} t_\alpha t_\beta = \mathbf{t}\boldsymbol{\sigma}\mathbf{t} = (t_1, t_2)\begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix}\begin{bmatrix} t_1 \\ t_2 \end{bmatrix}$$

i.e σ_t represents the stress component on the boundary tangential direction. The constant σ_{tm} is an average stress σ_t along the boundary S. Alternatively it could be incorporated as variable depending on the boundary unknown S in the objective function $J(S)$ by using the expression:

$$\sigma_{tm} = \frac{\oint_S \sigma_t(s) ds}{\oint_S ds} \qquad (9)$$

The unknown boundary S is assumed to be smooth and should fulfill the continuity requirements of the boundary Γ by imposing the pertinent continuity conditions at the contact points of the known fixed boundary $\Gamma - S$.

Several possibilities exist to solve the optimization problem defined by the objective function (8) and the corresponding elasticity problem represented by equations (1) to (7). The most simplest one consists in to describe the boundary S by a set of FE of continuity C^k and then the unknowns are the coordinates of the different nodes along the boundary curve. Once the problem has been discretized different optimization techniques can be applied. One very popular technique is the steepest descent gradient method which requires the gradient of the objective functional. However this procedure is very tedious and cumbersome from computational point of view. In fact for each set of different boundary coordinates a FE mesh has to be built, then a FE analysis has to be carried out and finally to find the value of the objective function. Then each coordinate has to be displaced in order to compute the corresponding sensibilities or gradients of the objective function. In this computation a FE mesh and the corresponding FE analysis has to be performed. Finally, it should be realized that the obtained results can be dependent on the refinement of the FE mesh.

The former approach can be considered as a discrete approach to the optimization problem. Here a continuous approach combined with an adjoint methodology to obtain the gradient is proposed. In order to describe the method it is convenience to go back to the continuous framework and introduce a more precise description of the admissible geometries for the unknown boundary S. Given an initial geometry S_0 and a parametrization of S_0, $\mathbf{r}_0 : (0, l) \rightarrow R^2$, the admissible boundaries S are parametrized as follows

$$\mathbf{r}(s) = \mathbf{r}_0(s) + \alpha(\mathbf{r}_0(s))$$

where $\boldsymbol{\alpha}$ is a profile vector field on S which satisfies $\boldsymbol{\alpha}(\mathbf{x}) = 0$ if the point x of S coincides with its boundary (extreme) points, i.e. $\mathbf{x} = (x_1, x_2) \in \partial S$. Thus, the set of admissible geometries is determined by the profile $\boldsymbol{\alpha}$ which describe the displacement of the initial geometry. The vector field $\boldsymbol{\alpha}$ is assumed to be smooth. In particular it is assumed $\boldsymbol{\alpha} \in U_{ad}$ where

$$U_{ad} = \{\boldsymbol{\alpha} \in (C^\infty(S))^2, \text{ with } \alpha(\mathbf{x}) = 0 \text{ if } \mathbf{x} \in \partial S\}.$$

Then the functional can be written in terms of $\boldsymbol{\alpha}$ as follows

$$J(\boldsymbol{\alpha}) = \frac{1}{2}\int_S (\sigma_{\alpha\beta} t_\alpha t_\beta - \sigma_{tm})^2 ds. \tag{10}$$

The optimization problem can be stated as follows:
Find $\boldsymbol{\alpha}_{\min} \in U_{ad}$ such that

$$J(\boldsymbol{\alpha}_{\min}) = \min_{\boldsymbol{\alpha} \in U_{ad}} J(\boldsymbol{\alpha}). \tag{11}$$

For this optimization problem a gradient type method is now implemented. Then a suitable discretization of the functional, the equations and the set of admissible boundaries, described by $\boldsymbol{\alpha} \in U_{ad}$, are introduced. This approach provides a corresponding discrete optimization problem for which a descent direction is required.

In this discretization procedure both the functional and the elasticity system can be obtained by usual numerical methods. In the following the introduction of a discretization of the admissible set U_{ad} is briefly described. It is assumed that α belongs to the finite dimensional space generated by some basis functions defined on S, $\{f_k(s)\}_{k=1,\ldots,m}$. In this way, the admissible profile functions α take the form

$$\alpha = \sum_{k=1}^{m} \alpha_k f_k(s)$$

where α_k are now some scalars that they are referred as discrete design variables.

In order to compute this descent direction the easiest procedure is to obtain the gradient by a finite difference approach. In this way, the partial derivative of J which respect to each one of the discrete design variables which describe $\alpha \in U_{ad}$ must be computed. If this set of design variables is large, this gradient requires a considerable computational effort.

In this work, a descent direction for the discrete functional is found in a different way. First, going back to the continuous formulation of the optimization problem the gradient of the continuous functional (10) is computed. Once this computation is performed the result is discretized and used it as a descent direction of the discrete optimization problem.

In order to compute the gradient of the functional (10), an adjoint procedure is applied. With this approach, the gradient is obtained by solving once a suitable adjoint problem, which is computationally much less expensive. Details of similar approaches applied to related problems can be seen in [3] or [7].

The method is mathematically described in the appendix. In the next section a short summary of the main computational steps is given. In the remaining sections some illustrative numerical examples are shown and finally some conclusions can be drawn.

3. Computational procedure

The following iterative procedure will be carried out in order to find the optimal boundary S.

1. The initial boundary S^0 is known. A suggested initial boundary should be a line intuitive closed to the solution.
2. It is assumed after k iterations the boundary S^k is known and it is represented by the set of coordinates (x_{1n}, x_{2n}), $n = 1, 2, \ldots, N$, of the N nodes of the boundary. Obtain for this boundary the following information:
   - A Finite Element mesh of the domain Ω for which the set of N nodes is located along the boundary S^k.
   - An interpolation curve $\mathbf{r}^k(s)$ connecting the N nodes such that $\mathbf{r}^k(s_n) = (x_{1n}, x_{2n})$ and s_n is the arch length of the curve.
   - The tangent $\mathbf{t}(t_1, t_2)$ and the outward normal $\mathbf{n}(n_1, n_2)$ vectors at the N nodes.
   - The curvatures κ of the curve S^k at the N nodes.
3. Compute displacements and stresses at the N nodes of the boundary S^k by means of a FE analysis with the domain Ω subjected to the given loads and to the imposed boundary conditions.
4. Obtain at each node n of the N nodes of S^k the values of the following function $H(s) = \sigma_{\alpha\beta} t_\alpha t_\beta - \sigma_{tm}$ where $\alpha, \beta = 1, 2$ and implicit summation convention is assumed.
5. Find the tangential derivative of the function $H(s)$ at the N nodes of S^k, i.e. $p_\alpha = \frac{dH}{ds} t_\alpha$, with $\alpha = 1, 2$.
6. Carry out a new FE analysis called elastic adjoint problem, that it will identified by the superscript $*$. This elastic adjoint problem is the same elastic problem with the domain Ω as the former FE analysis, but subjected to zero external loads and different boundary conditions along S^k. These new boundary conditions are the the following fictitious pressure loads $\sigma^*_{\alpha\beta} n_\beta = -\frac{E}{1-\nu^2} p_\alpha$. These loading conditions should be concentrated at the N nodes of the boundary, as it is standard in FE analysis.
7. Select from the FE analysis of the previous step the displacements at the N nodes of the boundary S^k, i.e. the values $\mathbf{u}^* = (u^*_{1n}, u^*_{2n})$.
8. Compute at the N nodes of the boundary S^k the function

$$M(s) = -\frac{\partial}{\partial s}\left(H\sigma_{\alpha\beta} n_\alpha t_\beta\right) + \frac{1}{2}\left[\kappa H^2 + \frac{\partial (H^2)}{\partial n}\right] + \frac{\partial}{\partial n}(\sigma_{\alpha\beta}) n_\beta u^*_\alpha + \frac{\partial}{\partial t}\left(\sigma_{\alpha\beta} t_\beta u^*_\alpha\right) - \frac{\nu}{1-\nu} H \frac{\partial}{\partial n}(\sigma_{\alpha\beta}) n_\alpha n_\beta \qquad (12)$$

   where $\frac{\partial}{\partial s}$ and $\frac{\partial}{\partial n}$ are the tangential and normal derivatives along the curve S^k.

9. Obtain tentative new boundary lines S_m^{k+1}, $(m = 1, 2, \ldots, M)$, using a search line technique given by the expression:

$$\mathbf{r}_l^{k+1}(s) = \mathbf{r}^k(s) - \mu_l M(s)\mathbf{n}, \quad l = 0, 1, 2, \ldots \tag{13}$$

where $\mathbf{n}$ is the normal unit vector of the curve $\mathbf{r}^k$ at point s and

$$\mu_l = 2l\frac{\varepsilon}{M_{max}}$$

The accuracy of the geometry of the boundary line S is expressed by the tolerance ε and the value M_{max} is given as the maximum of the absolute value $M(s)$, i.e.

$$M_{max} = \max_s[|M(s)|]$$

.

10. Select the curve S^{k+1} of the boundary, to be used for the new iteration step, from the set of the previous tentative boundary $\mathbf{r}_l^{k+1}(s)$ by applying the following line search criterion:

   - For each value l construct the curve S_l^{k+1} and compute $H(s)$ by application of steps 2, 3 and 4.
   - For each curve S_l^{k+1} compute the objective function

   $$OF(l) = \oint_{S_l^{k+1}} H(s)ds$$

   and find the value of l such that

   $$OF(l_{min}) = \min_l OF(l)$$

   and then the selected S^{k+1} is $S^{k+1} = S_{l_{min}}^{k+1}$

11. Compare S^{k+1} to S^k and if the difference is smaller than some given tolerance then the iteration stops. In the contrary case go to 2.

The proposed procedure corresponds to a rather simplified model. No smoothing of the different results has been contemplated. In this respect the boundary S is treated a set of points $P_n = (x_n, y_n)$ with $n = 1, 2, \ldots, N$. Under this assumption the steps 2 and 3 are carried out as follows. The arch length interval ds_n between P_n and P_{n+1} is

$$ds_n = \sqrt{(x_{n+1} - x_n)^2 + (y_{n+1} - y_n)^2}$$

The components of the unit tangent vector $\mathbf{t}$ are found according to the expressions, each of them valid in the interval (x_{n-1}, x_{n+1}):

$$\begin{aligned} y =& y_{n-1}\frac{(x - x_n)(x - x_{n+1})}{(x_{n-1} - x_n)(x_{n-1} - x_{n+1})} + y_n\frac{(x - x_{n-1})(x - x_{n+1})}{(x_n - x_{n-1})(x_n - x_{n+1})} \\ &+ y_{n+1}\frac{(x - x_{n-1})(x - x_n)}{(x_{n+1} - x_{n-1})(x_{n+1} - x_n)} \quad \text{with } n = 2, 3, \ldots, N-1 \end{aligned}$$

and the angle θ_n of the tangent with the axis x is given by the formula $\tan\theta_n = \frac{dy}{dx}\Big|_{x=x_n}$, i.e.

$$
\begin{aligned}
\tan\theta_n &= y_{n-1}\frac{x_n - x_{n+1}}{(x_{n-1}-x_n)(x_{n-1}-x_{n+1})} + y_n\frac{2x_n - x_{n-1} - x_{n+1}}{(x_n - x_{n-1})(x_n - x_{n+1})} \\
&\quad + y_{n+1}\frac{-x_{n-1}+x_n}{(x_{n+1}-x_{n-1})(x_{n+1}-x_n)} \quad \text{with } n = 2,3,\ldots,N-1 \\
\tan\theta_1 &= y_1\frac{2x_1 - x_2 - x_3}{(x_1-x_2)(x_1-x_3)} + y_2\frac{x_1 - x_3}{(x_2-x_1)(x_2-x_3)} + y_3\frac{x_1-x_2}{(x_3-x_1)(x_3-x_2)} \\
\tan\theta_N &= y_{N-2}\frac{x_N - x_{N-1}}{(x_{N-2}-x_N)(x_{N-2}-x_{N-1})} + y_{N-1}\frac{x_N - x_{N-2}}{(x_{N-1}-x_{N-2})(x_{N-1}-x_N)} \\
&\quad + y_N\frac{2x_N - x_{N-1} - x_{N-2}}{(x_N - x_{N-1})(x_N - x_{N-2})}
\end{aligned}
$$

Then, the tangent and normal vector components are

$$t_x(n) = \cos\theta_n, \quad t_y(n) = \sin\theta_n, \quad n_x(n) = -t_y(n) \text{ and } n_{xy}(n) = t_x(n)$$

Finally, the curvatures $\kappa(n)$ are obtained as follows

$$\kappa(n) = \frac{1}{\sqrt{a^2+b^2+2c}}, \quad \kappa(1) = \kappa(2), \kappa(N) = \kappa(N-1)$$

and the constants a, b and c are the solution of the linear system of equations

$$
\begin{bmatrix} x_{n-1} & y_{n-1} & 1 \\ x_n & y_n & 1 \\ x_{n+1} & y_{n+1} & 1 \end{bmatrix}
\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \frac{1}{2}
\begin{bmatrix} x_{n-1}^2 + y_{n-1}^2 \\ x_n^2 + y_n^2 \\ x_{n+1}^2 + y_{n+1}^2 \end{bmatrix}
$$

The former boundary discrete representation can be improved if a FE model representation for the boundary is used. In this way depending on the degree of continuity introduced in the FE formulation the boundary smoothness can be controlled, see the classical [9].

4. Illustrative example

The proposed approach to find the optimal geometry of a boundary in a plane elasticity problem can be applied to treat different problems, as finding the optimal geometry of a hole in a plate subjected to in plane forces or the shape of a tunnel vault under the earth weight and different live loads on the free surface of the mountain. This last example will be here discussed in order to grasp the main features of the proposed approach.

A simplified model of a tunnel can be to considered it as a hole passing through a mountain composed by an elastic material. Naturally, a more complex model should include the existence of a concrete arch along the perimeter of the hole to support the earth pressure. Also the elastic material should be replaced by other material with nonlinear properties as plasticity and creep among others.Finally, the construction procedure of the tunnel and the earth initial stresses should be also taken into account in a realistic analysis. In figure 1 a sketch of this complex problem is shown. The objective is to find the shape of the tunnel in order that the longitudinal stresses along the unknown tunnel boundary $A_1D_0A_2$ are close to an specified value c. As it was already commented there exists another possibility for the value c. It can be also obtained endogenously by the model assuming is the average value of the longitudinal stresses along the tunnel boundary. The width of the tunnel is specified value

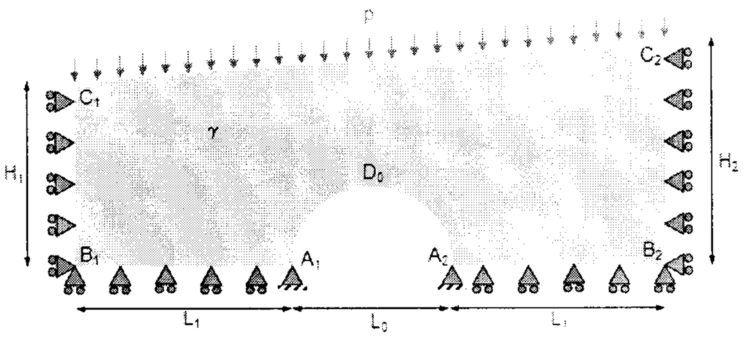

Figure 1: Illustrative example

L, however its height is unknown. The model of the mountain is defined by the following parameters L_1, H_1 and H_2. The terrain density is γ and the live load on the upper part of the mountain is a distributed loading of intensity p. The elastic constants of the terrain are defined by the Young modulus E and Poisson ratio ν.

The shape of the tunnel boundary $A_1D_0A_2$ is assumed initially to be a circle of radius R. The following particular case will be studied (units: kN, m):

$$L = 10, \qquad L_1 = 60, \quad H_1 = H_2 = 30$$
$$p = 10, \qquad \gamma = 20$$
$$E = 2 \times 10^5, \qquad \nu = 0.3$$

and the assumed initial circle radius is $R = \frac{L}{2} = 5$.

For the tunnel defined by the former data two cases have been analyzed. The first corresponds to the objective function (8) with the average compressive stress $\sigma_{tm} = -300$ given exogenously. The second case the mean compressive stress σ_{tm} is computed endogenously by the expression (9).

For the first case 200 steepest gradient steps have been carried out. Results of this case are summarized in figure 2. The initial tunnel geometry and the successive geometry changes of the tunnel according to the number of minimizing steps are depicted in figure 2a, where step 1 corresponds to initial tunnel geometry. It can be observed that after step 60 some discontinuities appear in the slope of tunnel geometry. This fact means the need to introduce more continuity requirements to the boundary function S than the ones used in this paper. A complementary information is given in figure 2b, in which the value of the function $H(s)$ and its variation with the number of minimizing step are shown. The way how this function

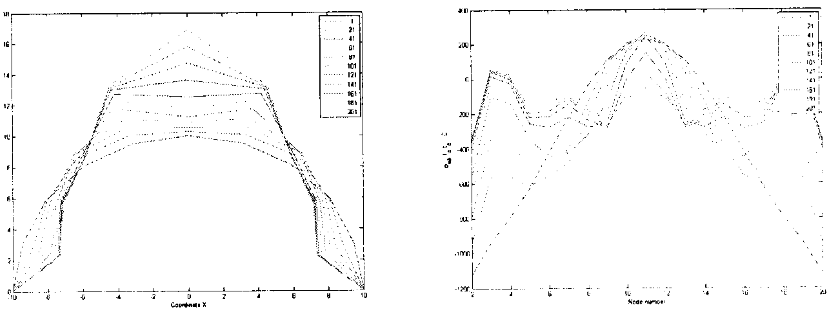

Figure 2: Illustrative example with σ_{tm} exogenous. (a) Geometry. (b) Function H

$H(s)$ varies and also the integral of its square value along the boundary S, i.e. the objective function, is reached can be seen in this figure 2b.

Similar results and comments to the previous ones can be drawn from the inspection of figure 3. Finally, in figure 4 the ratio $\frac{OF}{OF_{ini}}$ between the objective value and the initial objec-

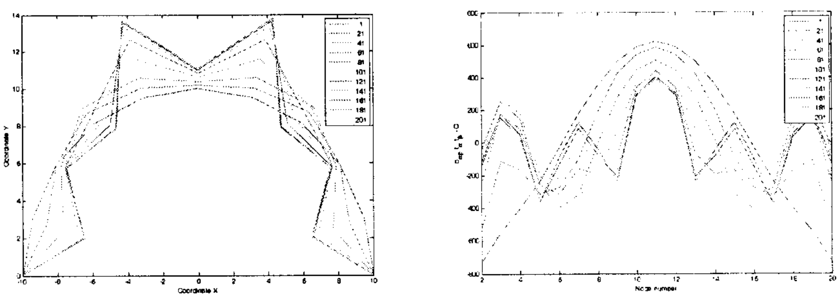

Figure 3: Illustrative example with σ_{tm} endogenous. (a) Geometry. (b) Function H

tive value versus the number of minimizing steps is shown. In this way the convergence rate to the the optimal value of the objective function OF can be compared for the two analyzed cases.

5. Conclusions

In this paper a computational efficient approach in order to find the optimal boundary of an elasticity problem has been presented. This approach is based on the solution of the adjoint problem. This solution permits to find a steepest gradient curve for the boundary such the objective function is minimized. This fact permits a drastic reduction of the computational

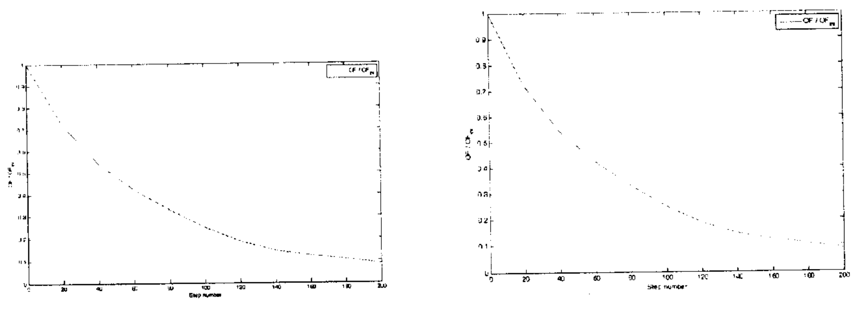

Figure 4: Convergence rate to the optimal OF. (a) σ_{tm} exogenous. (b) σ_{tm} endogenous

effort. As it is well known for this type of problems local optimal boundaries are reached, i.e. the obtained solution may be dependent on the selected initial boundary.

The application of the proposed approach can present some numerical instabilities near the optimal solution, particularly if the boundary analytical representation is not smooth enough. This difficulty could be solved used a FE geometrical representation of the boundary S.

The proposed approach can be extended to treat the boundary optimization in the framework of linear and nonlinear elasticity problems as well. In case of non linear elasticity the adjoint problem is linear. Finally, other maybe more realistic objective functions than the ones here discussed can be treated using a similar procedure.

Appendix

Gradient of J

In this section the shape derivative of the functional J at a geometry S described by $\boldsymbol{\alpha}$ is computed. It is well known that the displacements $\delta\boldsymbol{\alpha}$ can be restricted o be in the normal direction, i.e. the new geometry is

$$\boldsymbol{\alpha}_{new} = \boldsymbol{\alpha} + \delta\alpha\, \mathbf{n}$$

where $\mathbf{n}$ is the outward normal and $\delta\alpha$ is now a scalar function describing the displacement in the normal direction.

In order to clarify the calculus below the rest of this section will be presented in three subsections.

Gradient of the objective functional

It is well known that, for the functional

$$J(S) = \int_S j(u)ds,$$

the shape derivative of J in the direction $\delta\alpha\mathbf{n}$ is given by

$$\delta J = \int_S j'(u)u^* ds + \int_S [\kappa j(u) + \frac{\partial}{\partial n} j(u)]\delta\alpha ds \tag{14}$$

where κ is the curvature of S and δu is the shape derivative of u in the direction $\delta\alpha\mathbf{n}$.

In particular, applying this expression (14) to the functional (10), assuming σ_{tm} is a constant, it is obtained,

$$\begin{aligned}
\delta J &= \delta\left[\frac{1}{2}\int_S (\sigma_{\alpha\beta}t_\alpha t_\beta - \sigma_{tm})^2 ds\right] = \int_S (\sigma_{\alpha\beta}t_\alpha t_\beta - \sigma_{tm})\,\delta(\sigma_{\alpha\beta}t_\alpha t_\beta)\,ds \\
&+ \int_S \delta\alpha\left\{\kappa\frac{(\sigma_{\alpha\beta}t_\alpha t_\beta - \sigma_{tm})^2}{2} + \frac{\partial}{\partial n}\left[\frac{(\sigma_{\alpha\beta}t_\alpha t_\beta - \sigma_{tm})^2}{2}\right]\right\} ds \\
&= \int_S (\sigma_{\alpha\beta}t_\alpha t_\beta - \sigma_{tm})\,\delta\sigma_{\alpha\beta}t_\alpha t_\beta ds + 2\int_S (\sigma_{\alpha\beta}t_\alpha t_\beta - \sigma_{tm})\,\sigma_{\alpha\beta}\delta t_\alpha t_\beta ds \\
&+ \int_S \delta\alpha\left\{\kappa\frac{(\sigma_{\alpha\beta}t_\alpha t_\beta - \sigma_{tm})^2}{2} + \frac{\partial}{\partial n}\left[\frac{(\sigma_{\alpha\beta}t_\alpha t_\beta - \sigma_{tm})^2}{2}\right]\right\} ds \\
&= \int_S (\sigma_{\alpha\beta}t_\alpha t_\beta - \sigma_{tm})\,\delta\sigma_{\alpha\beta}t_\alpha t_\beta ds + 2\int_S (\sigma_{\alpha\beta}t_\alpha t_\beta - \sigma_{tm})\,\delta\alpha'\sigma_{\alpha\beta}n_\alpha t_\beta ds \\
&+ \int_S \delta\alpha\left\{\kappa\frac{(\sigma_{\alpha\beta}t_\alpha t_\beta - \sigma_{tm})^2}{2} + \frac{\partial}{\partial n}\left[\frac{(\sigma_{\alpha\beta}t_\alpha t_\beta - \sigma_{tm})^2}{2}\right]\right\} ds,
\end{aligned} \tag{15}$$

where κ is the curvature of S. Here the following identity

$$\delta\mathbf{t} = \delta\alpha'(s)\mathbf{n}.$$

has been used.

The variable $\delta\sigma_{ij}$ solves the system

$$\begin{cases}
\dfrac{\partial\delta\sigma_{\alpha\beta}}{\partial\beta} = 0, & \mathbf{x}\in\Omega, \quad \alpha,\beta = 1,2 \\
\delta\sigma_{33} = \nu(\delta\sigma_{11} + \delta\sigma_{22}), & \mathbf{x}\in\Omega, \\
\delta\varepsilon_{\alpha\beta} = \dfrac{1+\nu}{E}\delta\sigma_{\alpha\beta} - \dfrac{\nu}{E}\delta\sigma_{kk}\delta_{\alpha\beta}, & \\
\delta\varepsilon_{\alpha\beta} = \dfrac{1}{2}\left(\dfrac{\partial\delta u_\beta}{\partial\alpha} + \dfrac{\partial\delta u_\alpha}{\partial\beta}\right) & \\
\delta\varepsilon_{13} = \delta\varepsilon_{23} = \delta\varepsilon_{33} = 0. &
\end{cases} \tag{16}$$

Note that (16) is in fact the linearization of the elasticity system (1).

The boundary conditions are given by

$$\begin{aligned}
\delta u_\alpha &= 0, \quad \mathbf{x}\in\Gamma_1, \quad \alpha = 1,2 \\
\delta\sigma_{\alpha\beta}n_\beta &= 0, \quad \mathbf{x}\in\Gamma_2, \quad \alpha = 1,2 \\
\delta\sigma_{\alpha\beta}n_\beta + \frac{\partial}{\partial n}(\sigma_{\alpha\beta})\,n_\beta\delta\alpha - \sigma_{\alpha\beta}t_\beta\delta\alpha' &= 0, \quad \mathbf{x}\in S, \quad \alpha = 1,2
\end{aligned}$$

The main difficulty in the above expression (15) for δJ comes from the term

$$\int_S (\sigma_{ij}t_i t_j - \sigma_{tm})\, \delta\sigma_{\alpha\beta} t_\alpha t_\beta ds \tag{17}$$

since it requires to solve the linearized system, for each admissible variation. To simplify the calculus of this term the adjoint methodology described in the next subsection it is introduced.

Stresses sensitivity: Adjoint calculus

First of all, note that if the linearized stress-strain tensor relation on S is written with respect to the local system of coordinates associated to $\{\mathbf{t}, \mathbf{n}\}$ then the following expression for $\delta\sigma_{\alpha\beta}$ is obtained:

$$\begin{aligned} \delta\sigma_{\alpha\beta} t_\alpha t_\beta &= \frac{E}{1-\nu^2}\delta\varepsilon_{\alpha\beta} t_\alpha t_\beta + \frac{\nu}{1-\nu}\delta\sigma_{\alpha\beta} n_\alpha n_\beta \\ &= \frac{E}{1-\nu^2}\frac{\partial}{\partial t}(\delta\mathbf{u}\cdot\mathbf{t}) - \frac{\nu}{1-\nu}\frac{\partial}{\partial n}(\sigma_{\alpha\beta}) n_\alpha n_\beta \delta\alpha. \end{aligned} \tag{18}$$

In the last identity the boundary conditions to be satisfied for $\delta\mathbf{u}$ and $\mathbf{u}$ on S have been used.

Thus, the term in (17) is simplified as

$$\begin{aligned} \int_S (\sigma_{ij}t_i t_j - \sigma_{tm})\, \delta\sigma_{\alpha\beta} t_\alpha t_\beta ds &= \frac{E}{1-\nu^2}\int_S (\sigma_{ij}t_i t_j - \sigma_{tm})\, \frac{\partial}{\partial t}(\delta\mathbf{u}\cdot\mathbf{t}) ds \\ &\quad - \frac{\nu}{1-\nu}\int_S (\sigma_{ij}t_i t_j - \sigma_{tm})\, \frac{\partial}{\partial n}(\sigma_{\alpha\beta}) n_\alpha n_\beta \delta\alpha ds \\ &= -\frac{E}{1-\nu^2}\int_S \frac{\partial}{\partial t}(\sigma_{ij}t_i t_j - \sigma_{tm})\, (\delta u_\alpha t_\alpha) ds \\ &\quad - \frac{\nu}{1-\nu}\int_S (\sigma_{ij}t_i t_j - \sigma_{tm})\, \frac{\partial}{\partial n}(\sigma_{\alpha\beta}) n_\alpha n_\beta \delta\alpha ds. \end{aligned}$$

In order to eliminate the term δu_α the adjoint problem to the linearized system with suitable boundary conditions on S is introduced

$$\begin{aligned} \frac{\partial\sigma^*_{\alpha\beta}}{\partial\beta} &= 0, \quad x\in\Omega, \quad \alpha,\beta = 1,2 \\ \sigma^*_{33} &= \nu(\sigma^*_{11}+\sigma^*_{22}), \quad x\in\Omega, \\ \epsilon^*_{\alpha\beta} &= \frac{1+\nu}{E}\sigma^*_{\alpha\beta} - \frac{\nu}{E}\sigma^*_{kk}\delta_{\alpha\beta}, \\ \epsilon^*_{\alpha\beta} &= \frac{1}{2}\left(\frac{\partial u^*_\beta}{\partial\alpha} + \frac{\partial u^*_\alpha}{\partial\beta}\right) \\ \delta\epsilon^*_{13} &= \delta\epsilon^*_{23} = \delta\epsilon^*_{33} = 0. \end{aligned}$$

with the following suitable boundary conditions

$$\begin{aligned} u^*_\alpha &= 0, \quad x\in\Gamma_1, \quad \alpha = 1,2 \\ \sigma^*_{\alpha\beta} n_\beta &= 0, \quad x\in\Gamma_2, \quad \alpha = 1,2 \\ \sigma^*_{\alpha\beta} n_\beta &= \frac{-E}{1-\nu^2}\frac{\partial\left(\sigma_{ij}t_i t_j - \sigma_{tm}\right)}{\partial t} t_\alpha, \quad x\in S, \quad \alpha = 1,2. \end{aligned}$$

Multiplying the equations of the linearized system by u^*_α and integrating it is obtained

$$\begin{aligned} 0 &= \int_\Omega \frac{\partial(\delta\sigma_{\alpha\beta})}{\partial\beta} u^*_\alpha d\Omega = -\int_\Omega \delta\sigma_{\alpha\beta}\frac{\partial u^*_\alpha}{\partial\beta} d\Omega + \int_{\Gamma_1\cup\Gamma_2\cup S} \delta\sigma_{\alpha\beta} n_\beta u^*_\alpha ds \\ &= -\int_\Omega \delta\sigma_{\alpha\beta}\epsilon^*_{\alpha\beta} d\Omega + \int_S \delta\sigma_{\alpha\beta} n_\beta u^*_\alpha ds \\ &= -\int_\Omega \delta\sigma_{\alpha\beta}\epsilon^*_{\alpha\beta} d\Omega - \int_S \left(\frac{\partial\sigma_{\alpha\beta}}{\partial n} n_\beta \delta\alpha - \sigma_{\alpha\beta} t_\beta \delta\alpha'\right) u^*_\alpha ds \end{aligned} \tag{19}$$

Now, it is observed that the first term in this formula can be simplified as follows

$$\begin{aligned} \int_\Omega \delta\sigma_{\alpha\beta}\epsilon^*_{\alpha\beta} d\Omega &= \int_\Omega \delta\sigma_{\alpha\beta}\epsilon_{\alpha\beta} d\Omega = \int_\Omega \delta\sigma_{\alpha\beta}\left(\frac{1+\nu}{E}\sigma^*_{\alpha\beta} - \frac{\nu}{E}\sigma^*_{kk}\delta_{\alpha\beta}\right) ds \\ &= \int_\Omega \delta\sigma_{\alpha\beta}\left(\frac{1+\nu}{E}\sigma^*_{\alpha\beta} - \frac{\nu(1+\nu)}{E}\sigma^*_{\gamma\gamma}\delta_{\alpha\beta}\right) ds \\ &= \int_\Omega \sigma^*_{\alpha\beta}\left(\frac{1+\nu}{E}\delta\sigma_{\alpha\beta} - \frac{\nu(1+\nu)}{E}\delta\sigma_{\gamma\gamma}\delta_{\alpha\beta}\right) ds \\ &= \int_\Omega \sigma^*_{\alpha\beta}\left(\frac{1+\nu}{E}\sigma^*_{\alpha\beta} - \frac{\nu(1+\nu)}{E}\sigma^*_{\gamma\gamma}\delta_{\alpha\beta}\right) ds \\ &= \int_\Omega \sigma^*_{\alpha\beta}\delta\varepsilon_{\alpha\beta} ds \end{aligned} \tag{20}$$

Therefore, combining (19) and (20) it is reached

$$\begin{aligned} 0 &= -\int_\Omega \delta\varepsilon_{\alpha\beta}\sigma^*_{\alpha\beta} d\Omega - \int_S \left(\frac{\partial\sigma_{\alpha\beta}}{\partial n} n_\beta\delta\alpha - \sigma_{\alpha\beta}t_\beta\delta\alpha'\right) u^*_\alpha ds \\ &= \int_\Omega \delta u_\alpha \frac{\partial\sigma^*_{\alpha\beta}}{\partial\beta} d\Omega - \int_{\Gamma_1\cup\Gamma_2\cup S} \delta u_\alpha \sigma^*_{\alpha\beta} n_\beta ds - \int_S \left(\frac{\partial\sigma_{\alpha\beta}}{\partial n} n_\beta\delta\alpha - \sigma_{\alpha\beta}t_\beta\delta\alpha'\right) u^*_\alpha ds \\ &= \int_\Omega \delta u_\alpha \frac{\partial\sigma^*_{\alpha\beta}}{\partial\beta} d\Omega - \int_S \delta u_\alpha \sigma^*_{\alpha\beta} n_\beta ds - \int_S \left(\frac{\partial\sigma_{\alpha\beta}}{\partial n} n_\beta\delta\alpha - \sigma_{\alpha\beta}t_\beta\delta\alpha'\right) u^*_\alpha ds \\ &= \frac{E}{1-\nu^2}\int_S \frac{\partial\left(\sigma_{ij}t_it_j - \sigma_{tm}\right)}{\partial t}\delta u_\alpha t_\alpha ds - \int_S \left(\frac{\partial\sigma_{\alpha\beta}}{\partial n} n_\beta\delta\alpha - \sigma_{\alpha\beta}t_\beta\delta\alpha'\right) u^*_\alpha ds \end{aligned}$$

Thus,

$$\frac{E}{1-\nu^2}\int_S \frac{\partial}{\partial t}\left(\sigma_{ij}t_it_j - \sigma_{tm}\right)\delta u_\alpha t_\alpha ds = \int_S \left(\frac{\partial\sigma_{\alpha\beta}}{\partial n} n_\beta\delta\alpha - \sigma_{\alpha\beta}t_\beta\delta\alpha'\right) u^*_\alpha ds$$

Finally the following result is reached

$$\delta J = \int_S M(s)\delta\alpha \tag{21}$$

where

$$\begin{aligned} M(s) &= -\frac{\partial}{\partial t}(H\sigma_{ij}n_it_j) + \left[\kappa\frac{H^2}{2} + \frac{\partial}{\partial n}\left(\frac{H^2}{2}\right)\right] \\ &\quad + \frac{\partial}{\partial n}\left(\sigma_{\alpha\beta}\right) n_\beta u^*_\alpha + \frac{\partial}{\partial t}\left(\sigma_{\alpha\beta}t_\beta u^*_\alpha\right) - \frac{\nu}{1-\nu}H\frac{\partial\sigma_{\alpha\beta}}{\partial n} n_\alpha n_\beta \end{aligned}$$

and

$$H = H(s) = \sigma_{ij} t_i t_j - \sigma_{tm}$$

A descent direction for J at α is given by

$$\delta\alpha = -M.$$

The finite dimensional reduction

To compute the above gradient in practice a suitable finite dimensional approximation of the problem should be introduced. As it has been pointed out before, a natural way to obtain this finite dimensional problem is assuming that $\delta\alpha$ belongs to a finite dimensional space generated by some basis functions defined on S, $\{f_k(s)\}_{k=1,\ldots,m}$. In this way, the function $\delta\alpha$ takes the form

$$\delta\alpha = \sum_k \delta\alpha_k f_k(s)$$

where $\delta\alpha_k$ are now some scalars.

Now, formula (21) becomes

$$\delta J = \int_S M\delta\alpha = \sum_k \delta\alpha_k \int_S M f_k(s) ds \qquad (22)$$

and a descent direction, in this finite dimensional space, is given by

$$\delta\alpha_k = -\int_S M f_k(s) ds.$$

References

[1] Nocedal, S.J. and Wright, J. *Numerical Optimization*. Springer Verlag, Berlin, 1999.

[2] Bendsoe, M.P. and Sigmund, O. *Topology Optimization Theory, Methods and Applications*. Sprringer Verlag, 2003.

[3] Castro, C. and Lozano, C. and Palacios, F. and Zuazua, E. Systematic Continuous Adjoint Approach to Viscous Aerodynamic Design on Unstructured Grids. *AIAA Journal*, 45(9), 2007.

[4] De Salvo, G.J. *ANSYS Engineering Analysis System. Verification Manual*. Swanson Analysis System Inc., Houston, PA, 1985.

[5] Haftka, R. and Gürdal, Z. *Elements of Structural Optimization*. Kluwer Academic Publihers, Dordrecht, The Netherlands, 1992.

[6] Lu, Z. and Wang, M.Y. and Wei, P. A level set-based parametrization method for structural shape and topology optimization. *Intal. Journ. for Num. Meth. in Eng.*, 2007.

[7] Nadarajah, S.K. and Jameson, A. A Comparison of the Continuous and Discrete Adjoint Approach to Automatic aerodynamic Optimization. *AIAA Paper 2000-0667*, 2000.

[8] S.S. Rao. *Engineering Optimization -Theory and Practice*. John Wiley and Sons, New York, 1996.

[9] Zienckiewicz, O.C., Emson, C., and Bettess, P. A novel boundary infinite element. *International Journal of Numerical Methods in Engineering*, 19(November):393–404, 1983.

[10] Zienkiewicz, O.C. and Campell, J.S. *Shape optimization and sequential linear programming, Optimum Structural Design*. John Wiley and Sons, New York, 1973.