This paper appears in
Rueda S.L. (2012). Differential resultants of super essential systems of linear OD-polynomials. Libro de Resúmenes del XIII Encuentro de Álgebra Computacional y Aplicaciones EACA 2012, Obras Colectivas, Ciencas 09, Servicio de Publicaciones de la Universidad de Alcalá, 155-158, 2012. ISBN:978-84-8138-770-4. Editors: J. Rafael Sendra, C. Villarino

DIFFERENTIAL RESULTANTS OF SUPER ESSENTIAL SYSTEMS OF LINEAR OD-POLYNOMIALS

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ABSTRACT. The sparse differential resultant $\partial\text{Res}(\mathfrak{P})$ of an overdetermined system $\mathfrak{P}$ of generic nonhomogeneous ordinary differential polynomials, was formally defined recently by Li, Gao and Yuan (2011). In this note, a differential resultant formula $\partial\text{FRes}(\mathfrak{P})$ is defined and proved to be nonzero for linear "super essential" systems. In the linear case, $\partial\text{Res}(\mathfrak{P})$ is proved to be equal, up to a nonzero constant, to $\partial\text{FRes}(\mathfrak{P}^*)$ for the supper essential subsystem $\mathfrak{P}^*$ of $\mathfrak{P}$.

INTRODUCTION

Sparse algebraic resultants obtained great benefits from having close formulas for their computation [4]. These formulas provide bounds for the degree of the elimination output and ways of exploiting sparseness of the input polynomials on predicting the support of the output [1]. Thus reducing elimination to an interpolation problem in (numerical) linear algebra.

Let $\mathfrak{P}$ be an overdetermined system of generic sparse differential polynomials. It would be useful to represent the sparse differential resultant $\partial\text{Res}(\mathfrak{P})$, defined in [5], as the quotient of two determinants, as done for the algebraic case in [4]. A matrix representation of the sparse differential resultant is important because it is the basis for efficient computation algorithms and their study promises to have a grate contribution to the development and applicability of differential elimination techniques. In the differential case, so called Macaulay style formulas do not exist, even in the simplest situation. The matrices used in the algebraic case to define Macaulay style formulas are coefficient matrices of sets of polynomials obtained by multiplying the original ones by appropriate sets of monomials, [1]. In the differential case, in addition, derivatives of the original polynomials should be considered. The differential resultant formula defined by Carrà-Ferro in [2], is the algebraic resultant of Macaulay [6], of a set of derivatives of the ordinary differential polynomials in $\mathfrak{P}$. Already for linear differential polynomials these formulas vanish often, giving no information about the differential resultant $\partial\text{Res}(\mathfrak{P})$. The linear case can be seen as a previous stage to get ready to approach the nonlinear case, considering only the problem of taking the appropriate set of derivatives of the elements in $\mathfrak{P}$ for the moment.

In [9], the linear complete differential resultant $\partial\text{CRes}(\mathcal{P})$ of a set of linear differential polynomials $\mathcal{P}$ (non necessarily generic) was defined, as an improvement, in the linear case,

of the differential resultant formula given by Carrà-Ferro. Let us assume that $\mathfrak{P}$ is a system of linear generic sparse differential polynomials. Still, $\partial\text{CRes}(\mathfrak{P})$ is the determinant of a matrix having zero columns in many cases.
The linear differential polynomials in $\mathfrak{P}$ can be described via differential operators. We use appropriate bounds of the supports of those differential operators to decide on a convenient set $\text{ps}(\mathfrak{P})$ of derivatives of $\mathfrak{P}$, such that its coefficient matrix $\mathcal{M}(\mathfrak{P})$ is squared and has no zero columns. Furthermore, we can guarantee that the linear sparse differential resultant $\partial\text{Res}(\mathfrak{P})$ can always be computed (up to a nonzero constant) as the determinant of a matrix $\mathcal{M}(\mathfrak{P}^*)$, for a super essential subsystem $\mathfrak{P}^*$ of $\mathfrak{P}$, as defined in Section 2. A key fact is that not every polynomial in $\mathfrak{P}$ is involved in the computation of $\partial\text{Res}(\mathfrak{P})$, only those in a super essential subsystem $\mathfrak{P}^*$ of $\mathfrak{P}$ are, and $\mathfrak{P}^*$ is proved to exist in all cases. An extended version of the results presented can be found in [8].

1. Sparse linear differential resultant

Let us suppose that the field $\mathbb{Q}$ of rational numbers is a field of constants with respect to a derivation ∂. Let us consider the set $U = \{u_1, \ldots, u_{n-1}\}$ of differential indeterminates over $\mathbb{Q}$. By $\mathbb{N}_0$ we mean the natural numbers including 0. For $k \in \mathbb{N}_0$, we denote by $u_{j,k}$ the k-th derivative of u_j and for $u_{j,0}$ we simply write u_j. We denote by $\{U\}$ the set of derivatives of the elements of U.
For $i = 1, \ldots, n$ and $j = 1, \ldots, n-1$, let us consider subsets $\mathfrak{S}_{i,j}$ of $\mathbb{N}_0$ to be the supports of generic differential operators

$$\mathcal{G}_{i,j} := \begin{cases} \sum_{k\in\mathfrak{S}_{i,j}} c_{i,j,k}\partial^k & , \mathfrak{S}_{i,j} \neq \emptyset, \\ 0 & , \mathfrak{S}_{i,j} = \emptyset. \end{cases}$$

Let us consider the sets of differential indeterminates over $\mathbb{Q}$

$$C = \{c_1, \ldots, c_n\} \text{ and } \overline{C} := \cup_{i=1}^{n} \cup_{j=1}^{n-1} \{c_{i,j,k} \mid k \in \mathfrak{S}_{i,j}\}.$$

Let $\mathcal{K} = \mathbb{Q}\langle\overline{C}\rangle$, a differential field extension of $\mathbb{Q}$, and $\mathbb{D} = \mathcal{K}\{C\}$, a differential domain. Consider the set $\mathfrak{P} = \{\mathbb{F}_1, \ldots, \mathbb{F}_n\}$ of generic sparse linear differential polynomials in $\mathbb{D}\{U\}$ as follows

$$\mathbb{F}_i := c_i - \sum_{j=1}^{n-1} \mathcal{G}_{i,j}(u_j) = c_i - \sum_{j=1}^{n-1} \sum_{k\in\mathfrak{S}_{i,j}} c_{i,j,k} u_{j,k}, i = 1, \ldots, n.$$

Let $x_{i,j}$, $i = 1, \ldots, n$, $j = 1, \ldots, n-1$ be algebraic indeterminates over $\mathbb{Q}$. Let $X(\mathfrak{P}) = (X_{i,j})$ be the $n \times (n-1)$ matrix, such that

$$X_{i,j} := \begin{cases} x_{i,j} & , \mathcal{G}_{i,j} \neq 0, \\ 0 & , \mathcal{G}_{i,j} = 0. \end{cases}$$

The system $\mathfrak{P}$ is said to be differentially essential if $\text{rank}(X(\mathfrak{P})) = n-1$.
Let $[\mathfrak{P}]$ be the differential ideal generated by $\mathfrak{P}$ in $\mathbb{D}\{U\}$. By [5], Corollary 3.4, the dimension of the elimination ideal

$$\text{ID}(\mathfrak{P}) := [\mathfrak{P}] \cap \mathbb{D}$$

is $n-1$ if and only if $\mathfrak{P}$ is a differentially essential system. In such case, $\text{ID}(\mathfrak{P}) = \text{sat}(R)$, the saturation ideal of a unique (up to scaling) irreducible differential polynomial $R(c_1, \ldots, c_n)$ in $\mathbb{D}$. That is $\{R\}$ is a characteristic set w.r.t. any ranking on C, see [3], Section 4.2 and we can assume that $R \in \mathbb{Q}\{\overline{C}, C\}$ by clearing up denominators when necessary. By [5], Definition 3.5, R is the sparse differential resultant of $\mathfrak{P}$, we will denote it by $\partial\mathrm{Res}(\mathfrak{P})$.

2. Sparse differential resultant formula for supper essential systems

Let us assume that the order of $\mathbb{F}_i$ is $o_i \geq 0$, $i = 1, \ldots, n$. We define positive integers, to construct convenient intervals bounding the supports of the differential operators $\mathcal{G}_{i,j}$. Let $\mathrm{ldeg}(\mathcal{G}_{i,j}) := \min \mathfrak{S}_{i,j}$ and $\deg(\mathcal{G}_{i,j}) := \max \mathfrak{S}_{i,j}$. For $j = 1, \ldots, n-1$,

$$\overline{\gamma}_j(\mathfrak{P}) := \min\{o_i - \deg(\mathcal{G}_{i,j}) \mid \mathcal{G}_{i,j} \neq 0,\ i = 1, \ldots, n\},$$
$$\underline{\gamma}_j(\mathfrak{P}) := \min\{\mathrm{ldeg}(\mathcal{G}_{i,j}) \mid \mathcal{G}_{i,j} \neq 0,\ i = 1, \ldots, n\},$$

$$\gamma_j(\mathfrak{P}) := \underline{\gamma}_j(\mathfrak{P}) + \overline{\gamma}_j(\mathfrak{P}).$$

For $\mathcal{G}_{i,j} \neq 0$ the next set of lattice points contains $\mathfrak{S}_{i,j}$,

$$I_{i,j}(\mathfrak{P}) := [\underline{\gamma}_j(\mathfrak{P}), o_i - \overline{\gamma}_j(\mathfrak{P})] \cap \mathbb{Z}.$$

Finally,

$$\gamma(\mathfrak{P}) := \sum_{j=1}^{n-1} \gamma_j(\mathfrak{P}).$$

We denote by $X_i(\mathfrak{P})$, $i = 1, \ldots, n$, the submatrix of $X(\mathfrak{P})$ obtained by removing its ith row. The system $\mathfrak{P}$ is said to be supper essential if $\det(X_i(\mathfrak{P})) \neq 0$, $i = 1, \ldots, n$. Given $N := \sum_{i=1}^{n} o_i$, let

$$L_i := N - o_i - \gamma(\mathfrak{P}), i = 1, \ldots, n.$$

If $\mathfrak{P}$ is super essential then $L_i \geq 0$, $i = 1, \ldots, n$ and we can construct the set

$$\mathrm{ps}(\mathfrak{P}) := \{\partial^k \mathbb{F}_i \mid k \in [0, L_i] \cap \mathbb{Z},\ i = 1, \ldots, n\},$$

containing $L := \sum_{i=1}^{n}(L_i + 1)$ differential polynomials, in the set $\mathcal{V}$ of $L - 1$ differential indeterminates

$$\mathcal{V} := \{u_{j,k} \mid k \in [\underline{\gamma}_j(\mathfrak{P}), N - \overline{\gamma}_j(\mathfrak{P}) - \gamma(\mathfrak{P})] \cap \mathbb{Z},\ j = 1, \ldots, n-1\}.$$

The coefficient matrix $\mathcal{M}(\mathfrak{P})$ of the differential polynomials in $\mathrm{ps}(\mathfrak{P})$ as polynomials in $\mathbb{D}[\mathcal{V}]$ is an $L \times L$ matrix. We define a linear differential resultant formula for $\mathfrak{P}$, denoted by $\partial\mathrm{FRes}(\mathfrak{P})$, and equal to:

$$\partial\mathrm{FRes}(\mathfrak{P}) := \det(\mathcal{M}(\mathfrak{P})).$$

3. Main results

The implicitization of linear DPPEs (differential polynomial parametric equations) by differential resultant formulas was studied in [9] and [7]. In [8], some of the results in [7] are extended and used to obtain the next conclusions.

Given a differentially essential system $\mathfrak{P}$, $\mathrm{ID}(\mathfrak{P}) = [\partial\mathrm{Res}(\mathfrak{P})]_{\mathbb{D}}$, the differential ideal generated by $\partial\mathrm{Res}(\mathfrak{P})$ in $\mathbb{D}$. Furthermore, $R = \partial\mathrm{Res}(\mathfrak{P})$ is a linear differential polynomial verifying:

(1) $R = \sum_{i=1}^{n} \mathcal{L}_i(c_i)$, for some $\mathcal{L}_i \in \mathcal{K}[\partial]$ and a greatest common left divisor of $\mathcal{L}_1, \ldots, \mathcal{L}_n$ belongs to $\mathcal{K}$, that is R is ID-primitive.

(2) R belongs to $(\mathrm{ps}(\mathfrak{P})) \cap \mathbb{D}$, where $(\mathrm{ps}(\mathfrak{P}))$ is the algebraic ideal generated by $\mathrm{ps}(\mathfrak{P})$ in $\mathbb{D}[\mathcal{V}]$.

(3) The highest positive integer c such that $\partial^c R \in (\mathrm{ps}(\mathfrak{P}))$ is

$$c = |\mathrm{ps}(\mathfrak{P})| - 1 - \mathrm{rank}(\mathcal{M}(\mathcal{V})),$$

where $\mathcal{M}(\mathcal{V})$ is the submatrix of $\mathcal{M}(\mathfrak{P})$ of the $L-1$ columns indexed by the elements in $\mathcal{V}$.

Using these properties we can prove the next result.

Theorem 3.1. *Let $\mathfrak{P}$ be a system of generic sparse linear differential polynomials. If $\mathfrak{P}$ is super essential then $\partial\mathrm{FRes}(\mathfrak{P}) \neq 0$.*

Furthermore, if $\mathfrak{P}$ is not super essential, we can prove the existence of a super essential subsystem $\mathfrak{P}^*$ of $\mathfrak{P}$ and provide a computation method, [8], Section 4. Furthermore, $\mathfrak{P}$ is differentially essential if and only it has a unique super essential subsytem.

Theorem 3.2. *Let us consider a differentially essential system $\mathfrak{P}$, of generic sparse linear differential polynomials, and the super essential subsystem $\mathfrak{P}^*$ of $\mathfrak{P}$. There exists a nonzero constant $\alpha \in \mathcal{K}$ such that $\partial\mathrm{Res}(\mathfrak{P}) = \alpha\partial\mathrm{FRes}(\mathfrak{P}^*)$.*

The previous results, allow us to give a bound of the order of $\partial\mathrm{Res}(\mathfrak{P})$ in the differential indeterminates C. Namely, given $I^* := \{i \mid \mathbb{F}_i \in \mathfrak{P}^*\}$ and $i \in \{1, \ldots, n\}$

$$\mathrm{ord}(\partial\mathrm{Res}(\mathfrak{P}), c_i) = -1 \text{ if } i \notin I^*,$$
$$\mathrm{ord}(\partial\mathrm{Res}(\mathfrak{P}), c_i) = N^* - o_i - \gamma(\mathfrak{P}^*) \text{ if } i \in I^*,$$

with $N^* = \sum_{i \in I^*} o_i$ and equality holds for some $i \in I^*$.

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Dpto de Matemática Aplicada, E.T.S. Arquitectura, Universidad Politécnica de Madrid. Member of the Research Group ASYNACS (Ref. CCEE2011/, R34). Partially supported by the "Ministerio de Ciencia e Innovación" under the Project MTM2008-04699-C03-01 and by the "Ministerio de Economía y Competitividad" under the project MTM2011-25816-C02-01.

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