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Sparse differential resultant formulas: between the linear and the nonlinear case

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Abstract

A matrix representation of the sparse differential resultant is the basis for efficient computation algorithms, whose study promises a great contribution to the development and applicability of differential elimination techniques. It is shown how sparse linear differential resultant formulas provide bounds for the order of derivation, even in the nonlinear case, and they also provide (in many cases) the bridge with results in the nonlinear algebraic case.

Keywords

Differential resultant, sparse differential polynomial, super essential, algebraically essential

1 Introduction

Differential elimination is an important operation in differential algebraic geometry that, in theory, can be achieved through Gröbner bases, characteristic sets and differential resultants. For applications, sparse differential elimination is the operation that is naturally necessary. Sparse algebraic resultants have been broadly studied, regarding theory and computation (see [2], [3], [8] and references there in), meanwhile differential resultants were recently defined in [4] for sparse Laurent differential polynomials.

The computation and applicability of sparse algebraic resultants attained great benefits from having close formulas for their representation. Similar formulas in the differential case would improve the existing bounds for degree and order of the sparse differential resultant and therefore the existing algorithms for its computation. Matrix formulas would also contribute to the development of methods to predict the support of the sparse differential resultant, achieving similar benefits to the ones obtained in the algebraic case. In the differential case, these so called Macaulay style formulas do not exist. The differential resultant formula defined by Carrà-Ferro in [1], is the algebraic resultant of Macaulay, of a set of derivatives of the ordinary differential polynomials in $\mathfrak{P}$. Already in the linear sparse generic case, these formulas vanish often, giving no information about the differential resultant $\partial\mathrm{Res}(\mathfrak{P})$, and this was the starting point of my interest in this topic ([5], [6]).

In [7], determinantal formulas are provided for systems of n linear nonhomogeneous (non necessarily generic) differential polynomials $\mathcal{P}$ in a set U of $n-1$ differential indeterminates. These formulas are determinants of coefficient matrices of appropriate sets of derivatives of the differential polynomials in $\mathcal{P}$, or in a linear perturbation $\mathcal{P}_\varepsilon$ of $\mathcal{P}$, and allow the elimination of the differential variables in U from $\mathcal{P}$. In particular, the formula $\partial\mathrm{FRes}(\mathcal{P})$ is the determinant of a matrix $\mathcal{M}(\mathcal{P})$ having no zero columns if the system $\mathcal{P}$ is "super essential". As an application, if the system $\mathfrak{P}$ is sparse generic, such formulas can be used to compute the differential resultant $\partial\mathrm{Res}(\mathfrak{P})$ introduced in [4].

To approach the nonlinear case, one should observe that differential polynomials can be sparse in degree and in order of derivation. One can start with the problem of taking the appropriate set of derivatives of the elements in $\mathcal{P}$ to get a system of differential polynomials $\mathrm{ps}(\mathcal{P})$, that seen as algebraic, should have L polynomials in $L-1$ variables. For this purpose, we extend here the "super essential" condition to non linear polynomials, taking into consideration the sparsity in the order. Results obtained in the linear case can also be used to check, in some cases, the existence of the algebraic resultant of the generic system of algebraic polynomials whose specialization is $\mathrm{ps}(\mathcal{P})$, providing a link with the machinery available in the sparse algebraic case.

2 Sparse differential resultant

Let $\mathbb{D}$ be an ordinary differential domain with derivation ∂. Let $U = \{u_1, \ldots, u_{n-1}\}$ be a set of differential indeterminates over $\mathbb{D}$. By $\mathbb{N}$ we mean the natural numbers including 0. For $k \in \mathbb{N}$, we denote by $u_{j,k}$ the k-th derivative of u_j and for $u_{j,0}$ we simply write u_j. We denote by $\{U\}$ the set of derivatives of the elements of U, $\{U\} = \{\partial^k u \mid u \in U,\ k \in \mathbb{N}\}$, and by $\mathbb{D}\{U\}$ the ring of differential polynomials in the differential indeterminates U, which is a differential ring with derivation ∂. Given a subset $\mathcal{U} \subset \{U\}$, we denote by $\mathbb{D}[\mathcal{U}]$ the ring of polynomials in the indeterminates $\mathcal{U}$. Given $f \in \mathbb{D}\{U\}$ and $y \in U$, we denote by $\mathrm{ord}(f, y)$ the order of f in the variable y. If f does not have a term in y then we define $\mathrm{ord}(f, y) = -1$. The order of f equals $\max\{\mathrm{ord}(f, y) \mid y \in U\}$.

Let $\mathcal{P} := \{f_1, \ldots, f_n\}$ be a system of differential polynomials in $\mathbb{D}\{U\}$. We assume that:

($\mathcal{P}1$) The order of f_i is $o_i \geq 0$, $i = 1, \ldots, n$. So that no f_i belongs to $\mathbb{D}$.

($\mathcal{P}2$) $\mathcal{P}$ contains n distinct polynomials.

($\mathcal{P}3$) $\mathcal{P}$ is a nonhomogeneous system. At least one of the polynomials in $\mathcal{P}$ has nonzero degree zero term.

Let $[\mathcal{P}]$ denote the differential ideal generated by $\mathcal{P}$ in $\mathbb{D}\{U\}$. Our goal is to obtain elements of differential elimination ideal $[\mathcal{P}] \cap \mathbb{D}$, using differential resultant formulas.

Let us consider a generic system of nonhomogeneous sparse differential polynomials

$$\mathfrak{P} = \left\{ \mathbb{F}_i := c_i + \sum_{h=1}^{m_i} c_{i,h} M_{i,h} \mid i = 1, \ldots, n \right\},$$

c_i and $c_{i,h}$ are differential indeterminates over $\mathbb{Q}$, m_i is the number of monomials of $\mathbb{F}_i$, and $M_{i,h}$ are monomials in the variables $\{U\}$. Let us consider the differential field $\mathbb{K} = \mathbb{Q}\langle c_{i,h|i=1,\ldots,n,h=1,\ldots,m_i}\rangle$ and observe that $\mathfrak{P}$ is a system in $\mathbb{D}\{U\}$, with $\mathbb{D} = \mathbb{K}\{c_1, \ldots, c_n\}$. If the differential elimination ideal $[\mathfrak{P}] \cap \mathbb{D}$ has dimension $n - 1$ then $[\mathfrak{P}] \cap \mathbb{D} = \mathrm{sat}(\partial\mathrm{Res}(\mathfrak{P}))$, the saturated ideal determined by a differential polynomial $\partial\mathrm{Res}(\mathfrak{P})$, which is called the sparse differential resultant of $\mathfrak{P}$. Sparse differential resultants were defined in [4], were their existence is proved to be equivalent with the differentially essential condition on $\mathfrak{P}$.

3 A system $\mathrm{ps}(\mathcal{P})$ of L polynomials in $L-1$ algebraic variables

Given $f \in \mathbb{D}\{U\}$, let us denote the differential support in u_j of f by

$$\mathfrak{S}_j(f) = \{k \in \mathbb{N} \mid u_{j,k}/M \text{ for some monomial } M \text{ of } f\}.$$

Note that $\mathrm{ord}(f, u_j) := \max \mathfrak{S}_j(f)$ and define $\mathrm{lord}(f, u_j) := \min \mathfrak{S}_j(f)$. For $j = 1, \ldots, n-1$, we define the next positive integers, to construct convenient intervals bounding the differential support sets $\mathfrak{S}_j(f_i)$,

$$\begin{aligned} \overline{\gamma}_j(\mathcal{P}) &:= \min\{o_i - \mathrm{ord}(f_i, u_j) \mid \mathfrak{S}_j(f_i) \neq \emptyset,\ i = 1, \ldots, n\}, \\ \underline{\gamma}_j(\mathcal{P}) &:= \min\{\mathrm{lord}(f_i, u_j) \mid \mathfrak{S}_j(f_i) \neq \emptyset,\ i = 1, \ldots, n\}, \end{aligned} \tag{1}$$

Given $j \in \{1, \ldots, n-1\}$, observe that, for all i such that $\mathfrak{S}_j(f_i) \neq \emptyset$ we have

$$\mathfrak{S}_j(f_i) \subseteq [\underline{\gamma}_j(\mathcal{P}), o_i - \overline{\gamma}_j(\mathcal{P})]. \tag{2}$$

Finally, $\gamma(\mathcal{P}) := \sum_{j=1}^{n-1} \gamma_j(\mathcal{P})$, with $\gamma_j(\mathcal{P}) := \underline{\gamma}_j(\mathcal{P}) + \overline{\gamma}_j(\mathcal{P})$.

Let $N := \sum_{i=1}^{n} o_i$. If $N - o_i - \gamma(\mathcal{P}) \geq 0$, $i = 1, \ldots, n$, the sets of lattice points $\mathbb{I}_i := [0, N - o_i - \gamma(\mathcal{P})] \cap \mathbb{N}$ are non empty. We define the set of differential polynomials

$$\mathrm{ps}(\mathcal{P}) := \{\partial^k f_i \mid k \in \mathbb{I}_i,\ i = 1, \ldots, n\}, \tag{3}$$

containing $L := \sum_{i=1}^{n} (N - o_i - \gamma(\mathcal{P}) + 1)$ differential polynomials, whose variables belong to the set $\mathcal{V}$ of $L - 1$ differential indeterminates

$$\mathcal{V} := \{u_{j,k} \mid k \in [\underline{\gamma}_j(\mathcal{P}), N - \overline{\gamma}_j(\mathcal{P}) - \gamma(\mathcal{P})] \cap \mathbb{N},\ j = 1, \ldots, n-1\}.$$

In general, given $j \in \{1, \ldots, n-1\}$ we have

$$\cup_{f \in \mathrm{ps}(\mathcal{P})} \mathfrak{S}_j(f) \subseteq [\underline{\gamma}_j(\mathcal{P}), N - \overline{\gamma}_j(\mathcal{P}) - \gamma(\mathcal{P})] \cap \mathbb{N}, \tag{4}$$

and we cannot guarantee that the equality holds. If there exists j such that (4) is not an equality, we will say that the system $\mathcal{P}$ is sparse in the order.

Let $x_{i,j}$, $i = 1, \ldots, n$, $j = 1, \ldots, n-1$ be algebraic indeterminates over $\mathbb{Q}$, the field of rational numbers. Let $X(\mathcal{P}) = (X_{i,j})$ be the $n \times (n-1)$ matrix, such that

$$X_{i,j} := \begin{cases} x_{i,j}, & \mathfrak{S}_j(f_i) \neq \emptyset, \\ 0, & \mathfrak{S}_j(f_i) = \emptyset. \end{cases} \tag{5}$$

We denote by $X_i(\mathcal{P})$, $i = 1, \ldots, n$, the submatrix of $X(\mathcal{P})$ obtained by removing its ith row. Thus $X(\mathcal{P})$ is an $n \times (n-1)$ matrix with entries in the field $\mathbb{K} := \mathbb{Q}(X_{i,j} \mid X_{i,j} \neq 0)$.

The notion of super essential system of differential polynomials was introduced in [7], for systems of linear differential polynomials and it is extended here to the nonlinear case.

Definition 3.1 *The system $\mathcal{P}$ is called* super essential *if* $\det(X_i(\mathcal{P})) \neq 0$, $i = 1, \ldots, n$.

Given a super essential system $\mathcal{P}$ (non necessarily linear), it can be proved as in [7], Lemma 3.6 that $N - o_i - \gamma(\mathcal{P}) \geq 0$, $i = 1, \ldots, n$. Furthermore, the next result can be shown adapting the proof of [7], Theorem 3.11 to the nonlinear case.

Theorem 3.2 *If $\mathcal{P}$ is super essential then*

$$\cup_{f \in \mathrm{ps}(\mathcal{P})} \mathfrak{S}_j(f) = [\underline{\gamma}_j(\mathcal{P}), N - \overline{\gamma}_j(\mathcal{P}) - \gamma(\mathcal{P})] \cap \mathbb{N}, \ \ j = 1, \ldots, n-1.$$

That is, $\mathcal{P}$ is a system of L polynomials in $L-1$ algebraic indeterminates.

It can be proved as in [7], Section 4 that every system $\mathcal{P}$ contains a super essential subsystem $\mathcal{P}^*$ and if $\mathrm{rank}(X(\mathcal{P})) = n-1$ then $\mathcal{P}^*$ is unique.

Example 3.3 *Let us consider the systems $\mathcal{P}_1 = \{f_1, f_2, f_3, f_4\}$ and $\mathcal{P}_2 = \{f_1, f_2, f_3, f_5\}$ with*

$$f_1 = 2 + u_1 u_{1,1} + u_{1,2}, f_2 = u_1 u_{1,2}, f_3 = u_2 u_{3,1}, f_4 = u_{1,1} u_2, f_5 = u_{1,2},$$

$$X(\mathcal{P}_1) = \begin{pmatrix} x_{1,1} & 0 & 0 \\ x_{2,1} & 0 & 0 \\ 0 & x_{3,2} & x_{3,3} \\ x_{4,1} & x_{4,2} & 0 \end{pmatrix} \text{ and } X(\mathcal{P}_2) = \begin{pmatrix} x_{1,1} & 0 & 0 \\ x_{2,1} & 0 & 0 \\ 0 & x_{3,2} & x_{3,3} \\ x_{4,1} & 0 & 0 \end{pmatrix}.$$

$\mathcal{P}_1$ is not super essential but since $\mathrm{rank}(X(P_1)) = 3$, *it has a unique super essential subsystem, which is $\{f_1, f_2\}$. $\mathcal{P}_2$ is not super essential and* $\mathrm{rank}(X(P_1)) < 3$, *super essential subsystems are $\{f_1, f_2\}$, $\{f_1, f_3\}$ and $\{f_2, f_3\}$.*

4 Associated sparse algebraic resultant

We can establish a bijection between $\mathcal{V}$ and the set $Y = \{y_1, \ldots, y_{L-1}\}$ of $L-1$ algebraic indeterminates. This can be extended to a ring homomorphism $\beta : \mathbb{D}[\mathcal{V}] \to \mathbb{D}[Y]$. Monomials in $\mathbb{D}[Y]$ are $Y^\alpha = y_1^{\alpha_1} \cdots y_{L-1}^{\alpha_{L-1}}$ with $\alpha = (\alpha_1, \ldots, \alpha_{L-1}) \in \mathbb{N}^{L-1}$. Given $f \in \mathbb{D}[\mathcal{V}]$, we denote the algebraic support $\mathcal{A}(f)$ of f, with $\beta(f) = \sum_{\alpha \in \mathbb{N}^{L-1}} a_\alpha Y^\alpha$, as $\mathcal{A}(f) := \left\{\alpha \in \mathbb{N}^{L-1} \mid a_\alpha \neq 0\right\}$.

We define the algebraic system of generic polynomials associated to $\mathcal{P}$ as

$$\mathrm{ags}(\mathcal{P}) = \left\{ \sum_{\alpha \in \mathcal{A}(f)} c_\alpha(f) Y^\alpha \mid f \in \mathrm{ps}(\mathcal{P}) \right\},$$

where $c_\alpha(f)$ are algebraic indeterminates over $\mathbb{Q}$. Let us denote $c(f) := c_{\overline{0}}(f)$, $f \in \mathrm{ps}(\mathcal{P})$, where $\overline{0}$ is the zero of $\mathbb{N}^{L-1}$.

Given a subsystem $\mathcal{S} \subseteq \mathrm{ags}(\mathcal{P})$, let us define the fields

$$\mathcal{E} := \mathbb{Q}\left(c_\alpha(f) \mid f \in \mathrm{ps}(\mathcal{P}), \alpha \in \mathcal{A}(f)\backslash\{\overline{0}\}\right), \qquad \mathcal{E}_{\mathcal{S}} := \mathcal{E}\left(f - c(f) \mid f \in \mathcal{S}\right).$$

As in [4], a subsystem of polynomials $\mathcal{S}$ of $\mathrm{ags}(\mathcal{P})$ is said to be algebraically independent if the transcendence degree, of $\mathcal{E}_{\mathcal{S}}$ over $\mathcal{E}$, $\mathrm{trdeg}(\mathcal{E}_{\mathcal{S}}/\mathcal{E}) = |\mathcal{S}|$, otherwise it is said to be algebraically dependent. A subsystem of polynomials $\mathcal{S}$ of $\mathrm{ags}(\mathcal{P})$ is said to be algebraically essential if $\mathcal{S}$ is algebraically dependent and every proper subsystem $\mathcal{S}'$ of $\mathcal{S}$ is algebraically independent.

Assuming that $\cup_{f\in\mathrm{ps}(\mathcal{P})}\mathcal{A}(f)\backslash\{\overline{0}\}$ spans $\mathbb{Z}^{L-1}$, it was proved in [8] that, a necessary and sufficient condition for the existence of the algebraic resultant R of $\mathrm{ags}(\mathcal{P})$ is the existence of a unique algebraically essential subsystem of $\mathrm{ags}(\mathcal{P})$.

Example 4.1 *Let us consider the system $\mathcal{P} = \{f_1, f_2\}$ in $\mathbb{D}\{u\}$,*

$$\begin{aligned} f_1 &= a_2x + (a_1 + a_4x)u + u' + (a_3 + a_6x)u^2 + a_5u^3, \\ f_2 &= x' + (b_1 + b_3x)u + (b_2 + b_5x)u^2 + b_4u^3, \end{aligned}$$

with a_i, b_j algebraic indeterminates over $\mathbb{Q}$, $\mathbb{D} = \mathbb{Q}(t)[a_i, b_j]\{x\}$ and $\partial = \frac{\partial}{\partial t}$. Since $\mathrm{ps}(\mathcal{P}) = \{f_1, f_2, \partial f_2\}$, with $\partial f_2 = x'' + b_3x'u + (b_3x + b_1)u' + b_5x'u^2 + (2b_5x + 2b_2)uu' + 3b_4u^2u'$ and $\mathcal{V} = \{u, u'\}$, we have the following system of algebraic generic polynomials in y_1, y_2

$$\mathrm{ags}(\mathcal{P}) = \left\{ \begin{array}{l} P_1 = c^1_{(0,0)} + c^1_{(1,0)}y_1 + c^1_{(0,1)}y_2 + c^1_{(2,0)}y_1^2 + c^1_{(3,0)}y_1^3, \\ P_2 = c^2_{(0,0)} + c^2_{(1,0)}y_1 + c^2_{(2,0)}y_1^2 + c^2_{(3,0)}y_1^3, \\ P_3 = c^3_{(0,0)} + c^3_{(1,0)}y_1 + c^3_{(0,1)}y_2 + c^3_{(2,0)}y_1^2 + c^3_{(1,1)}y_1y_2 + c^3_{(2,1)}y_1^2y_2 \end{array} \right\},$$

where c_{α,f_1}, c_{α,f_2} and $c_{\alpha,\partial f_2}$ are denoted by c^1_α, c^2_α and c^3_α respectively, $\alpha \in \mathbb{N}^2$. Observe that $\mathrm{ags}(\mathcal{P})$ is algebraically essential because the linear part of the polynomials in $\mathrm{ags}(\mathcal{P})$, $\{c^1_{(0,0)} + c^1_{(1,0)}y_1 + c^1_{(0,1)}y_2, c^2_{(0,0)} + c^2_{(1,0)}y_1, c^3_{(0,0)} + c^3_{(1,0)}y_1 + c^3_{(0,1)}y_2\}$ is an algebraically essential system. Thus the algebraic resultant R of $\mathrm{ags}(\mathcal{P})$ exists and it generates the algebraic ideal $(\mathrm{ags}(\mathcal{P})) \cap \mathbb{Q}[c^i_\alpha \mid \alpha \in \mathcal{A}(f_i), i = 1, 2, \alpha \in \mathcal{A}(\partial f_2), i = 3] = (R)$. Using "toricres04", Maple 9 code for sparse (toric) resultant matrices by I.Z. Emiris, [2], we obtain a matrix M whose determinant is $c^3_{(0,0)}R$. This matrix is the coefficient matrix of the polynomials

$$y_1P_1, y_1y_2P_1, y_1y_2^2P_1, y_1^2P_2, y_1y_2P_2, y_1^2y_2P_2, y_1y_2^2P_2, y_1^2y_2^2P_2, y_1P_3, y_1y_2P_3, y_1y_2^2P_3, y_1y_2^3P_3$$

in the monomials $y_1, y_1^2, y_1y_2, y_1^2y_2, y_1y_2^2, y_1^2y_2^2, y_1y_2^3, y_1^2y_2^3, y_1y_2^4, y_1^2y_2^4, y_1y_2^5, y_1^2y_2^5$. The specialization of the algebraic indeterminates $\{c^i_\alpha \mid \alpha \in \mathcal{A}(f_i), i = 1, 2, \alpha \in \mathcal{A}(\partial f_2), i = 3\}$ in R, to the corresponding coefficients of $\mathrm{ps}(\mathcal{P})$, gives a nonzero differential polynomial $\overline{R}$ in the differential elimination ideal $[\mathcal{P}] \cap \mathbb{D}$.

5 Some consequences from results in the linear case

Given a linear system $\mathcal{P}$, differential resultant formulas were defined in [7], see also [5] and [6]. In particular, if $N - o_i - \gamma \geq 0$, $i = 1, \ldots, n$, the formula $\partial\mathrm{FRes}(\mathcal{P})$ is the determinant of the $L \times L$ coefficient matrix $\mathcal{M}(\mathcal{P})$ of the set of polynomials $\mathrm{ps}(\mathcal{P})$ in the set of variables $\mathcal{V}$. Furthermore, if $\mathcal{P}$ is super essential, by Theorem 3.2 (which is [7], Theorem 3.11 in the linear case), the matrix $\mathcal{M}(\mathcal{P})$ has no zero columns.

Let $\mathcal{P}$ be a linear system and let $\mathcal{S} = \mathrm{ags}(\mathcal{P})$, which is also a linear system. For every subsystem $\mathcal{S}'$ of $\mathcal{S}$, let $C(\mathcal{S}')$ be the coefficient matrix of the homogeneous part of the polynomials in $\mathcal{S}'$ in the variables Y, this is a $|\mathcal{S}'| \times L - 1$ matrix. Adapting the results in [7], Section 4 the next proposition is proved.

Proposition 5.1 *Let $\mathcal{P}$ be a super essential linear system and let $\mathcal{S} = \mathrm{ags}(\mathcal{P})$. The following statements hold:*

1. *Let $\mathcal{S}_l$ be the subsystem of $\mathcal{S}$ obtained by removing its lth polynomial, $l = 1, \ldots, L$. $\mathcal{S}$ is algebraically essential if and only if $\det(C(\mathcal{S}_l)) \neq 0$, $l = 1, \ldots, L$.*

2. *There exists an algebraically essential subsystem of $\mathcal{S}$.*

3. $\mathrm{rank}(C(\mathcal{S})) = |\mathcal{S}| - 1 = L - 1$ *if and only if there exists a unique algebraically essential subsystem* $\mathcal{S}^*$ *of* $\mathcal{S}$.

Let $\mathfrak{P}$ be a generic system of sparse linear differential polynomials. As a consequence of the previous result, if $\partial\mathrm{FRes}(\mathfrak{P}) \neq 0$ then $\mathrm{ags}(\mathfrak{P})$ contains a unique algebraically essential subsystem $\mathcal{S}^*$, which corresponds to a subsystem of $\mathrm{ps}(\mathfrak{P})$ that we call S^*. Let $\mathcal{M}(S^*)$ be the coefficient matrix of S^*, which is $|S^*| \times |S^*|$. The rows and columns of $\mathcal{M}(\mathfrak{P})$ can be reorganized to obtain a matrix

$$\begin{bmatrix} E & * \\ 0 & \mathcal{M}(S^*) \end{bmatrix}, \text{ such that } \partial\mathrm{FRes}(\mathfrak{P}) = \pm\det(E)\det(\mathcal{M}(S^*)), \text{ and } \partial\mathrm{Res}(\mathfrak{P}) = \det(\mathcal{M}(S^*)).$$

Using the previous results, a family of systems $\mathcal{F}$ of generic sparse differential polynomials can be obtained, so that degree bounds of the sparse differential resultant can be given in terms of mixed volumes. In [4], such bound was given for the case of generic non sparse differential polynomials. Let $\mathrm{lin}(\mathfrak{P})$ be the system of the linear parts of the polynomials in $\mathfrak{P}$. We define $\mathcal{F}$ as the family of all systems of generic sparse differential polynomials in the variables $\{U\}$ such that the supports of the polynomials in $\mathrm{ps}(\mathrm{lin}(\mathfrak{P}))$ jointly span $\mathbb{Z}^{L-1}$ and $\partial\mathrm{FRes}(\mathrm{lin}(\mathfrak{P})) \neq 0$, see Example 4.1. For every $\mathfrak{P}$ in $\mathcal{F}$, $\mathrm{ags}(\mathrm{lin}(\mathfrak{P}))$ is algebraically essential and furthermore $\mathrm{ags}(\mathfrak{P})$ is algebraically essential, thus the algebraic resultant of $\mathrm{ags}(\mathfrak{P})$ exists and it can be used to give bounds of the degrees in terms of mixed volumes.

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