

Accuracy estimates based on multifield variational principles

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ABSTRACT. — Recent developments to fit the so called Free Formulation into a variational framework have suggested the possibility of introducing a new category of error estimates for finite element computations. Such error estimates are based on differences between certain multifield functionals, which give the same value for the true solution. In the present paper the formulation of some estimates of this kind is introduced for elasticity and plate bending problems, and several examples of their performance are discussed.

The observed numerical behavior of the new accuracy measures seems to be acceptable from an engineering point of view. However, further numerical experimentation is still needed to establish practical tolerance levels for real problems.

1. Introduction

After a history of more than thirty years, the finite element method has become a widely used engineering tool, particularly in Structural and Solid Mechanics. At the present stage, the role of *a posteriori* accuracy estimates for finite element computations is considered essential; since engineers with little or no knowledge of the foundations of the numerical tool can now use finite elements to model a structure and produce results. In this situation, the accuracy estimates provide the analyst with objective measures of quality. They allow the setting up of tolerance levels to consider a particular solution as acceptable. A natural consequence is that whenever the estimated accuracy is not enough, the discretization will have to be improved until the selected measure meets the standard. If the finite element method tends to be used as a "black box" (e. g. within integrated CAD/FEA/CAM packages) the importance of developing automatic solution improvement processes becomes apparent. This is the idea behind adaptative procedures and it must be realized that the key element of such methods is a reliable accuracy estimate.

The importance of developing reliable and economical accuracy (or error) estimates has been recognized by a large number of researchers during the past decade. A good survey of their work can be found in [Babuska *et al.*, 1986], though this reference should be completed by more recent papers [Zienkiewicz & Zhu, 1987]. The same ideas have also been applied to other numerical methods [Alarcón & Reverter, 1986]. The error

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estimates proposed in the literature generally fall into one of the following wide categories:

1) Estimates computed from the residuals corresponding to the finite element solution [Kelly *et al.*, 1983; Zienkiewicz *et al.*, 1983; Kelly, 1984].

2) Estimates based on the extrapolation of successive approximate solutions [Szabo, 1986].

3) Estimates that take advantage of the differences between the finite element stress field (stress in a generalized sense) and a smoothed or projected stress field, that is considered to have a higher order of accuracy [Z & Z, 1987].

Estimates within the first category were introduced before the others [Babuska, 1975]. The residuals result from inserting the finite element solution into the differential equations that govern the problem. Generally, their numerical computation is very expensive and, for that reason, the estimates based on the residuals often require an out of proportion computational effort. However, very reliable accuracy estimates can be obtained.

In order to compute the estimates of the second category, it is assumed that the energy norms of the approximate solutions converge to the exact energy norm in a predetermined way. Thus, if a number of finite element solutions are known, extrapolation can be used to make a prediction of the true energy norm. The practical drawback is precisely the need of computing several solutions, but the method is very well suited for adaptive processes with complete refinement.

Estimates in the third category have been introduced more recently and they are easily implemented within the usual structure of a finite element code. The main difficulty in this case is to obtain the "smooth" stress field in a way consistent with the assumption that it has a higher degree of accuracy.

On the other hand, recent developments by Felippa [Felippa, 1989 *a*; Felippa, 1989 *b*; Felippa, 1989 *c*] to fit the Free Formulation of Bergan and Nygard [Bergan & Nygard, 1984] in a variational framework, have suggested the possibility of introducing a new category of error estimates. The main concept is as follows. If a particular problem can be solved using a family of variational principles derived one from another by the Lagrange multiplier method, then all the associated functionals yield the same value at the exact solution of the problem [Courant & Hilbert, 1953]. This important property suggests that, given an approximate solution, a certain difference between functionals might be used as an estimate of the accuracy. The difficulty is to identify such a difference, so that it is easily computed within the usual structure of a finite element code and it gives a reliable estimate, at least from an engineering point of view. In this sense, the classical dual analysis principle [Fraeijs de Veubeke, 1965], as a method to compute upper and lower bounds, is not allowed, since it involves two completely different finite element models, even though it yields a very reliable error estimate. The goal is to develop estimates usable within a single finite element approach. This will lead to the use of mixed or hybrid elements.

The authors have worked in the above direction, looking for error estimates for elasticity problems (C^0 problems) and plate bending problems (C^1 problems) [Beltrán, 1990], although the idea can probably be generalized to any problem with a variational

foundation. The estimates within the proposed new category are difficult to implement if elements formulated in the standard way (variational principle with only one independent field) are used. However, it must be noted that nowadays the research effort to develop new elements, mainly shell and plate elements, is strongly based on techniques associated implicitly or explicitly with multifield variational principles. Examples of that are the Free Formulation itself or the Assumed Natural Strain [MacNeal, 1978; Park & Stanley, 1986; Bathe & Dvorkin, 1985; Simó & Hughes, 1986] and the Assumed Natural Deviatoric Strain [Militello & Felippa, 1989] formulations. It is in this context where the proposed estimates would be useful.

In the following paragraphs the work the authors in elasticity and plate bending problems is summarized. In both cases the proposed error estimate is introduced and the main aspects of the elements developed to be used with it are described. Also, in order to give an idea of the numerical behavior of these accuracy estimates, some examples with known analytical solution are presented.

2. Elasticity

2.1. NOTATION

In order to set the notation, the linear elasticity problem is stated here.

Consider a linearly elastic body under static loading that occupies a domain $\Omega \subset \mathbb{R}^3$. The boundary of the body is a surface S , $S = S_d \cup S_r$. The outward unit normal on S is \mathbf{n} .

On S_d the displacements $\bar{\mathbf{d}}$ are known, whereas surface tractions $\bar{\mathbf{t}}$ are imposed on S_r . The volume force field \mathbf{b} in Ω is also given. The unknowns of the problem are: the displacement field \mathbf{u} in Ω , the infinitesimal strain field \mathbf{e} in Ω and the stress field $\boldsymbol{\sigma}$ in Ω .

When variational principles are used to solve the problem, two types of fields are involved: independent or primary fields, which are subject to variations, and secondary or derived fields, which are obtained from primary fields. The solution is determined by taking variations with respect to the independent fields. In the present work the notation from Felippa [F. 1989 c] will be adopted to distinguish between dependent and independent fields. An independently varied field will be denoted by a tilde “ \sim ” over its symbol, e.g. $\tilde{\boldsymbol{\sigma}}$, $\tilde{\mathbf{u}}$... For a dependent field the dependence will be identified by writing the independent field symbol as a superscript. For example:

$$(1) \quad \mathbf{e}^u = \frac{1}{2}(\nabla + \nabla^t)\tilde{\mathbf{u}} \quad \text{and} \quad \boldsymbol{\sigma}^u = \mathbf{D}\mathbf{e}^u$$

where ∇ represents the gradient operator and \mathbf{D} is the tensor of elastic constants. In this notation the symbols without tilde or superscript \mathbf{u} , \mathbf{e} , $\boldsymbol{\sigma}$ are reserved for the exact solution fields.

The writing of volume and surface integrals will be abbreviated by placing the integrand between domain-subscripted parentheses and square brackets respectively:

$$(2) \quad (f)_{\Omega} \equiv \int_{\Omega} f d\Omega. \quad [f]_{S_i} \equiv \int_{S_i} f dS$$

If \mathbf{f} and \mathbf{g} are tensor functions, the following notation is defined:

$$(3) \quad (\mathbf{f}, \mathbf{g})_{\Omega} \equiv \int_{\Omega} \mathbf{f} : \mathbf{g} d\Omega$$

and similarly for surface integrals, in which case square brackets are used.

2.2. VARIATIONAL APPROACH

The most common variational principle in solving the elasticity problem is the Principle of Minimum Potential Energy. It states that from all the displacement fields $\tilde{\mathbf{u}}$ satisfying the kinematical boundary condition $\tilde{\mathbf{u}} = \bar{\mathbf{d}}$ on S_d , the solution \mathbf{u} of the problem is the one that makes the total potential energy Π_p :

$$(4) \quad \Pi_p(\tilde{\mathbf{u}}) = \frac{1}{2}(\boldsymbol{\sigma}^u, \mathbf{e}^u)_{\Omega} - (\mathbf{b}, \tilde{\mathbf{u}})_{\Omega} - [\bar{\mathbf{t}}, \tilde{\mathbf{u}}]_{S_t}$$

be a minimum.

The Principle of Minimum Potential Energy can be generalized using the Lagrange multiplier method [C & H, 1953; Jones, 1964; Washizu, 1974]. The application of this technique to a variational principle allows the suppression of constraints for the primary field(s) in the associated functional by introducing new fields: the Lagrange multipliers. Starting from the Principle of Minimum Potential Energy, a whole family of derived variational principles can be obtained. In general, the solution of the elasticity problem makes the functionals associated to those principles stationary (not necessarily a minimum or a maximum).

One of the most general principles that can be derived in the way described above is the Hu-Washizu principle [W, 1974]. It states that the solution fields \mathbf{u} , \mathbf{e} , $\boldsymbol{\sigma}$, \mathbf{t} , where \mathbf{t} is the surface tractions field on S_d , make the functional:

$$(5) \quad \Pi_w(\tilde{\mathbf{u}}, \tilde{\mathbf{e}}, \tilde{\boldsymbol{\sigma}}, \tilde{\mathbf{t}}) = \frac{1}{2}(\boldsymbol{\sigma}^e, \tilde{\mathbf{e}})_{\Omega} + (\tilde{\boldsymbol{\sigma}}, \mathbf{e}^u - \tilde{\mathbf{e}})_{\Omega} - (\mathbf{b}, \tilde{\mathbf{u}})_{\Omega} - [\tilde{\mathbf{t}}, \tilde{\mathbf{u}}]_{S_t} + [\tilde{\mathbf{t}}, \bar{\mathbf{d}} - \tilde{\mathbf{u}}]_{S_d}$$

stationary without any constraint for the four independent fields.

In the functional Π_w , fields $\tilde{\mathbf{e}}$ can be introduced that are not independent but derived from the primary stress fields $\tilde{\boldsymbol{\sigma}}$ through the tensor of elastic constants:

$$(6) \quad \tilde{\mathbf{e}} = \mathbf{e}^{\sigma} = \mathbf{D}^{-1} \tilde{\boldsymbol{\sigma}} \quad \text{in } \Omega$$

Also, the fields $\tilde{\boldsymbol{\sigma}}$ and $\tilde{\mathbf{t}}$ can be related using:

$$(7) \quad \tilde{\mathbf{t}} = \mathbf{t}^{\sigma} = \tilde{\boldsymbol{\sigma}} \mathbf{n} \quad \text{on } S_d$$

Introducing (6) and (7) into (5) yields the functional:

$$(8a) \quad \Pi_R(\tilde{\mathbf{u}}, \tilde{\boldsymbol{\sigma}}) = -\frac{1}{2}(\tilde{\boldsymbol{\sigma}}, \mathbf{e}^\sigma)_\Omega + (\tilde{\boldsymbol{\sigma}}, \mathbf{e}^u)_\Omega - (\mathbf{b}, \tilde{\mathbf{u}})_\Omega - [\tilde{\mathbf{t}}, \tilde{\mathbf{u}}]_{S_t} + [\tilde{\boldsymbol{\sigma}} \mathbf{n}, \bar{\mathbf{d}} - \tilde{\mathbf{u}}]_{S_d}$$

The displacement fields $\tilde{\mathbf{u}}$ can be taken in a way they meet the boundary condition $\tilde{\mathbf{u}} = \bar{\mathbf{d}}$ on S_d , so that the functional Π_R is written as:

$$(8b) \quad \Pi_R(\tilde{\mathbf{u}}, \tilde{\boldsymbol{\sigma}}) = -\frac{1}{2}(\tilde{\boldsymbol{\sigma}}, \mathbf{e}^\sigma)_\Omega + (\tilde{\boldsymbol{\sigma}}, \mathbf{e}^u)_\Omega - (\mathbf{b}, \tilde{\mathbf{u}})_\Omega - [\tilde{\mathbf{t}}, \tilde{\mathbf{u}}]_{S_t}$$

and then the solution fields $\mathbf{u}, \boldsymbol{\sigma}$ make (8b) stationary if the functional is defined on fields $\tilde{\mathbf{u}}$ satisfying the boundary condition on S_d . This is the variational principle of Hellinger-Reissner [Reissner, 1950].

Note that the total potential energy functional (4) and the Hellinger-Reissner functional (8b) are special cases of the following parametrized form:

$$(9) \quad \begin{aligned} \Pi_\gamma(\tilde{\mathbf{u}}, \tilde{\boldsymbol{\sigma}}) &= (1-\gamma)\Pi_P(\tilde{\mathbf{u}}) + \gamma\Pi_R(\tilde{\mathbf{u}}, \tilde{\boldsymbol{\sigma}}) \\ &= (1-\gamma)\frac{1}{2}(\boldsymbol{\sigma}^u, \mathbf{e}^u)_\Omega - \gamma\frac{1}{2}(\tilde{\boldsymbol{\sigma}}, \mathbf{e}^\sigma)_\Omega + \gamma(\tilde{\boldsymbol{\sigma}}, \mathbf{e}^u)_\Omega - (\mathbf{b}, \tilde{\mathbf{u}})_\Omega - [\tilde{\mathbf{t}}, \tilde{\mathbf{u}}]_{S_t} \end{aligned}$$

with $\gamma \in \mathbb{R}$, $0 \leq \gamma \leq 1$.

Then, it is conceivable a parametrized variational principle intermediate between the Minimum Potential Energy Principle and the Hellinger-Reissner Principle: the solution fields $\mathbf{u}, \boldsymbol{\sigma}$ will make stationary the functional (9) if primary displacement fields $\tilde{\mathbf{u}}$ satisfy $\tilde{\mathbf{u}} = \bar{\mathbf{d}}$ on S_d .

Parametrized functionals of the kind of (9) were introduced by Felippa [F, 1989c]. For elasticity, variational principles associated with functionals including up to three independent parameters can be stated [Felippa & Militello, 1989]. If these principles are used to obtain approximate solutions by the finite element method, it is likely that certain combinations of the parameters yield better results than the classical principles. In fact, parametrized principles are being used successfully in the formulation of high performance elements [F & M, 1989].

2.3. ERROR ESTIMATE

All functionals presented in the previous paragraphs yield the same value when the exact solution fields $\mathbf{u}, \boldsymbol{\sigma}, \mathbf{e}, \mathbf{t}$ are introduced into them. This is a general property of the Lagrange multiplier method [C & H, 1953]. Hence, it can be written:

$$(10) \quad \Pi_P(\mathbf{u}) = \Pi_W(\mathbf{u}, \boldsymbol{\sigma}, \mathbf{e}, \mathbf{t}) = \Pi_R(\mathbf{u}, \boldsymbol{\sigma}) = \Pi_\gamma(\mathbf{u}, \boldsymbol{\sigma}) = \Pi$$

where Π is the exact potential for the problem.

The question is to find out whether the differences between the functionals for approximate solutions can be used as estimates of the discretization error.

As a first step, the measure of the discretization error to be estimated should be defined. When the solution of an elasticity problem is approximated using the Minimum Potential Energy Principle, there is only one unknown field: the displacement field. Then, a field of errors \mathbf{er} can be defined as:

$$(11) \quad \mathbf{er} = \tilde{\mathbf{u}} - \mathbf{u}$$

where $\tilde{\mathbf{u}}$ represents the approximate solution for the displacements and \mathbf{u} is the true solution.

The magnitude of the error field \mathbf{er} is usually measured by its energy norm [K *et al.*, 1983]:

$$(12) \quad \|\mathbf{er}\|^2 = (\boldsymbol{\sigma}^{er}, \mathbf{e}^{er})_{\Omega}$$

and, in most cases, this norm is the one that conventional error estimates try to approximate. On the other hand, $\tilde{\mathbf{u}}$ is a kinematically admissible field:

$$(13) \quad \tilde{\mathbf{u}} = \bar{\mathbf{d}} \quad \text{on } S_d$$

then [F, 1989 *c*; B, 1990]:

$$(14) \quad \Pi_p(\tilde{\mathbf{u}}) - \Pi = \frac{1}{2} \|\mathbf{er}\|^2$$

that is, the magnitude of the error field \mathbf{er} , represented by its energy norm, can be obtained as a difference between the exact potential and the approximate functional.

In a similar way, when multifield variational principles are used, e. g. the one associated with $\Pi_{\gamma}(\tilde{\mathbf{u}}, \tilde{\boldsymbol{\sigma}})$, the error magnitude can be defined as the difference $\Pi_{\gamma}(\tilde{\mathbf{u}}, \tilde{\boldsymbol{\sigma}}) - \Pi$ between the value of the functional for the approximate solution and the potential. The difference $\Pi_p(\tilde{\mathbf{u}}) - \Pi$ between the total potential and the true potential can still be utilized in this case. The error measures defined by those differences are the values to be approximated by an estimate.

It is proposed as an estimate of the discretization error the value:

$$(15) \quad \varepsilon = \Pi_p(\tilde{\mathbf{u}}) - \Pi_{\gamma}(\tilde{\mathbf{u}}, \tilde{\boldsymbol{\sigma}})$$

associated with the approximate fields $\tilde{\mathbf{u}}$ y $\tilde{\boldsymbol{\sigma}}$, where $\tilde{\mathbf{u}}$ is a kinematically admissible field.

Note that the presence of two independent fields $\tilde{\mathbf{u}}$, $\tilde{\boldsymbol{\sigma}}$ in (15) will generally force the use of mixed formulations, so that the two fields are available for computing the estimate. However, the difference (15) is relatively easy to compute since:

$$(16) \quad \begin{aligned} \Pi_p(\tilde{\mathbf{u}}) - \Pi_{\gamma}(\tilde{\mathbf{u}}, \tilde{\boldsymbol{\sigma}}) &= \frac{1}{2}(\boldsymbol{\sigma}^u, \mathbf{e}^u)_{\Omega} - \frac{1}{2}(1-\gamma)(\boldsymbol{\sigma}^u, \mathbf{e}^u)_{\Omega} + \frac{1}{2}\gamma(\tilde{\boldsymbol{\sigma}}, \mathbf{e}^{\sigma})_{\Omega} - \gamma(\tilde{\boldsymbol{\sigma}}, \mathbf{e}^u)_{\Omega} \\ &= \frac{1}{2}\gamma(\boldsymbol{\sigma}^u, \mathbf{e}^u)_{\Omega} + \frac{1}{2}\gamma(\tilde{\boldsymbol{\sigma}}, \mathbf{e}^{\sigma} - \mathbf{e}^u)_{\Omega} - \frac{1}{2}\gamma(\tilde{\boldsymbol{\sigma}}, \mathbf{e}^u)_{\Omega} = \frac{1}{2}\gamma(\boldsymbol{\sigma}^u - \tilde{\boldsymbol{\sigma}}, \mathbf{e}^u - \mathbf{e}^{\sigma})_{\Omega} \end{aligned}$$

if the reciprocity relationship $(\bar{\boldsymbol{\sigma}}, \mathbf{e}^u)_\Omega = (\boldsymbol{\sigma}^u, \mathbf{e}^\sigma)_\Omega$ is taken into account.

That is, it suffices an integration over the domain Ω : integration that can be performed by adding the contributions from the subdomains or elements Ω_i into which Ω is divided:

$$(17) \quad \varepsilon = \sum_i \varepsilon_i = \sum_i \frac{1}{2} \gamma (\boldsymbol{\sigma}^u - \bar{\boldsymbol{\sigma}}, \mathbf{e}^u - \mathbf{e}^\sigma)_{\Omega_i}$$

The contributions ε_i are estimates for the local error. In a finite element discretization the values:

$$(18) \quad \eta_i = \frac{\varepsilon_i}{\text{vol}_i}$$

where vol_i is the volume of element i , represent error densities that can be used as indicators of the elements in the mesh where a refinement would be more profitable.

Finally, it must be noted that, according to (16), if the stress-strain relationship is positive definite and the parameter γ is $\gamma > 0$, then the proposed estimate ε is $\varepsilon \geq 0$.

2.4. FINITE ELEMENT DISCRETIZATION

The implementation of the error estimate introduced in the previous section is relatively easy when the variational principle associated to the parametrized functional $\Pi_\gamma(\bar{\mathbf{u}}, \bar{\boldsymbol{\sigma}})$ is used to obtain an approximate solution. If the procedure is the finite element method, that implies the use of elements based on the parametrized variational principle. In the present work, the choice has been to develop first such elements since, once they are available, the computation of the error estimate is immediate. In the following paragraphs the guidelines of the element formulation and the requirements for obtaining acceptable estimates are given. The details can be found in [B, 1990].

Following standard practice in finite element literature, the components of stresses and strains are arranged as column vectors, whereas the elastic coefficients are arranged as a square symmetric matrix; *i. e.* $\boldsymbol{\sigma}$, \mathbf{e} will represent column vectors from this point onwards, and \mathbf{D} will be a square symmetric matrix.

If the domain Ω is considered to be divided into a number of subdomains or elements Ω_i , the basic finite elements assumption is that the displacement field $\bar{\mathbf{u}}$ and the stress field $\bar{\boldsymbol{\sigma}}$ within an element can be expressed as a linear combination of displacement modes and stress modes respectively:

$$(19) \quad \bar{\mathbf{u}} = \mathbf{N} \mathbf{q} \quad \text{in } \Omega_i \quad \text{and}$$

$$(20) \quad \bar{\boldsymbol{\sigma}} = \mathbf{A} \mathbf{a} \quad \text{in } \Omega_i$$

where matrices \mathbf{N} and \mathbf{A} collect generalized-displacement shape functions and internal stress modes, and column vectors \mathbf{q} and \mathbf{a} gather generalized displacements and stress mode amplitudes.

In the present work, the number of internal displacement modes (*i. e.* the number of vector \mathbf{q} components) should be equal to the number n of external (nodal) degrees of

freedom in the element. The vector that gathers these n nodal displacements will be denoted by \mathbf{v} . The kinematical relationship between the amplitudes of internal displacement modes, \mathbf{q} , and the nodal displacements, \mathbf{v} , is easily obtained collocating (19) for each node:

$$(21) \quad \mathbf{v} = \mathbf{G} \mathbf{q}$$

where \mathbf{G} is a square matrix of order n .

Internal displacement modes in \mathbf{N} should be linearly independent, so that matrix \mathbf{G} has an inverse \mathbf{H} :

$$(22) \quad \mathbf{q} = \mathbf{H} \mathbf{v}$$

The fields derived from $\tilde{\mathbf{u}} \gamma \tilde{\boldsymbol{\sigma}}$ are:

$$(23) \quad \mathbf{e}^u = \frac{1}{2}(\mathbf{V} + \mathbf{V}') \mathbf{N} \mathbf{q} = \mathbf{B} \mathbf{q} \quad \text{in } \Omega_i$$

$$(24) \quad \boldsymbol{\sigma}^u = \mathbf{D} \mathbf{B} \mathbf{q} \quad \text{in } \Omega_i \quad \text{and}$$

$$(25) \quad \mathbf{e}^\sigma = \mathbf{D}^{-1} \boldsymbol{\sigma} = \mathbf{D}^{-1} \mathbf{A} \mathbf{a} \quad \text{in } \Omega_i$$

If (19), (20) and (23) to (25) are introduced into the parametrized functional (9), its discrete or algebraic form is obtained:

$$(26) \quad \Pi_\gamma(\tilde{\mathbf{u}}, \tilde{\boldsymbol{\sigma}}) = \sum_i \Pi_{\gamma_i} = \frac{1}{2}(1-\gamma) \sum_i \mathbf{q}^t (\mathbf{B}^t \mathbf{D} \mathbf{B})_{\Omega_i} \mathbf{q} \\ - \frac{1}{2} \gamma \sum_i \mathbf{a}^t (\mathbf{A}^t \mathbf{D}^{-1} \mathbf{A})_{\Omega_i} \mathbf{a} + \gamma \sum_i \mathbf{a}^t (\mathbf{A}^t \mathbf{B})_{\Omega_i} \mathbf{q} - \sum_i (\mathbf{b}^t \mathbf{N})_{\Omega_i} \mathbf{q} - \sum_i [\bar{\mathbf{f}}^t \mathbf{N}]_{S_i} \mathbf{q}$$

where \mathbf{q} and \mathbf{a} are the displacement and stress amplitudes within element i .

It should be noted that the computation of the functional Π_γ as a sum of values Π_{γ_i} corresponding to the subdomains Ω_i (Eq. (26)) assumes implicitly that fields $\tilde{\boldsymbol{\sigma}}$ and $\boldsymbol{\sigma}^u$ are finite on interelement boundaries. In this sense, (26) is written assuming that the field $\tilde{\mathbf{u}}$ is continuous across those boundaries.

If the following definitions are introduced to represent the integrals in (26):

$$(27) \quad \left\{ \begin{array}{lll} \mathbf{K}_u = (\mathbf{B}^t \mathbf{D} \mathbf{B})_{\Omega_i}, & \mathbf{C} = (\mathbf{A}^t \mathbf{D}^{-1} \mathbf{A})_{\Omega_i} \\ \mathbf{Q}^t = (\mathbf{A}^t \mathbf{B})_{\Omega_i}, & \mathbf{f}'_q = (\mathbf{b}^t \mathbf{N})_{\Omega_i}, & \mathbf{f}'_s = [\bar{\mathbf{f}}^t \mathbf{N}]_{S_i} \end{array} \right.$$

the fraction of the functional corresponding to a generic element is:

$$(28) \quad \Pi_{\gamma_i}(\mathbf{q}, \mathbf{a}) = \frac{1}{2}(1-\gamma) \mathbf{q}^t \mathbf{K}_u \mathbf{q} - \frac{1}{2} \gamma \mathbf{a}^t \mathbf{C} \mathbf{a} + \gamma \mathbf{a}^t \mathbf{Q}^t \mathbf{q} - \mathbf{f}'_q \mathbf{q} - \mathbf{f}'_s \mathbf{q}$$

And making Π_{γ_i} stationary for the variations of \mathbf{q} and \mathbf{a} yields the two following matrix equations:

$$(29) \quad 1) \quad \left\{ \begin{array}{l} \frac{\partial \Pi_{\gamma_i}}{\partial \mathbf{q}} = \mathbf{0} \quad (\text{equilibrium}) \\ (1-\gamma)\mathbf{K}_u \mathbf{q} + \gamma \mathbf{Q} \mathbf{a} - \mathbf{f}_q - \mathbf{f}_s = \mathbf{0} \end{array} \right.$$

$$(30) \quad 2) \quad \left\{ \begin{array}{l} \frac{\partial \Pi_{\gamma_i}}{\partial \mathbf{a}} = \mathbf{0} \quad (\text{internal compatibility}) \\ \mathbf{C} \mathbf{a} = \mathbf{Q}^t \mathbf{q} \end{array} \right.$$

From (29) and (30), using also (22), it can be stated:

$$(31) \quad \left\{ (1-\gamma) \mathbf{H}^t \mathbf{K}_u \mathbf{H} + \gamma \mathbf{H}^t \mathbf{Q} \mathbf{C}^{-1} \mathbf{Q}^t \mathbf{H} \right\} \mathbf{v} = \mathbf{H}^t (\mathbf{f}_q + \mathbf{f}_s)$$

where the matrix between brackets on the left hand side is the conventional element stiffness matrix; and the vector on the right hand side is the element load vector.

In the discretization of the parametrized variational principle described above, the displacement modes in \mathbf{N} and the stress modes in \mathbf{A} should meet some requirements. In addition to the continuity of the displacement field $\tilde{\mathbf{u}}$ across interelement boundaries and to the availability of constant stress and rigid body movement modes, the limitation principle of Fraeijns de Veubeke [F, 1965] should be taken into account. This principle applies when the modes \mathbf{DB} corresponding to the derived stress field σ^u :

$$(32) \quad \sigma^u = \mathbf{DB} \mathbf{q}$$

can be obtained as a linear combination of the stress modes \mathbf{A} used to represent the independent stress field $\tilde{\sigma}$. In such cases the limitation principle states that the approximation to the stress field given by $\tilde{\sigma}$ can not be better than the one obtained in σ^u , even though the same or more modes have been used to construct the former [F, 1989 a]. It can be seen then [B, 1990] that the element stiffness matrix in (31) becomes independent of γ and the proposed error estimate is always zero.

2.5. DEVELOPED ELEMENTS

Following the lines presented in the previous paragraphs, four elements for two-dimensional elasticity have been developed (Fig. 1). In those elements the internal displacement modes \mathbf{N} are divided into three categories:

$$(33) \quad \mathbf{N} = [\mathbf{N}_r \quad \mathbf{N}_m \quad \mathbf{N}_h]$$

where:

\mathbf{N}_r = rigid body movement modes.

\mathbf{N}_m = complete polynomials up to order m (m depends on the particular element, $1 \leq m \leq 4$).

\mathbf{N}_h = incomplete polynomials of order greater than m .

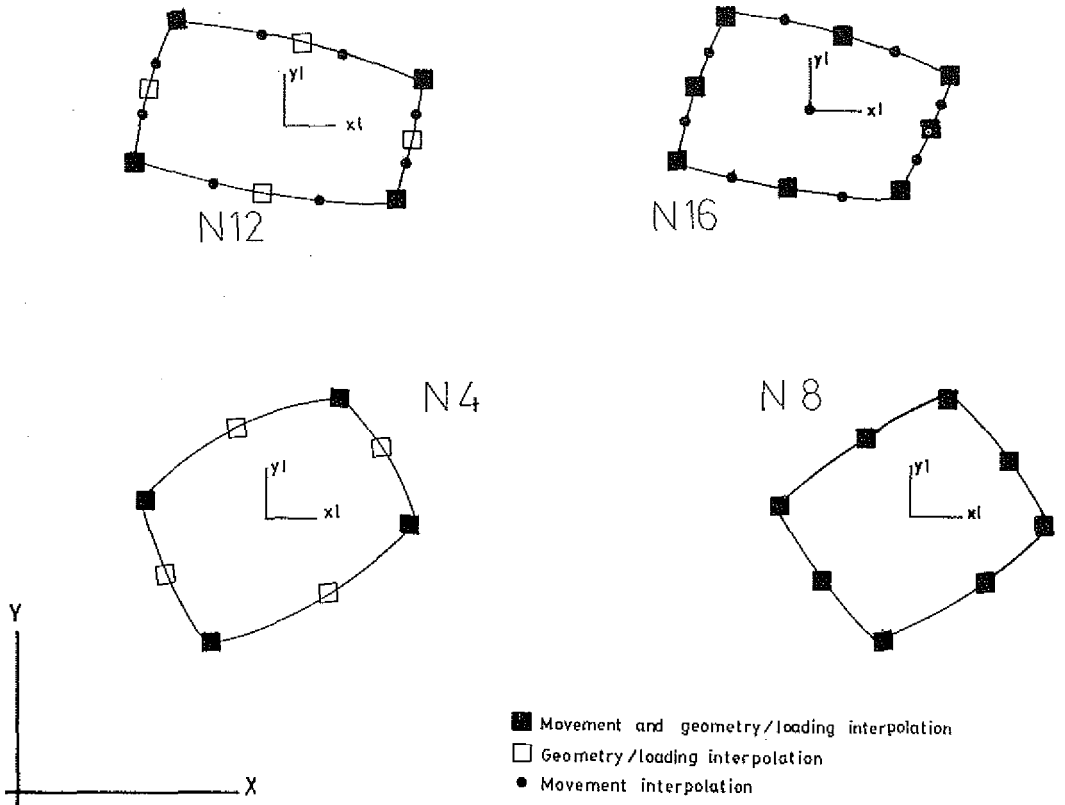


Fig. 1. - Elements formulated for 2D elasticity.

The latter modes are introduced so that the number of internal degrees of freedom in \mathbf{q} equals the number of nodal degrees of freedom in \mathbf{v} .

This partition of the internal displacement modes leads to a parallel partition of the strain modes \mathbf{B} :

$$(34) \quad \mathbf{B} = \frac{1}{2}(\nabla + \nabla') \mathbf{N} = [\mathbf{0} \quad \mathbf{B}_m \quad \mathbf{B}_h]$$

where the columns corresponding to \mathbf{N}_r are zero.

The field $\boldsymbol{\sigma}^u$ will be then:

$$(35) \quad \boldsymbol{\sigma}^u = \mathbf{D} \mathbf{B} \mathbf{q} = \mathbf{D} [\mathbf{B}_m \quad \mathbf{B}_h] \mathbf{q}$$

For the independent stress modes in \mathbf{A} , it is formally taken:

$$(36) \quad \mathbf{A} = \mathbf{D} \mathbf{B}_m$$

choice that avoids the consequences of the limitation principle.

A substantial simplification is gained if the strain modes in \mathbf{B}_m are orthogonal with respect to the energy norm:

$$(37) \quad (\mathbf{B}_m^t \mathbf{D} \mathbf{B}_m)_{\Omega_i} = \int_{\Omega_i} \mathbf{B}_m^t \mathbf{D} \mathbf{B}_m d\Omega = \text{diagonal matrix}$$

and if, besides, the modes in \mathbf{B}_m are orthogonal to the modes in \mathbf{B}_h :

$$(38) \quad (\mathbf{B}_m^t \mathbf{D} \mathbf{B}_h)_{\Omega_i} = \int_{\Omega_i} \mathbf{B}_m^t \mathbf{D} \mathbf{B}_h d\Omega = 0$$

In general, the election of \mathbf{N}_m and \mathbf{N}_h will not satisfy (37) and (38), but the internal displacement modes can be modified element by element applying a Gram-Schmidt orthogonalization process to the strain modes \mathbf{B}_m , \mathbf{B}_h [B, 1990]. The verification of (37) has some additional advantages when p -adaptive processes are implemented using these elements [B, 1990].

With the above development, the element stiffness matrix is:

$$(39) \quad \mathbf{K} = \mathbf{H}_m^t \mathbf{K}_{um} \mathbf{H}_m + (1 - \gamma) \mathbf{H}_h^t \mathbf{K}_{uh} \mathbf{H}_h = \mathbf{K}_b + \mathbf{K}_h \quad \text{where:}$$

$$\mathbf{H}^t = [\mathbf{H}_r^t \quad \mathbf{H}_m^t \quad \mathbf{H}_h^t], \quad \mathbf{K}_{um} = (\mathbf{B}_m^t \mathbf{D} \mathbf{B}_m)_{\Omega_i}, \quad \mathbf{K}_{uh} = (\mathbf{B}_h^t \mathbf{D} \mathbf{B}_h)_{\Omega_i}$$

Note that the stiffness matrix is obtained as a sum of a basic matrix \mathbf{K}_b , independent of γ , and a higher order matrix \mathbf{K}_h , dependent on γ .

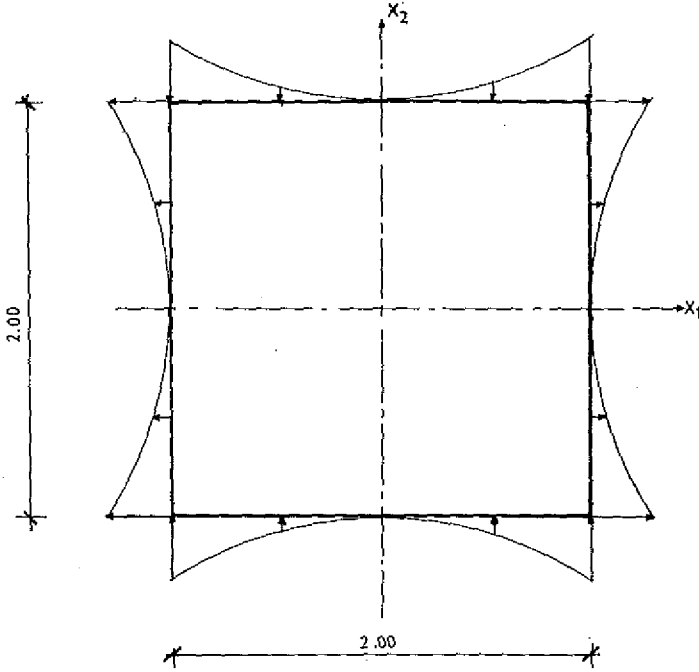
The proposed error estimate has the expression:

$$(40) \quad \varepsilon = \sum_i \varepsilon_i = \sum_i \frac{\gamma}{2} \mathbf{q}_h^t \mathbf{K}_{uh} \mathbf{q}_h$$

ε_i represents the elastic energy associated to the higher order strain modes \mathbf{B}_h multiplied by the parameter γ . In this sense ε_i is similar to the heuristic estimate introduced by Melosh and Marcal [Melosh & Marcal, 1977].

2.6. NUMERICAL EXAMPLE

A numerical example is presented here to illustrate the behavior of the error estimate proposed in the previous paragraphs. Necessarily, a simple problem has to be chosen, as the real error measures should be known in order to compare with the estimate.



$E = 1.00$ $\nu = 0.00$ thickness = 1.00	displacements	$\begin{cases} U_1 = \frac{2}{E} x_1 x_2^2 \\ U_2 = -\frac{2}{E} x_1^2 x_2 \end{cases}$
Stresses	$\begin{cases} \sigma_1 = 2 x_2^2 \\ \sigma_2 = 2 x_1^2 \\ \tau_{12} = 0 \end{cases}$	elastic energy = $\frac{16}{5E}$

Fig. 2. - Example 1. Plate with quadratic stresses.

Figure 2 shows a square plate under quadratic tractions on its boundaries. Double symmetry allows to study only one quarter of the plate. The example has been solved using a h -adaptive process and, also, a p -adaptive technique with complete refinement. In both cases the base mesh is formed with a single N4 element, which is the element of lowest order among the four developed. For the h refinement more N4 elements are used, whereas in the p refinement the order of the interpolation in the element is automatically increased using successively elements N8 and N12. The element N12 gives the exact solution, as its formulation includes complete cubic polynomials for displacement approximation.

In Figure 3 the base mesh and the evolution of h refinement is represented. The refinement is controlled by the local error indicators defined in (18): the rule has been to divide the elements in which the indicator was higher. Note that eventually an optimal

MULTIFIELD VARIATIONAL PRINCIPLES

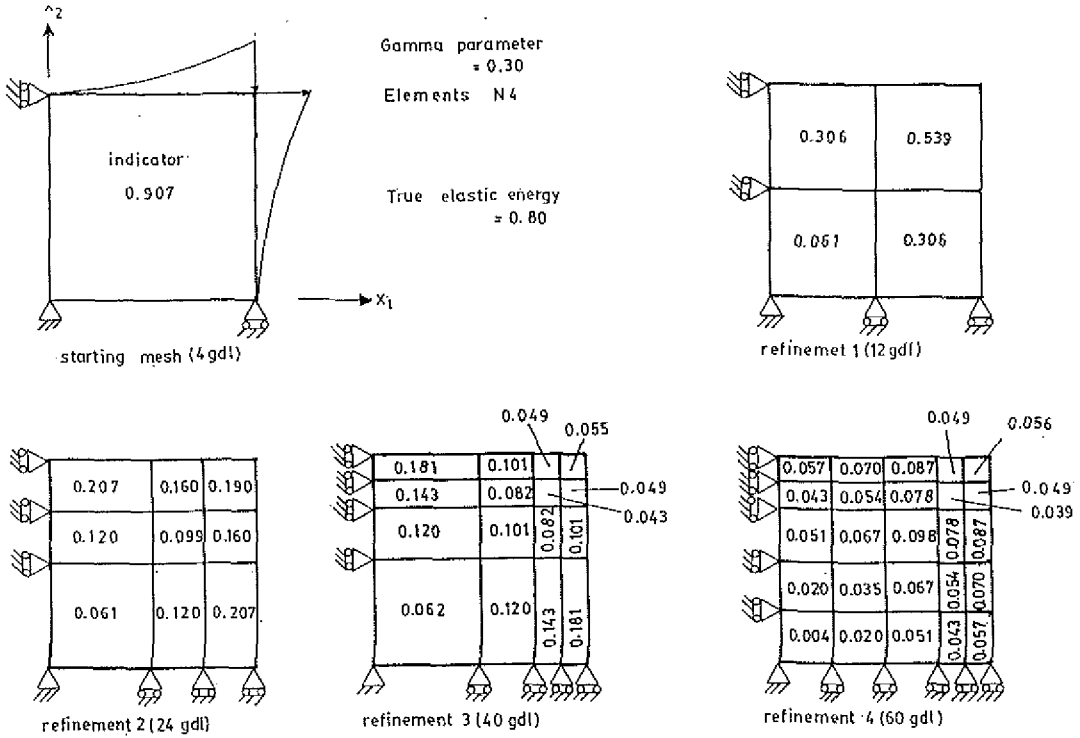


Fig. 3. - Example 1. H refinement.

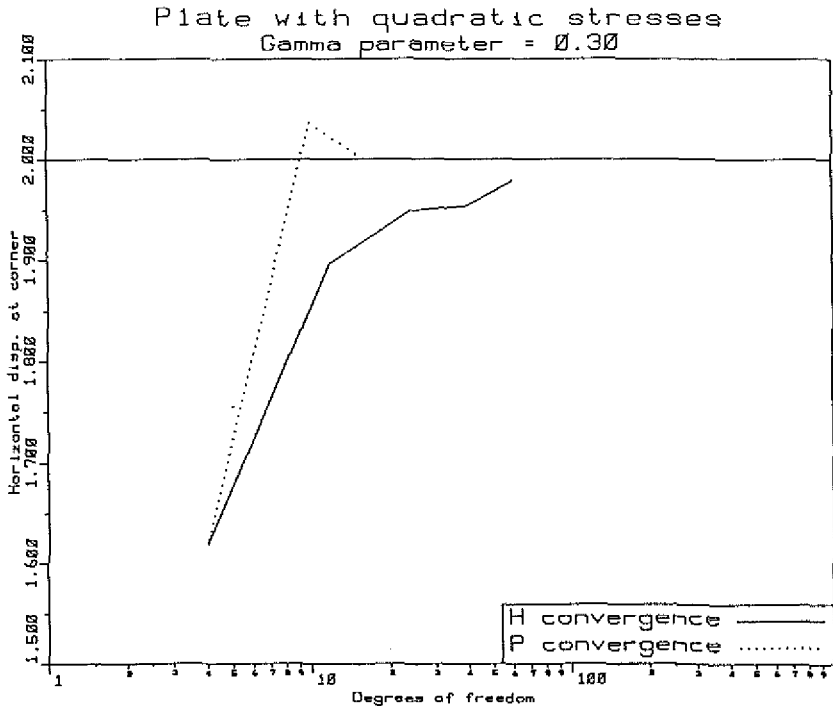


Fig. 4. - Example 1. Horizontal displacement at corner.

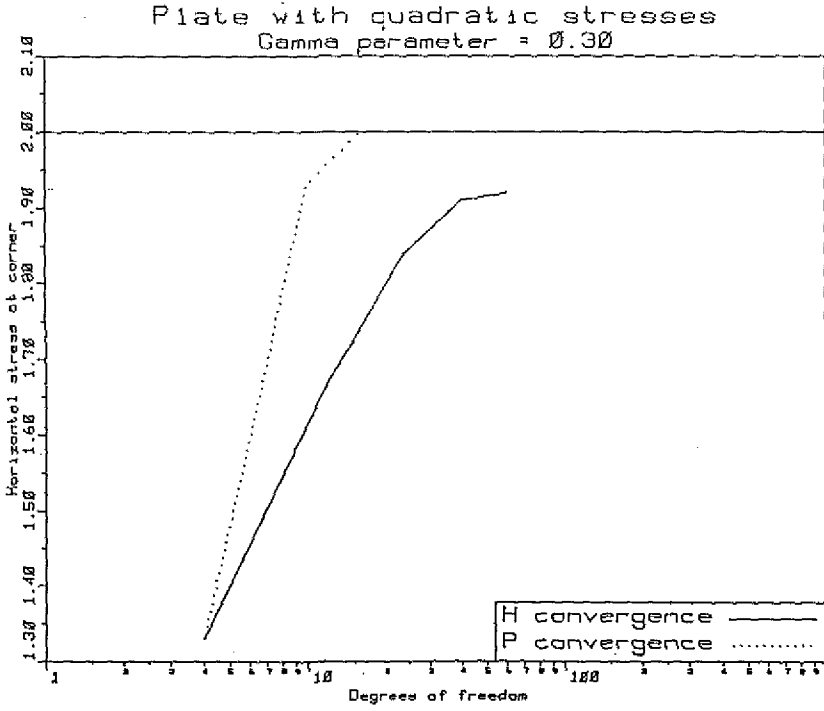


Fig. 5. - Example 1. Horizontal stress at corner.

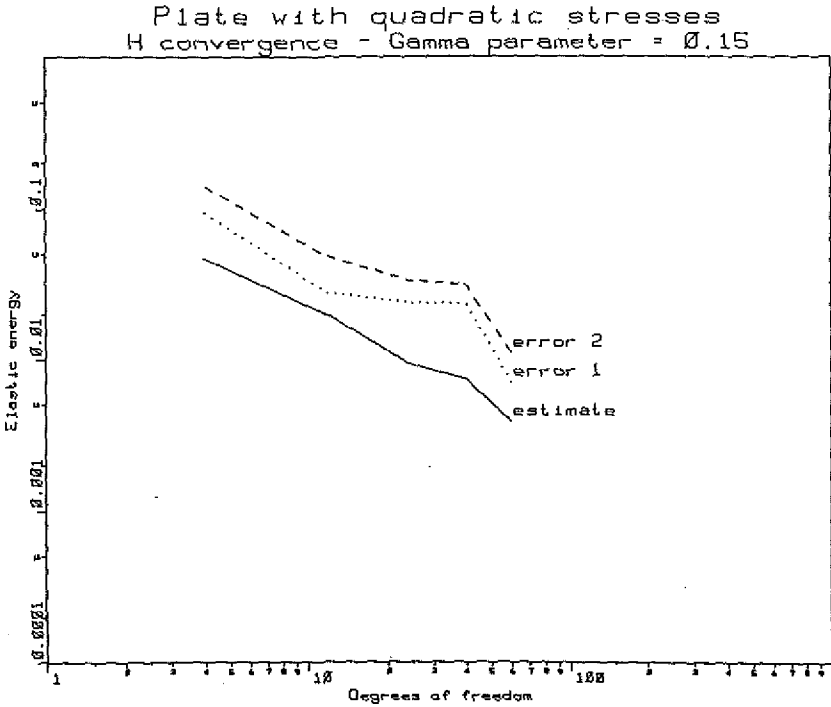


Fig. 6. - Example 1. Error estimate evolution (I).

MULTIFIELD VARIATIONAL PRINCIPLES

Plate with quadratic stresses
H convergence - Gamma parameter = 0.30

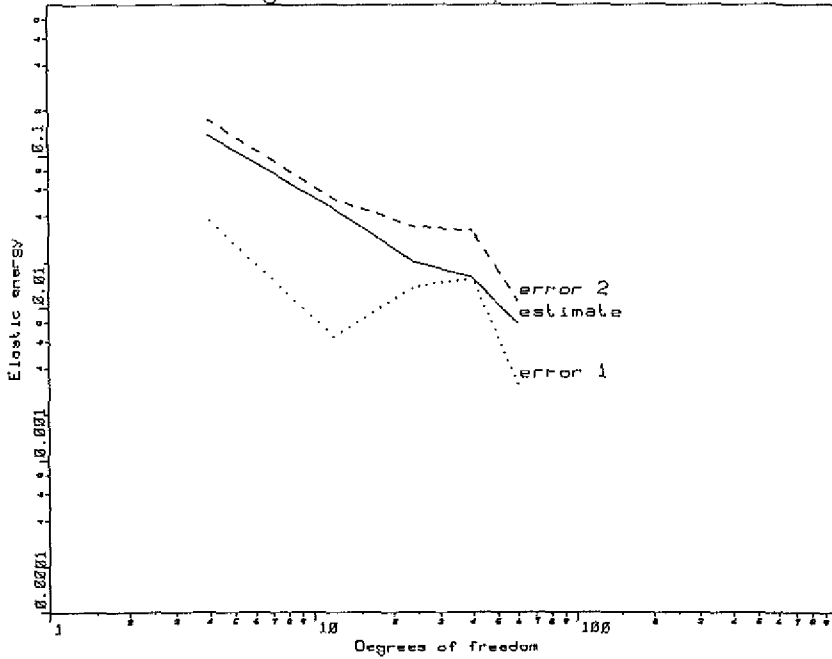


Fig. 7. - Example I. Error estimate evolution (II).

Plate with quadratic stresses
H convergence - Gamma parameter = 0.80

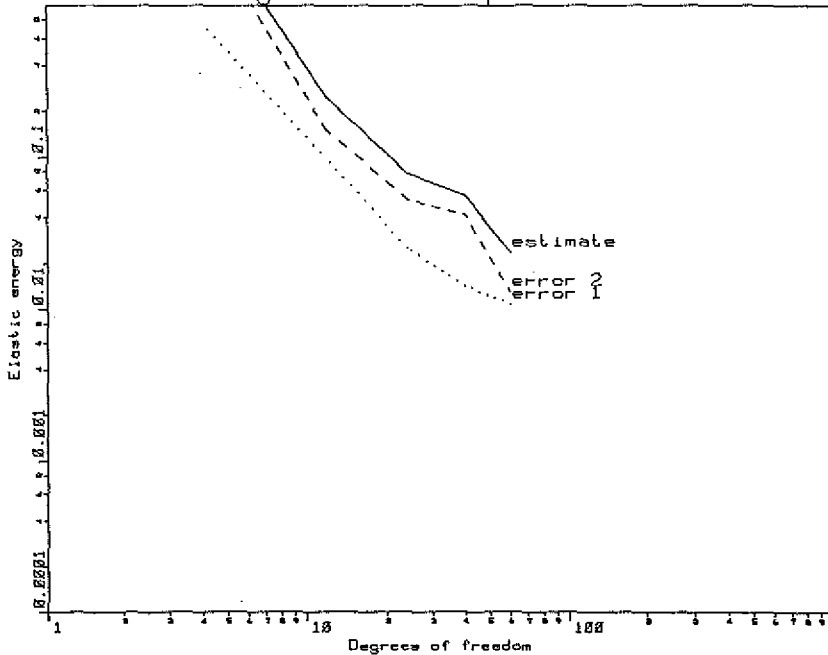


Fig. 8. - Example I. Error estimate evolution (III).

mesh is reached, in the sense that the indicators have approximately the same value in all elements. The behavior of this local measure seems to be good and it does not vary significantly for other values of the parameter.

Figures 4 and 5 show the computed horizontal displacement u_1 and the horizontal stress σ_1 at the plate corner during the h and p refinements. Note the power of p refinement; it yields the true solution with only 16 degrees of freedom.

The evolution of the proposed error estimate has been studied for different values of the parameter γ . Figures 6 to 8 summarize the results obtained during the h refinement. In these figures "error 1" means the absolute value of the difference between the true potential Π and the parametrized functional Π_γ , "Error 2" means the difference between the total potential Π_p and the exact potential Π . The values of the estimate, error 1 and error 2 are represented against the number of degrees of freedom. It can be seen how the evolution of the estimate and that of the error 2 ($\Pi_p - \Pi$) are very similar in all cases.

The values of γ has an important influence not only in the magnitude of the errors but also in the relative magnitude of the estimate with respect to the errors. Low values of γ (less than 0.20) produce a parametrized functional Π_γ not very different from the total Π_p , thus approximate solutions of the same sort as those given by conventional elements can be expected. The proposed estimate understates the errors, though it has their same tendency (Fig. 6).

For $\gamma=0.30$ (Fig. 7), amazingly low values of the difference $|\Pi_\gamma - \Pi|$ are obtained. However, note that this difference not always diminishes when the number of degrees of freedom is increased. In this case the proposed estimate slightly underestimates the difference $\Pi_p - \Pi$ (error 2).

Finally, for values of γ above 0.50 (Fig. 8) the proposed estimate has an excellent behavior, as it takes values slightly greater than both errors and it has their same tendency. On the other side, note that the magnitude of the errors is higher than for lower values of γ , at least when the number of degrees of freedom is small.

3. Plate bending

3.1. NOTATION

The statement of the plate bending problem according to the hypotheses of Kirchhoff [Timoshenko & Woinowski-Krieger, 1959] will serve to introduce the notation.

Consider a plate with thickness h (Fig. 9). A reference system is used whose X and Y axes are on the midplane of the plate, while the Z axis is normal to it. The lateral surface of the plate is supposed to be cylindrical, *i.e.* parallel to Z axis. The midplane of the plate is a domain $P \subset \mathbb{R}^2$ with boundary $C = C_d \cup C_r$. The outward unit normal on C is $\mathbf{n} \equiv (n_x, n_y)$. On C_d the movements $\bar{\mathbf{d}}$ (Fig. 10) are given, whereas on C_r the stress resultant (forces and moments) per unit length $\bar{\mathbf{m}}$ (Fig. 11) are known. It is also known the field q of distributed normal load per unit area in P .

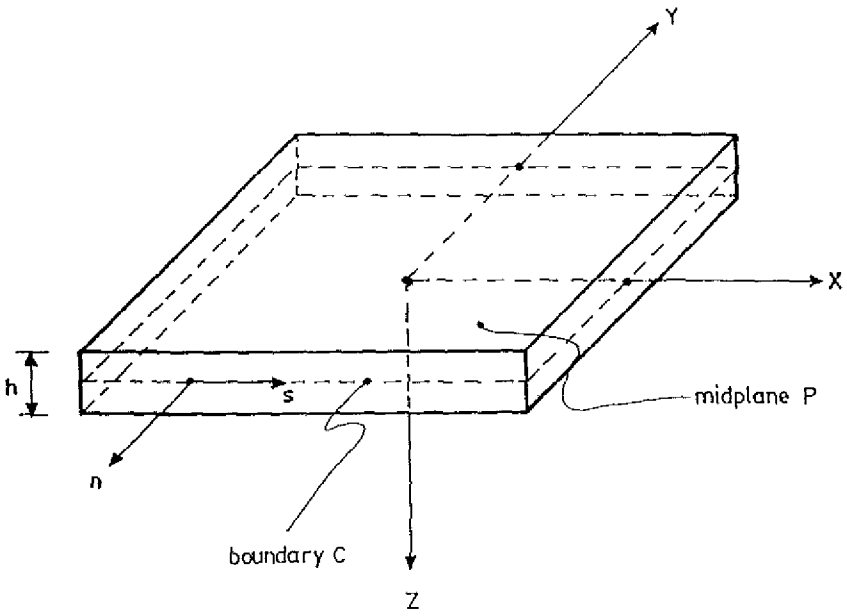


Fig. 9. — Reference system for a plate.

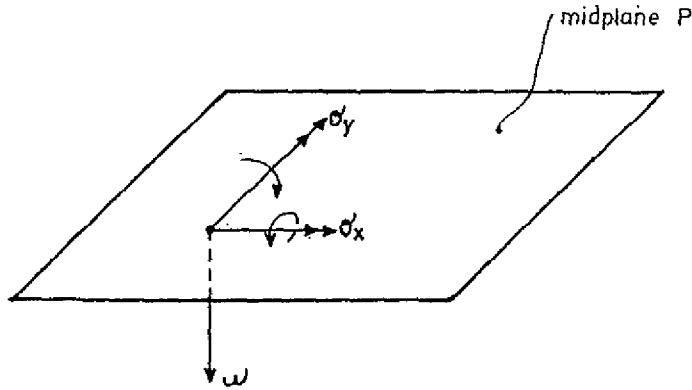


Fig. 10. — Midplane movements in plate bending.

In the classical theory of plate bending the unknowns are the field of transverse displacements or deflections ω in P , the field of moments or generalized stresses M in P and the field of curvatures or generalized strains \varkappa in P . Note the general parallelism with the elasticity problem.

Along the following paragraphs the previous notation to distinguish between primary and secondary fields will be used. A field that can be varied independently will be denoted with a tilde “ \sim ” over its symbol: in the dependent fields the dependence will be

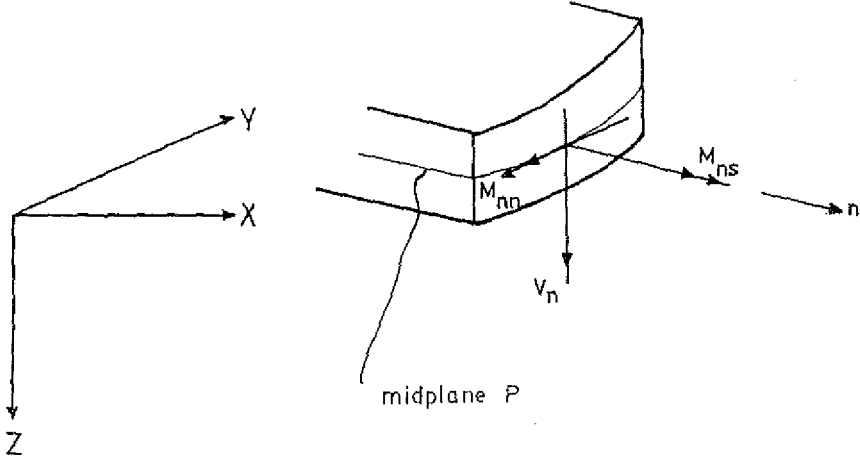


Fig. 11. - Boundary stress resultants per unit length.

identified by setting as a superscript the symbol of the primary field. For example:

$$(41) \quad \left\{ \begin{aligned} \kappa_{xx}^{\omega} &= -\frac{\partial^2 \omega}{\partial x^2}, & \kappa_{yy}^{\omega} &= -\frac{\partial^2 \omega}{\partial y^2}, & \kappa_{xy}^{\omega} &= 2\frac{\partial^2 \omega}{\partial x \partial y} \\ \mathbf{M}^{\omega} &= \mathbf{D} \boldsymbol{\kappa}^{\omega} \end{aligned} \right.$$

where \mathbf{D} is a tensor of elastic constants.

The writing of surface and line integrals will be generally abbreviated by placing the integrand between domain-subscripted parentheses and square brackets respectively.

3.2. VARIATIONAL APPROACH

The bending of a plate is nothing more than an elasticity problem. If some assumptions on the displacements, stresses or strains are made, the dimension of the problem can be reduced from three to two, and the problem is restricted to the study of the midplane. Thus, the conventional variational principles for plate bending problems can be obtained from the principles used in elasticity by integration through the thickness.

The minimum Potential Energy principle for plate bending can be stated as that the solution field ω for the deflections makes the functional:

$$(42) \quad \Pi_P(\tilde{\omega}) = \frac{1}{2} (\mathbf{M}^{\omega}, \boldsymbol{\kappa}^{\omega})_P - (q, \tilde{\omega})_P - (\tilde{\omega}, \tilde{\mathbf{m}})_C,$$

a minimum when it is defined amongst the fields $\tilde{\omega}$ that satisfy the kinematical boundary conditions: $\tilde{\omega} = \tilde{\mathbf{d}}$ en C_d . In (42) the vector field $\tilde{\omega}$ is:

$$(43) \quad \tilde{\omega} = [\tilde{\omega}, \theta_x^{\omega}, \theta_y^{\omega}]$$

where, according to Kirchhoff's hypotheses:

$$(44) \quad \theta_x^o = -\frac{\partial \tilde{\omega}}{\partial y}, \quad \theta_y^o = \frac{\partial \tilde{\omega}}{\partial x}$$

From this principle of Minimum Potential Energy the other conventional variational principles (Hu-Washizu, Hellinger-Reissner...) can be derived using the Lagrange multiplier method, as it has been done in the sections devoted to elasticity problems.

Consider now the domain $P \subset \mathbb{R}^2$ of the midplane of the plate divided into a number of subdomains P_j (Fig. 12): $C_d \cup C_e$ is the exterior boundary of P , while C_i will represent the interior boundaries of the subdomains. Let $\tilde{\omega}^{(j)}$ be a field of movements defined in P_j independently from other similar fields defined in the rest of subdomains. All these fields $\tilde{\omega}^{(j)}$ will constitute a global field $\tilde{\omega}$ defined over the whole domain P .

Consider also a field of movements $\tilde{\mathbf{d}}$ defined only on the external and internal boundaries $C = C_d \cup C_e \cup C_i$. If the following conditions are satisfied:

- a) The field $\tilde{\omega}^{(j)}$ is continuous and single-valued within the domain P_j .
- b) The field $\tilde{\omega}^{(j)}$ takes the same values as the field $\tilde{\mathbf{d}}$ on the boundary of the domain P_j .
- c) The field $\tilde{\mathbf{d}}$ is kinematically admissible, i. e. $\tilde{\mathbf{d}} = \bar{\mathbf{d}}$ on C_d .

then the total potential energy can be written as:

$$(45) \quad \Pi_P(\tilde{\omega}) = \sum_j \Pi_{P_j}(\tilde{\omega}^{(j)}) = \sum_j \left\{ \frac{1}{2} (\mathbf{M}^o, \boldsymbol{\kappa}^o)_{P_j} - (q, \tilde{\omega}^{(j)})_{P_j} - [\bar{\mathbf{m}}, \tilde{\omega}^{(j)}]_{C_{i_j}} \right\}$$

The condition *b* above can be introduced into the functional (45) by means of a Lagrange multiplier field $\tilde{\boldsymbol{\lambda}}$ defined on the generalized boundary C . Thus, the fields $\tilde{\mathbf{d}}$ y $\tilde{\omega}^{(j)}$ would be independent from each other. Variational principles in which the relation $\tilde{\mathbf{d}} = \tilde{\omega}$ is not forced *a priori* on C , are known as displacement generalized or *d*-generalized principles [F, 1989 *b*].

The *d*-generalized potential energy functional is then:

$$(46) \quad \Pi_P^d(\tilde{\omega}, \tilde{\mathbf{d}}, \tilde{\boldsymbol{\lambda}}) = \sum_j \left\{ \frac{1}{2} (\mathbf{M}^o, \boldsymbol{\kappa}^o)_{P_j} - (q, \tilde{\omega}^{(j)})_{P_j} - [\bar{\mathbf{m}}, \tilde{\mathbf{d}}]_{C_{i_j}} + [\tilde{\boldsymbol{\lambda}}, \tilde{\mathbf{d}} - \tilde{\omega}^{(j)}]_{C_j} \right\}$$

where C_j represents the boundary of the subdomain P_j .

The solution of the plate bending problem will be obtained by making the functional $\Pi_P^d(\tilde{\omega}, \tilde{\mathbf{d}}, \tilde{\boldsymbol{\lambda}})$ stationary, with the restriction $\tilde{\mathbf{d}} = \bar{\mathbf{d}}$ on C_d . If the associated Euler equations and the natural boundary conditions are obtained, the Lagrange multiplier field $\tilde{\boldsymbol{\lambda}}$ can be identified with a field $\tilde{\mathbf{m}}$ of stress resultants (moments and forces) per unit length on the generalized boundary C [B, 1990]. Thus, the *d*-generalized total potential Π_P^d is:

$$(47) \quad \Pi_P^d(\tilde{\omega}, \tilde{\mathbf{d}}, \tilde{\mathbf{m}}) = \sum_j \left\{ \frac{1}{2} (\mathbf{M}^o, \boldsymbol{\kappa}^o)_{P_j} - (q, \tilde{\omega}^{(j)})_{P_j} - [\bar{\mathbf{m}}, \tilde{\mathbf{d}}]_{C_{i_j}} + [\tilde{\mathbf{m}}, \tilde{\mathbf{d}} - \tilde{\omega}^{(j)}]_{C_j} \right\}$$

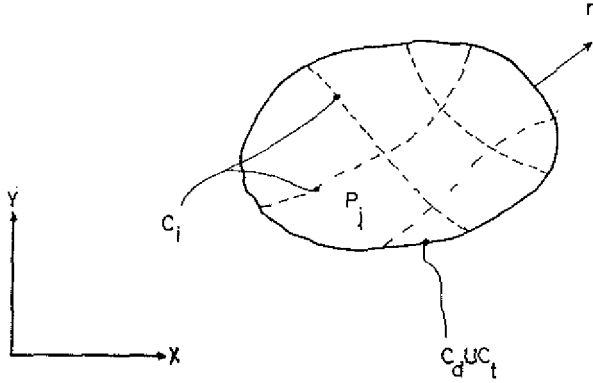


Fig. 12. - P domain subdivision.

Other d -generalized variational principles can be found in recent papers of Felippa [F, 1989 *c*; F & M, 1989]. For our purposes, it has a particular interest a d -generalized Hellinger-Reissner principle. The associated functional is in this case:

$$(48) \quad \Pi_R^d(\tilde{\omega}, \tilde{\mathbf{d}}, \tilde{\mathbf{M}}) = \sum_j \left\{ -\frac{1}{2}(\tilde{\mathbf{M}}, \boldsymbol{\kappa}^M)_{P_j} + (\tilde{\mathbf{M}}, \boldsymbol{\kappa}^o)_{P_j} - (q, \tilde{\omega}^{(j)})_{P_j} - [\tilde{\mathbf{m}}, \tilde{\mathbf{d}}]_{C_j} + [\tilde{\mathbf{M}}_n, \tilde{\mathbf{d}} - \tilde{\omega}^{(j)}]_{C_j} \right\}$$

where $\tilde{\mathbf{M}}$ is an independent field of moments and $\tilde{\mathbf{M}}_n$ is the vector field of stress resultants per unit length on C_j , derived from $\tilde{\mathbf{M}}$. The solution fields make Π_R^d stationary if this functional is defined over fields $\tilde{\mathbf{d}}$ with $\tilde{\mathbf{d}} = \bar{\mathbf{d}}$ on C_d .

In a way absolutely similar to the ideas exposed above for elasticity problems, note that the functionals Π_p^d and Π_R^d are special cases of the parametrized functional:

$$(49) \quad \begin{aligned} \Pi_\gamma^d(\tilde{\omega}, \tilde{\mathbf{d}}, \tilde{\mathbf{M}}) &= (1-\gamma)\Pi_p^d(\tilde{\omega}, \tilde{\mathbf{d}}, \tilde{\mathbf{M}}_n) + \gamma\Pi_R^d(\tilde{\omega}, \tilde{\mathbf{d}}, \tilde{\mathbf{M}}) \\ &= \sum_j \left\{ \frac{1}{2}(1-\gamma)(\mathbf{M}^o, \boldsymbol{\kappa}^o)_{P_j} - \frac{1}{2}\gamma(\tilde{\mathbf{M}}, \boldsymbol{\kappa}^M)_{P_j} + \gamma(\tilde{\mathbf{M}}, \boldsymbol{\kappa}^o)_{P_j} \right. \\ &\quad \left. - (q, \tilde{\omega}^{(j)})_{P_j} - [\tilde{\mathbf{m}}, \tilde{\mathbf{d}}]_{C_j} + [\tilde{\mathbf{M}}_n, \tilde{\mathbf{d}} - \tilde{\omega}^{(j)}]_{C_j} \right\} \\ &\quad \gamma \in \mathbb{R}, \quad 0 \leq \gamma \leq 1 \end{aligned}$$

and the solution fields make Π_γ^d stationary when it is defined over fields $\tilde{\mathbf{d}}$ such that $\tilde{\mathbf{d}} = \bar{\mathbf{d}}$ on C_d .

3.3. ERROR ESTIMATE

As in the case of elasticity, all the functionals introduced above for plate bending problems take the same value Π when the exact solution fields ω , \mathbf{M} , $\boldsymbol{\kappa}$, \mathbf{d} are introduced into them.

Again, the question of the possibility of using some difference between functionals as an energy error estimate is arisen. Following the same reasoning as before, it turns out that if Π_p^d and Π_γ^d are the values of the potential energy functional and the parametrized functional for an approximate solution of the problem, then the differences $|\Pi_p^d - \Pi|$ and $|\Pi_\gamma^d - \Pi|$ are energy measures of the discretization error magnitude.

As an estimate of those energy measures, the difference:

$$(50) \quad \varepsilon = \Pi_p^d(\tilde{\omega}, \tilde{\mathbf{d}}, \tilde{\mathbf{M}}_n) - \Pi_\gamma^d(\tilde{\omega}, \tilde{\mathbf{d}}, \tilde{\mathbf{M}})$$

is proposed.

In this case three fields are needed to compute the estimate. Two of them ($\tilde{\omega}$ and $\tilde{\mathbf{M}}$) defined over the domain P and the other defined on the generalized boundary C . If the three fields are available (e. g. in a hybrid formulation) the difference (50) is evaluated easily since:

$$(51) \quad \varepsilon = \sum_j \varepsilon_j = \sum_j \left\{ \frac{1}{2} \gamma (\mathbf{M}^o, \boldsymbol{\kappa}^o)_{P_j} + \frac{1}{2} \gamma (\tilde{\mathbf{M}}, \boldsymbol{\kappa}^M)_{P_j} - \gamma (\tilde{\mathbf{M}}, \boldsymbol{\kappa}^o)_{P_j} \right\} \\ = \frac{\gamma}{2} \sum_j \{ (\mathbf{M}^o - \tilde{\mathbf{M}}, \boldsymbol{\kappa}^o - \boldsymbol{\kappa}^M)_{P_j} \}$$

with j from 1 to the number of subdomains P_j .

The contributions ε_j are, at the same time, indicators of the local error. Each ε_j , once divided by the area of the subdomain P_j , is an "error density" and those error densities can be used to guide the refinement in an adaptive scheme. Note on the other hand that if the relationship between moments and curvatures is positive definite and $\gamma \geq 0$, then the estimate ε is also $\varepsilon \geq 0$.

The similitude of (51) and the expression (16) for elasticity problems is apparent. In both cases the estimate is associated with the differences in energy between two primary fields within the parametrized functional.

3.4. FINITE ELEMENT DISCRETIZATION

As in the elasticity case, the implementation of the proposed estimate is easy if the principle associated with the parametrized functional $\Pi_\gamma^d(\tilde{\omega}, \tilde{\mathbf{d}}, \tilde{\mathbf{M}})$ is used. This principle yields hybrid finite elements whose connecting variables are the nodal movements. In the present work the formulation of such elements has been established introducing an additional kinematical constraint between the fields $\tilde{\omega}$ and $\tilde{\mathbf{d}}$. The resultant elements satisfy the patch test *a priori* and the Free Formulation of Bergan [B & N, 1984] turns out to be a particular case of the development. The main points of the elements's formulation are presented in the following paragraphs; the details can be found in [B, 1990].

In the usual way, the three components of the field of moments $\tilde{\mathbf{M}}$ are arranged into a column vector. The same is done with the three curvatures in $\tilde{\boldsymbol{\kappa}}$. The tensor \mathbf{D} of elastic coefficients becomes a symmetric square matrix.

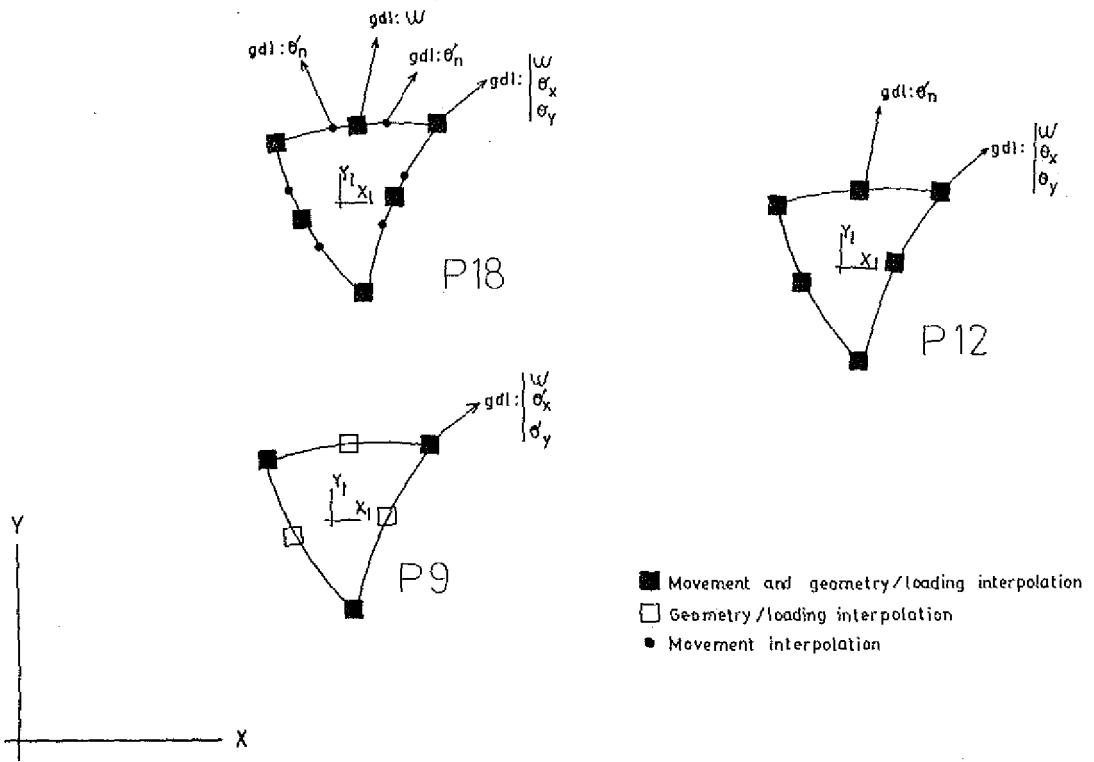


Fig. 13. - Elements formulated for plate bending.

The assumption on which the finite element discretization is based is that the field of movements $\tilde{\omega}$ and the field of moments \tilde{M} can be written within each element or subdomain P_j as:

$$(52) \quad \tilde{\omega} \equiv \begin{Bmatrix} \tilde{\omega} \\ -\tilde{\omega}_{,y} \\ \tilde{\omega}_{,x} \end{Bmatrix} \equiv \begin{Bmatrix} \tilde{\omega} \\ \theta_x^e \\ \theta_y^e \end{Bmatrix} = \mathbf{N} \mathbf{q} \quad \text{in } P_j \quad \text{and}$$

$$(53) \quad \tilde{M} \equiv \begin{Bmatrix} \tilde{M}_{xx} \\ \tilde{M}_{yy} \\ \tilde{M}_{xy} \end{Bmatrix} = \mathbf{A} \mathbf{a} \quad \text{in } P_j$$

It is assumed also that the field of boundary movements $\tilde{\mathbf{d}}$ can be obtained on the element boundary as:

$$(54) \quad \tilde{\mathbf{d}} \equiv \begin{Bmatrix} \tilde{\omega}_n \\ \tilde{\theta}_{nm} \\ \theta_{ns}^e \end{Bmatrix} = \mathbf{V} \mathbf{v} \quad \text{on } C_j$$

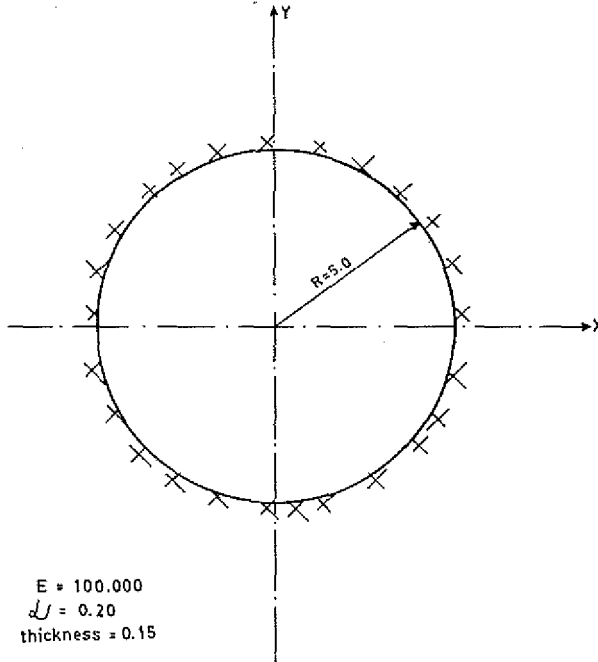


Fig. 14. — Example 2. Circular plate under concentrated load.

Now the continuity of the internal movements $\tilde{\omega}$ across the element boundaries is not required. Hence, the modes in \mathbf{N} can be chosen with freedom from conformity requirements. On the other hand, the interface movement field $\tilde{\mathbf{u}}$ should be continuous on the generalized boundary $C = C_t \cup C_d \cup C_i$.

The total number of internal movement modes (*i.e.* the number of components of vector \mathbf{q}) is taken the same as the number of interface movement modes (*i.e.* the number of components of vector \mathbf{v}). The secondary fields derived from $\tilde{\omega}$ and $\tilde{\mathbf{M}}$ are:

$$(55) \quad \mathbf{x}^{\omega} = \left\{ \begin{array}{ccc} -\frac{\partial^2}{\partial x^2} & 0 & 0 \\ -\frac{\partial^2}{\partial y^2} & 0 & 0 \\ 2\frac{\partial^2}{\partial x \partial y} & 0 & 0 \end{array} \right\} \mathbf{Nq} = \mathbf{Bq} \quad \text{in } P_j$$

$$(56) \quad \mathbf{M}^{\omega} = \mathbf{DBq} \quad \text{in } P_j$$

$$(57) \quad \mathbf{x}^{\mathbf{M}} = \mathbf{D}^{-1} \mathbf{Aa} \quad \text{in } P_j$$

The Eq. (52) to (57) can be used to produce the discretized form of the parametrized functional:

$$(58) \quad \Pi_\gamma(\tilde{\omega}, \tilde{\mathbf{M}}, \tilde{\mathbf{d}}) = \sum_j \Pi_{\gamma_j}$$

$$= \sum_j \left\{ \frac{1}{2} (1-\gamma) \mathbf{q}' (\mathbf{B}' \mathbf{D} \mathbf{B})_{P_j} \mathbf{q} - \frac{1}{2} \gamma \mathbf{a}' (\mathbf{A}' \mathbf{D}^{-1} \mathbf{A})_{P_j} \mathbf{a} \right. \\ \left. + \gamma \mathbf{a}' (\mathbf{A}' \mathbf{B})_{P_j} \mathbf{q} - (\mathbf{p}' \mathbf{N})_{P_j} \mathbf{q} - [\tilde{\mathbf{m}}' \mathbf{V}]_{C_j} \mathbf{v} \right. \\ \left. + \mathbf{a}' [\mathbf{A}'_n \mathbf{V}]_{C_j} \mathbf{v} - \mathbf{a}' [\mathbf{A}'_n \mathbf{N}_n]_{C_j} \mathbf{q} \right\}$$

where $\mathbf{p}' = [q \ 0 \ 0]$ is the vector of loads per unit surface; matrix \mathbf{A}_n is:

$$\mathbf{A}_n = \left\{ \begin{array}{ccc} n_x \frac{\partial}{\partial x} & n_y \frac{\partial}{\partial y} & - \left(n_x \frac{\partial}{\partial y} + n_y \frac{\partial}{\partial x} \right) \\ n_x^2 & n_y^2 & -2n_x n_y \\ -n_x n_y & n_x n_y & (n_y^2 - n_x^2) \end{array} \right\} \mathbf{A}$$

and matrix \mathbf{N}_n is:

$$\mathbf{N}_n = \left\{ \begin{array}{ccc} 1 & 0 & 0 \\ 0 & n_y & -n_x \\ 0 & n_x & n_y \end{array} \right\} \mathbf{N}$$

If the following matrices are defined:

$$(59) \quad \left\{ \begin{array}{llll} \mathbf{K}_u = (\mathbf{B}' \mathbf{D} \mathbf{B})_{P_j} & \mathbf{C} = (\mathbf{A}' \mathbf{D}^{-1} \mathbf{A})_{P_j} & \mathbf{Q}' = (\mathbf{A}' \mathbf{B})_{P_j} \\ \mathbf{L}' = [\mathbf{A}'_n \mathbf{V}]_{C_j} & \mathbf{P}' = [\mathbf{A}'_n \mathbf{N}_n]_{C_j} & \mathbf{f}'_q = (\mathbf{p}' \mathbf{N})_{P_j} & \mathbf{f}'_v = [\tilde{\mathbf{m}}' \mathbf{V}]_{C_j} \end{array} \right.$$

the fraction of the functional that corresponds to a particular element is:

$$(60) \quad \Pi_{\gamma_j} = \frac{1}{2} (1-\gamma) \mathbf{q}' \mathbf{K}_u \mathbf{q} - \frac{1}{2} \gamma \mathbf{a}' \mathbf{C} \mathbf{a} + \gamma \mathbf{a}' \mathbf{Q}' \mathbf{q} + \mathbf{a}' \mathbf{L}' \mathbf{v} - \mathbf{a}' \mathbf{P}' \mathbf{q} - \mathbf{f}'_q \mathbf{q} - \mathbf{f}'_v \mathbf{v}$$

Making (60) stationary yields three matrix equations per element:

$$(61) \quad 1) \quad \left\{ \begin{array}{l} \frac{\partial \Pi_{\gamma_j}}{\partial \mathbf{a}} = \mathbf{0} \quad (\text{compatibility}) \\ -\gamma \mathbf{C} \mathbf{a} + \gamma \mathbf{Q}' \mathbf{q} - \mathbf{P}' \mathbf{q} + \mathbf{L}' \mathbf{v} = \mathbf{0} \end{array} \right.$$

$$(62) \quad 2) \quad \left\{ \begin{array}{l} \frac{\partial \Pi_{\gamma_j}}{\partial \mathbf{q}} = \mathbf{0} \quad (\text{domain and external boundary equilibrium}) \\ (1-\gamma) \mathbf{K}_u \mathbf{q} + \gamma \mathbf{Q} \mathbf{a} - \mathbf{P} \mathbf{a} = \mathbf{f}_q \end{array} \right.$$

$$(63) \quad \left\{ \begin{array}{l} \frac{\partial \Pi_{ij}}{\partial \mathbf{v}} = \mathbf{0} \quad (\text{internal boundaries equilibrium}) \\ \mathbf{L} \mathbf{a} = \mathbf{f}_v \end{array} \right.$$

Equations (61) to (63) are the discrete form of the stationarity condition. From this system the internal degrees of freedom \mathbf{a} and \mathbf{q} should be eliminated in order to obtain a stiffness matrix utilizable within a conventional assembly process.

If \mathbf{a} and \mathbf{q} are simply statically condensed in (61)-(63), a γ independent stiffness matrix is produced. Then, the solution for the fields $\tilde{\mathbf{M}}$ and \mathbf{M}° is the same in all elements and the proposed estimate is always zero: it is a situation similar to that described when the *Limitation Principle* was discussed in the section devoted to elasticity.

The above difficulty disappears if the degrees of freedom in \mathbf{q} are eliminated using an additional kinematical constraint:

$$(64) \quad [\tilde{\mathbf{M}}_n, \tilde{\mathbf{d}} - \tilde{\omega}]_{C_j} = 0$$

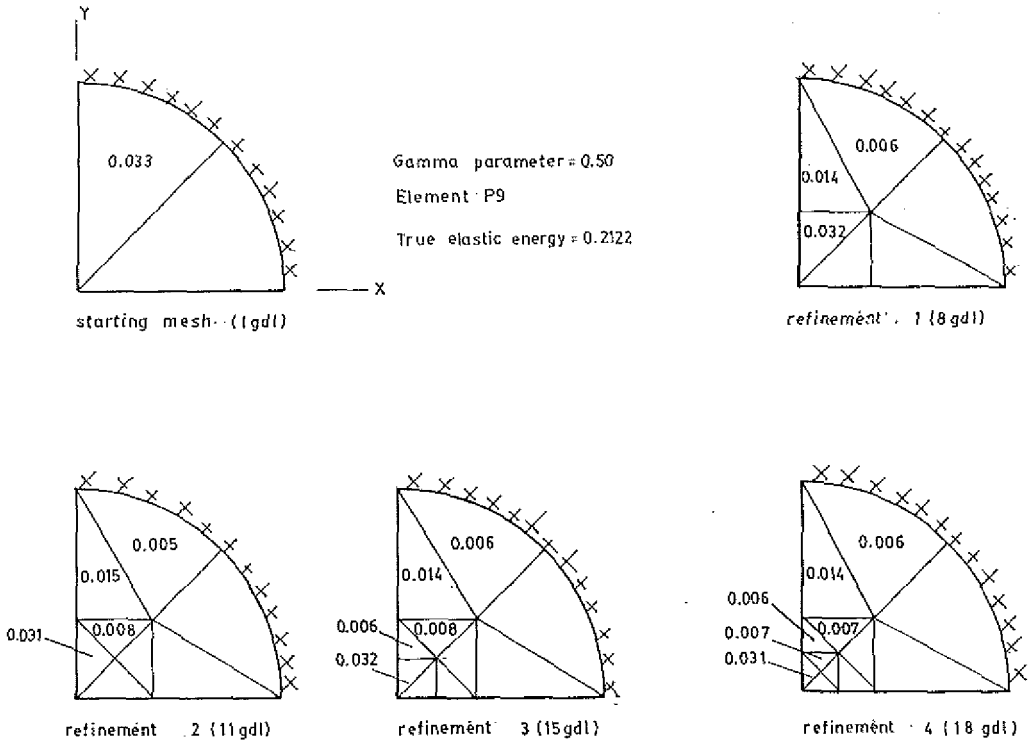


Fig. 15. - Example 2, H refinement evolution.

that imposes weak compatibility on fields $\tilde{\mathbf{d}}$ and $\tilde{\omega}$ along the boundary C_j of the element.

In discrete terms, constraint (64) is satisfied if the modes in \mathbf{N} , \mathbf{V} , \mathbf{A} are chosen so that:

$$(65) \quad [\tilde{\mathbf{M}}_n, \tilde{\mathbf{d}} - \tilde{\omega}]_{C_j} = \mathbf{a}^t [\mathbf{A}_n^t \mathbf{V}]_{C_j} \mathbf{v} - \mathbf{a}^t [\mathbf{A}_n^t \mathbf{N}_n]_{C_j} \mathbf{q} = \mathbf{0} \quad \text{or:}$$

$$(66) \quad \mathbf{a}^t \mathbf{L}^t \mathbf{v} - \mathbf{a}^t \mathbf{P}^t \mathbf{q} = 0$$

condition that is fulfilled if:

$$(67) \quad \mathbf{L}^t \mathbf{v} = \mathbf{P}^t \mathbf{q}$$

Eq. (67) requires that if the mode amplitudes \mathbf{v} and \mathbf{q} are linearly related by a matrix \mathbf{H} :

$$(68) \quad \mathbf{q} = \mathbf{H} \mathbf{v}$$

then the matrix \mathbf{H} should be:

$$(69) \quad \mathbf{P}^t \mathbf{H} = \mathbf{L}^t$$

In practice the path is reversed: relations (68) and (69) are imposed so that (67) is verified automatically. The way of constructing matrix \mathbf{H} numerically is presented in [B, 1990].

With all the above, static condensation of \mathbf{a} in (61)-(63) together with the use of (68) and (69) to eliminate \mathbf{q} , yields:

$$(70) \quad \{(1 - \gamma) \mathbf{H}^t \mathbf{K}_u \mathbf{H} + \gamma \mathbf{H}^t \mathbf{Q} \mathbf{C}^{-1} \mathbf{Q}^t \mathbf{H}\} \mathbf{v} = \mathbf{H}^t \mathbf{f}_q + \mathbf{f}_v$$

where the matrix between brackets on the left hand side is the element stiffness matrix and the vector on the right hand side is the element load vector. The formal similitude with (31) is apparent.

3.5. DEVELOPED ELEMENTS

When elements based on the formulation presented above are to be developed, it seems appropriate to take as external or connecting degrees of freedom \mathbf{v} the movements of nodes located on the element boundary. Thus, if the nodal values are used to interpolate the field $\tilde{\mathbf{d}}$, this field will be the same along the common side of any two elements.

During the present work three Kirchhoff plate bending elements have been formulated (Fig. 13). As in the elasticity case, the internal movement modes have been divided into three categories:

$$(71) \quad \mathbf{N} = [\mathbf{N}_r \quad \mathbf{N}_m \quad \mathbf{N}_h]$$

where \mathbf{N}_r , \mathbf{N}_m and \mathbf{N}_h have the same meaning as before. In this case the value m (maximum order of the complete polynomials present in the interpolation of the deflections $\tilde{\omega}$) varies between 2 and 4, depending on the particular element. The partition of internal movement modes leads to a parallel partition of curvature modes:

$$(72) \quad \mathbf{B} = [\mathbf{0} \quad \mathbf{B}_m \quad \mathbf{B}_h]$$

The field \mathbf{M}^0 will be then:

$$(73) \quad \mathbf{M}^0 = \mathbf{D} \mathbf{B} \mathbf{q} = \mathbf{D} [\mathbf{B}_m \quad \mathbf{B}_h] \mathbf{q}$$

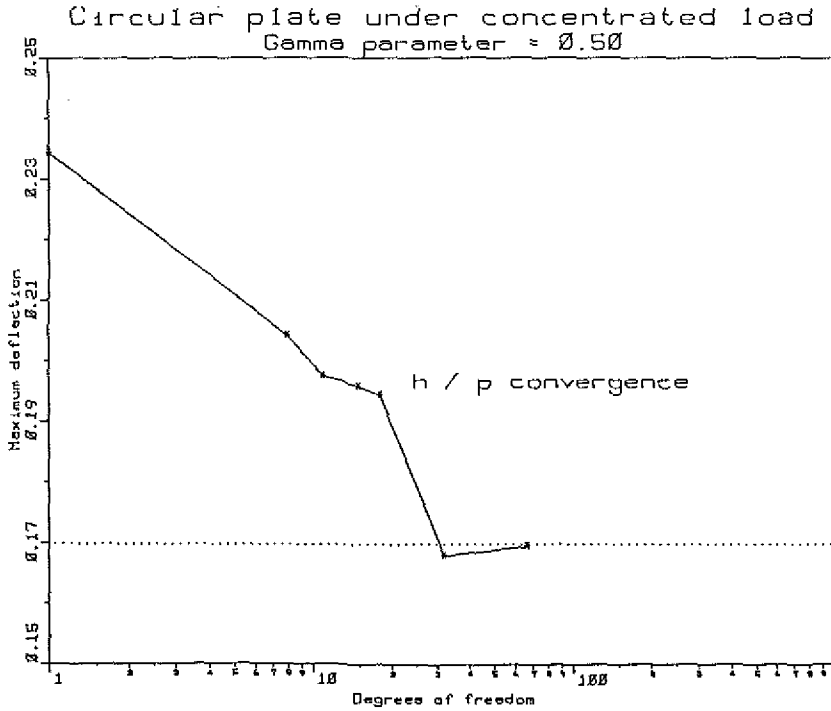


Fig. 16. - Example 2. Maximum deflection evolution.

For the independent moment modes it is formally taken:

$$(74) \quad A = DB_m$$

Two orthogonality conditions for the curvature modes are forced using a Gram-Schmidt process:

$$(75) \quad (B_m^t DB_m)_{P_j} \equiv \int_{P_j} B_m^t DB_m dP = \text{diagonal matrix, and:}$$

$$(76) \quad (B_m^t DB_h)_{P_j} \equiv \int_{P_j} B_m^t DB_h dP = 0$$

If (71) to (74) are then introduced into the element stiffness matrix given in (70), it turns out that the stiffness matrix is formally identical to the one derived for elasticity elements:

$$(77) \quad K = H_m^t K_{um} H_m + (1 - \gamma) H_h^t K_{uh} H_h \quad \text{with:}$$

$$H^t = [H_r^t H_m^t H_h^t], \quad K_{um} = (B_m^t DB_m)_{P_j}, \quad K_{uh} = (B_h^t DB_h)_{P_j}$$

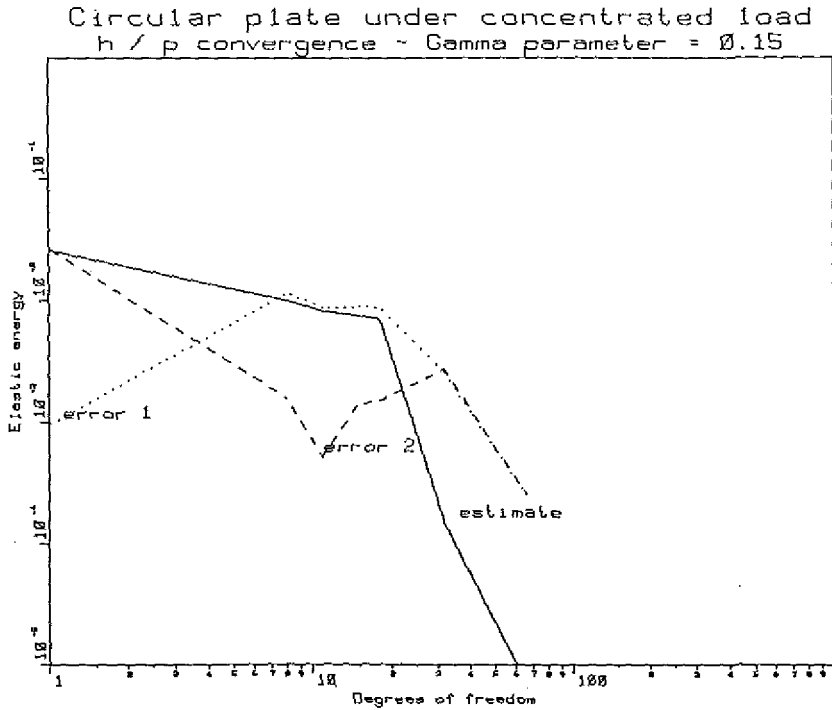


Fig. 17. - Example 2. Error estimate evolution (I).

The resultant expression of the proposed estimate is:

$$(78) \quad \epsilon = \sum_j \epsilon_j = \sum_j \frac{\gamma}{2} \mathbf{q}_h^t \mathbf{K}_{uh} \mathbf{q}_h$$

value that represents the elastic energy associated with the higher order curvature modes \mathbf{B}_h multiplied by the parameter γ .

3.6. NUMERICAL EXAMPLE

Figure 14 illustrates a clamped circular plate that is loaded with a 10 unit force at the center. The analytical solution of this problem is known [T & W, 1959] and it displays a singularity in the moments at the application point of the force. The symmetry allows to study only one quarter of the plate and characteristic values of the exact solution are the maximum deflection (0.1697 units) and the strain energy (0.2122 units for 1/4 of the plate).

The problem has been approached combining a selective h -adaptive process with a complete p refinement. The h process starts from a very coarse mesh (two curved triangles) and, using low order elements P9, produces a mesh adequate to start the p

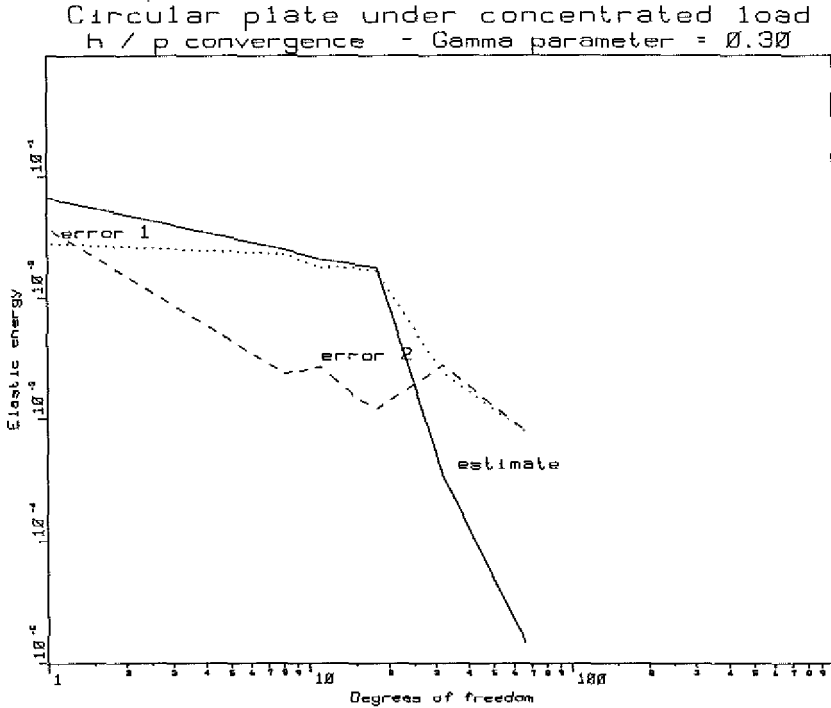


Fig. 18. - Example 2. Error estimate evolution (II).

refinement. This p refinement has two steps, the first with elements P12 and the second with elements P18.

Figure 15 shows the evolution of the mesh during the h refinement, including the values of the error densities, when the parameter γ is 0.50. The error densities (error indicators) “point” to the singularity at the center of the plate. For other values of γ the behavior of these indicators is very similar, they lead to the same results when they are used to “guide” the refinement.

The mesh with 18 degrees of freedom was considered appropriate to be the base mesh for the p refinement. Figure 16 shows the evolution of the maximum deflection during the $h-p$ process. Note the convergence acceleration due to the p refinement.

Figures 17 to 20 present the variation of the proposed error estimate in the $h-p$ process for different values of the parameter γ . The variation of the differences $|\Pi - \Pi_h^d|$ (error 1) and $|\Pi - \Pi_p^d|$ (error 2) is also represented to compare with the estimate. It is apparent in all the figures the strong acceleration of convergence when p refinement is introduced. The best behavior of the estimate occurs for high values of γ (0.50, 0.80), where it follows closely the true energy errors. However, it must be pointed out that the errors are lower for smaller values of γ , though their evolution is more unpredictable.

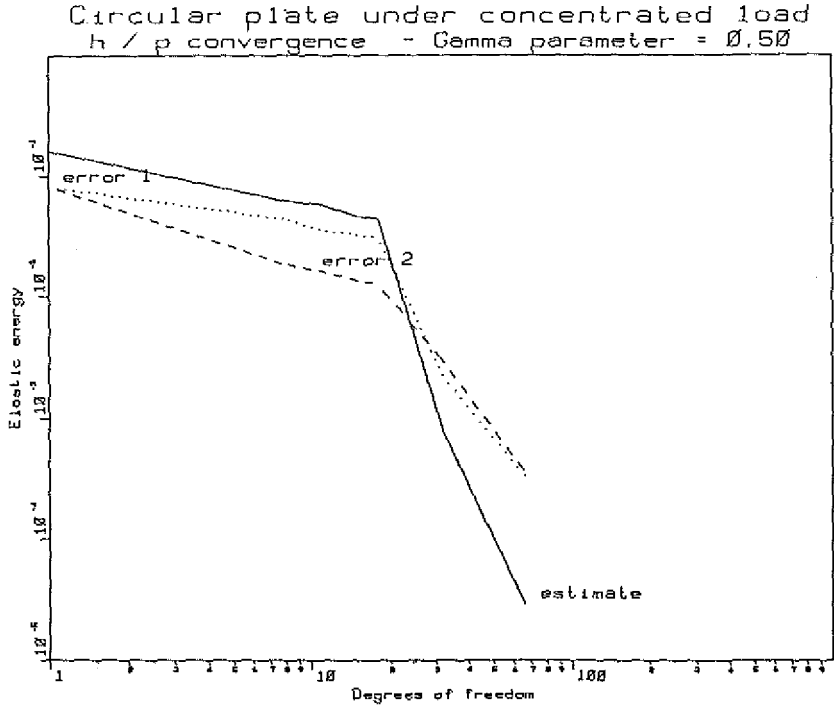


Fig. 19. - Example 2. Error estimate evolution (III).

4. Conclusions

Due to the great progress in computing hardware, the finite element method shows a tendency to become a "black box" from which novice users can take results without a clear understanding of the foundations of the numerical tool. At this stage, the introduction of reliable accuracy estimates is considered essential if the quality of approximate solutions is going to be controlled in a systematic and objective way, even by non expert users.

In the present paper the theoretical basis, numerical formulation and practical evaluation of a new category of error estimates for elasticity (C^0) and plate bending (C^1) problems have been presented. The estimates are computed as a difference between two multifield functionals, one of them parametrized. The main advantage of the new estimates is that they are computed element by element, and for each element only the information corresponding to that element is needed. This feature makes the new estimates very suited to parallel processing and avoids the difficulties associated to natural discontinuities (e. g. stress jumps) in the conventional error estimation procedures.

The observed numerical behavior of the proposed estimate seems to be acceptable from an engineering point of view. However, numerical experimentation is still needed to establish some practical tolerance levels for real problems.

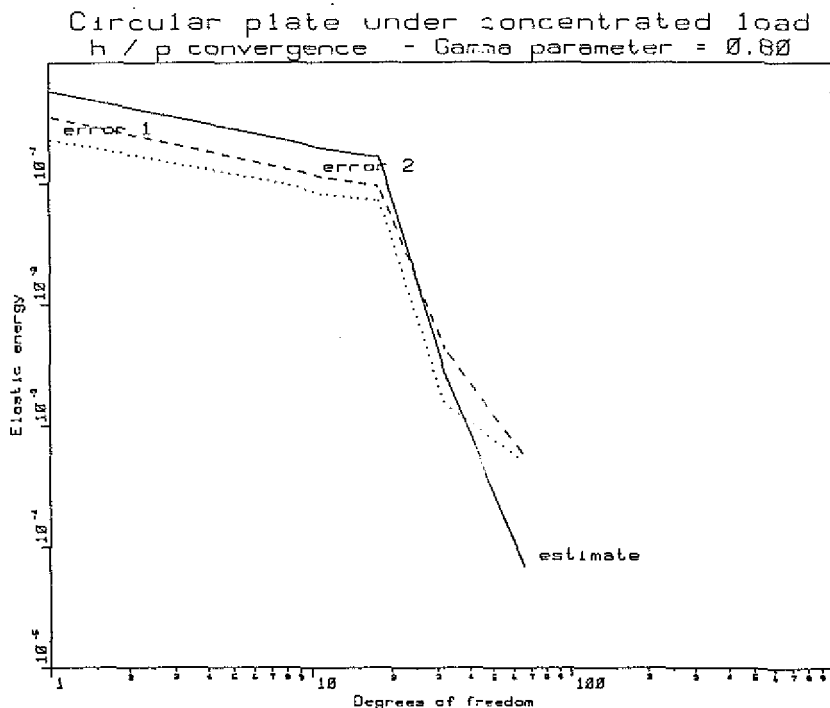


Fig. 20. — Example 2. Error estimate evolution (IV).

The idea on which the proposed estimates are based, *i.e.* the difference between two functionals that have the same value for the true solution, is quite general. Estimates of this kind could be obtained for any problem amenable to treatment by multifield functionals.

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